Large random resource allocation problems : some lessons from the statistical physics of stylized models

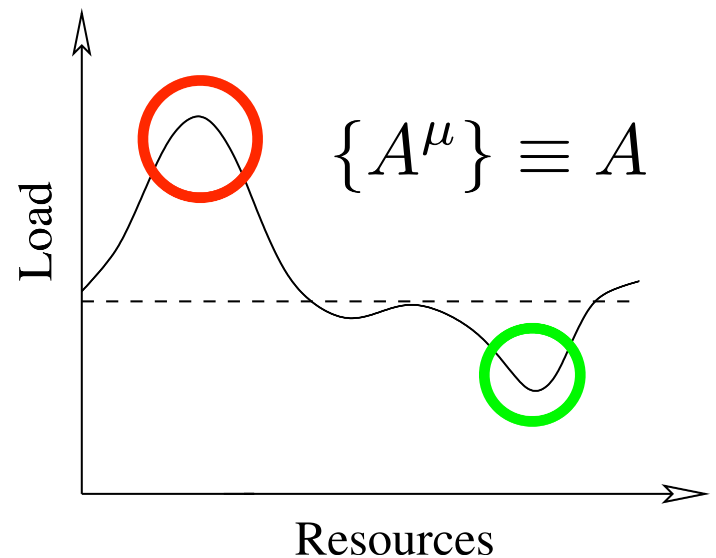
- General framework
- Examples
- Why bother?
- Market impact
- Stability/Role of diversity
- Macroscopic structure of GE
- (Dynamics)

General framework

N agents, M resources and a matrix of dependencies $\{\xi_i^\mu\}$

- Agents are
  - heterogeneous
  - selfish
- Resources are
  - equivalent
  - scarce
  - (distributed)
- No direct interaction

$\Longrightarrow$



Typical macroscopic properties : random $\{\xi_i^\mu\}$ $+$ $\begin{matrix} N \to \infty \\ M = \alpha N \end{matrix}$

Typical properties

Sample-independent in the limit of infinite N

$$\frac{\langle\langle X^2 \rangle\rangle - \langle\langle X \rangle\rangle^2}{\langle\langle X \rangle\rangle^2} \to 0 \quad \text{for } N \to \infty$$

GE for linear production economies

[Lancaster]
[De M.-Marsili-Perez Castillo]

N firms, M commodities and an input-output matrix $\{\xi_i^\mu\}$

Variables : $\{s_i \geq 0\} \equiv s \ , \ \{x^\mu \geq 0\} \equiv x \ , \ \{p^\mu\} \equiv p$

Find (s^*, x^*, p^*) such that

$$s_i^* = \arg \max_{s_i \geq 0} \left[s_i \sum_\mu p^\mu \xi_i^\mu \right]$$

$$x^* = \arg \max_{x \in B} U(x)$$

+ b.c.

$$x^\mu = y^\mu + \sum_i s_i \xi_i^\mu \qquad \forall \mu$$

1. Distributions of s, x, p, ...
2. Uniqueness/stability
3. GDP vs M/N

Lotka-Volterra [May] [?]
[De M.-Marsili]

N species, M resources and a matrix of dependencies $\{\xi_i^\mu\}$

Variables : $\{n_i \geq 0\}$

$$\frac{\partial n_i}{\partial t} = n_i(t)\left[\lambda_i + \sum_\mu \xi_i^\mu A^\mu(t)\right]$$

$$A^\mu(t) = A_0^\mu - \sum_j \xi_j^\mu n_j(t)$$

1. Uniqueness/stability of the steady state
2. How many species can be supported?

GCMG [Challet-Marsili]

N agents, M information bits and a matrix of strategies $\{a_i^\mu = \pm 1\}$

Variables : $\{n_i = 0/1\}$

$$U_i(t+1) - U_i(t) = -\overline{a_i A(t)} - \epsilon_i$$

$$n_i(t) = \theta[U_i(t)] \quad , \quad A^\mu(t) = \sum_i a_i^\mu n_i(t)$$

$$\overline{X} = \sum_\mu X^\mu / M$$

1. Statistics of A (returns)
2. Finite N effects

MG [Arthur]
[Challet-Zhang]

N agents, M information bits and S strategy matrices $\{a_{ig}^{\mu} = \pm 1\}$

Variables : $\{g_i = 1, \ldots, S\} \to \{s_i = \pm 1\} \quad (S = 2)$

$$U_{ig}(t+1) - U_{ig}(t) = -\overline{a_{ig} A(t)}$$

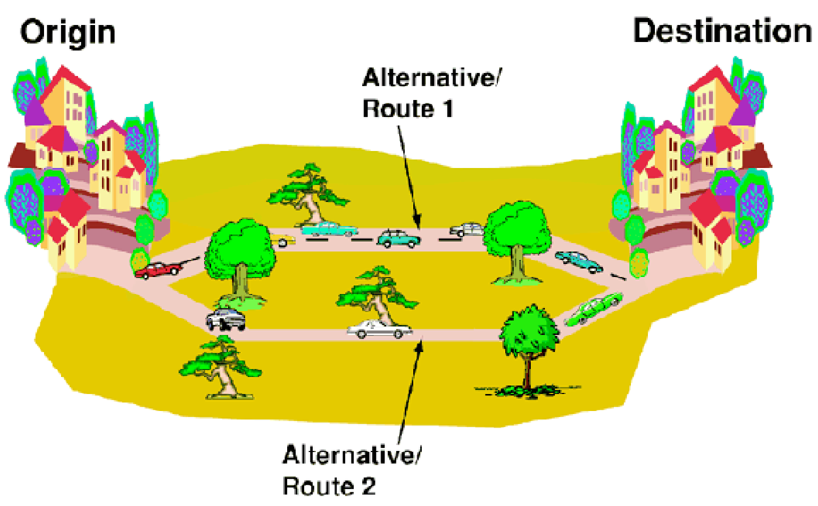
$$g_i(t) = \arg\max_g U_{ig}(t) \quad , \quad A^{\mu}(t) = \sum_i a^{\mu}_{i,g_i(t)}$$

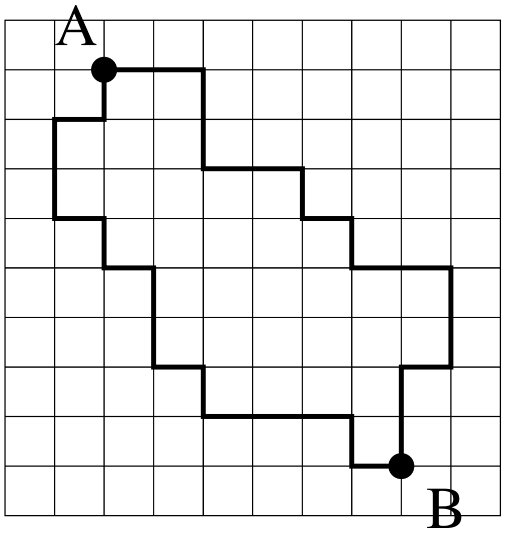
1. Statistics of A (returns)
2. ...

Route choice game

[Selten et al.] [Helbing et al.]
[De M.-Marsili-Mulet]

N drivers on a grid of M streets





1. Statistics of traffic loads
2. ...

Also

1. Market organization [Weisbuch-Kirman-Herreiner] [De M.-Marsili]
2. Round-trip transaction models [Challet]
3. Auctions [Vohra-De Vries]
4. ...

Not Majority Games

Why bother?

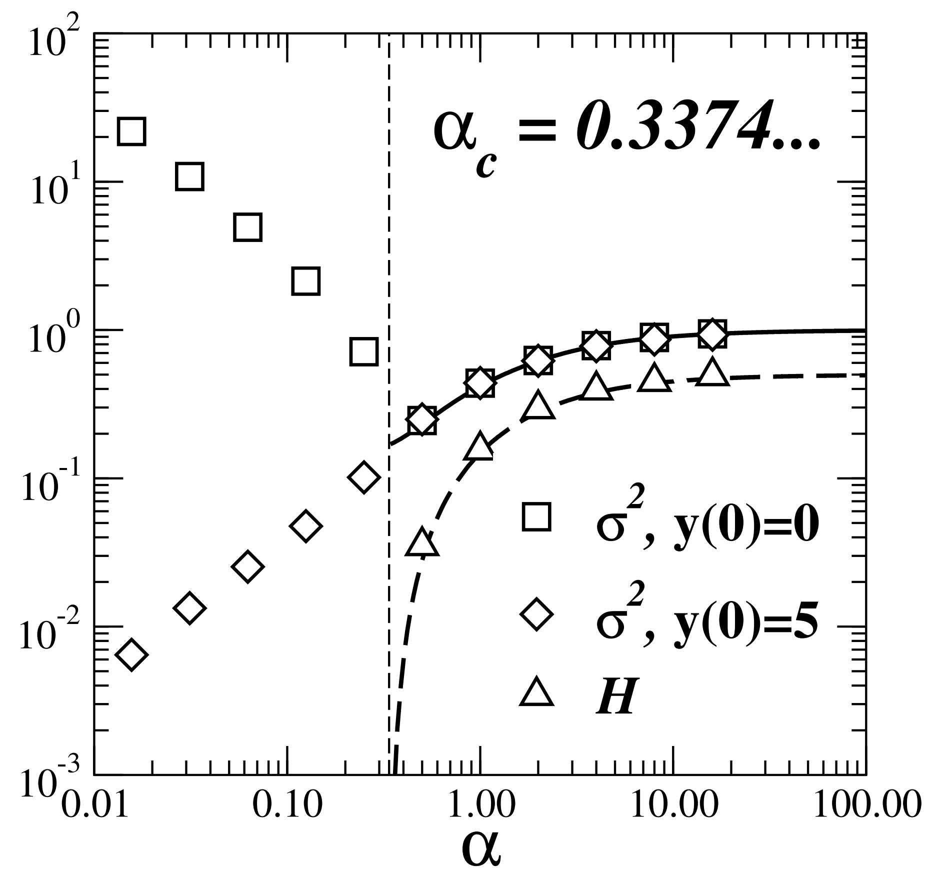
1. Dream : identify the microscopic ingredients of "stylized facts"
2. Compute emerging macroscopics without RA [Kirman]
3. Systems with direct interaction
   a. social [Bala-Goyal] [Vega Redondo]
   b. combinatorial auctions [Galla et al.]
   c. ...
4. (Rigorously)

Generic result on P(A)

A : average load in the steady state

H : variance of P(A)

σ^2 : fluctuations



Market impact : basic idea

Agent watching a MG wants to evaluate how good his trading strategies are

$$\pi_g^{\text{out}} = -\overline{a_g \langle A \rangle}$$

$\langle X \rangle$ (time avg)

Then goes in

$$\langle A \rangle \to \langle A + a_{g'} \rangle$$

$$\pi_{g'}^{\text{in}} \simeq -\overline{a_{g'} \langle A \rangle} - \overline{a_{g'} a_{g'}} = \pi_{g'}^{\text{out}} - 1$$

$$\pi_{g''}^{\text{in}} \simeq -\overline{a_{g''} \langle A \rangle} - \overline{a_{g'} a_{g''}} \simeq \pi_{g''}^{\text{out}}$$

$$v_g \equiv \langle U_g(t+1) - U_g(t) \rangle = \pi_g^{\text{in}} + 1 - f_g$$

Reducing the effects of MI

$$v_g \equiv \langle U_g(t+1) - U_g(t) \rangle = \pi_g^{\text{in}} + 1 - f_g$$

$$\Downarrow$$

$$v_g = \pi_g^{\text{in}} + 1 - f_g + \eta f_g$$

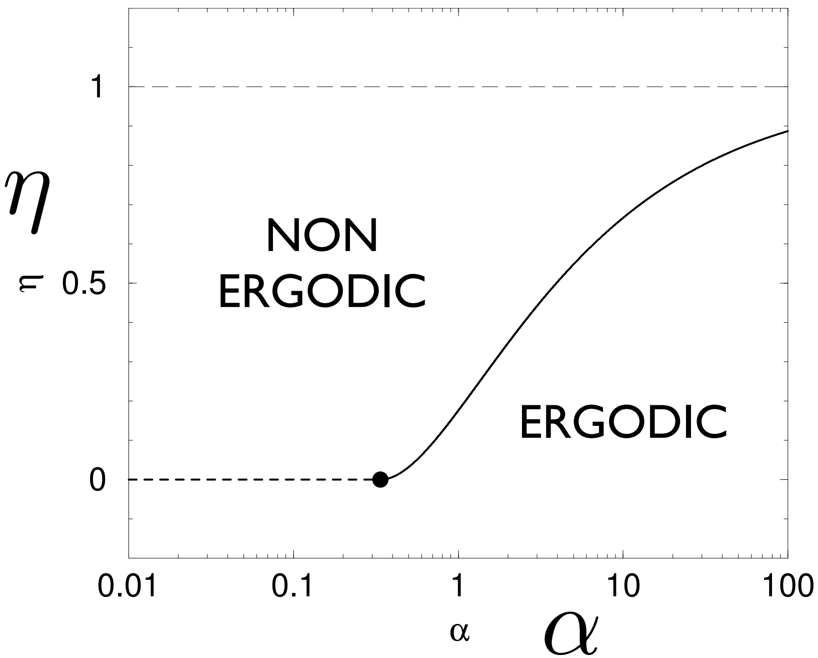
Modified learning [Challet-Marsili-Zecchina]

$$U_{ig}(t+1) - U_{ig}(t) = -\frac{\overline{a_{ig} A(t)}}{N} + \frac{\eta}{N} \delta_{g,g_i(t)}$$

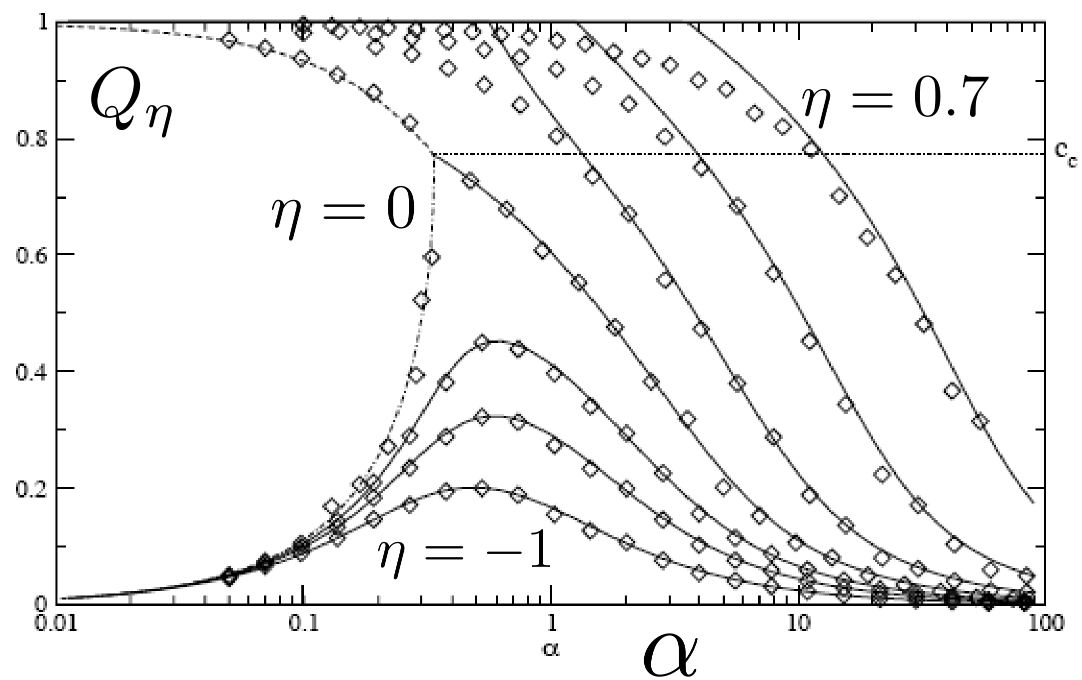
Steady state [De M.-Marsili] [Heimel-De M.]

2 strategies per agent

$s_i(t) = \pm 1$



$$Q_\eta = \frac{1}{N} \sum_i \langle s_i \rangle^2$$



Agents behave stochastically for $\eta < 1$

Steady state

[De M.-Marsili]
[Heimel-De M.]

2 strategies per agent

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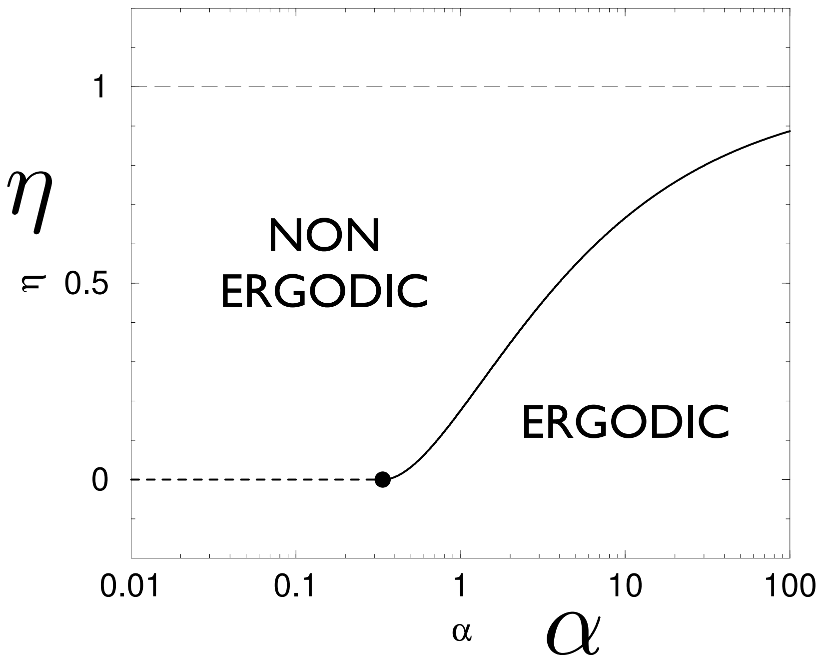
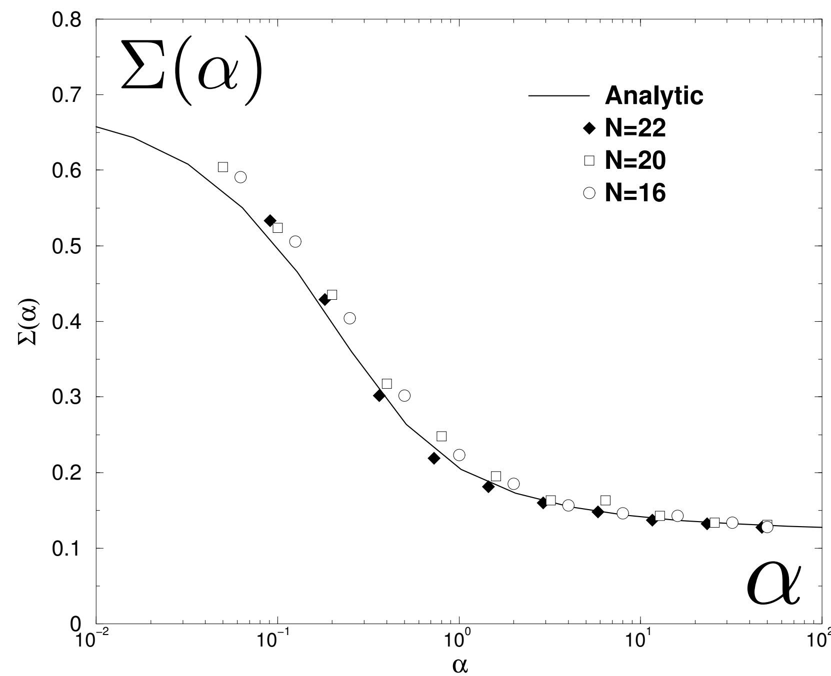


$\eta = 1$

Freezing



$\mathcal{N} \sim e^{N\Sigma(\alpha)}$

$\eta = 1 \;:\; \frac{1}{N}\sum_i \langle s_i \rangle^2 = 1 \;\Rightarrow$ each agent uses one strategy

Remark

$\mathcal{N}$ is not self-averaging

$$\langle\!\langle \log \mathcal{N} \rangle\!\rangle = \langle\!\langle \log[\langle\!\langle \mathcal{N} \rangle\!\rangle + \mathcal{N} - \langle\!\langle \mathcal{N} \rangle\!\rangle] \rangle\!\rangle$$

$$= \log \langle\!\langle \mathcal{N} \rangle\!\rangle - \frac{\langle\!\langle \mathcal{N}^2 \rangle\!\rangle - \langle\!\langle \mathcal{N} \rangle\!\rangle^2}{2 \langle\!\langle \mathcal{N} \rangle\!\rangle^2} + \cdots$$

Summary : steady state

	$\eta = 1$	$\eta = 0$
Number	$\mathcal{N} \sim e^{N\Sigma(\alpha)}$	Unique for $\alpha > \alpha_c$ Degenerate for $\alpha < \alpha_c$
Geometry	Discrete	(Continuous)
Dynamics	Not ergodic	Ergodic for $\alpha > \alpha_c$ Not ergodic for $\alpha < \alpha_c$
Agents	Deterministic	Stochastic
Convergence	(Fast)	Fast
P(A)	$H \neq 0$	$H = 0 \text{ for } \alpha < \alpha_c$

Note

$$A(t) \equiv A^{\mu(t)}$$

$$U_{ig}(t+1) - U_{ig}(t) = \underbrace{-a_{ig}^{\mu(t)} \frac{A(t)}{N}}_{O(N^{-1/2})} + \underbrace{\frac{\eta}{N} \delta_{g,g_i(t)}}_{O(1/N)}$$

But in the long term (average over information)

$$\overline{A} = O(1)$$

Lotka-Volterra

[May] [?]
[De M.-Marsili]

N species, M resources and a matrix of dependencies $\{\xi_i^\mu\}$

Variables : $\{n_i \geq 0\}$

$$\frac{\partial n_i}{\partial t} = n_i(t)\left[\lambda_i + \sum_\mu \xi_i^\mu A^\mu(t)\right]$$

$$A^\mu(t) = A_0^\mu - \sum_j \xi_j^\mu n_j(t)$$

1. Uniqueness/stability of the steady state
2. How many species can be supported?

Model definition

$$\frac{\partial n_i}{\partial t} = n_i(t)\left[\lambda_i + \sum_\mu \xi_i^\mu A^\mu(t)\right]$$

$$A^\mu(t) = A_0^\mu - \sum_j \xi_j^\mu n_j(t)$$

$$A_0^\mu = M + s\sqrt{M}x^\mu$$

$$\langle x^\mu \rangle = 0 \qquad \langle x^\mu x^\nu \rangle = \delta_{\mu\nu}$$

$$\langle\langle \xi_i^\mu \rangle\rangle = \xi \qquad \langle\langle (\xi_i^\mu - \xi)^2 \rangle\rangle = 1$$

Parameters :

$$s > 0 \ , \ \xi \ , \ \alpha$$

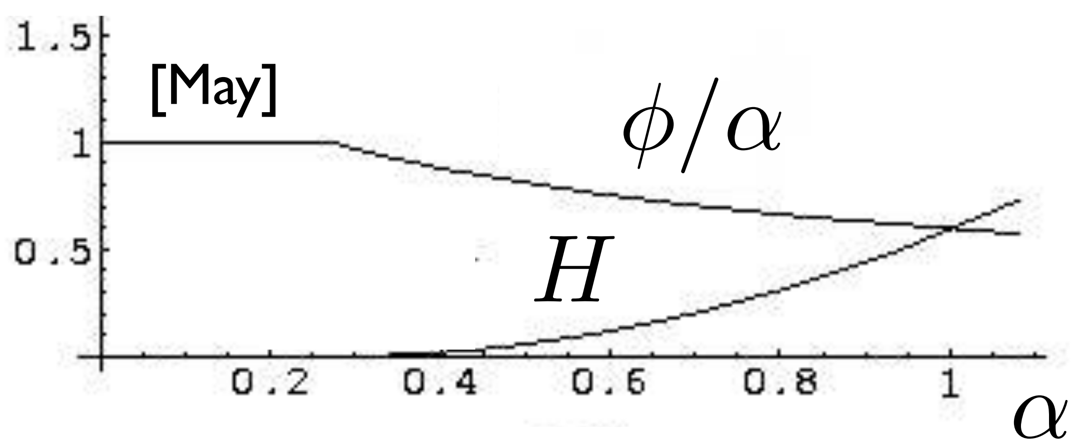
Results

Fraction of surviving species

$$\phi \begin{cases} = \alpha & \text{for } \alpha < \alpha_c \\ < \alpha & \text{for } \alpha > \alpha_c \end{cases}$$

Variance of P(A)

$$H \begin{cases} = 0 & \text{for } \alpha < \alpha_c \\ > 0 & \text{for } \alpha > \alpha_c \end{cases}$$



$$s\xi = 1$$

Results

$$A_0^\mu = M + s\sqrt{M}x^\mu$$

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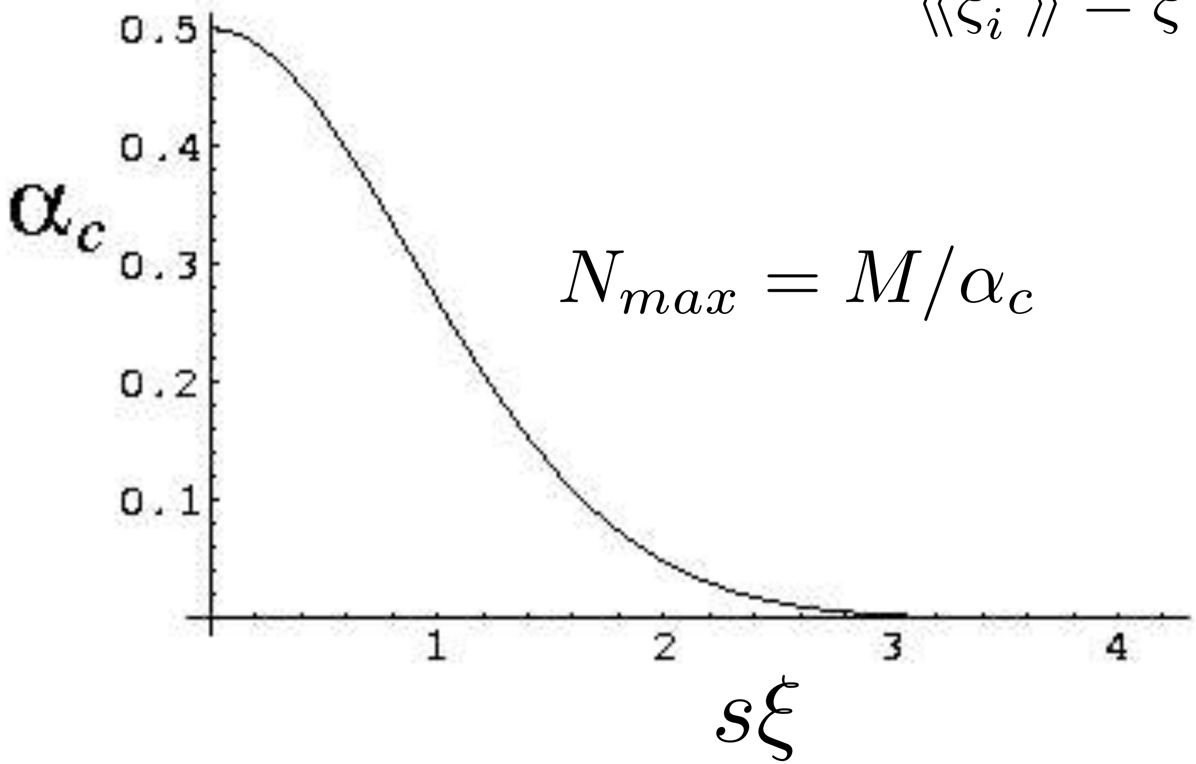


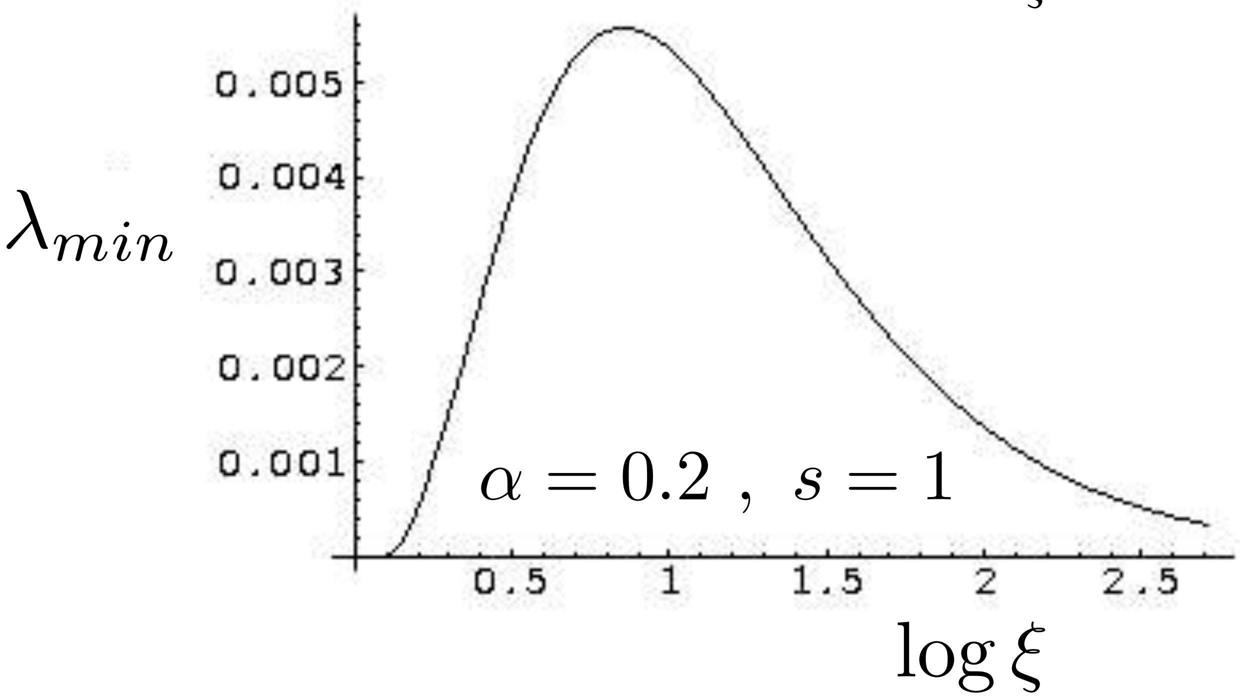
$$N_{max} = M/\alpha_c$$

Increasing s, more species can be sustained

But also increasing ξ

Stability

$$\lambda_{min} = \frac{1}{\xi}(\sqrt{\alpha} - \sqrt{\phi})$$



Note : adding species decreases stability (saturation)

GELPE

N firms, M commodities and an input-output matrix $\{\xi_i^\mu\}$

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$$x^\mu = y^\mu + \sum_i s_i \xi_i^\mu \qquad \forall \mu$$

$$\sum_\mu \xi_i^\mu = -\eta_i$$

$$U(x) = \sum_\mu \log x^\mu$$

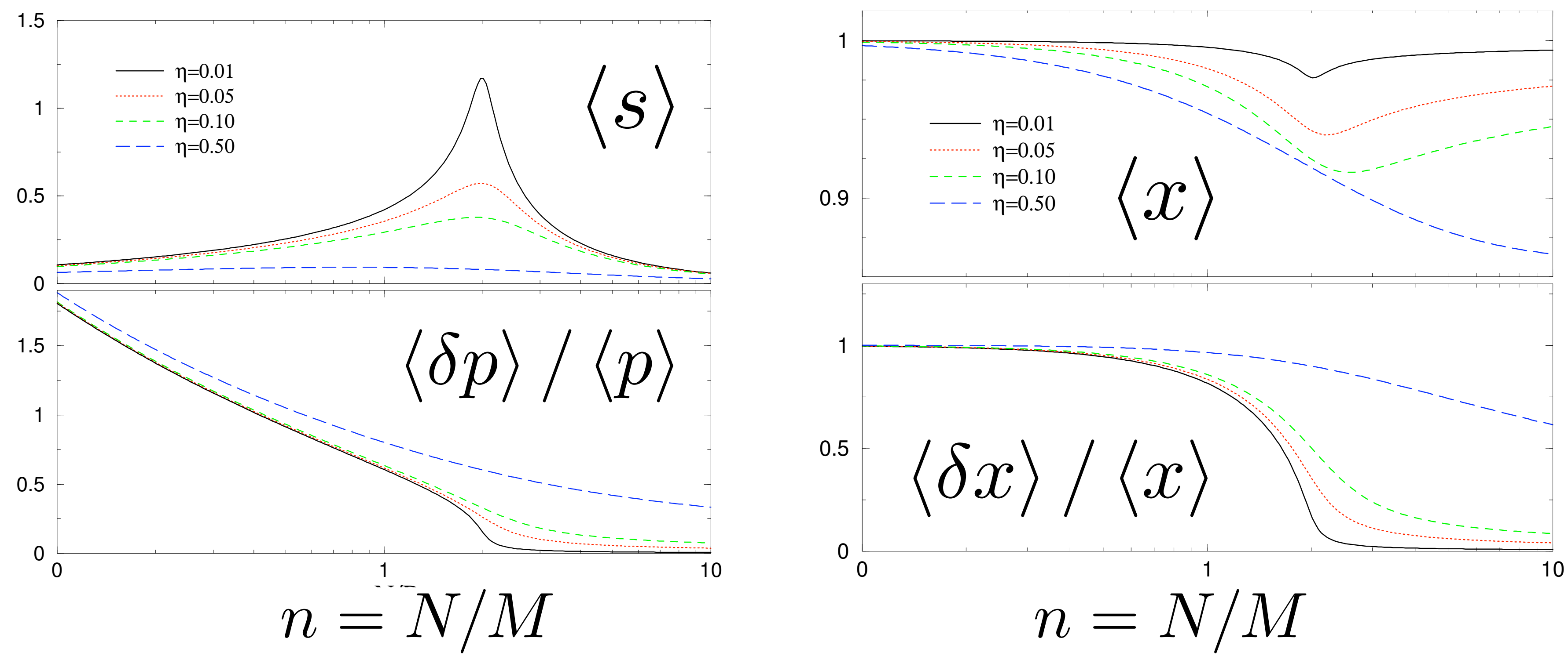
$$y^\mu \leftarrow \rho(y)$$

Basic idea

$$U(x) = \sum_{\mu} \log x^{\mu} \Rightarrow \text{optimize on the initial imbalance of commodities}$$

$$\rho(y) = \delta(y - \overline{y}) \Rightarrow \text{no activity at all : } P(s) = \delta(s)$$

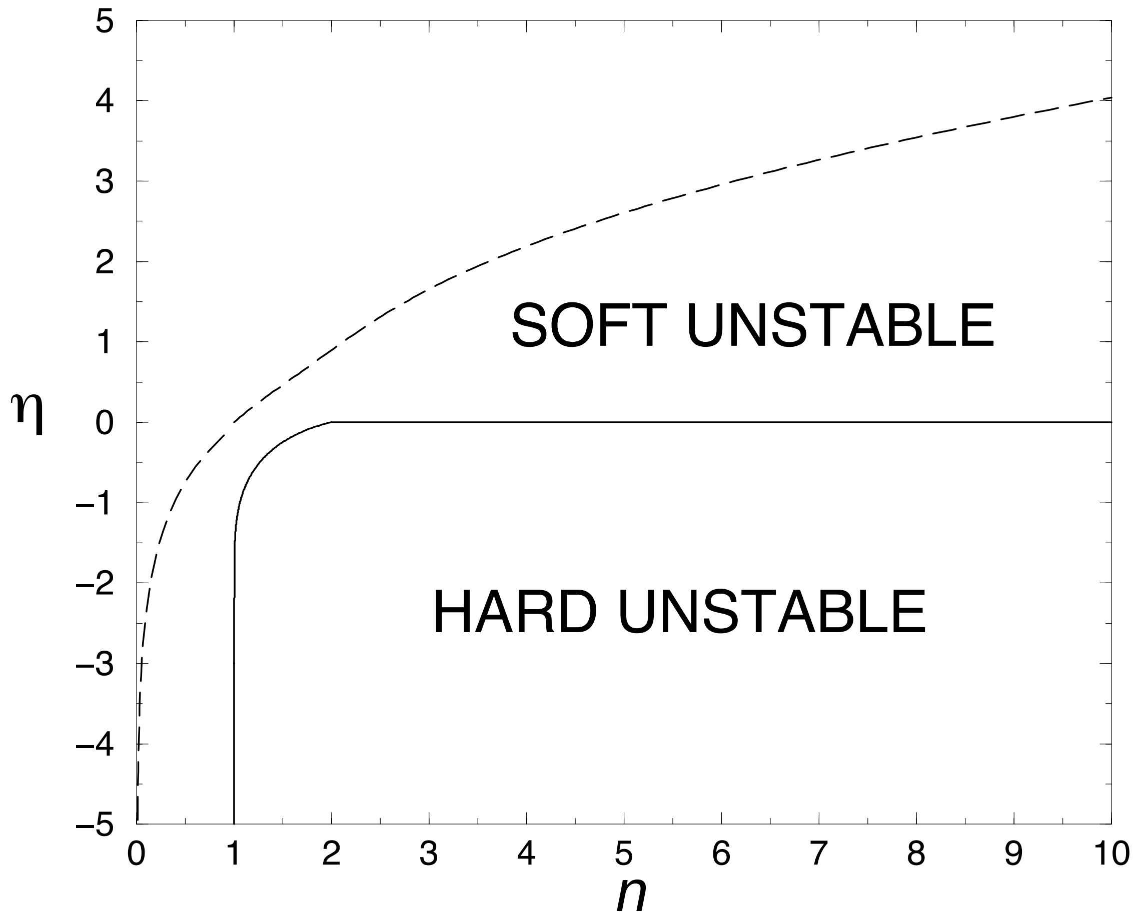
Typical properties [De M.-Marsili-Perez Castillo]



$$P(s) \sim s^{-\gamma}$$

$$\eta \to 0 \;:\; \langle s \rangle \sim |n - n_c|^{-1/2}$$

Stability [De M.-Marsili-Perez Castillo]

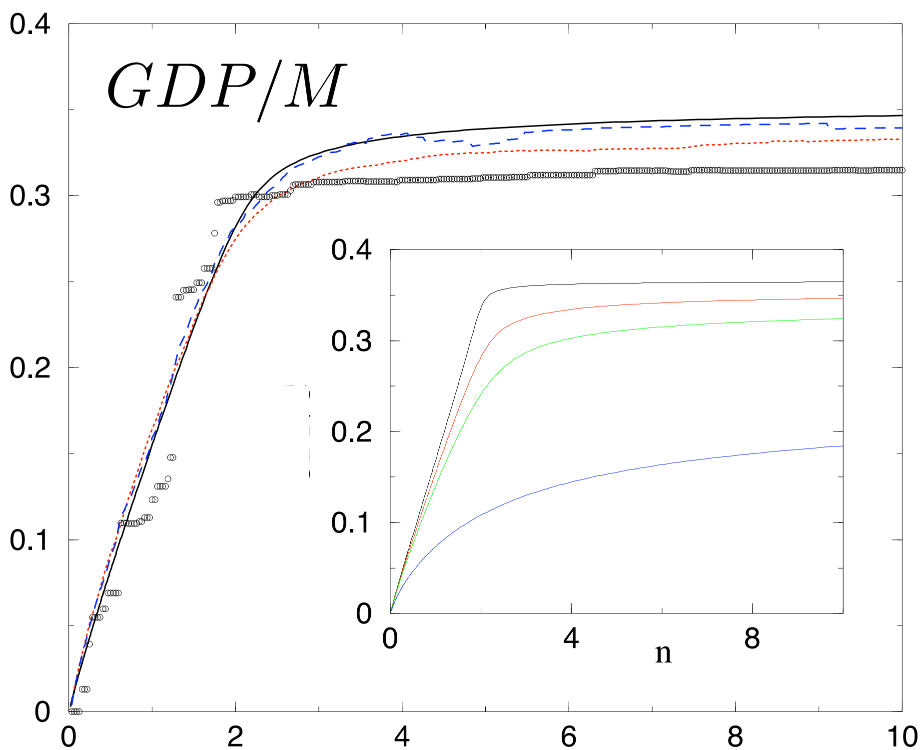


$$\langle\langle \xi_i^\mu \rangle\rangle = -\eta_i/M$$

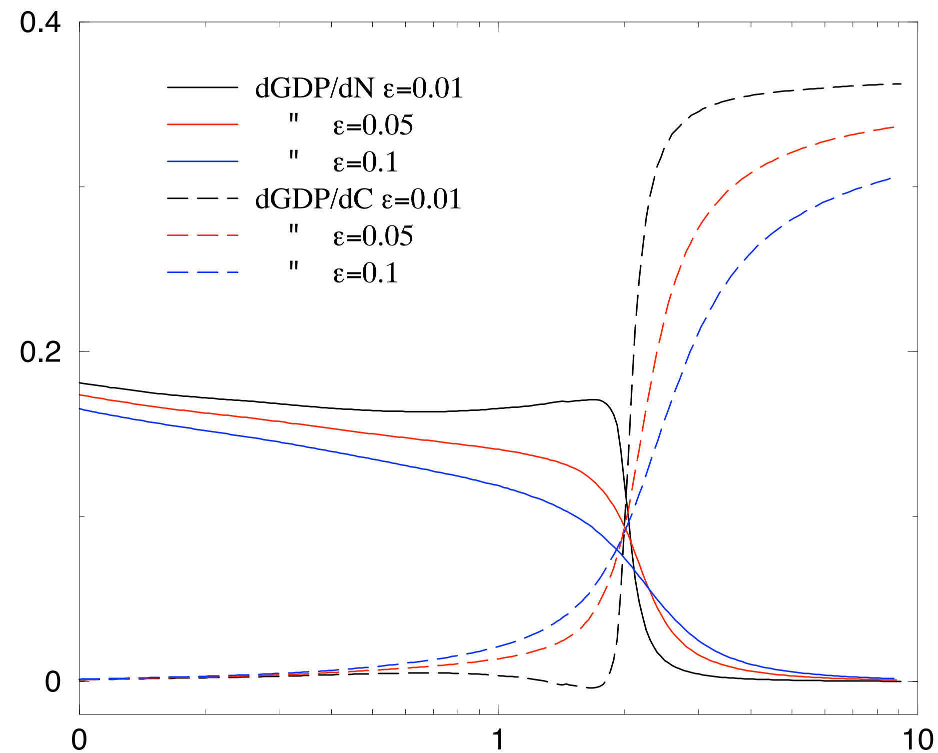
$$\sum_\mu \xi_i^\mu = -\eta_i$$

GDP [De M.-Marsili-Perez Castillo]

$$GDP = M\frac{\sum_\mu |x^\mu - y^\mu| p^\mu}{2\sum_\mu p^\mu}$$



$$n = N/M$$



$$n = N/M$$