Statistical mechanics of systems of heterogeneous interacting agents

- Some empirical facts (=why bother?)
  - > Financial/Commodity markets, traffic, ...
- Models
  - > Minority Games
  - > (Phenomenological models)
- Theory (some key points)
- General framework

http://chimera.roma1.infn.it/ANDREA

Why bother?

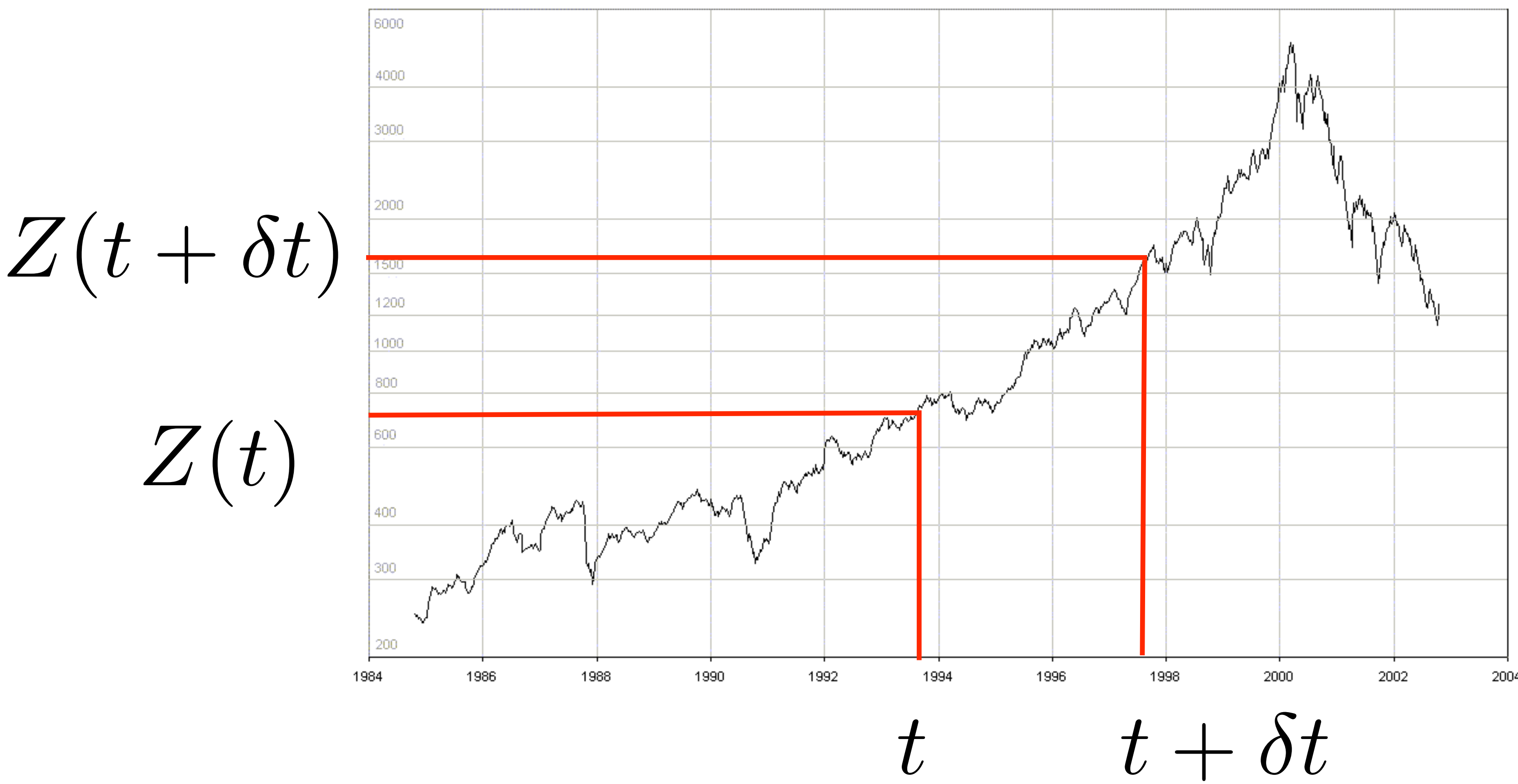
1) Understanding stylized facts of financial markets from a microscopic viewpoint

(ocean of data to search through)

Dow Jones Industrial Avg (1895-2003)

S&P Index (1927-2003)

Nasdaq composite (1985-2003)



$$r_{\delta t}(t) = \log Z(t+\delta t) - \log Z(t) \simeq \frac{Z(t+\delta t) - Z(t)}{Z(t)}$$

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$$\delta t = 10s, \ldots, 1 \text{ month}$$

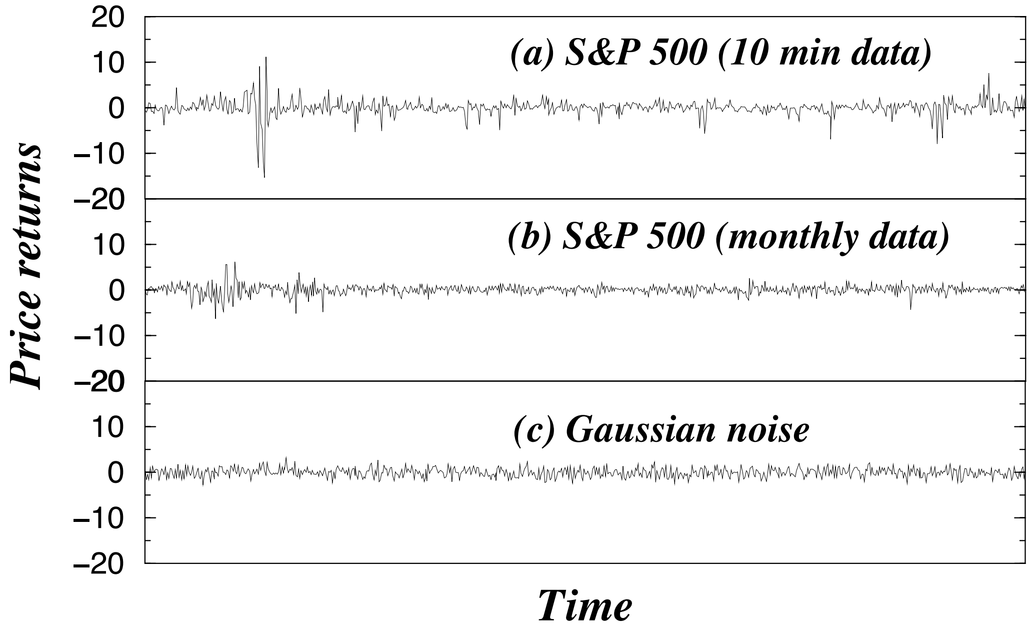
Question : how likely is it to observe a large fluctuation?

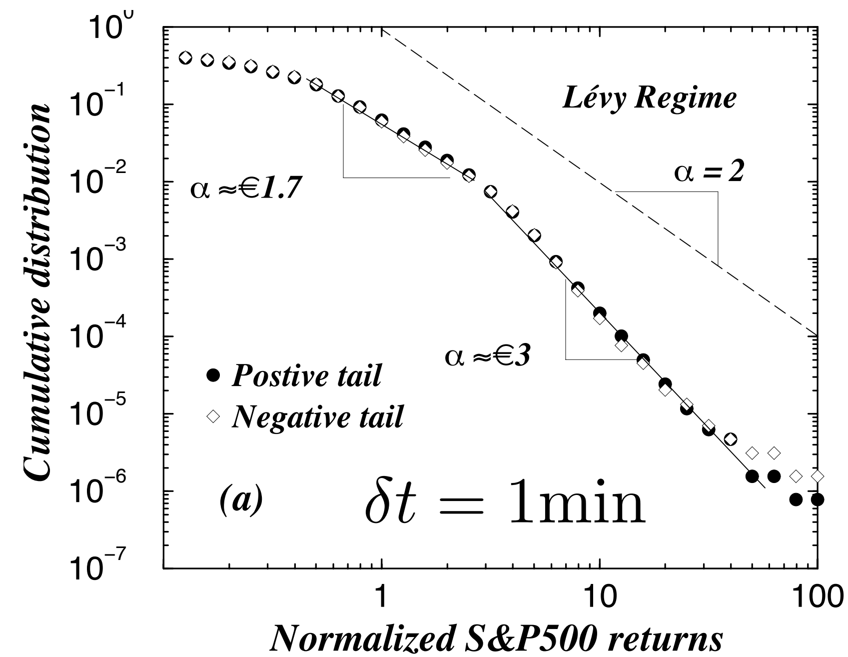
$$\delta t \text{ large } : \ P\{r(t) > r\} \propto e^{-r^2}$$

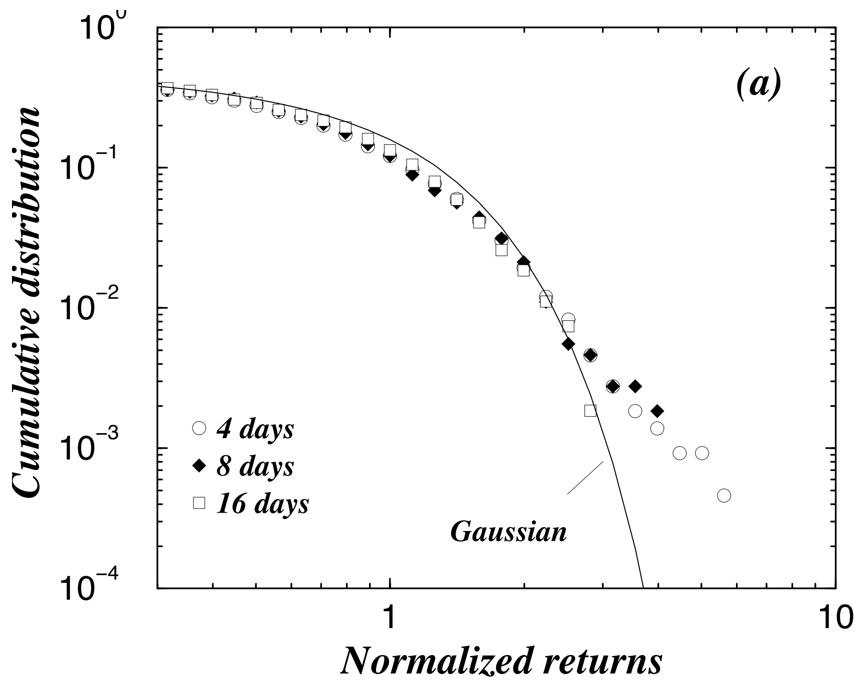
$$\delta t \text{ small } : \ P\{r(t) > r\} \sim r^{-\alpha} \qquad \alpha \simeq 3$$

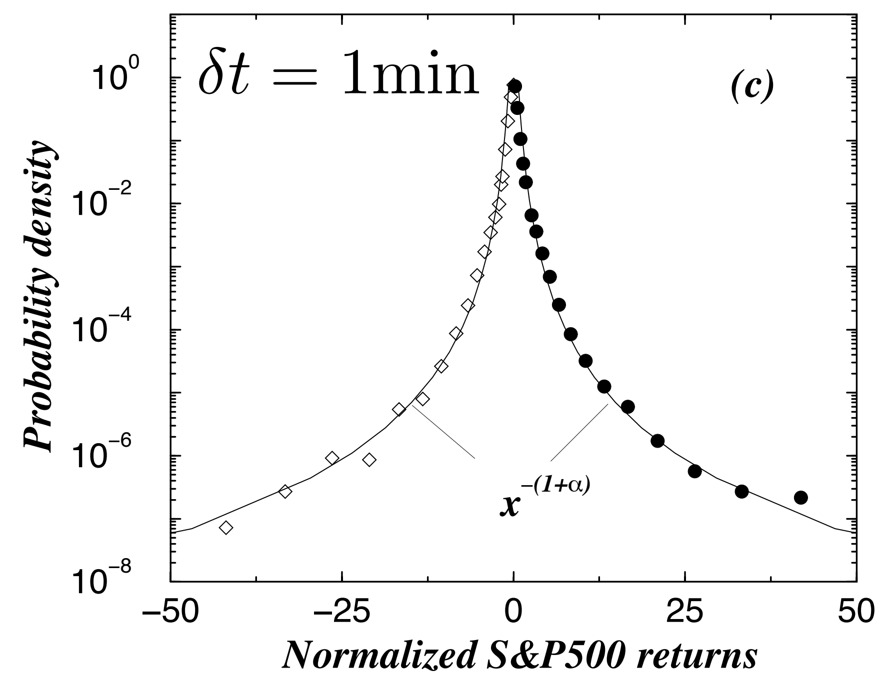
(similar for negative fluctuations)

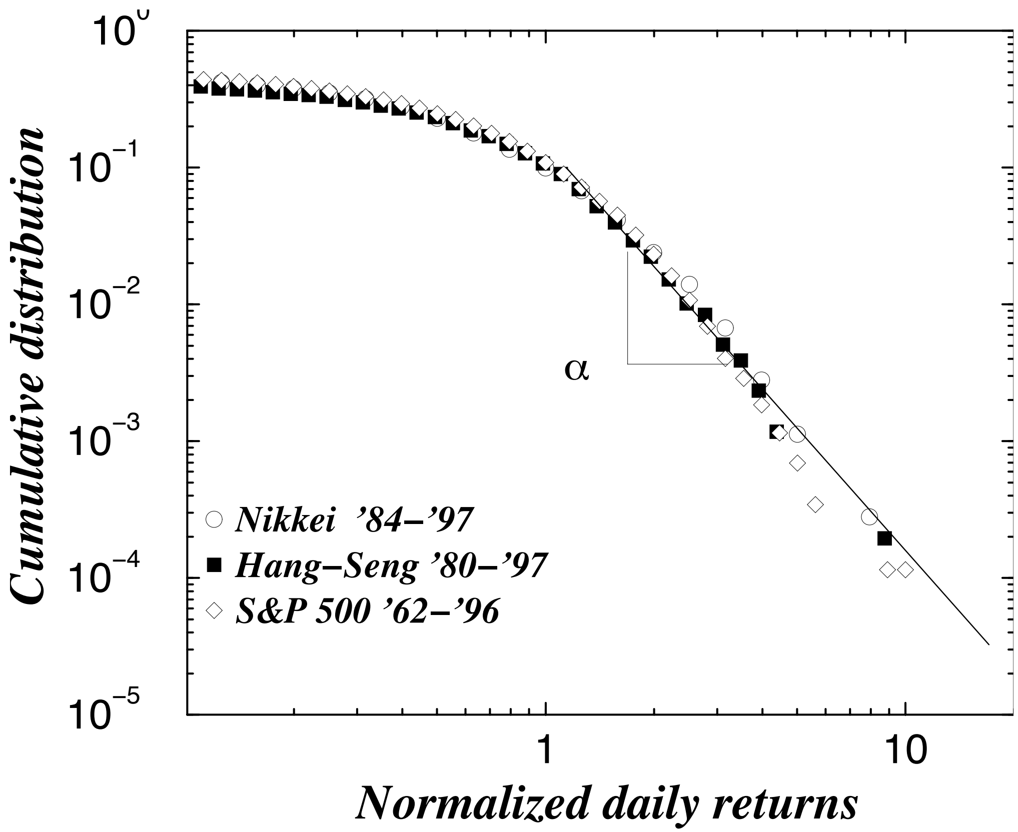
[Gopikrishnan et al. 1999]

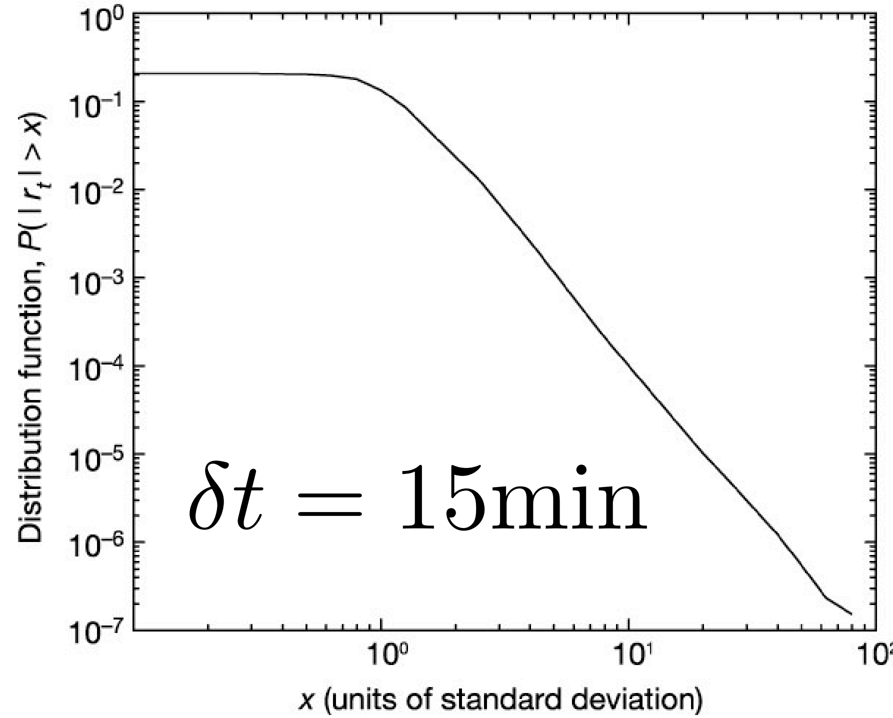








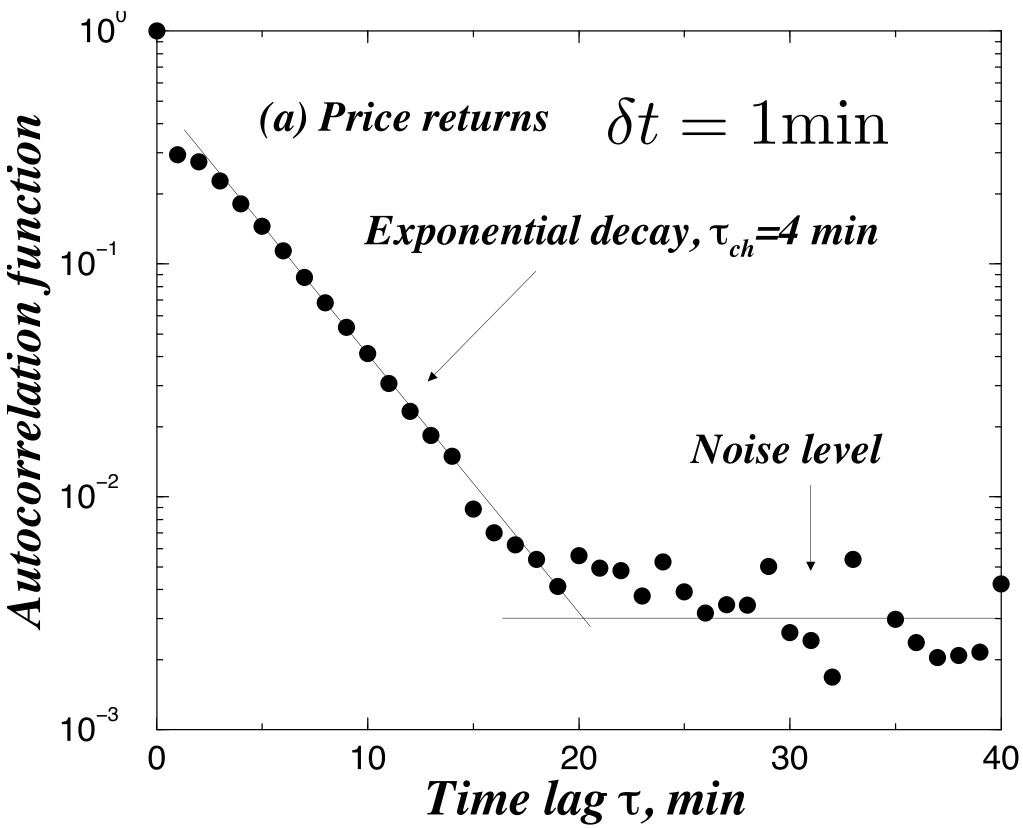


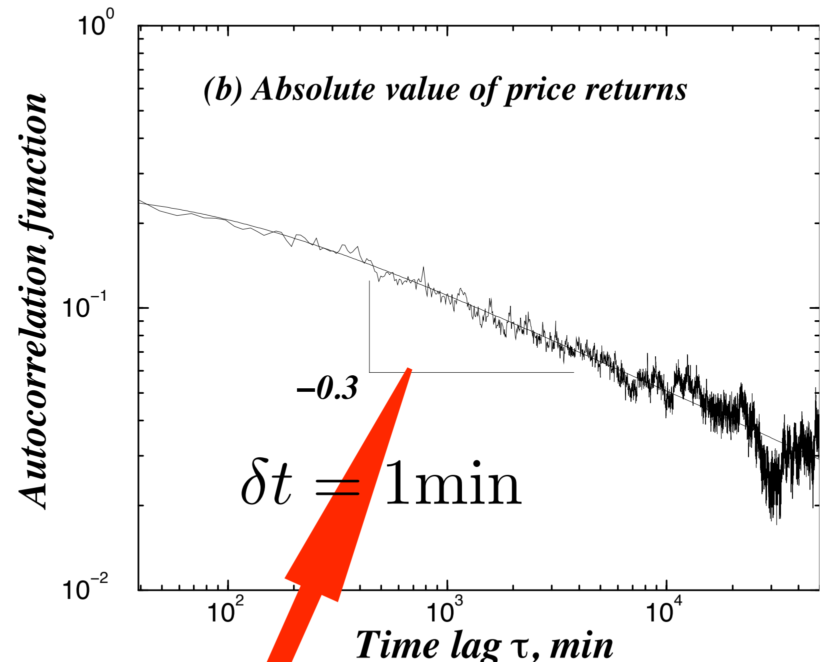


1000 largest stocks in TAQ

[Gopikrishnan et al. 1999]

Volatility clustering (intermittency)





$$\langle |r(t)||r(t+\tau)| \rangle_t \sim \tau^{-\gamma} \qquad \gamma \simeq 0.3$$

Individual stocks

Important differences between high- and low-volume stocks [Farmer-Lillo]

Order book

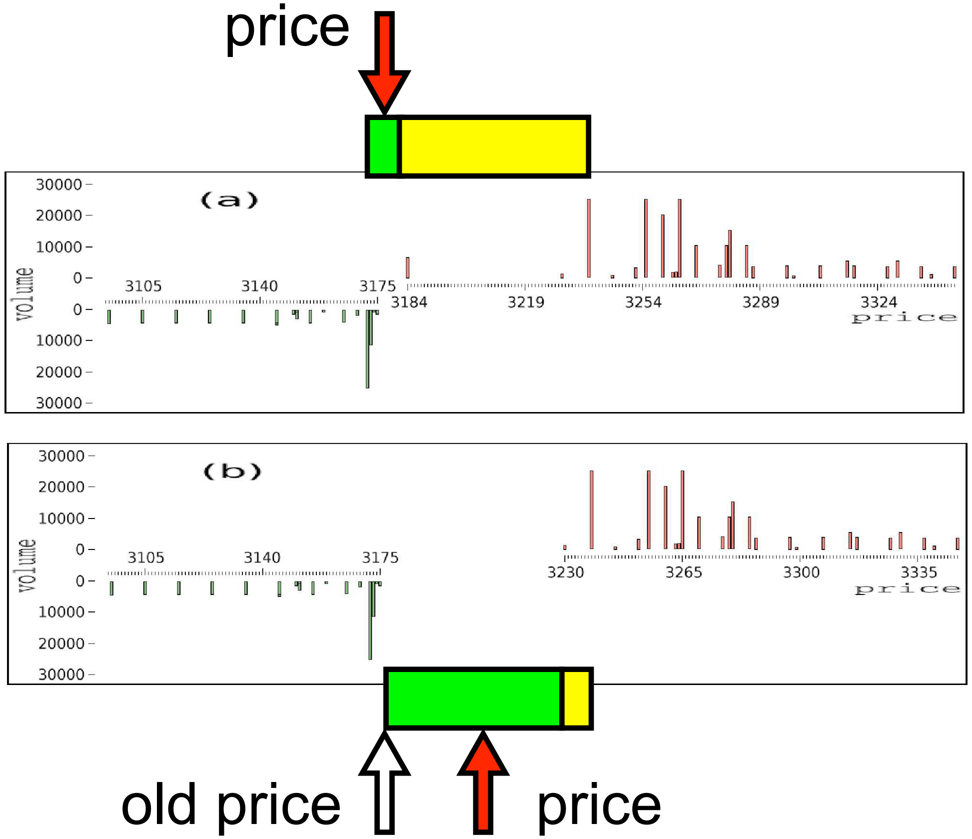
Continuous Double Auction

study order book dynamics

More stlylized facts?

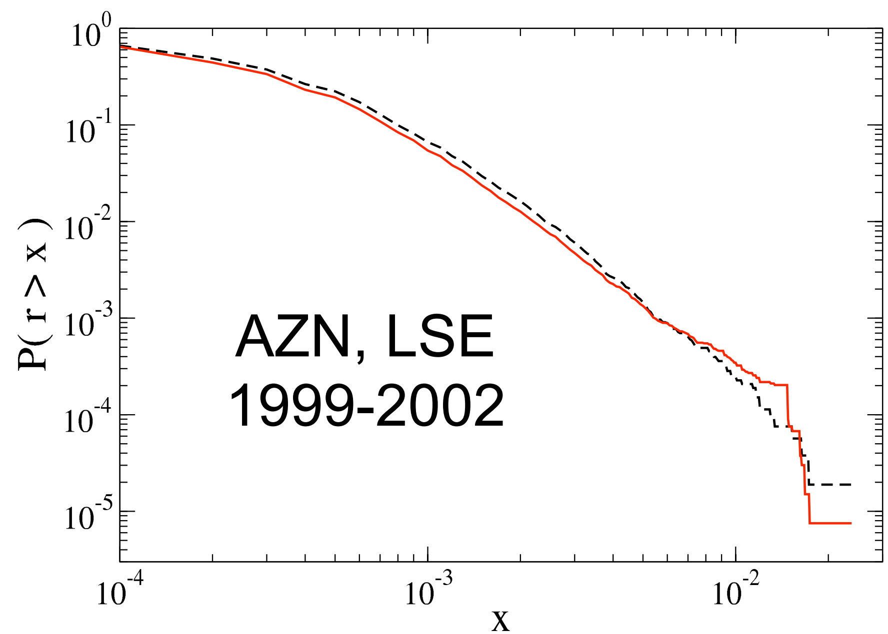
buy/sell difference

first gap



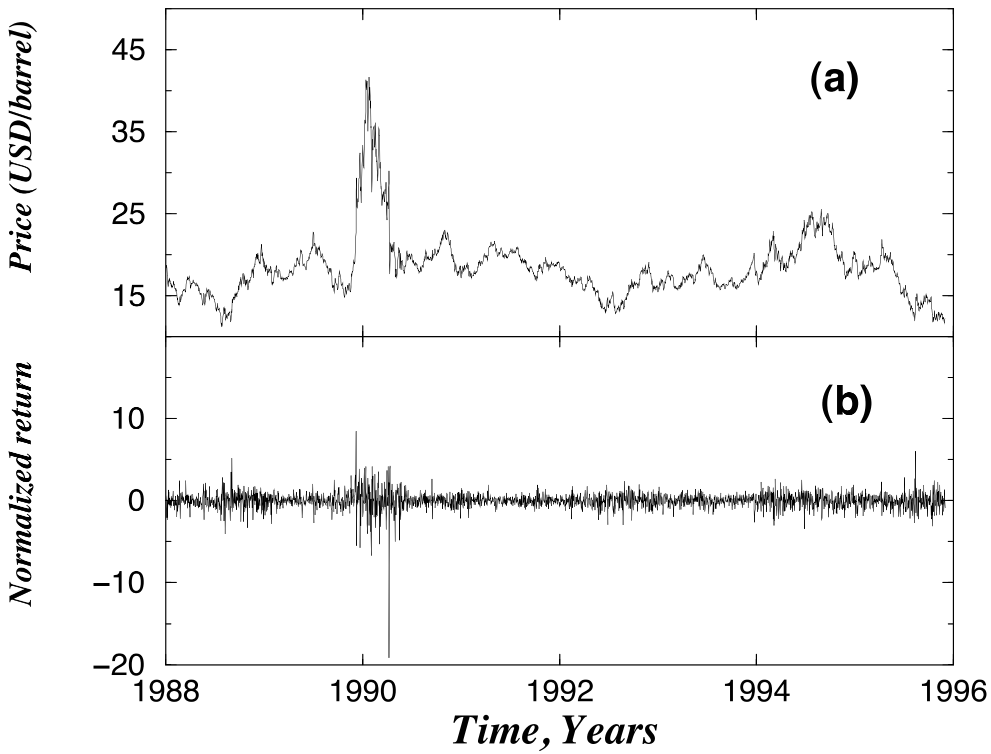
Large price variation =
change of the first gap

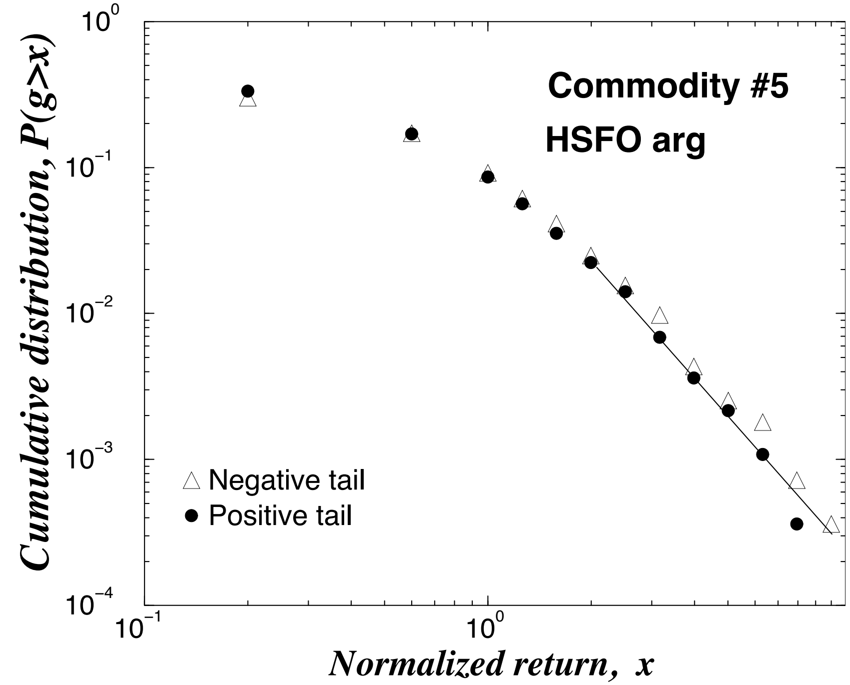
[Farmer et al 2004]

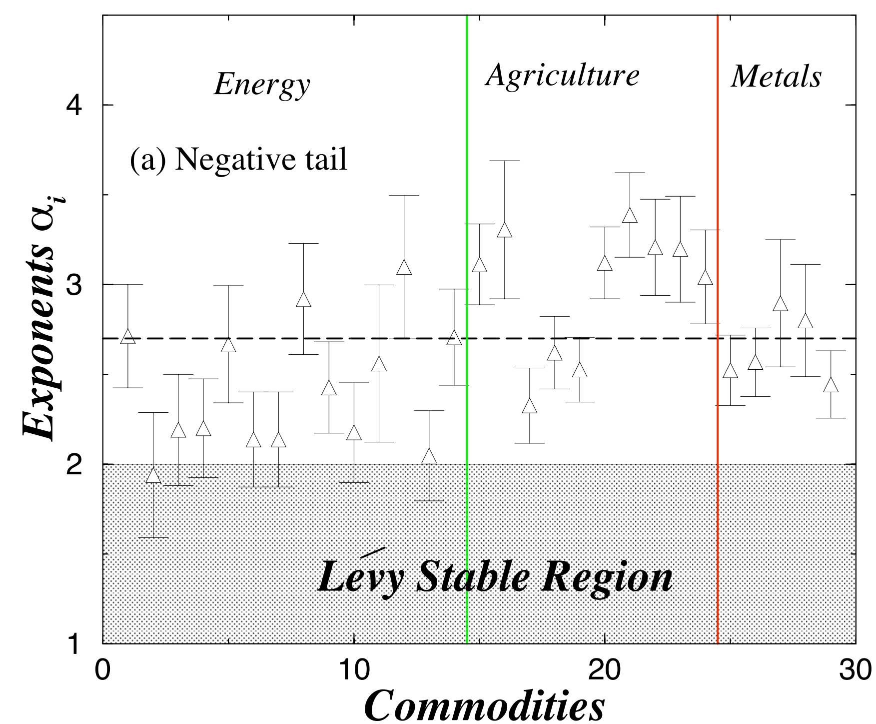


Very detailed models
are required

Commodities







[Matia et al 2002]

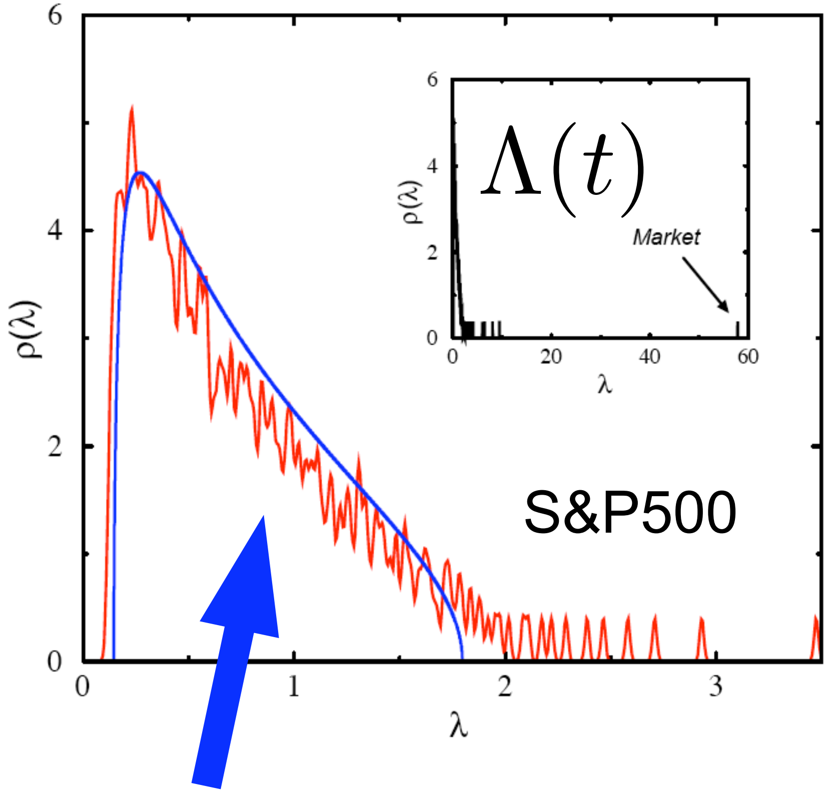
Why bother?

2) Origin of the correlation structure in financial markets (difficult)

Object to look at : covariance matrix of asset returns at e.g. daily scale

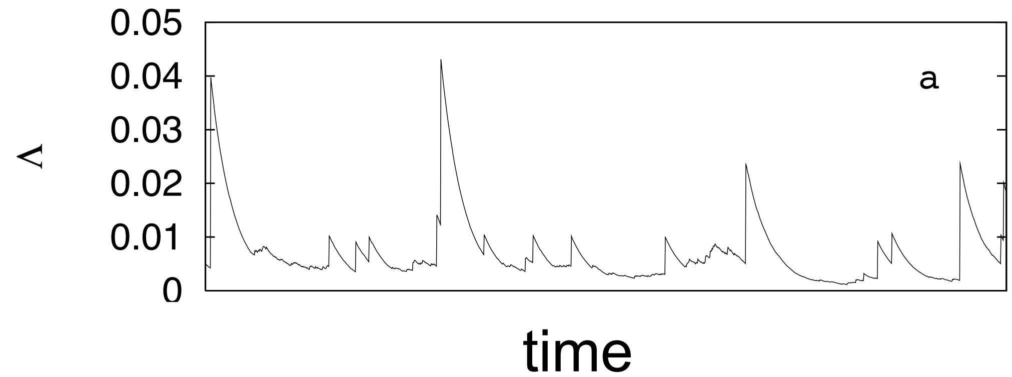
$$C_{ij} = \frac{1}{T}\sum_{t=1}^{T} r_i(t) r_j(t)$$

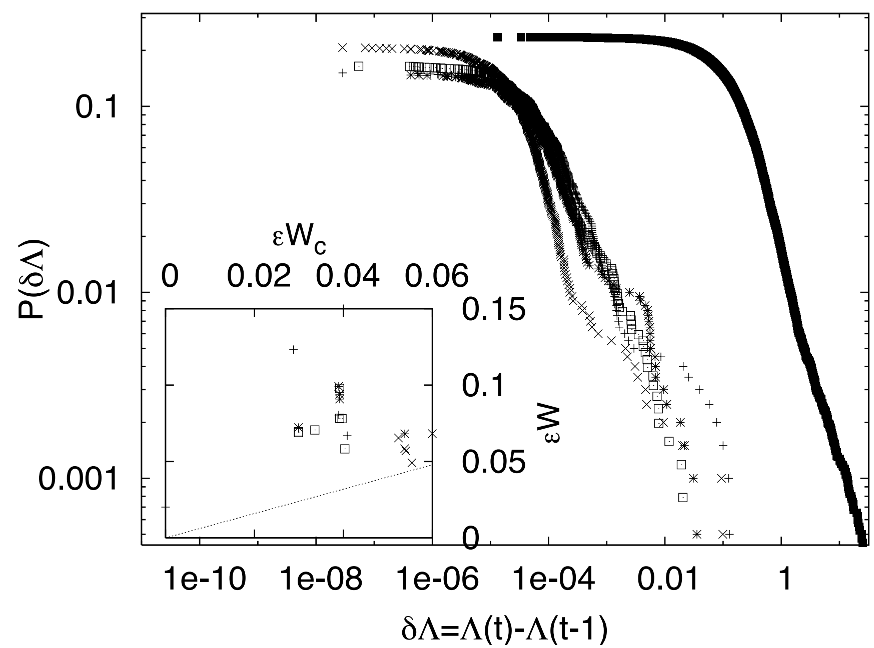
[Potters et al.]



Marcenko-Pastur spectrum

$$r_i(t) = \beta_i \Lambda(t) + \eta_i(t)$$





Market mode dynamics
[Raffaelli-Marsili]

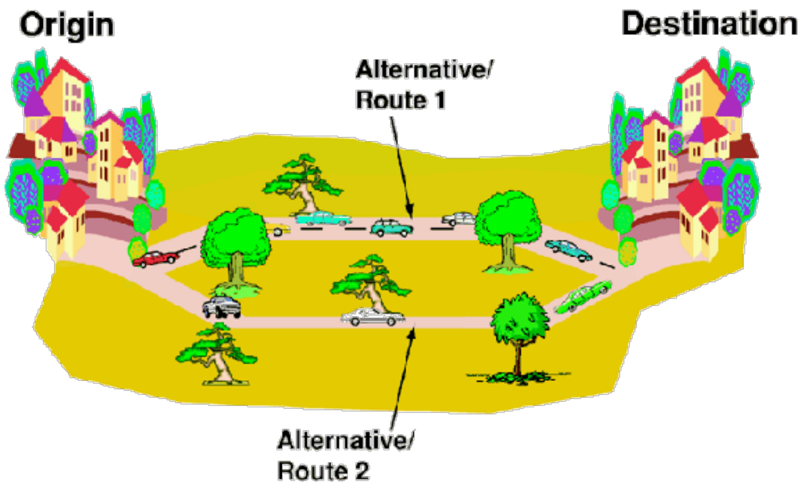
Why bother?

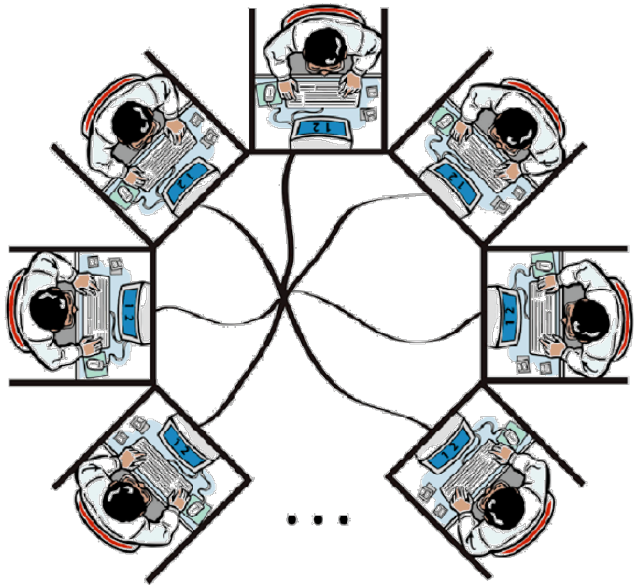
3) Understanding over-reaction

very few data from finance, but...

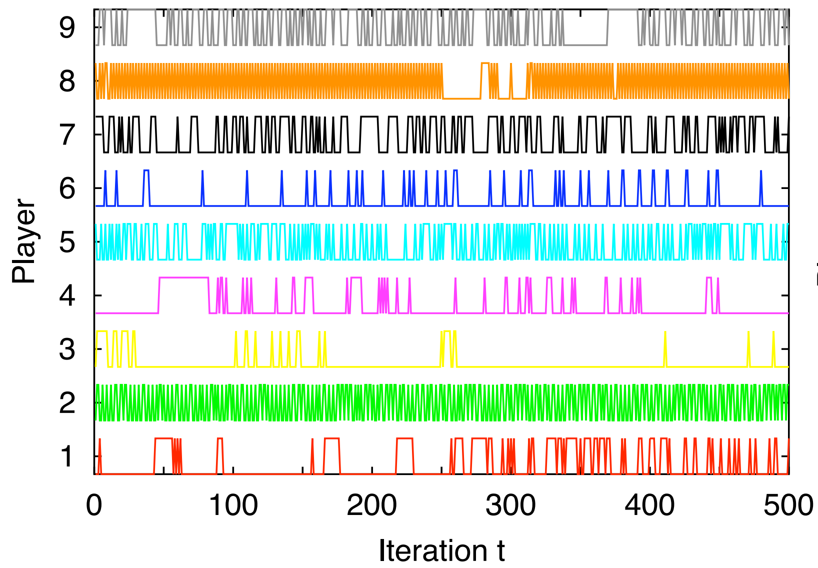
Experiments on humans

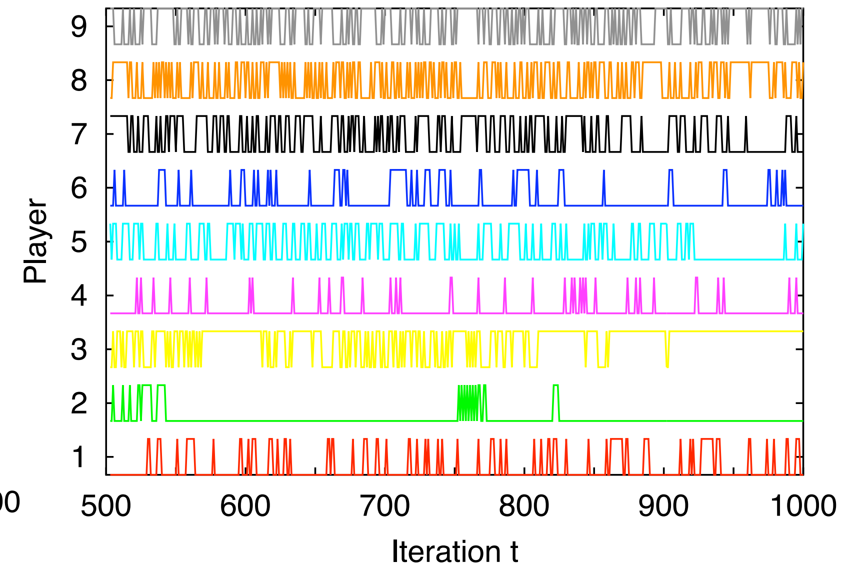
[Helbing et al. 2002] [Selten+8 2004]

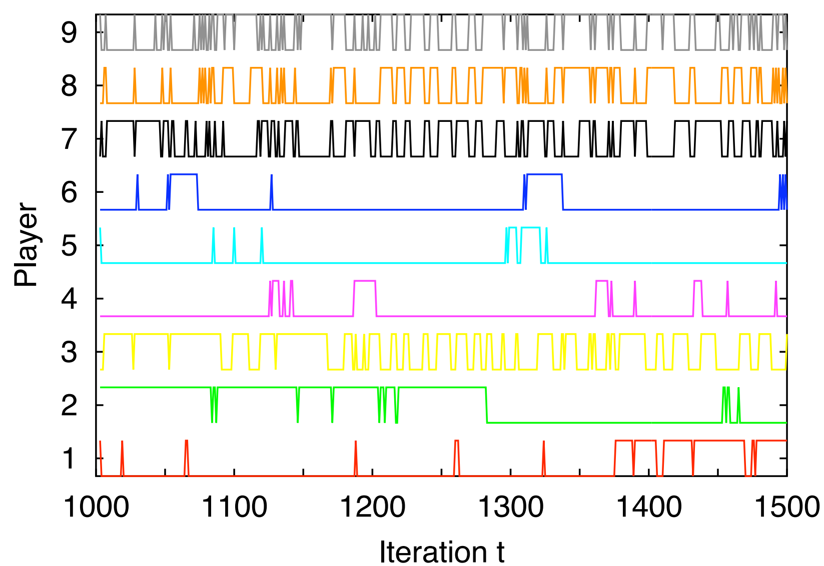




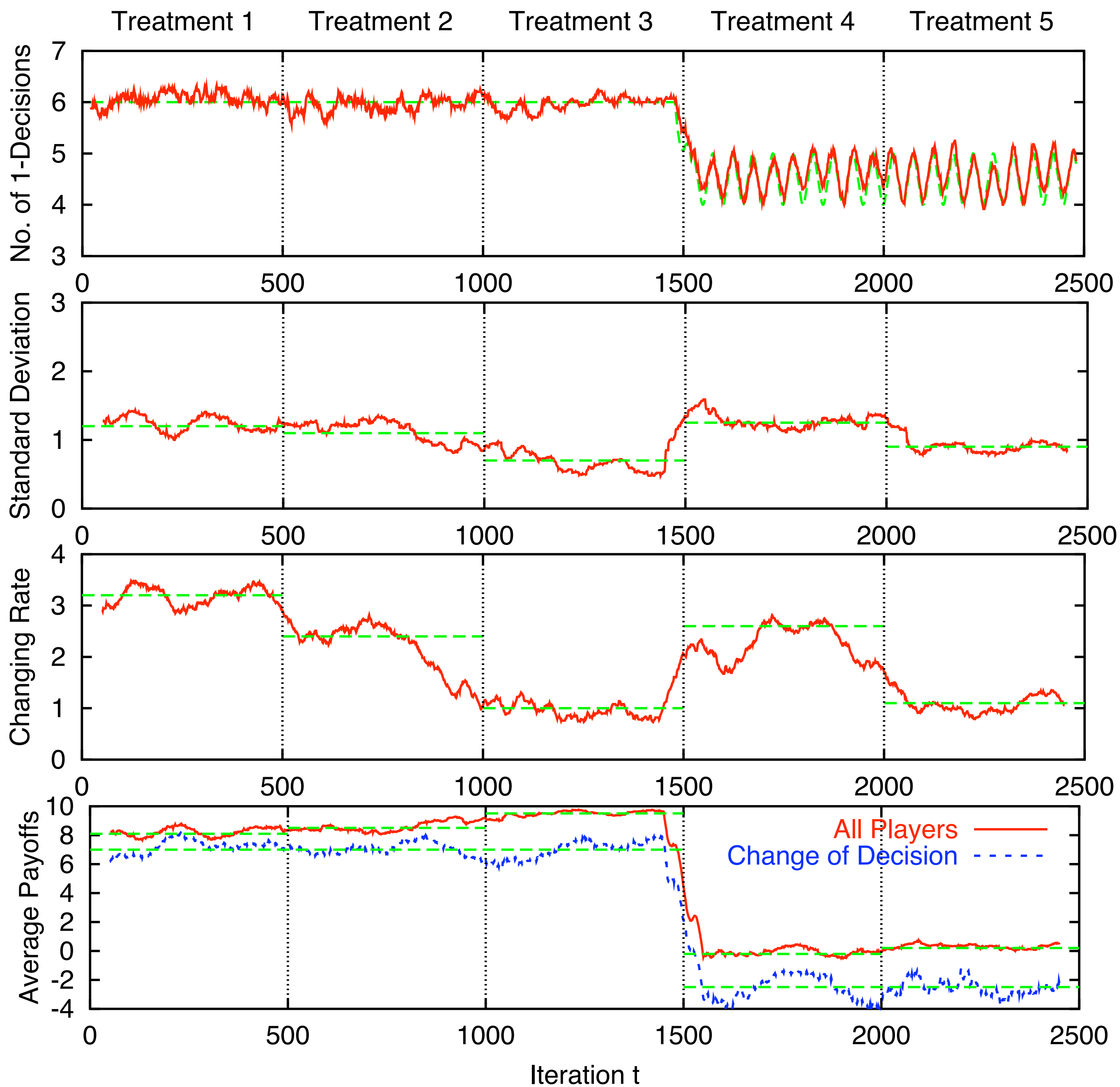
$$P_1(n_1) = P_1^0 - P_1^1 n_1$$
$$P_2(n_2) = P_2^0 - P_2^1 n_2 \quad \Rightarrow \quad f_1^{eq} = \frac{P_2^1}{P_1^1 + P_2^1} + \frac{1}{N}\frac{P_1^0 - P_2^0}{P_1^1 + P_2^1}$$







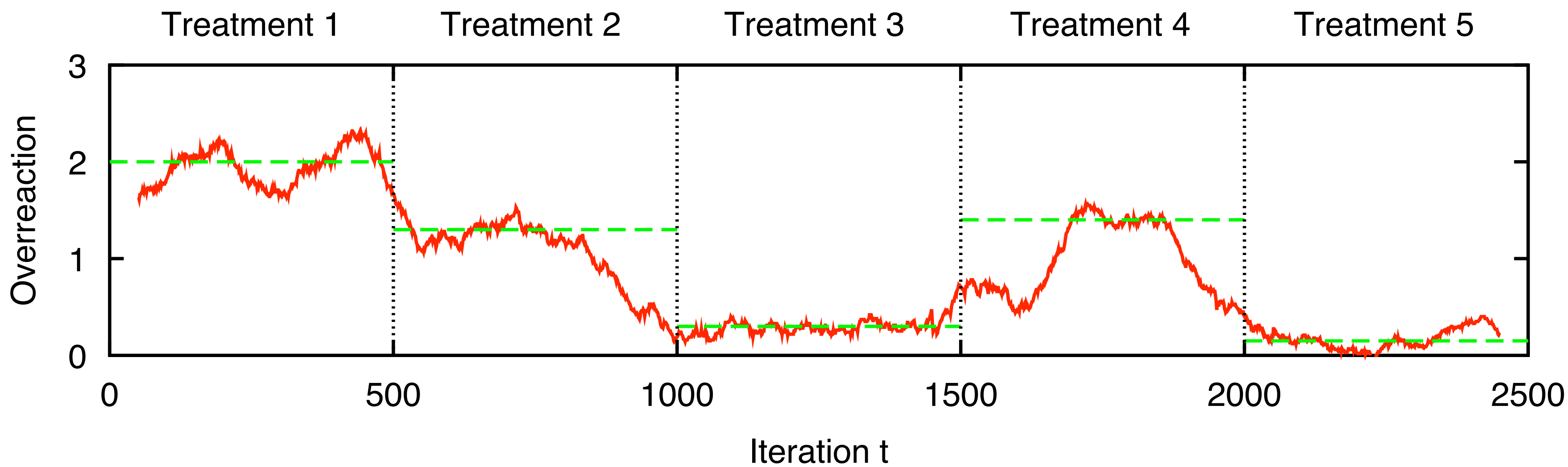
Human specimens adapt well to equilibrium but fluctuations persisted under several different information schemes



Treatment 1
Treatment 2
Treatment 3
Treatment 4
Treatment 5
No. of 1-Decisions
Standard Deviation
Changing Rate
Average Payoffs
All Players
Change of Decision
Iteration t

Over-reaction

i.e. difference between the actual changing rate and the required one, namely the standard deviation from equilibrium



Just their own payoff

Both payoffs

With impact correction

Same but more complicated

User-specific

Modeling : key ingredients

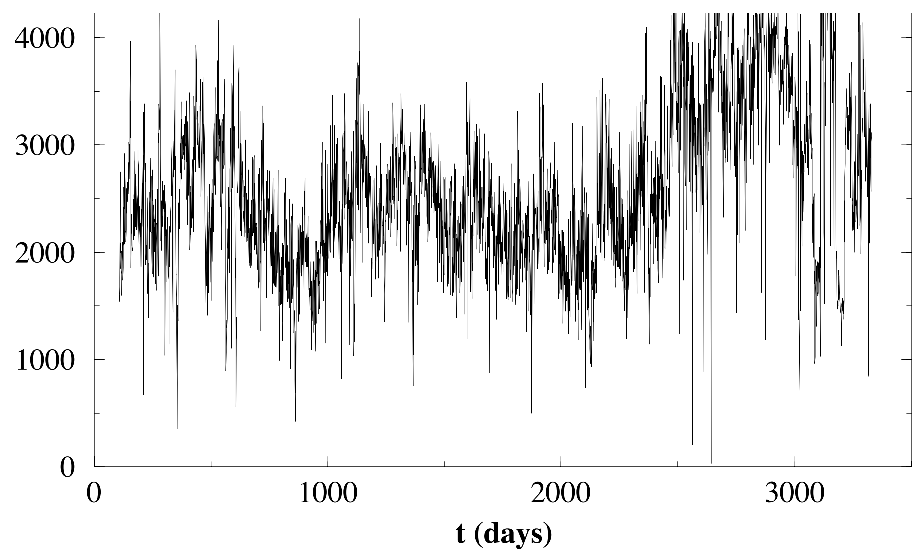
1) Agents react to the receipt of information

2) Heterogeneity : they react differently from each other

3) Frustration : not everybody can win

Volume (nr of trades)

S&P500 futures 1985-1998



[Bouchaud et al 2001]

4) Volume (=number of agents) fluctuates in time

Frustration : the minority mechanism

[Challet-Zhang 1997]

N people must choose between two alternatives
(e.g. buy/sell), the minority group wins

(cfr law of supply and demand)

realistic whenever agents compete for a scarce resource
which may be distributed or not
(e.g. money, space, food, CPU time, ...)

Information : $\mu \in \{1, \ldots, P\}$ public (same for everybody)

$$t \to \mu(t) \in \{1, \ldots, P\} \qquad \text{random(uniform)}$$

Heterogeneity : agent i reacts to information in a prescribed fixed way (`strategy')

$$\mu \to a_i^\mu \in \{-1, 1\} \quad \text{random (uniform, quenched)}$$

Minority rule : payoff to i is $\pi_i(t) = -a_i^{\mu(t)} A(t)$, $A(t) = \sum_j a_j^{\mu(t)}$

Volume fluctuations : at each time step agent i can either respond to the information (play) or not

Note : no price + agents are fully connected

(Grand-canonical) Minority Game

N agents, P possible information patterns, discrete time t=0,1,...

Each agent has a *strategy* and a *score function* : $\{a_i^\mu\}_{\mu=1}^P \ , \ U_i(t)$

Important assumption : $N = nP \ (N \to \infty)$

$$t \to \mu(t) \in \{1, \ldots, P\} \qquad \text{random(uniform)}$$

$$\phi_i(t) = \Theta[U_i(t)] = \begin{cases} 1 & \text{if } U_i(t) > 0 \ (i \text{ active}) \\ 0 & \text{otherwise } (i \text{ inactive}) \end{cases}$$

Update scores : $U_i(t+1) - U_i(t) = -a_i^{\mu(t)} A(t) - \epsilon$

$$A(t) = \sum_j \phi_j(t) a_j^{\mu(t)}$$

Initial condition : $U_i(0)$

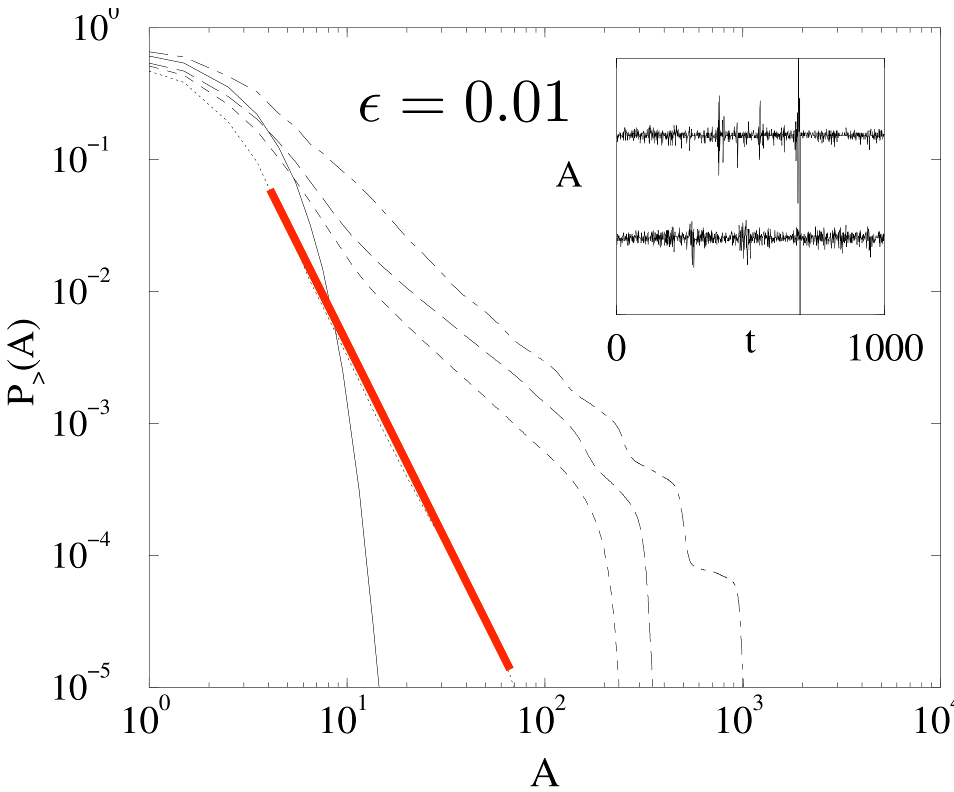
Parameters : $\epsilon \ , \ n$

$P\{A(t) > A\}$ Gaussian for small n but ...

Simulations

(finite size : N<1000)

$$\epsilon \in [-0.1, 0.1]$$



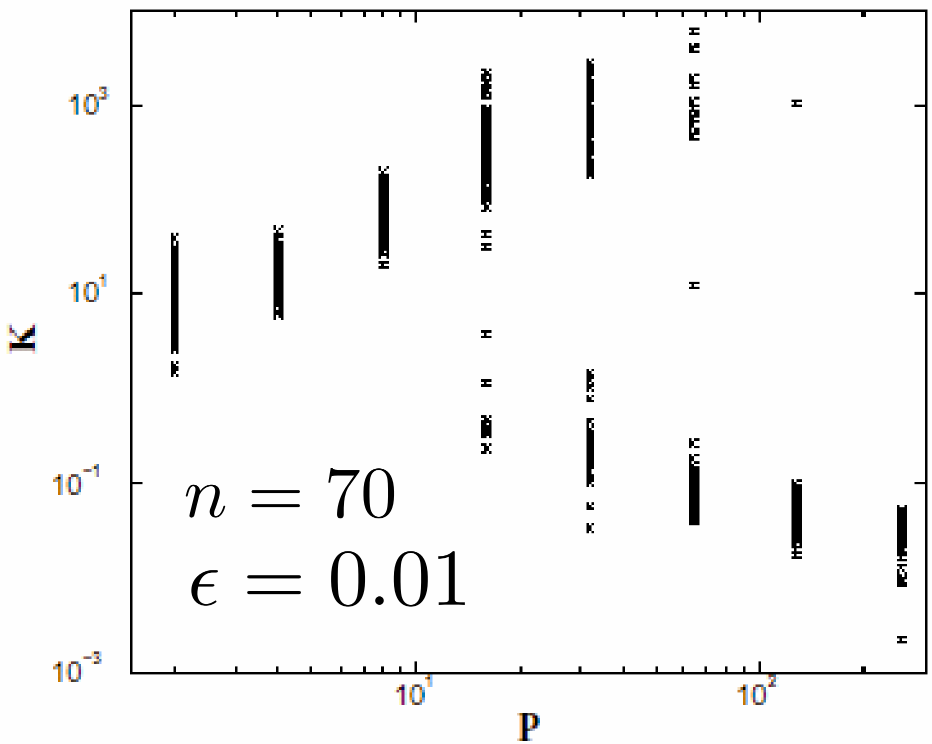
$$n = N/P$$

$$P\{A(t) > A\} \simeq A^{-\gamma}$$

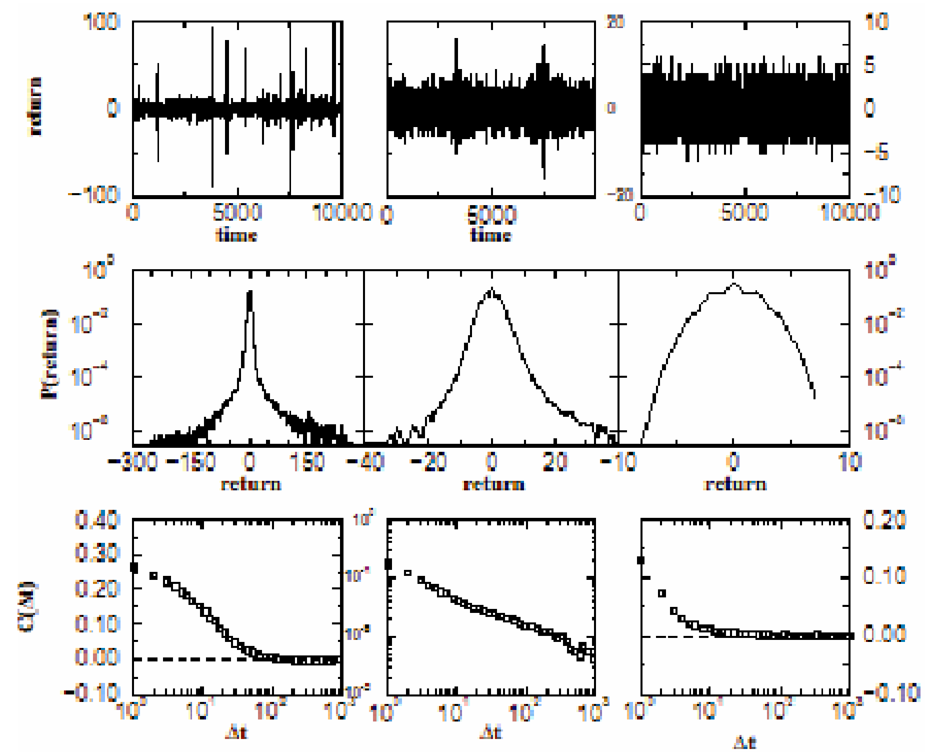
$$\gamma \simeq 2.8 \text{ for } n = 20$$

A closer look : finite size effects, sample dependence

increase size (P or N=nP)



3 different samples

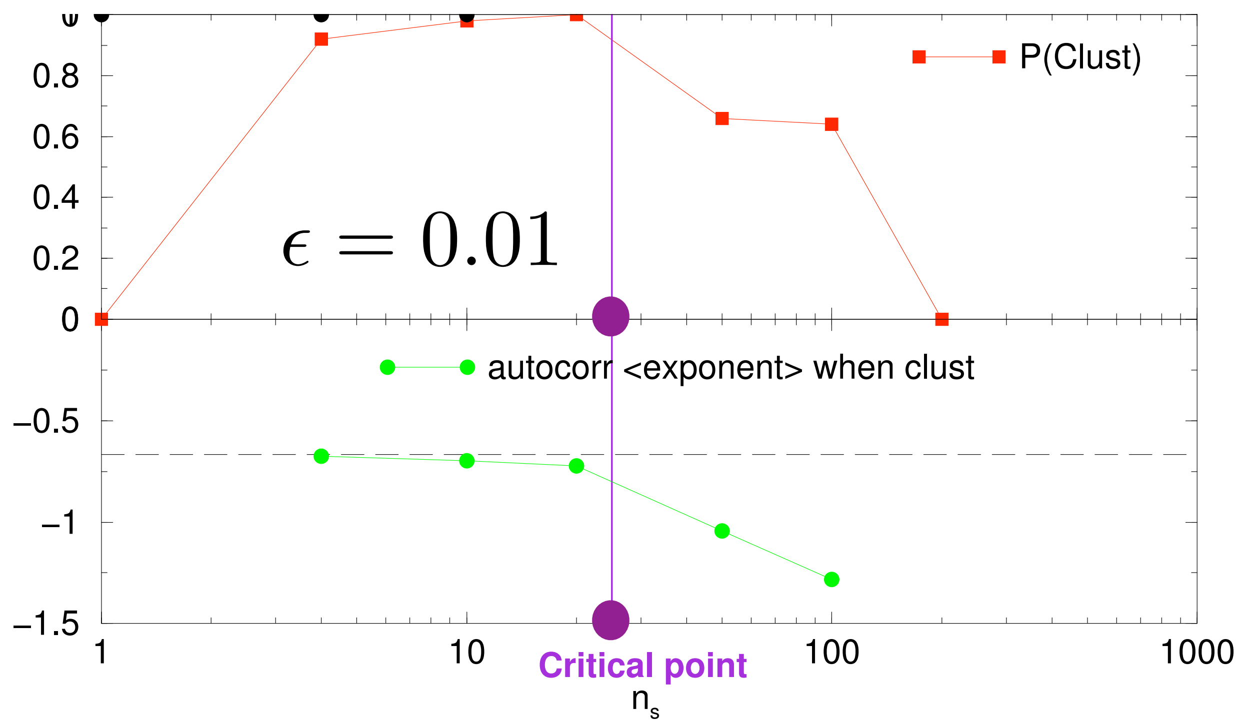


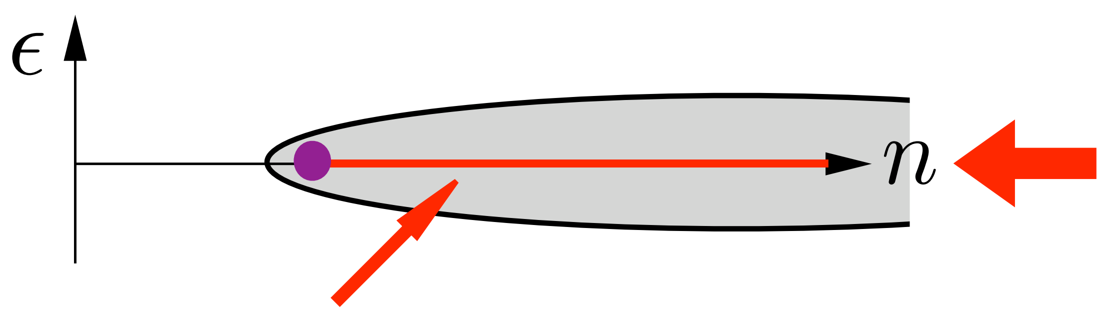
$$K = \frac{\langle A^4 \rangle}{\langle A^2 \rangle^2} - 3$$

"kurtosis excess" (K=0 means Gaussian)

What is so special about those values of n?

They are close to a theoretical critical point $(N \to \infty)$





Critical line

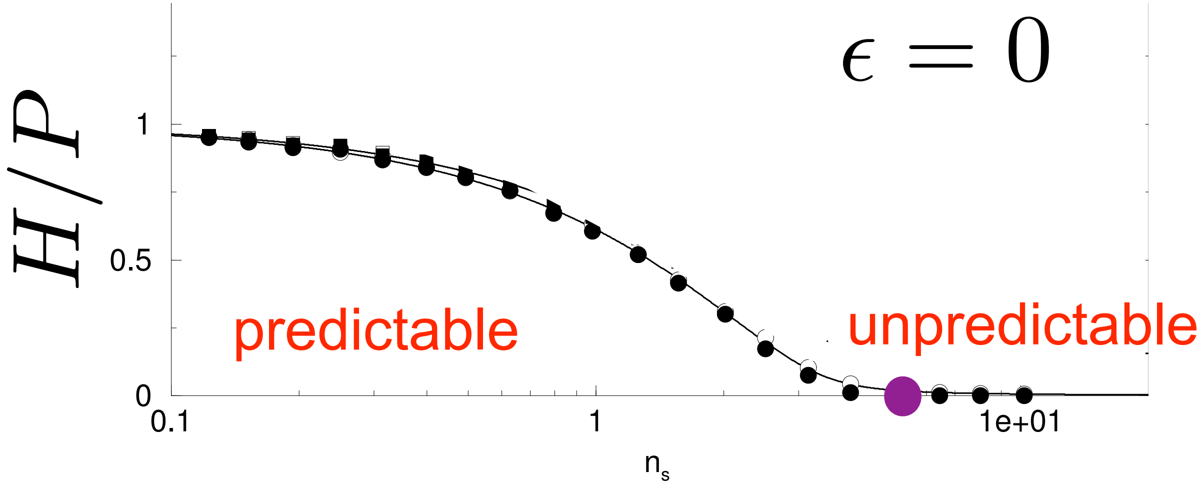
"Window" where stylized facts are observed shrinks as N increases

What are the phases here?

$$\langle A|\mu\rangle \begin{cases} = 0 & A(t) \text{ stat. unpredictable based on } \mu \\ \neq 0 & A(t) \text{ stat. predictable based on } \mu \end{cases}$$

Order parameter

$$H = \frac{1}{P}\sum_{\mu}\langle A|\mu\rangle^2$$

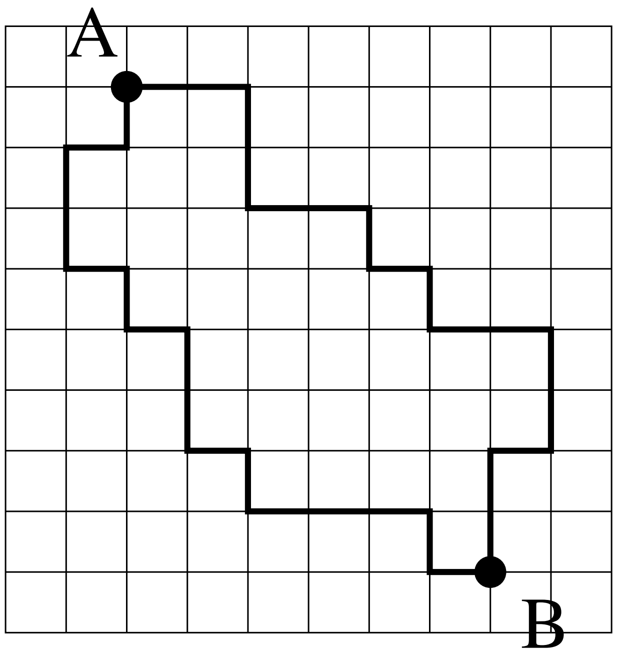


Do real markets operate close to a critical point?

Modeling the traffic experiment = MG with resources distributed on a lattice

$$\{\mu\} = \text{edges}$$

(streets)



Each driver must go from A to B every day
(A and B depend on the driver)

strategies = different possible routes to go from A to B

$$q_{ig}^{\mu} \in \{0,1\} \quad (i \text{ doesn't pass/passes through } \mu \text{ in route } g)$$

N drivers, P streets, discrete time n=0,1,...

Each driver has S *strategies* and S *score functions* : $\{q^\mu_{ig}\}\ ,\ U_{ig}(n)$

Important assumption : $N = cP\ (c = \mathrm{car\ density})$

$$\widetilde{g}_i(n) = \arg\max_g U_{ig}(n)$$

$$U_{ig}(n+1) = U_{ig}(n) - \frac{1}{P}\sum_{\mu=1}^{P} q^\mu_{ig} Q^\mu(n) + \frac{1}{2}(1-\delta_{g,\widetilde{g}_i(n)})\zeta_{ig}(n)$$

Traffic load on μ
(Minority term)

$$\boxed{Q^\mu(n) = \sum_{i=1}^{N} q^\mu_{i,\widetilde{g}_i(n)}}$$

Information noise
(only for $g \neq \widetilde{g}_i(n)$)

Information noise : describes different `treatments'

$$\langle \zeta_{ig}(n) \rangle = \eta$$
$$\langle \zeta_{ig}(n)\zeta_{jh}(m) \rangle = \Delta \delta_{ij}\delta_{gh}\delta_{nm}$$

$$U_{ig}(n+1) = U_{ig}(n) - \frac{1}{P}\sum_{\mu=1}^{P} q_{ig}^{\mu} Q^{\mu}(n) + \frac{1}{2}(1-\delta_{g,\widetilde{g}_i(n)})\zeta_{ig}(n)$$

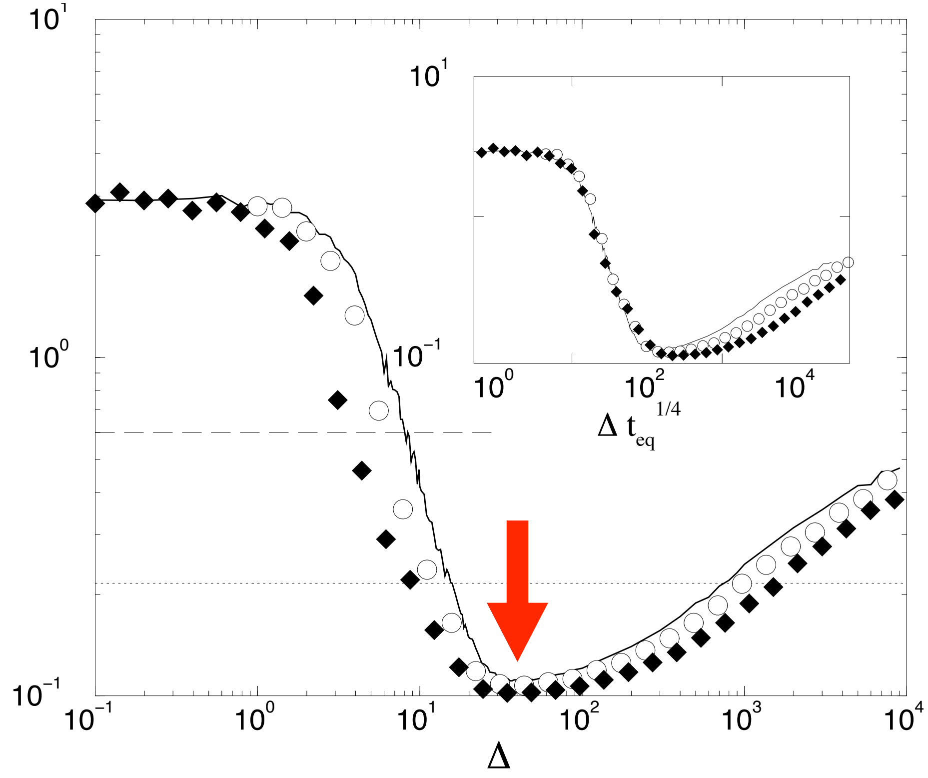
$\eta = 0$: unbiased information
$\eta > 0$: overestimates unused routes
$\eta < 0$: underestimates unused routes

$\Delta = 0$: same information (biased or not) for all drivers
$\Delta > 0$: user-specific information

Relevant quantity : traffic load fluctuations

$$\sigma^2 = \frac{1}{P}\sum_\mu [\langle (Q^\mu)^2\rangle - \langle Q^\mu\rangle^2]$$

$$\eta = 0$$

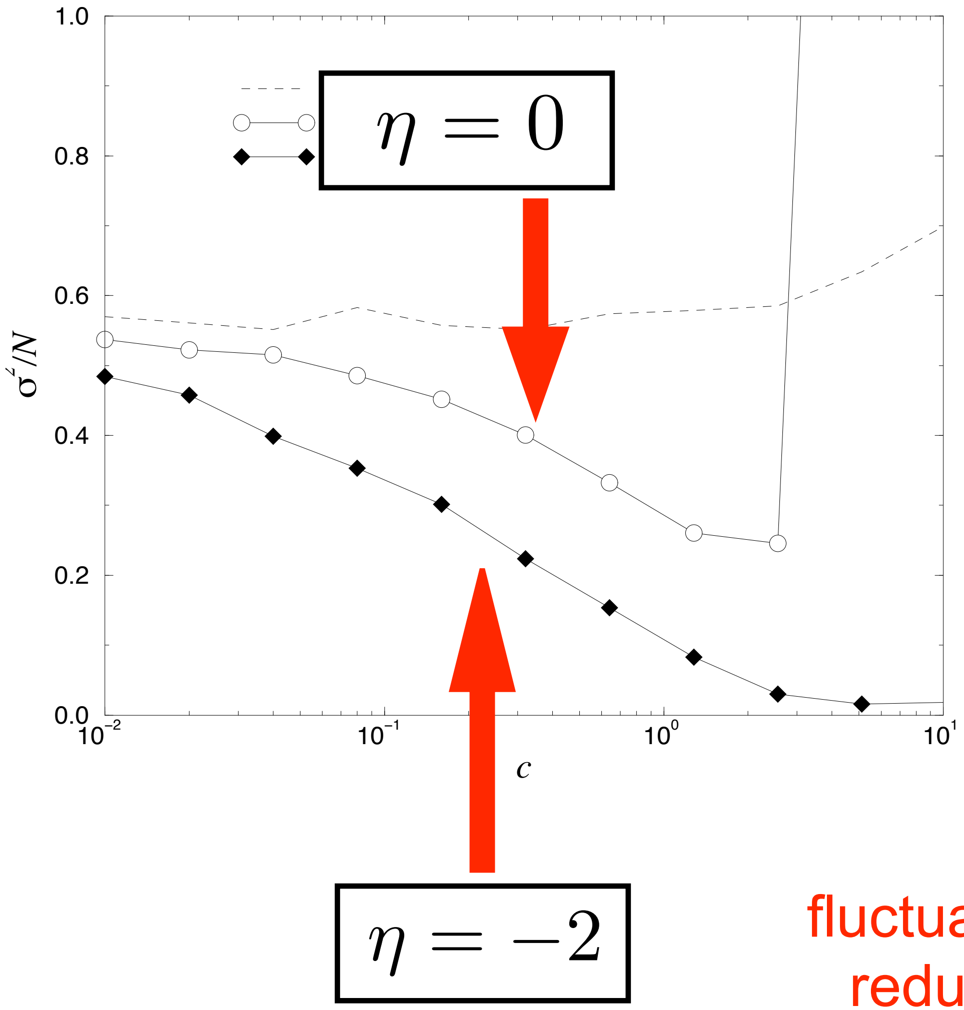


high car density
(c=N/P>3)

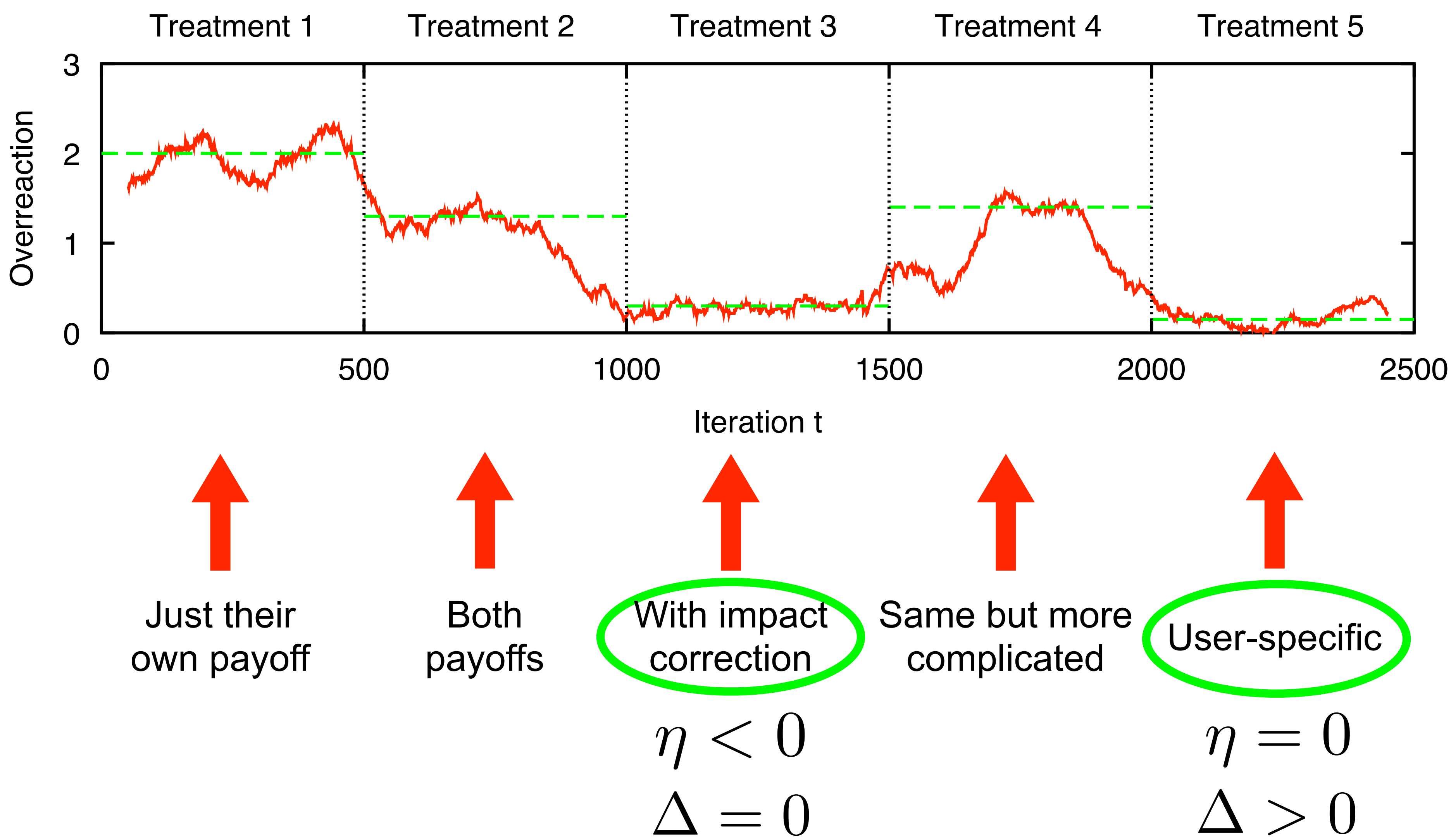
calibrated
user-specific
noise reduces
fluctuations

(joint work with R Mulet)

$\Delta = 0$



fluctuations are drastically reduced also when the information is biased `pessimistically'


Treatment 1
Treatment 2
Treatment 3
Treatment 4
Treatment 5
Overreaction
0
1
2
3
0
500
1000
1500
2000
2500
Iteration t
Just their own payoff
Both payoffs
With impact correction
Same but more complicated
User-specific
$\eta < 0$
$\Delta = 0$
$\eta = 0$
$\Delta > 0$

strict theory

these systems are out of equilibrium
(microscopic dynamics violates detailed balance)

$$U_i(t+1) - U_i(t) = -a_i^{\mu(t)} A(t) - \epsilon \;\neq -\frac{\delta H}{\delta U_i(t)} + \text{noise}$$

$$A(t) = \sum_j \phi_j(t) a_j^{\mu(t)} \equiv A(\{\phi_i(t)\}) \quad , \quad \boxed{\phi_i(t) = \Theta(U_i(t))}$$

$\Rightarrow$ no Hamiltonian H, study dynamics
(dynamical generating functionals
a.k.a. path integrals)

Luckily some important things can be understood without path integrals

References

Review : DM-Marsili, physics/0606107 [J Phys A 2006]

Basic probability : Chapter 1 of Bouchaud-Potters, *Theory of financial risks*

Stylized facts+correlations in finance : A Pagan, J Empirical Finance **3** 15 (1999) ; Laloux et al., cond-mat/9810255 [PRL,1999] ; Mantegna-Stanley, *Introduction to econophysics*

Traffic experiments : Helbing et al., cond-mat/0112479 [New J Phys 2002]

Minority Game : Challet-Marsili-Zhang, *Minority Games*

Economists : Alan Kirman, Mauro Gallegati, Terrance Odean, Fernando Vega Redondo