# Minority Games

1. Context
2. basic facts (experiments/theory)
3. MGs and financial markets: stylized facts
4. Inverse problem: optimal information schemes?
5. open problems

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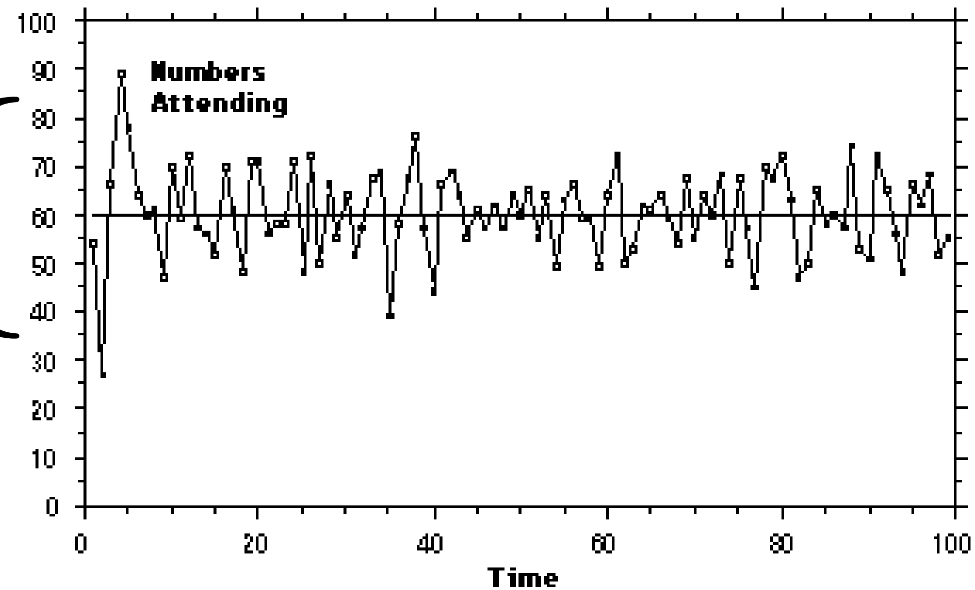
Arthur 94 : how well can inductive agents exploit some public endogenous information to forecast a signal they contribute to create?

volatility




inductive agents self-organize around the comfort level (on average)

string of past m attendances (memory)

key ingredients :

1) agents (N) react to the receipt of information
2) heterogeneity (they react differently from each other)
3) frustration (not everybody can win)

heavy strategic interdependence → bounded rationality, learning predictors/strategies/lookup tables/...

$$2^m = P$$

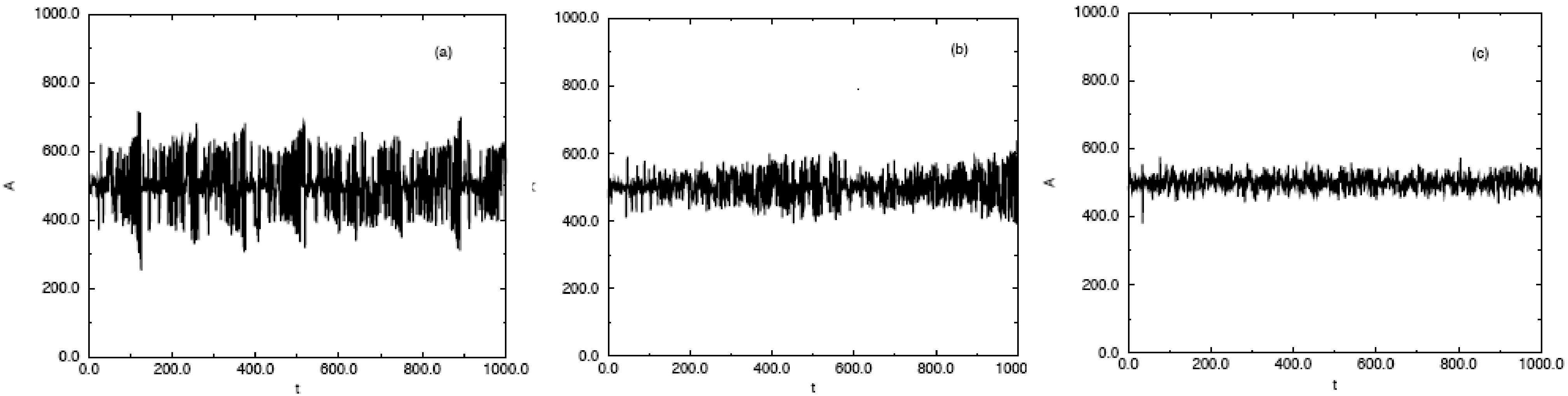
| Past | Prediction |
|---|---|
| 0 0 0 | 1 |
| 0 0 1 | 1 |
| 0 1 0 | 0 |
| 0 1 1 | 1 |
| 1 0 0 | 0 |
| 1 0 1 | 0 |
| 1 1 0 | 0 |
| 1 1 1 | 1 |

few (S) strategies per agent (reduce computational costs)

choose the one that proved more successful in the past

Challet and Zhang 97

comfort level at 50% (the minority wins) : look at fluctuations around the comfort level (average)



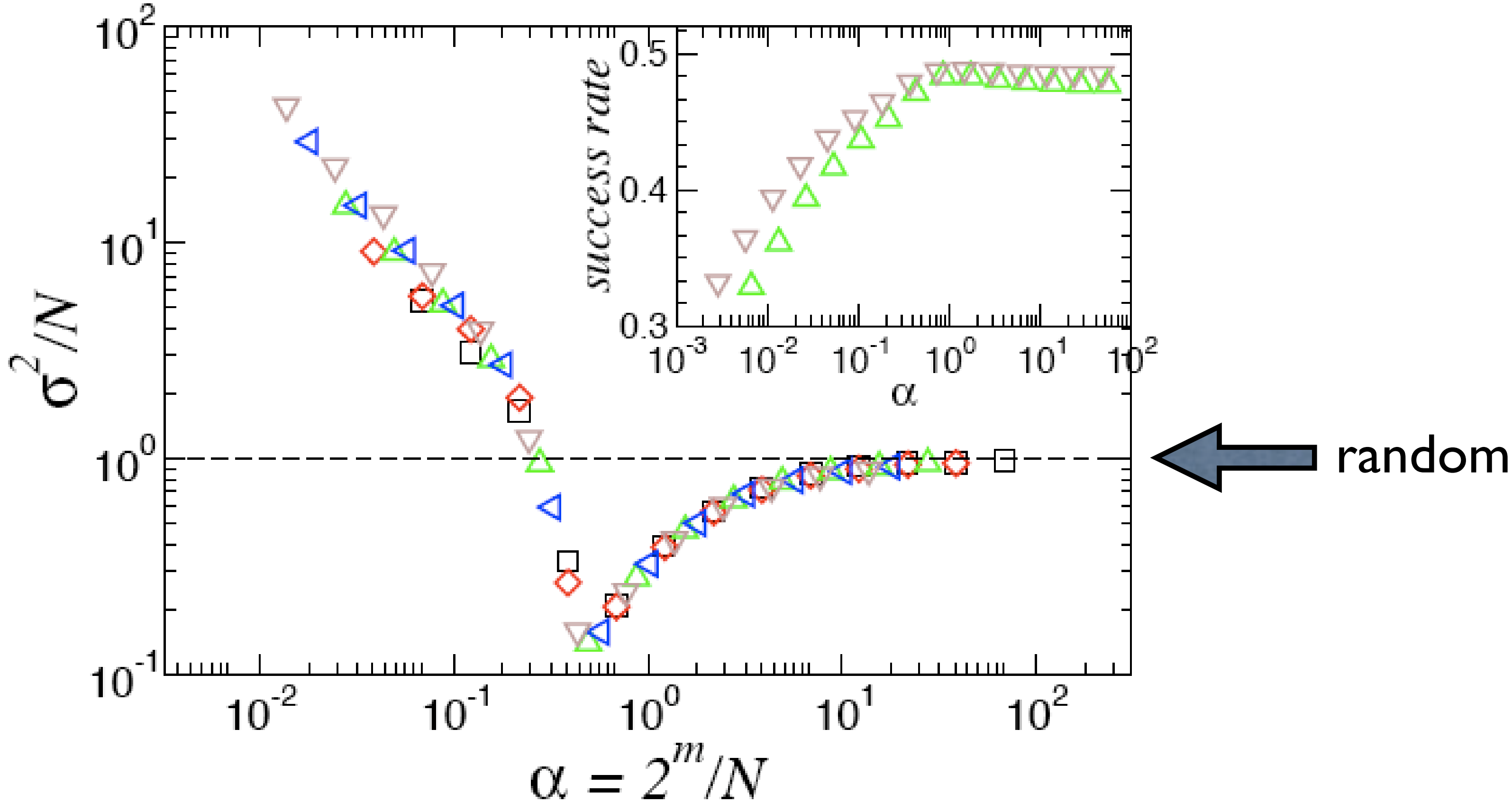

increasing m (at fixed N)

increasing m = increasing the amount/diversity of receivable information

"good" states have the correct average and "small" fluctuations (e.g. better than random)

Savit et al 99

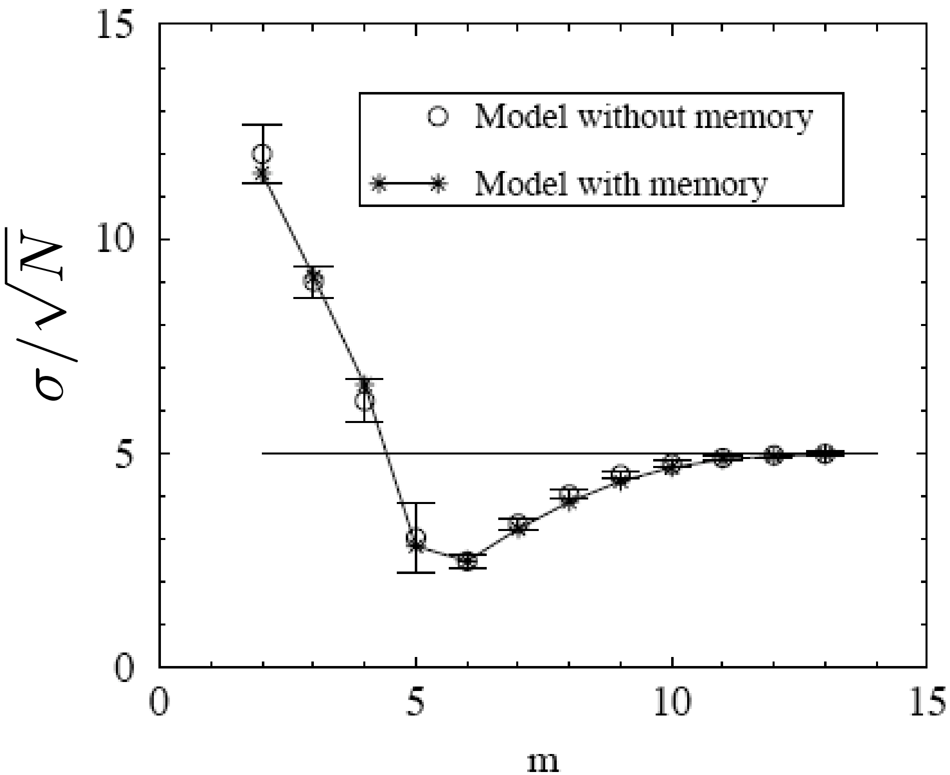
$$N \to \infty \ , \ \alpha \text{ fixed}$$



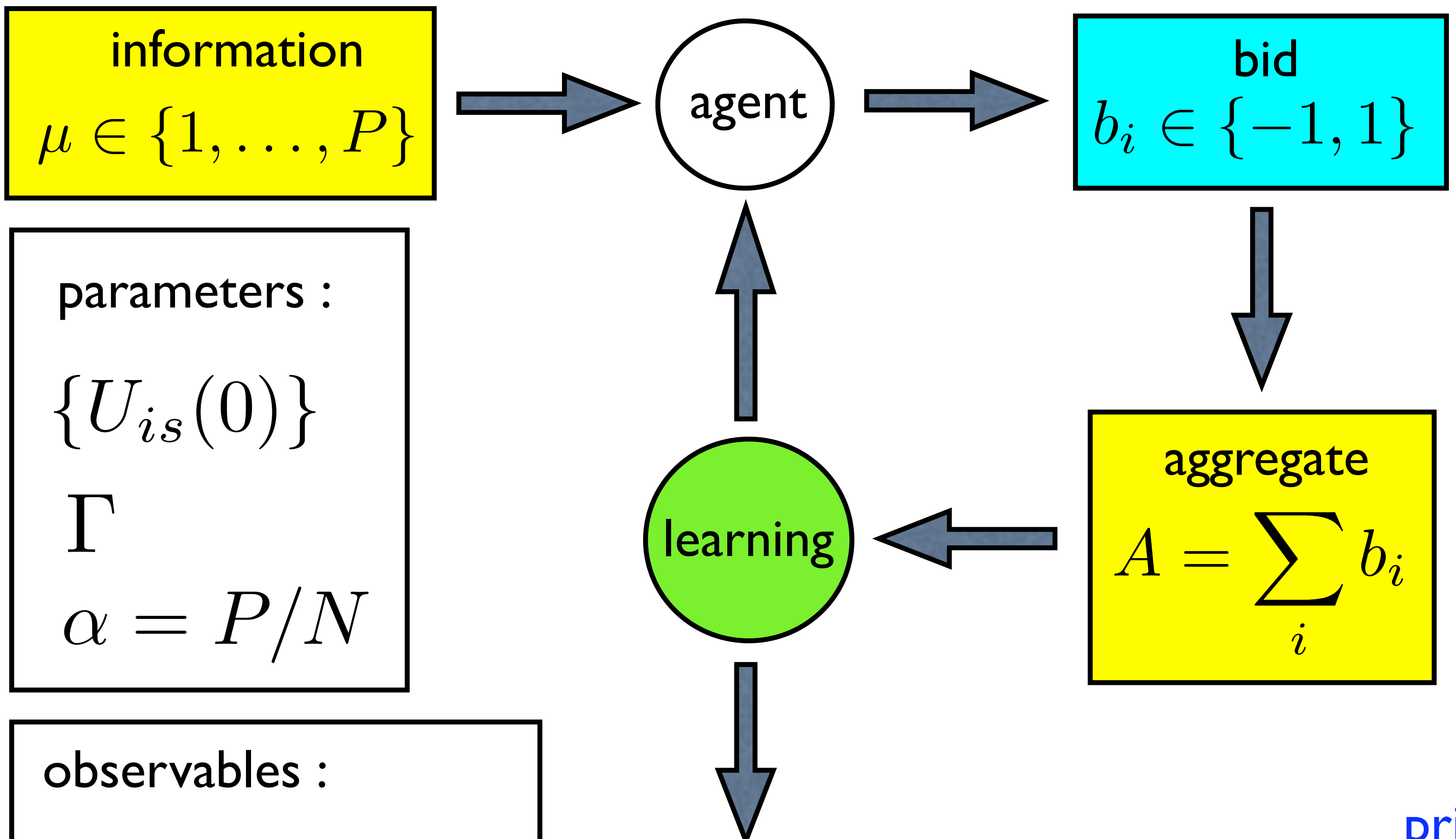

$$b_i(t) \in \{-1, 1\} \ , \ A(t) = \sum_i b_i(t) \ \to \ A(t) \propto \sqrt{N} \ \to \ \sigma^2 \propto N$$

Cavagna 99

no change when information is not endogenous but exogenous (uniform rv)




(wrong, but triggered analytical solution)




no direct interaction

observables :

$$\sigma^2 = \langle A^2 \rangle$$

$$H = \frac{1}{P} \sum_{\mu} \langle A|\mu \rangle^2$$

$$\langle A \rangle = 0$$

price-taking

$$U_{is}(t+1) - U_{is}(t) = -a_{is}^{\mu(t)} A(t)/N$$

$$\mathrm{Prob}\{s_i(t) = s\} \propto e^{\Gamma U_i(t)}$$

$$b_i(t) = a_{i,s_i(t)}^{\mu(t)}$$

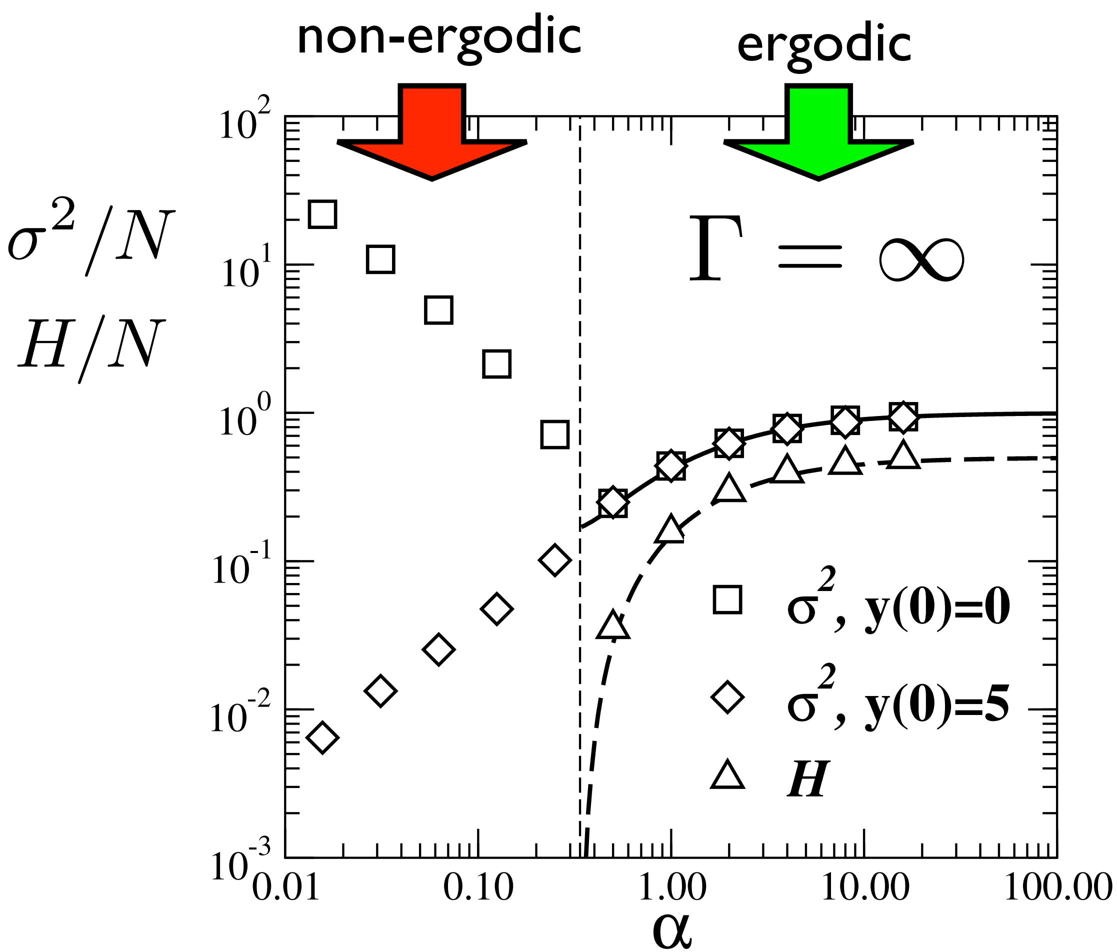
## informational efficiency

$$H = \frac{1}{P}\sum_{\mu}\langle A|\mu\rangle^2$$

$$\langle A|\mu\rangle \begin{cases} = 0 & A(t) \text{ stat. unpredictable based on } \mu \\ \neq 0 & A(t) \text{ stat. predictable based on } \mu \end{cases}$$

$$H \begin{cases} = 0 & \text{outcome stat. unpredictable} \\ > 0 & \text{outcome stat. predictable} \end{cases}$$

existence of a critical point




$$\frac{\sigma^2}{N} = \begin{cases} f(\alpha, \Gamma, \text{i.c.}) & \text{for } \alpha < \alpha_c \\ g(\alpha) & \text{for } \alpha > \alpha_c \end{cases}$$

$$\frac{H}{N} = \begin{cases} 0 & \text{for } \alpha < \alpha_c \\ h(\alpha) & \text{for } \alpha > \alpha_c \end{cases}$$

do markets fluctuate around a critical point?

## theory (key steps) : statistical approach

1) thermodynamic limit

$$N \to \infty \ , \ P \to \infty \ , \ \alpha = P/N \text{ fixed}$$

2) focus on self-averaging quantities (they don't fluctuate from sample to sample in the thermodynamic limit)

$$\mathrm{Prob}\left\{ \left| \sigma^2(\text{sample}) - \overline{\sigma^2} \right| \right\} \to 0 \quad \text{as } N \to \infty$$

$\Rightarrow$ "typical properties"

3) link with financial markets

$$A(t) = \sum_i b_i(t) \quad \to \quad \log p(t+1) - \log p(t) \propto A(t)$$

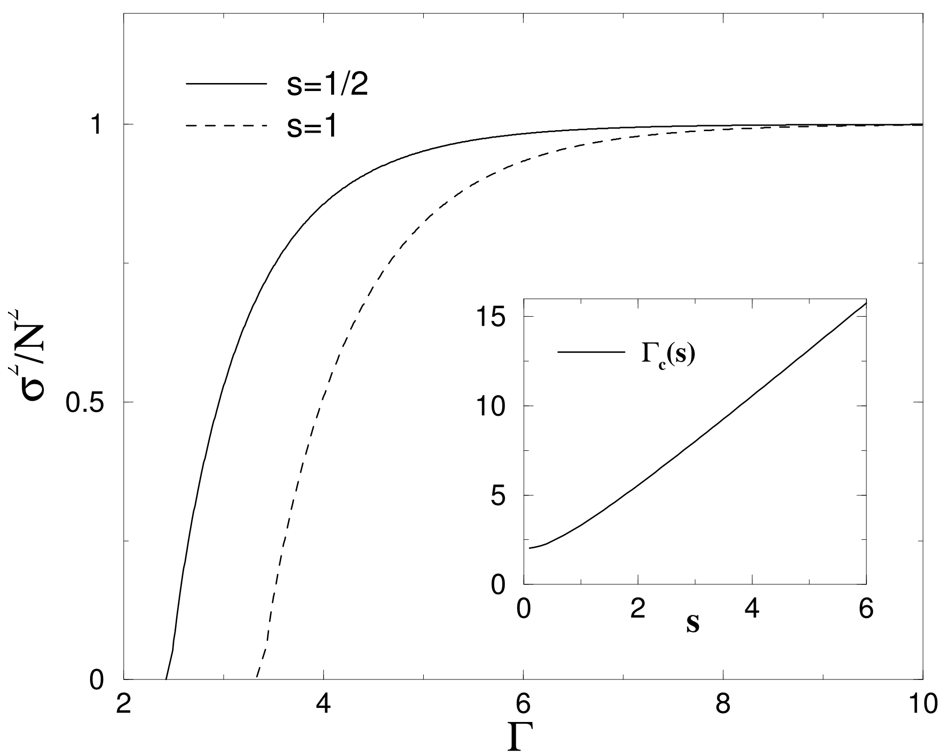
$$\mathrm{Prob}\{a_i(t) = a\} \propto e^{aU_i(t)} \quad , \quad a = \pm 1$$

$$\boxed{\pi_i(t) = -a_i(t)A(t)}$$

$$U_i(t+1) - U_i(t) = -\Gamma A(t)/N \quad , \quad A(t) = \sum_j a_j(t) \ , \ \Gamma > 0$$

$U_i(0)$ randomly sampled from $q(U) = \mathcal{G}(0, s^2)$

$$\text{Fluctuations : } \sigma^2 = \sum_i (\langle a_i^2 \rangle - \langle a_i \rangle^2) = \sum_i (1 - \langle a_i \rangle^2)$$




simplest MG [Marsili 01]

$$\Gamma < \Gamma_c \;\Rightarrow\; \sigma^2 \propto N$$

$$\Gamma > \Gamma_c \;\Rightarrow\; \sigma^2 \propto N^2$$

Lesson is : the larger the spread of i.c.'s (heterogeneity), the smaller are the fluctuations and the more stable is the fixed point, but fluctuations are horrible in fact : the steady state is not a Nash equilibrium

Optimal state : $\begin{cases} \frac{N-1}{2} & \text{do } a \\ \frac{N+1}{2} & \text{do } -a \end{cases}$

$$\text{Fluctuations : } \sigma^2 = \sum_i (\langle a_i^2 \rangle - \langle a_i \rangle^2) = \sum_i (1 - \langle a_i \rangle^2)$$

$$\text{for this Nash eq.} \;\; \Rightarrow \sigma^2 = 1$$

$$\mathcal{N}_{\text{Nash}} \sim e^{N\Sigma} \;, \;\; \Sigma > 0$$

Another Nash eq. : $\text{Prob}\{a_i = \pm 1\} = 1/2 \;\; \Rightarrow \sigma^2 = N$

i is in here

## Why can't they get to Nash?

$$U_i(t+1) - U_i(t) = -\Gamma A(t)/N \quad , \quad A(t) = \sum_j a_j(t) \ , \ \Gamma > 0$$

remove self-interaction (market impact)

$$U_i(t+1) - U_i(t) = -\frac{\Gamma}{N}[A(t) - \eta a_i(t)] \qquad \eta \in \{0,1\}$$

$$\langle U_i(t+1)\rangle - \langle U_i(t)\rangle = -\frac{\Gamma}{N}[\sum_j m_j - \eta m_i] = -\frac{\Gamma}{N}\frac{\partial H}{\partial m_i}$$

$$H = \frac{1}{2}\Big(\sum_i m_i\Big)^2 - \frac{\eta}{2}\sum_i m_i^2 \qquad m_i = \langle a_i \rangle$$

$$-1 \leq m_i \leq 1$$

minima : $\eta = 0 \ \Rightarrow \ m_i = 0 \quad \Rightarrow \sigma^2 = N$

$\eta = 1 \ : \ H$ is harmonic $\Rightarrow \ m_i = \pm 1$

$\Rightarrow \sigma^2 = 1$ (odd $N$)

[De Martino-Marsili 01]

Nasdaq composite
(1985-2003)

$Z(t+\delta t)$

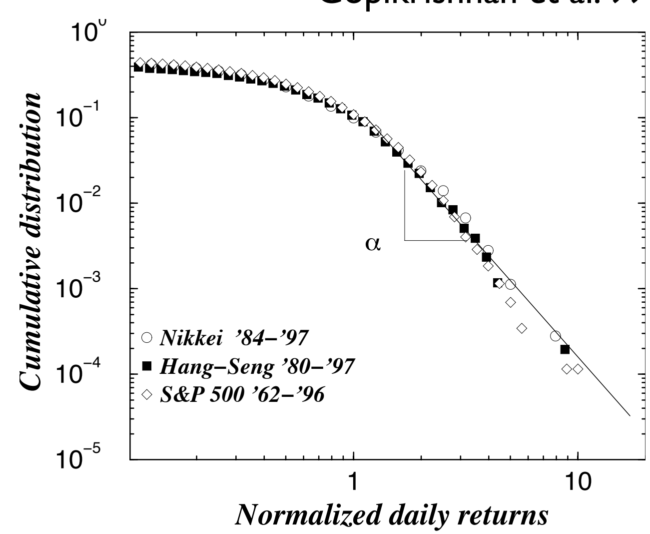
$Z(t)$


$t \qquad t+\delta t$

$$\delta t = 10s, \ldots, 1 \text{ month}$$

$$r_{\delta t}(t) = \log Z(t+\delta t) - \log Z(t) \simeq \frac{Z(t+\delta t) - Z(t)}{Z(t)}$$

Gopikrishnan et al. 99




Question : how likely are large fluctuations?

$$\delta t \text{ large} \;:\; P\{r(t) > r\} \propto e^{-r^2}$$

$$\delta t \text{ small} \;:\; P\{r(t) > r\} \sim r^{-\alpha} \qquad \alpha \simeq 3$$

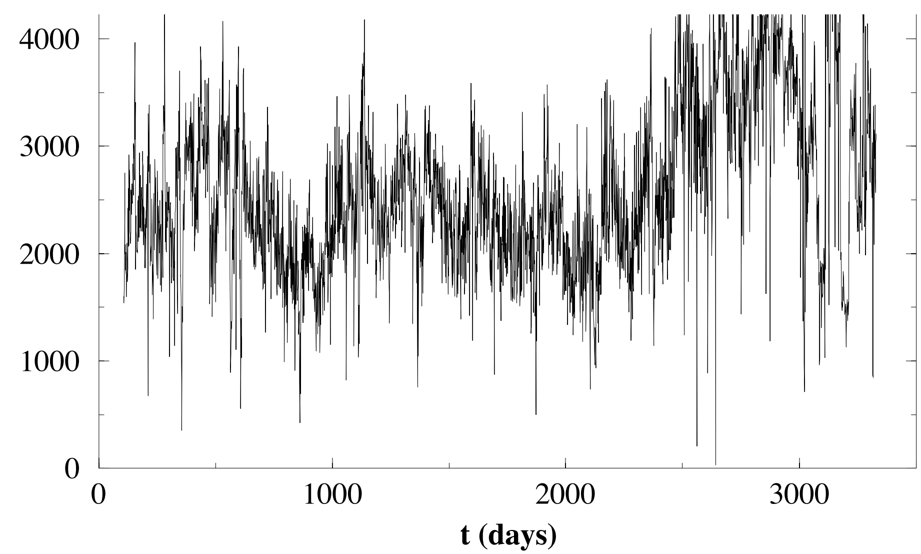
key ingredients :

1) agents (N) react to the receipt of information
2) heterogeneity (they react differently from each other)
3) frustration (not everybody can win)

Bouchaud et al 01

Volume
(nr of trades)

S&P500 futures
1985-1998



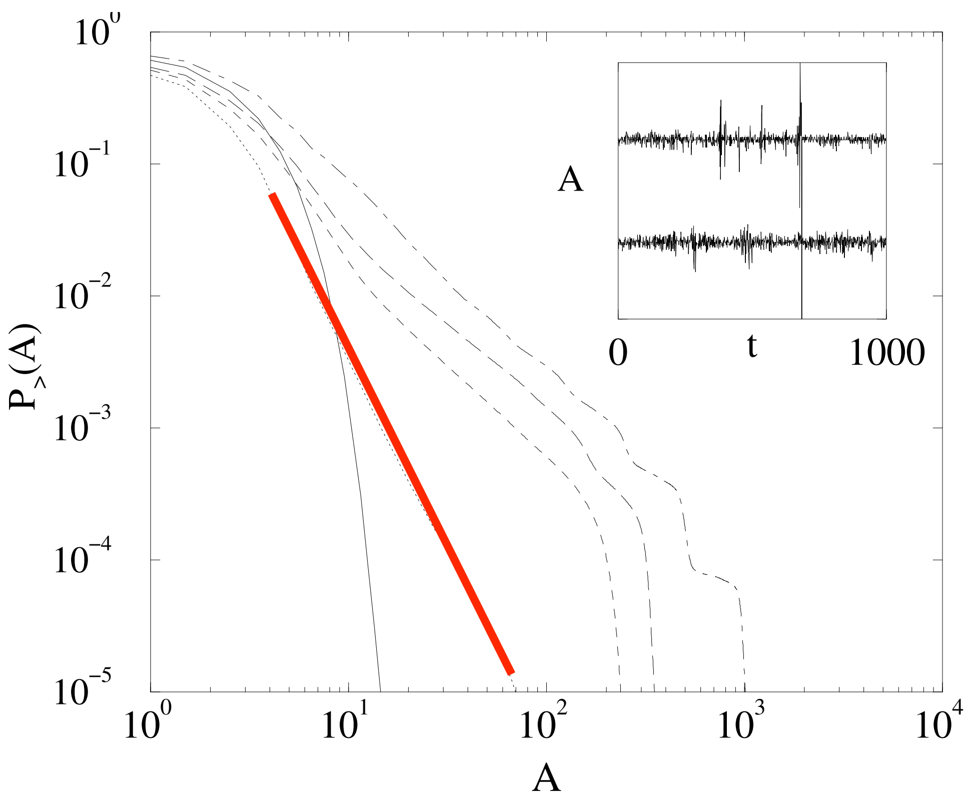

4) Volume (=number of agents) fluctuates in time

“grand-canonical” MG
[Challet-Marsili 03]
[Challet et al 05+06]

(finite size : N<1000)

$n = N/P$

$P\{A(t) > A\}$ Gaussian for small $n$ but ...



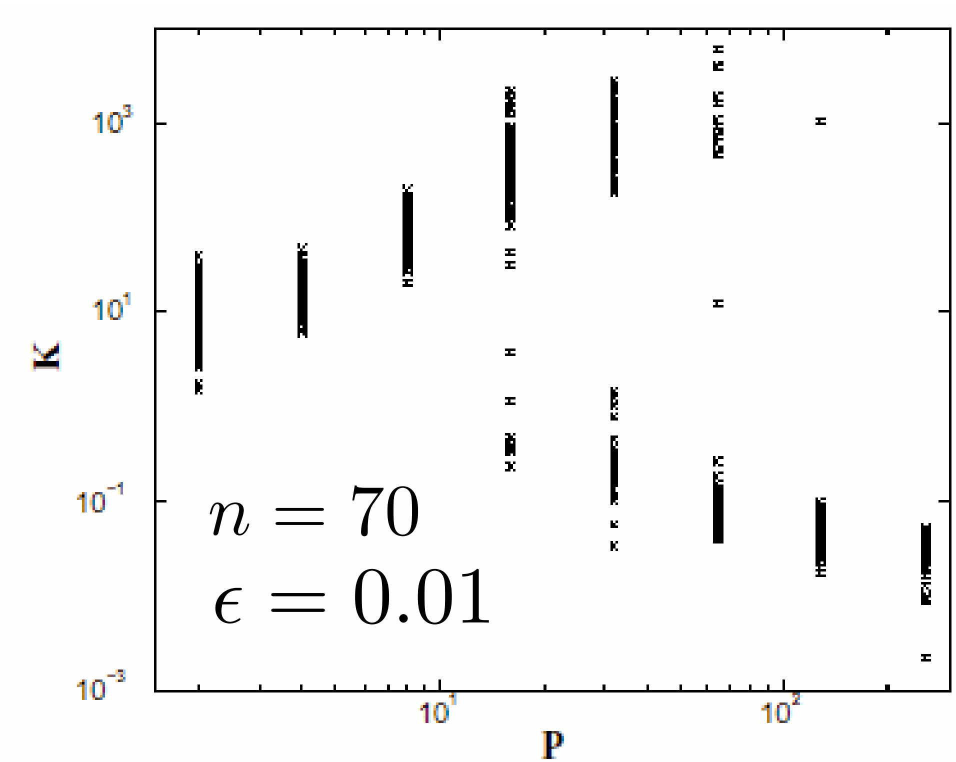

$$P\{A(t) > A\} \simeq A^{-\gamma}$$

$\gamma \simeq 2.8$ for $n = 20$

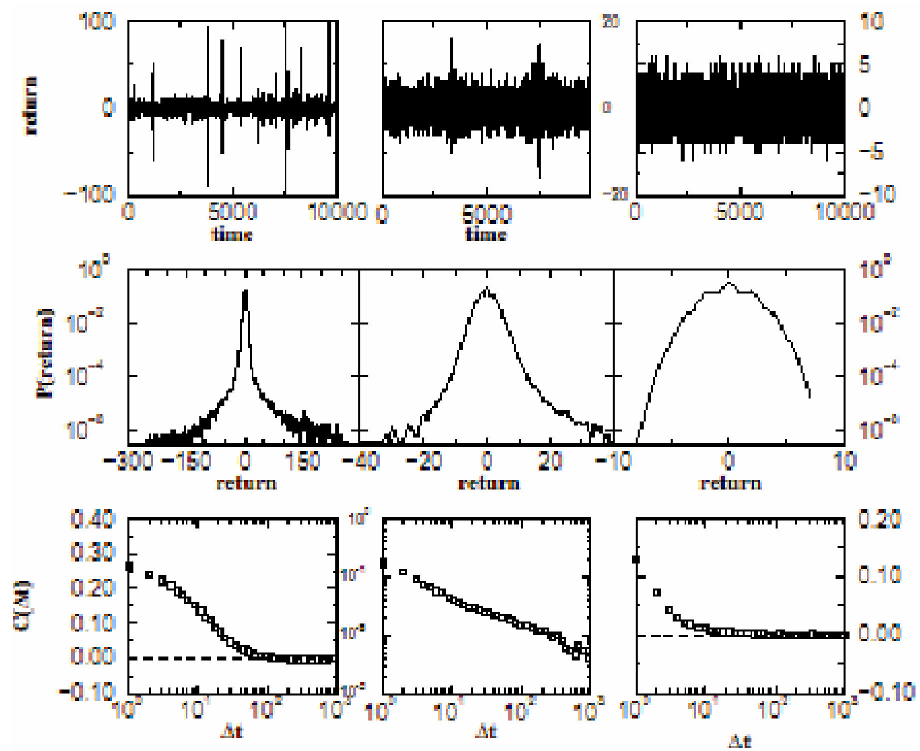
around the
critical point!

# A closer look : finite size effects, sample dependence

increase size (P or N=nP)




3 different samples

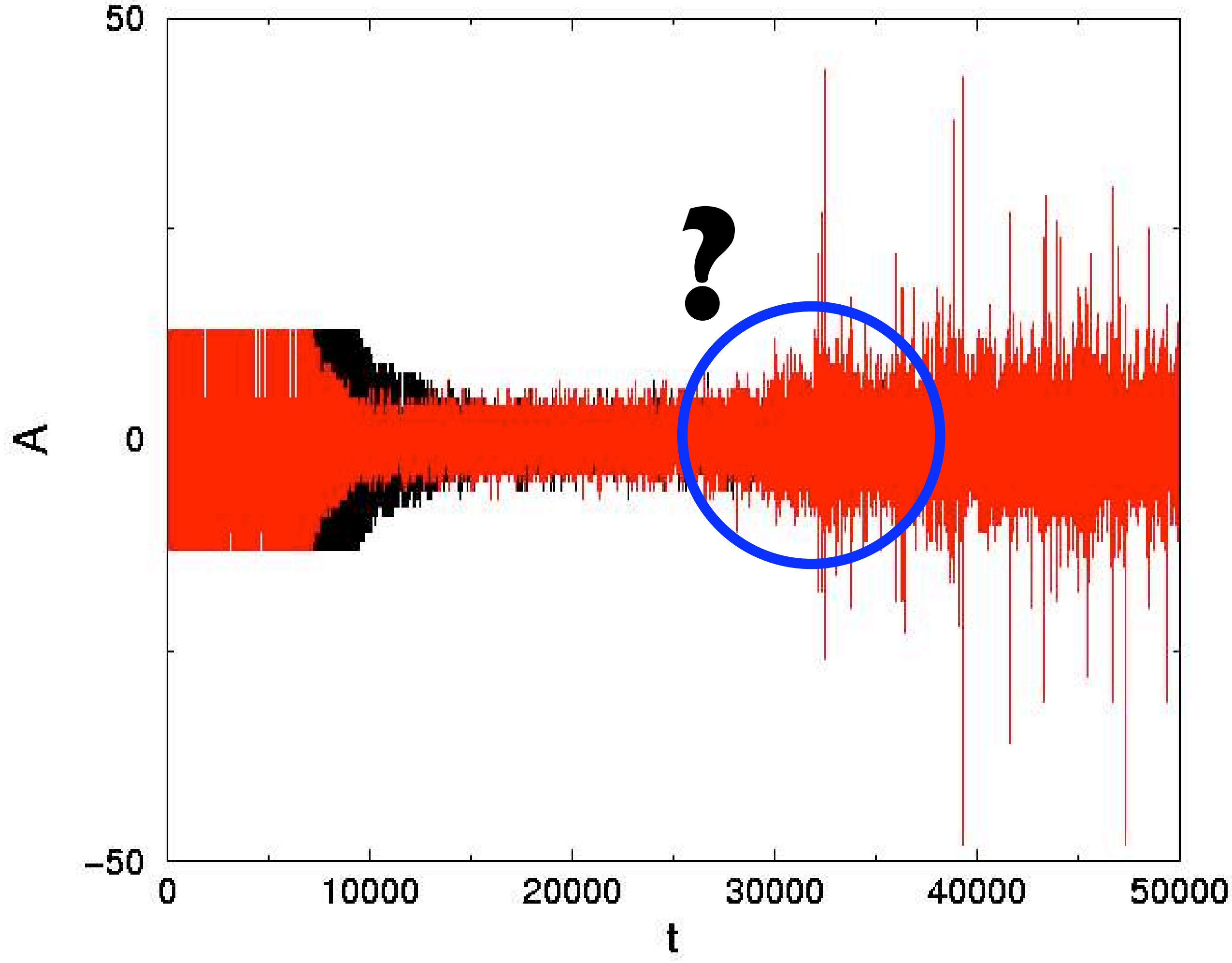



$$K = \frac{\langle A^4 \rangle}{\langle A^2 \rangle^2} - 3$$

"kurtosis excess" (K=0 means Gaussian)

MGs with finite score memory
[Challet et al 05+06]

# dynamical instability



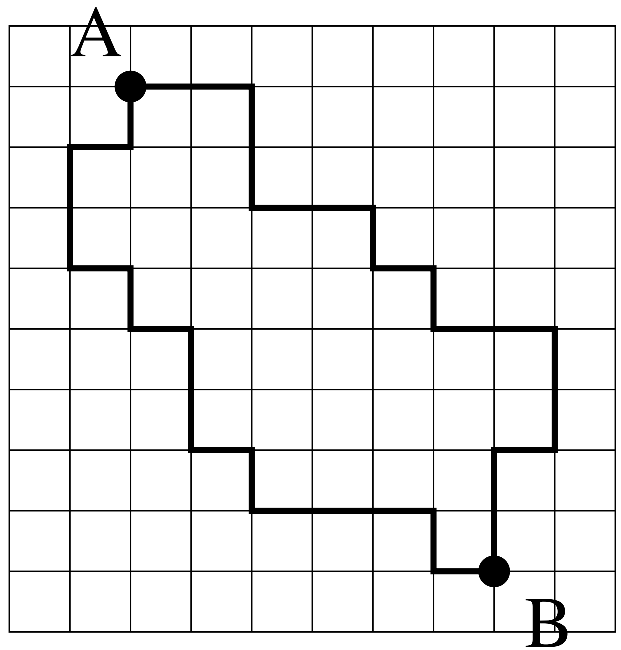
## more generalizations

1) majority and mixed games [Kozlowski-Marsili 03] [De Martino et al 03]
2) heterogeneous time scales [Piai-Marsili 02] [De Martino 03]
3) private information [Berg et al 01] [De Martino-Galla 05]
4) risk-sensitive payoff [De Martino et al 04+05]
5) many assets [Bianconi et al 07]
6) solution in the ergodic regime [De Martino et al 07]
7) Market ecology [Challet et al 02]
8) ...

## open issues

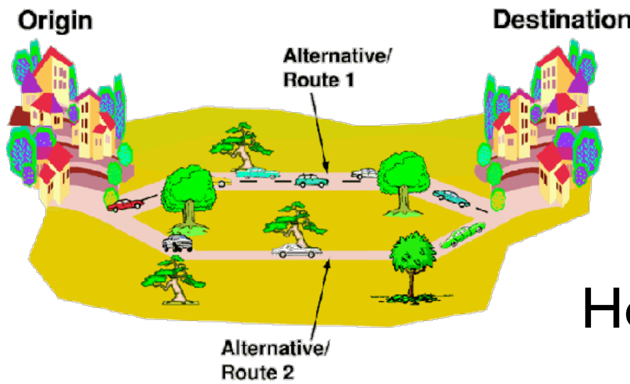
1) self-organization around the critical point?
2) evolving strategies?
3) analytics in the non-ergodic regime?
4) volatility?
5) buy-hold-sell models? [Challet 06]
6) reaction to an isolated shock? [Virasoro]
7) many signals? [Virasoro]
8) ...

# Another MG

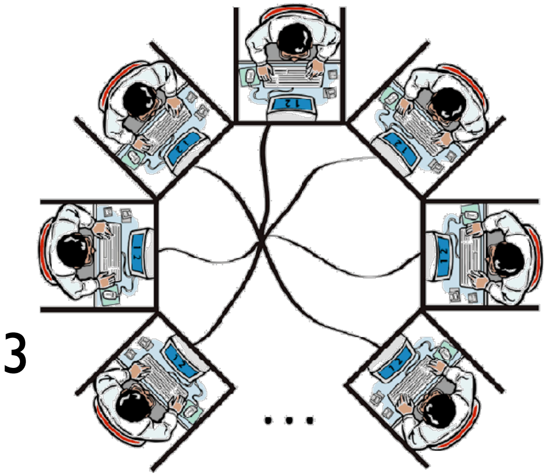


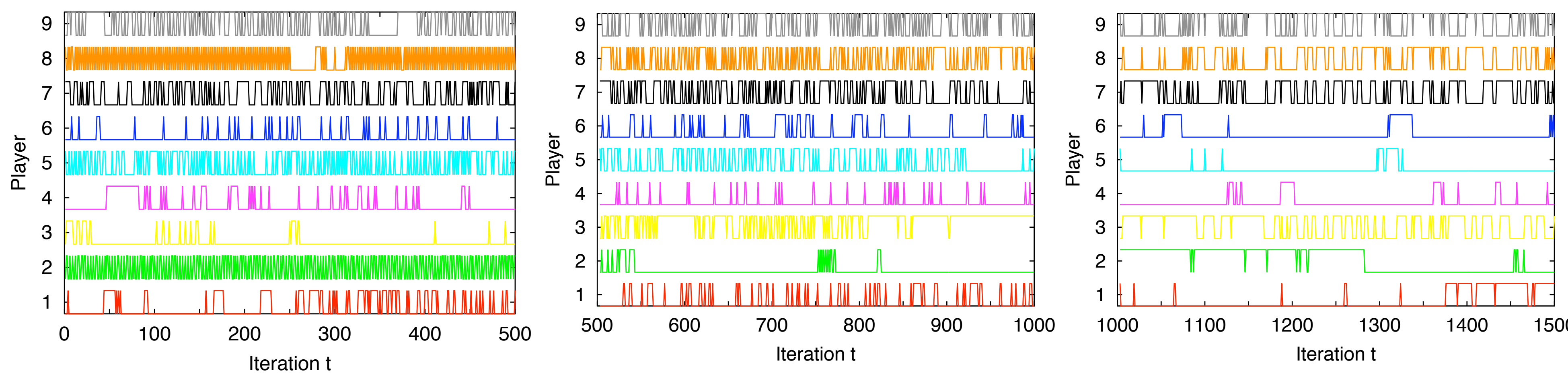

Each driver must go a point A to a point B every day
(A and B depend on the driver)

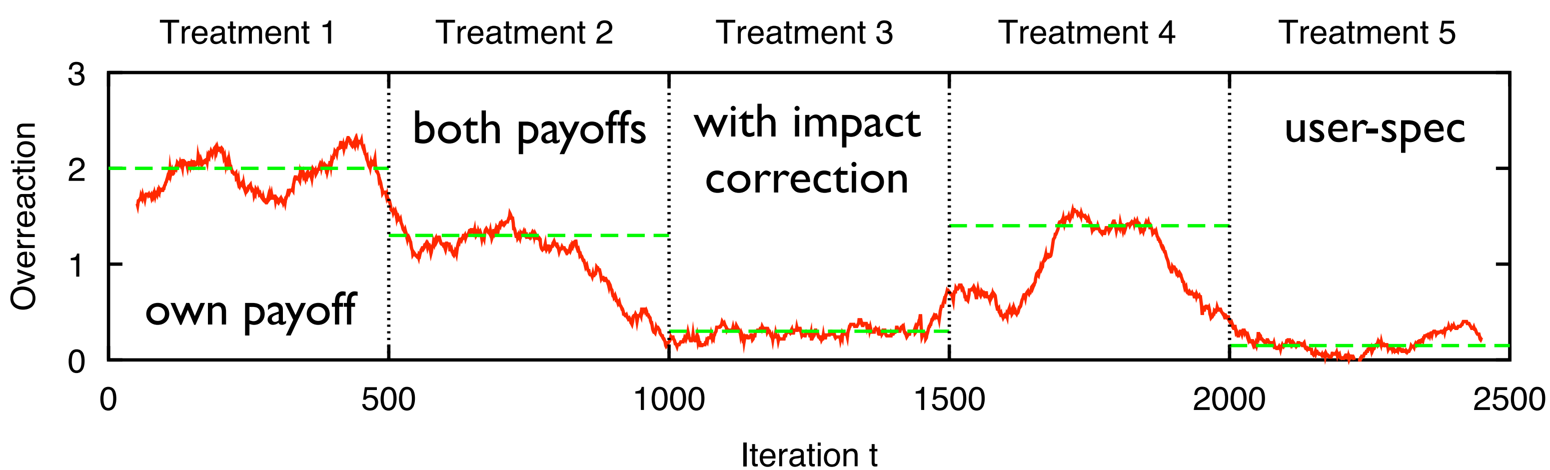
strategies = different possible routes to go from A to B




Helbing et al 03





N drivers, P streets, discrete time n=0,1,...

Each driver has S *strategies* and S *score functions*

$\{\mu\} = \text{edges}$

(streets)

Assume $N = cP \ (c = \text{car density})$

$$\widetilde{g}_i(n) = \arg\max_g U_{ig}(n)$$

$$U_{ig}(n+1) = U_{ig}(n) - \frac{1}{P}\sum_{\mu=1}^{P} q^{\mu}_{ig} Q^{\mu}(n) + \frac{1}{2}(1 - \delta_{g,\widetilde{g}_i(n)})\zeta_{ig}(n)$$

MG $$\boxed{Q^{\mu}(n) = \sum_{i=1}^{N} q^{\mu}_{i,\widetilde{g}_i(n)}}$$

Information noise

Information noise :
describes different
`treatments'

$$\langle \zeta_{ig}(n) \rangle = \eta$$
$$\langle \zeta_{ig}(n)\zeta_{jh}(m) \rangle = \Delta \delta_{ij}\delta_{gh}\delta_{nm}$$

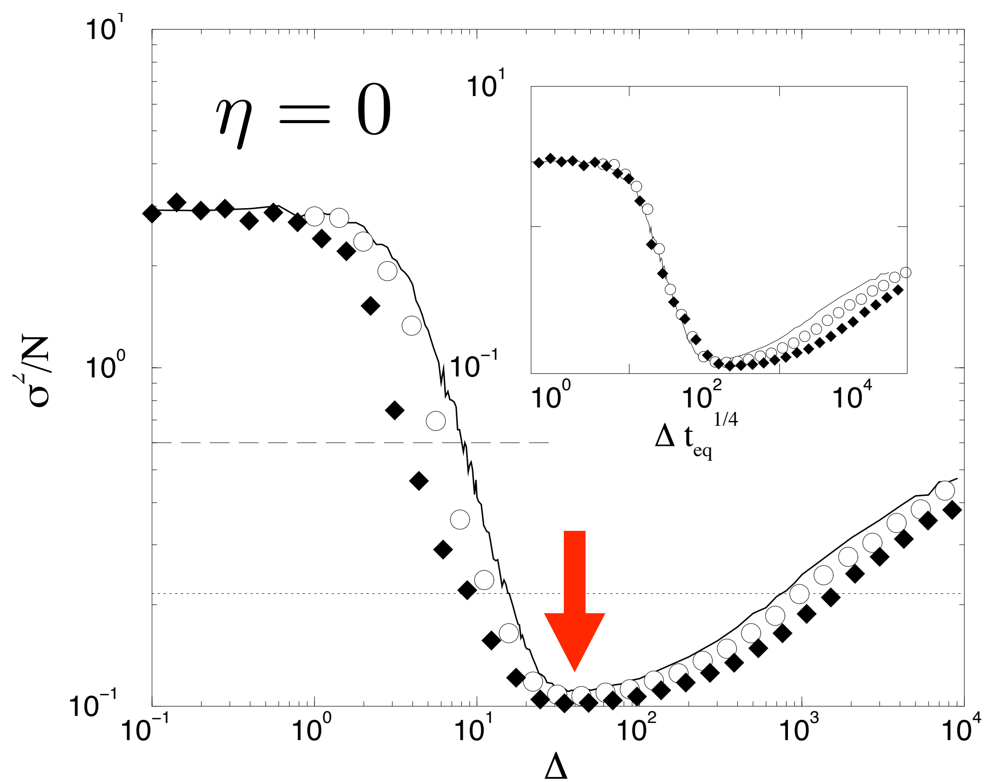
$$U_{ig}(n+1) = U_{ig}(n) - \frac{1}{P}\sum_{\mu=1}^{P} q_{ig}^{\mu} Q^{\mu}(n) + \frac{1}{2}(1-\delta_{g,\widetilde{g}_i(n)})\zeta_{ig}(n)$$

$\eta = 0$ : unbiased information
$\eta > 0$ : overestimates unused routes
$\eta < 0$ : underestimates unused routes

$\Delta = 0$ : same information (biased or not) for all drivers
$\Delta > 0$ : user-specific information

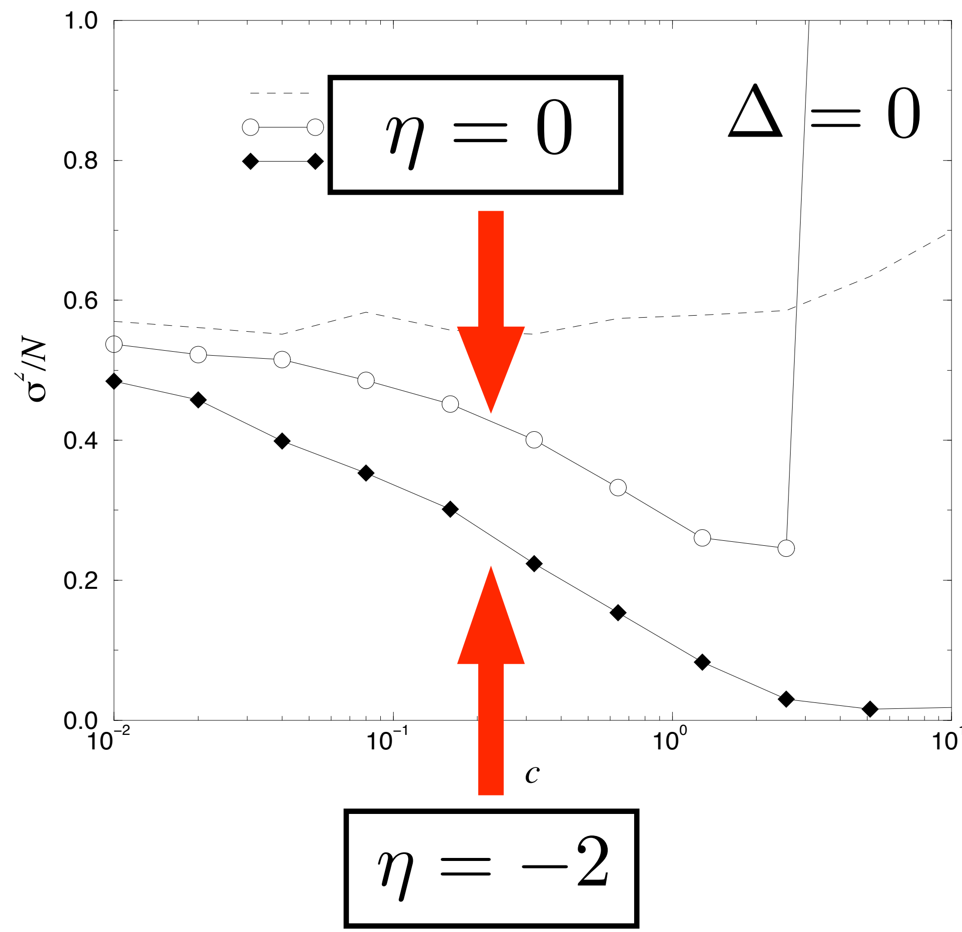
# Relevant quantity : traffic load fluctuations

$$\sigma^2 = \frac{1}{P} \sum_{\mu} [\langle (Q^{\mu})^2 \rangle - \langle Q^{\mu} \rangle^2]$$




high car density
(hard phase)

calibrated user-specific noise
reduces fluctuations




`pessimist' information bias
reduces fluctuations

## 4 references


(a) D Challet, M Marsili and YC Zhang : *Minority Games* (Oxford UP)

(b) ACC Coolen : *The mathematical theory of Minority Games* (Oxford UP)

(c) A De Martino and M Marsili : *Statistical mechanics of socio-economic systems with heterogeneous agents* , J. Physics A: Math. Gen. **39** R465 (2006)

(d) http://www.unifr.ch/econophysics/minority/


http://chimera.roma1.infn.it/ANDREA/