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Bond diluted Levy spin-glass model and a new finite-size scaling method to determine a phase transition

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A spin-glass transition occurs both within and outside of the limits of validity of mean-field theory for a diluted one-dimensional chain of Ising spins where exchange bonds occur with a probability decaying as the inverse power of the distance. Varying the power in this long-range model corresponds, in a one-to-one relationship, to changing the dimension in spin-glass short-range models. Using different finite-size scaling methods, evidence for a spin-glass transition is also found for systems whose equivalent dimension is below the upper critical dimension at zero magnetic field. The application of a new method is discussed, which can be exported to systems in a magnetic field.

Keywords: spin glasses; phase transition; finite-size scaling; disordered systems; statistical mechanics; thermodynamics

1. Introduction

Long-range (LR) models are such that their lower critical dimension is lower than that of the corresponding short-range (SR) model. In particular, one can have a phase transition even in one-dimensional systems, provided the range of interaction is large enough. One-dimensional models with power-law decaying interactions actually allow us to explore both LR and SR regimes by changing the power, and enable us to compare the ordered phase within and outside of the range of validity of the mean-field approximation. This is very useful for spin-glass models that are known to have a rather complex state structure in the low-temperature phase in mean-field theory [1,2]. Whether this structure exists in finite-dimensional models with short-range interactions is, though, still a matter of debate. Theories alternative to the mean-field theory have been proposed [3,4], but SR systems are very tough to study analytically. Numerical simulations have, thus, been extensively employed, though with no conclusive indication of the nature of the spin-glass (SG) phase in finite dimension nor of the existence of a thermodynamic SG phase in the presence of an external magnetic field.

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A one-dimensional spin-glass model with power-law decaying interactions [5–11] makes it possible to probe larger linear sizes. Moreover, if bonds are diluted [12–15] the run time in numerical simulations grows simply as the size L of the system, rather than proportionally to L^2, as in fully connected systems. This is a fundamental issue because finite-volume effects are strong in these models, so much so that the very existence of the transition can be rather difficult to establish with canonical methods, e.g. when the interaction has a rapid decay and/or an external magnetic field is applied.

2. The one-dimensional Levy spin-glass model

The model investigated is a one-dimensional chain of L Ising spins ($\sigma_i = \pm 1$) and Hamiltonian

$$\mathcal{H} = -\sum_{ij} J_{ij}\sigma_i\sigma_j. \tag{1}$$

The quenched random couplings J_{ij} are independent and identically distributed random variables taking a non-zero value with a probability decaying with the distance between spins σ_i and σ_j, $r_{ij} = |i-j| \bmod (L/2)$, as

$$\mathbf{P}[J_{ij} \neq 0] \propto r_{ij}^{-\rho} \quad \text{for} \quad r_{ij} \gg 1. \tag{2}$$

Non-zero couplings take the value ± 1 with equal probability. We use periodic boundary conditions and a $z = 6$ average coordination number. Links are generated by repeating the following process $zL/2$ times: randomly choose two spins at distance r with probability $r^{-\rho}/\sum_{i=1}^{L/2} i^{-\rho}$; if they are already connected, repeat the process, otherwise connect them.

As the power ρ varies, this model is known to display different statistical mechanical behaviour [5–8]. For the diluted case [12] this behaviour is reported in Table 1. The equivalence between one-dimensional systems with power-law decaying interactions and D-dimensional systems with short-range (nearest neighbour) interactions can be approximately written as [14]

$$\rho - 1 = \frac{2-\eta}{D}, \tag{3}$$

Table 1. From infinite-range to short-range behaviour of the SG model defined in Equations (1) and (2).

ρ	$D(\rho)$	Transition type
≤ 1	∞	Bethe lattice like
$]1:4/3]$	$[6:\infty[$	Second order, MF
$]4/3:2]$	$[2.5:6[$	Second order, non-MF
2	2.5	Kosterlitz–Thouless or $T=0$-like
>2	<2.5	None

where η is the critical exponent of the space correlation function for the short-range model. The relationship is exact at the mean-field threshold $\rho = 4/3$ ($D = 6$, $\eta = 0$) and is approximated below. The analogue of a three-dimensional spin glass in zero magnetic field (with $-0.384(9) < \eta < -0.337(15)$ [16–20]) would then be a system with $\rho \simeq 1.8$. We will focus on this value in this paper to introduce and test our method. Other model cases have been considered in [12,13].

3. Numerical simulations

We simulate two replicas $\sigma_i^{(1,2)}$ using the parallel tempering (PT) algorithm [21]. The simulated sizes are $L = 2^\kappa$, with $\kappa = 6, 8, 10, 12$. The interval between temperatures in the PT evolution is $\Delta T = 0.05$. The number of samples is $N_J = 6400$, 25,600, 82,752, 89,600 for, respectively, $\kappa = 6$, 8, 10, 12. All data used for our analysis are thermalised. Thermalisation has been checked by measuring all observables on exponentially growing time windows until the last two points coincide within the statistical error.

4. Critical point

The key observable to approach the critical behaviour in disordered systems is the four-spin correlation function:

$$C_4(x) = \frac{1}{L}\sum_{i=1}^{L} \overline{\left\langle \sigma_i^{(1)}\sigma_i^{(2)}\sigma_{i+x}^{(1)}\sigma_{i+x}^{(2)} \right\rangle} \tag{4}$$

and its Fourier transform $\tilde{C}_4(k)$. In order to determine the critical point, a correlation length-like observable is usually defined on the one-dimensional lattice as [10,11,22]

$$\xi = \frac{1}{2\sin k_1/2}\left[\frac{\tilde{C}_4(0)}{\tilde{C}_4(k_1)} - 1\right]^{1/(\rho-1)} \tag{5}$$

with $k_1 = 2\pi/L$. In Figure 1, we present the ξ/L curves whose crossing point should tend, as $L \to \infty$, to T_c. In the inset, we also show the behaviour of

$$\chi_{SG} = L\tilde{C}_4(0), \tag{6}$$

another finite-size scaling (FSS) function for the present model (in which $\eta = \eta_{MF} = \rho - 1$ also for $\rho > \rho_{MF}$). Due to the statistical error it is not straightforward to identify clear crossing points for ξ/L. Moreover, in both the above mentioned cases, to extrapolate a clear limit of T_c as $L \to \infty$ with a FSS interpolating function like $a + bL^{-c}$, cf. Figure 4 below, three degrees of freedom are not enough and the interpolations are thus just indicative (see the following).

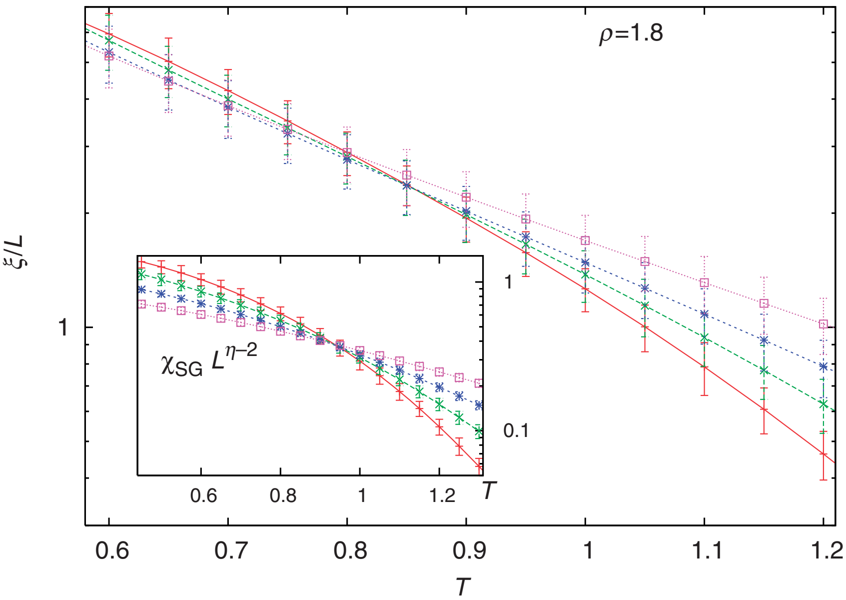

Figure 1. FSS function ξ/L vs. T for different simulated sizes at $\rho = 1.8$ for $L = 2^{6,\ 8,\ 10,\ 12}$. Inset: $\chi_{SG}L^{\eta-2}$ vs. T for the same sizes.

5. A novel method

We can, otherwise, use all of the information provided by $\tilde{C}_4(k)$. In Figure 2, we plot $1/\tilde{C}_4(k)$ vs. $[\sin(k/2)/\pi]^{\rho-1}$ for $\rho = 1.8$ for all simulated sizes both at temperatures above T_c (right) and at $T \simeq T_c$ (left). We observe that finite-size effects act in opposite ways on the value of $\tilde{C}_4(0)$ (i.e. χ_{SG}) and on the rest of the function $\tilde{C}_4(k > 0)$ (see the insets of Figure 2): while $1/\tilde{C}_4(0)$ tends to its thermodynamic limit from above, $1/\tilde{C}_4(k > 0)$ and its interpolation at $k = 0$ tend to the thermodynamic limit from below. Even though $\tilde{C}_4(0)$ and $\lim_{k\to 0}\tilde{C}_4(k)$ are the same object for $L \to \infty$, their FSS scaling is qualitatively different.

Interpolating $1/\tilde{C}_4(k)$ for small k at a given size and temperature as

$$F^{\rm fit}(k) = A(L, T) + B(L, T)[\sin(k/2)/\pi]^{0.8}, \tag{7}$$

we can analyse the L and T dependence of

$$A(L, T) \equiv \lim_{k\to 0} \tilde{C}_4(k) \tag{8}$$

and determine the transition from the FSS analysis of the points at which $A(L, T) = 0$, rather than using the FSS of the crossing points of Equation (5) (or $\tilde{C}_4(0)L^{\eta-1}$). For $\rho = 1.8$, the behaviour of A in L and T is plotted in Figure 3.

This method has the advantage of using high-temperature data and one only needs to simulate systems down to the candidate T_c. As $A(\infty, T)$ becomes negative, indeed, the functional form of the propagator in the paramagnetic phase breaks down and this provides evidence for a phase transition. In Figure 4, we plot the finite-size values of T_c obtained by this method and we compare them with the estimates derived from FSS functions ξ and χ_{SG}. At $\rho = 1.8$ we find $T_c = 1.060(7)$

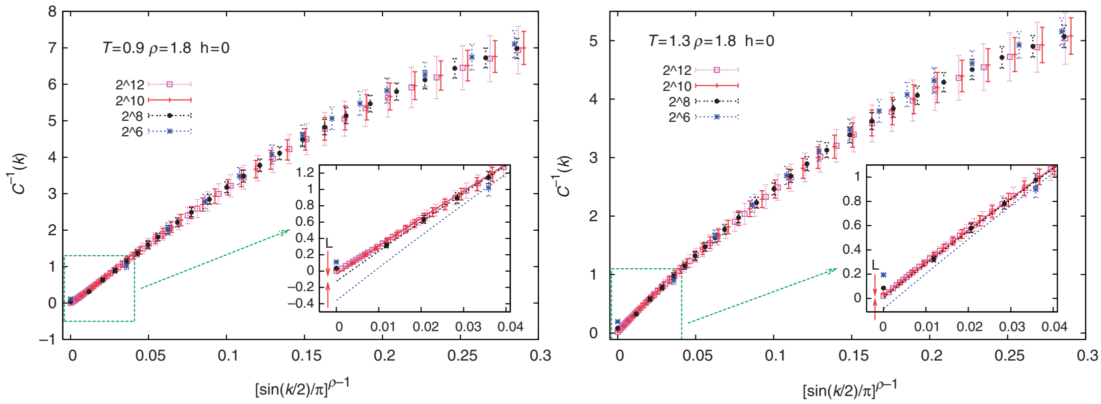

Figure 2. Average $\tilde{C}_4^{-1}(k)$ vs. $[\sin(k/2)/\pi]^{0.8}$ for systems of size $L=2^{6,8,10,12}$. Left: $T=0.9$, slightly below the critical region. Right: $T=1.3>T_c$. Insets: detail for low k (points) and comparison of the values for $k=0$ with the $k\to 0$ limit of the interpolation $A+B[\sin(k/2)/\pi]^{0.8}$ (lines).

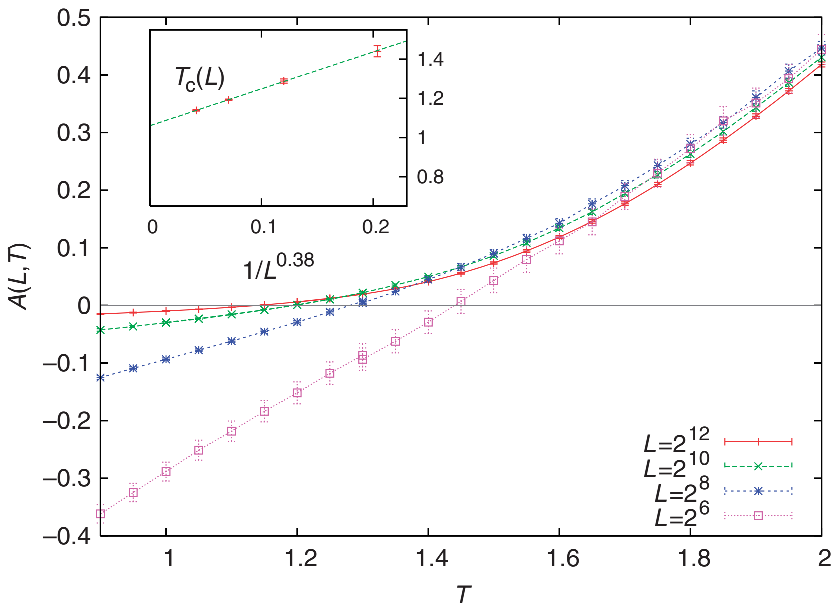

Figure 3. Behaviour of $A(L,T)$, cf. Equation (8), for $L=2^{6,8,10,12}$. The points at which $A=0$ are finite-size estimates of T_c.

and $1/\nu=0.38(2)$ with a $\chi^2=0.075$ on the same data as the previous analysis (the same statistics, same thermalisation times, same sizes).

We stress that also the definition of ξ as a correlation length in Equation (5) is valid only in the paramagnetic phase and that below T_c this is just a scaling function without physical meaning. In that approach, though, in order to appreciate crossings of $\xi_L(T)/L$ curves at different L one also has to simulate the system at temperatures below T_c, where thermalisation times increase, and for at least five different sizes in order to provide enough points for a well-defined FSS interpolation.

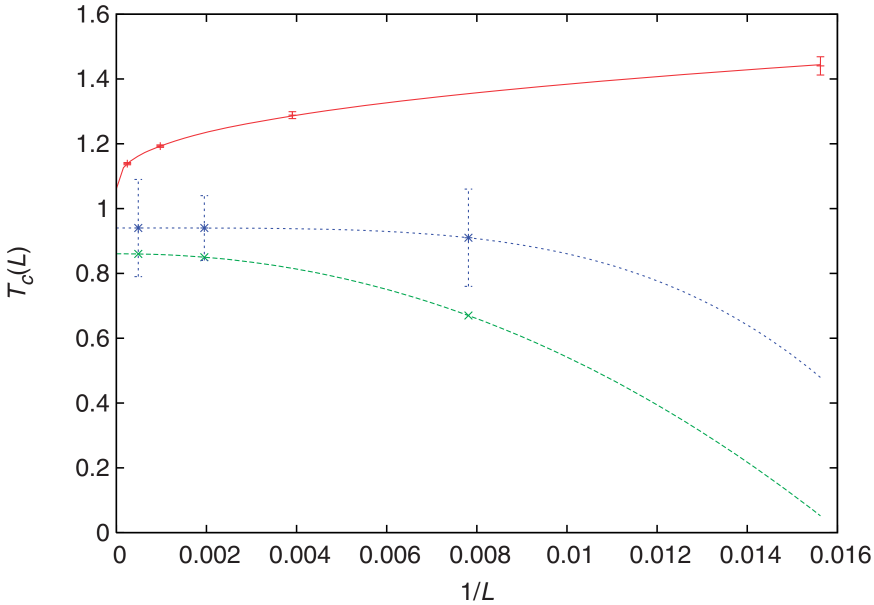

Figure 4. Comparison between the finite-size estimates of $T_c(L)$ by means of the canonical ‘crossing’ methods (of the FSS functions ξ/L and $\chi_{SG}L^{\eta-2}$) and of the $A(L, T)=0$ method, cf. Equation (8). The fit for T_c from $A(L, T)=0$ has a $\chi^2=0.075$ (full/red curve), while the other two interpolating curves (dashed/green for ξ/L, dotted/blues for $\chi_{SG}L^{\eta-2}$) are only indicative (not enough d.o.f. to provide an estimate of statistical errors, no error bars defined on the $T_c(L)$ from ξ/L crossings).

6. Correlation length estimate

We are interested in characterising the above mentioned critical behaviour by means of a growing correlation length, as it happens in ordinary continuous phase transitions in systems without quenched disorder. Also for this analysis our starting point is the four-spin correlation function, this time in position space. As one can notice from Figure 5, we can identify two different decays of the correlation at a given $T > T_c$, if we are able to study long enough Levy glass chains. We observe a slower power-law decay as $x \ll L$, $x^{-\alpha}$, and a faster decay as $x \sim L/2$. Contrary to what happens in short-range models at $D \geq 2.5$ [16,23], this second decay is also power-law (with power equal to ρ) because the interaction correlation decays – by construction – as $x^{-\rho}$ and, therefore, $C_4(x)$ cannot decay any faster.

From the definition of η as exponent of the anomalous decay of the correlation function as $x^{-d+2-\eta}$, with $\eta = 3-\rho$, we obtain the relation $\alpha+\rho = d+1 = 2$.[1] We can thus think about interpolating the whole $C_4(x)$ behaviour above the critical point as

$$C_4^{\text{fit}}(x) \equiv Kx^{-\alpha}\left[1+\left(\frac{x}{\ell}\right)^{2\delta(\rho-\alpha)}\right]^{-1/\delta} \tag{9}$$

with $\rho = 1.8$ and $\alpha = 2-\rho = 0.2$.

From this interpolation we can look at the temperature behaviour of the length-like parameter ℓ, which is an estimate of the correlation length of the system, as long as the second power-law decay is observable ($T > T_c$). In Figure 6, we display the

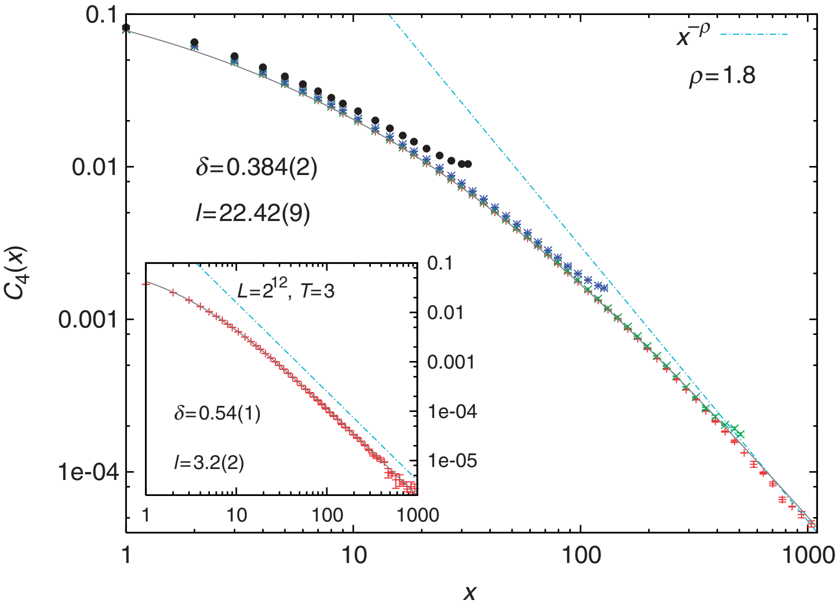

Figure 5. Four-spin correlation function at high temperature ($T = 2 \simeq 1.9\,T_c$) on a log–log scale for all simulated sizes $L = 2^{6,\ 8,\ 10,\ 12}$. Dotted lines: interpolation by means of Equation (9). Dashed-dotted line: $x^{-\rho}$. No finite-size effects are present, apart from the values at $x \simeq L/2$, and a crossover from a power law with exponent $\alpha \simeq 0.2$ to a power law with $\rho = 1.8$ can be identified for $L > 10$. Inset: $C_4(x)$ for $L = 2^{12}$ at $T = 3$ compared to $x^{-1.8}$.

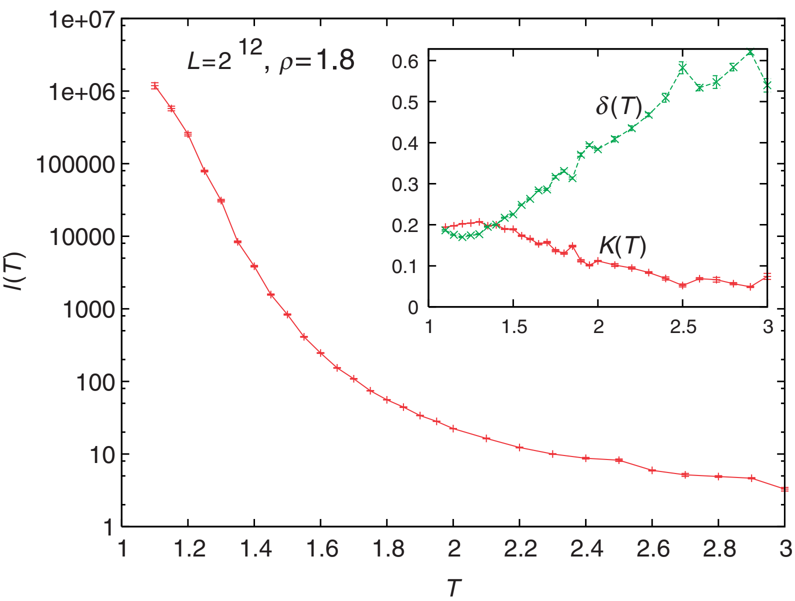

Figure 6. Correlation length parameter ℓ of the interpolating function $C_4^{\rm fit}(x)$ vs. T on a log scale for $L = 2^{12}$. Inset: T behaviour of fit parameters K and δ.

behaviour of $\ell(T)$ (and K and δ in the inset) as T is lowered down to the estimated T_c. Fits with Equation (9) are reasonable down to $T = 1.1$.

As the temperature approaches the critical value from above, the simulated systems are too small to appreciate the existence of a crossover length $\ell \gg L$ and,

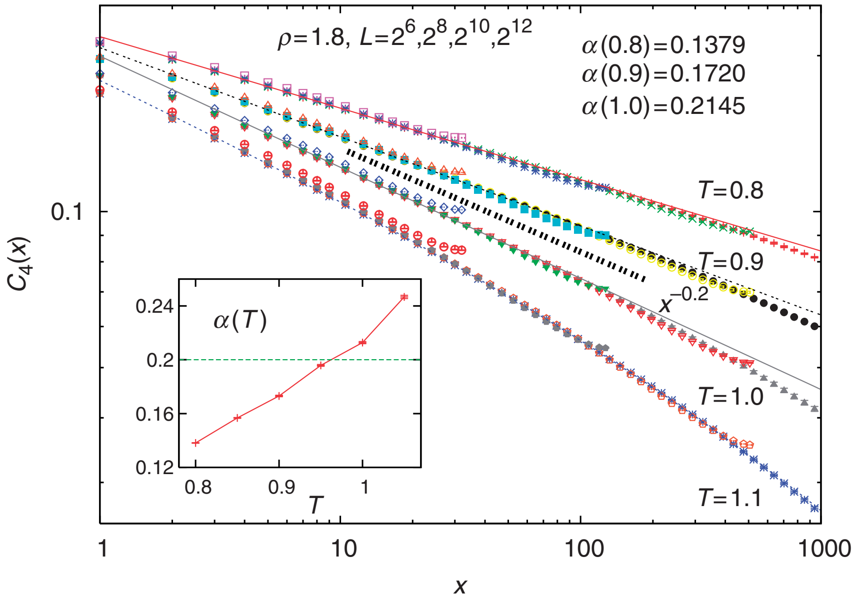

Figure 7. Four-spin correlation function across the critical region ($T = 0.8, 0.9, 1, 1.1$) on a log–log scale for all simulated sizes $L = 2^{6,8,10,12}$ and relative interpolating functions. For all T, finite-size effects are seemingly small. Dotted lines: interpolation by means of Equation (9) at $T = 1.1$; for lower T the fit function $Ax^{-\alpha}$ is used and no crossover to a power law with $\rho = 1.8$ is observed for any L, in agreement with $\ell \gg L$. The T-dependence of the value of α is shown in the inset.

thus, the $C_4(x)$ decay appears as a single power law. The same behaviour remains below T_c, as shown in Figure 7. This is incompatible with the onset of a plateau at any x, whereas it is consistent with the clustering properties of the mean-field theory for spin glasses.

7. Conclusions

We have introduced a new FSS method to determine, by numerical simulations, the existence of a critical point in finite-dimensional systems. This method employs high-temperature data. Thus, it requires lower thermalisation times and disordered sample statistics with respect to canonical methods. It works well if one has a sufficient number of points of the four-spin correlation function at low wavelength numbers k, that is, if the interaction range is not too broad (i.e. preferably outside of the mean field) and the linear size is long enough.

We have tested the method in the case of a bond-diluted one-dimensional Levy spin glass with a power-law decaying interaction outside the limit of validity of the mean-field approximation and equivalent to a three-dimensional nearest-neighbour interacting system on a cubic lattice. We have compared the results with those obtained by canonical FSS analysis on the same set of data and shown that they are compatible, cf. Figure 4.

In position space, at $T > T_c$ we have identified a crossover in the (power-law) relaxation decay of the four-spin correlation function from slow to fast (i.e. $x^{-\rho}$)

relaxation, cf. Equation (9). The crossover takes place at a correlation length that grows exponentially as T decreases. At $T < T_c$ the decay is well interpolated by a simple power law, providing no evidence for the plateau predicted by droplet [3] and TNT theories [4].

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Note

1. This can be seen, e.g. in Figure 6 of [12] for $\rho = 1.5$. In the inset of figure 6, α was erroneously misplaced with δ.

References

[1] G. Parisi, J. Phys. A 13 (1980) p.L115.
[2] M. Mezard, G. Parisi and M. Virasoro, *Spin Glass Theory and Beyond*, World Scientific, Singapore, 1987.
[3] D.S. Fisher and D.A. Huse, Phys. Rev. Lett. 56 (1986) p.1601.
[4] F. Krzakala and O.C. Martin, Phys. Rev. Lett. 85 (2000) p.3013.
[5] A.C.D. van Enter and J.L. van Hemmen, J. Stat. Phys. 32 (1983) p.141.
[6] A.C.D. van Enter and J.L. van Hemmen, J. Stat. Phys. 39 (1985) p.1.
[7] G. Kotliar, P.W. Anderson and D.L. Stein, Phys. Rev. B 27 (1983) p.602.
[8] M. Campanino, E. Olivieri and A.C.D. van Enter, Commun. Math. Phys. 108 (1987) p.241.
[9] L. Leuzzi, J. Phys. A 32 (1999) p.1417.
[10] H.G. Katzgraber and A.P. Young, Phys. Rev. B 67 (2003) p.134410.
[11] H.G. Katzgraber and A.P. Young, Phys. Rev. B 68 (2003) p.224408.
[12] L. Leuzzi, G. Parisi, F. Ricci-Tersenghi and J.J. Ruiz-Lorenzo, Phys. Lett. Rev. 101 (2008) p.107203.
[13] L. Leuzzi, G. Parisi, F. Ricci-Tersenghi and J.J. Ruiz-Lorenzo, Phys. Rev. Lett. 103 (2009) p.267201.
[14] H.G. Katzgraber, D. Larson and A.P. Young, Phys. Rev. Lett. 102 (2009) p.177205.
[15] D. Larson, H.G. Katzgraber, M.A. Moore and A.P. Young, Phys. Rev. B 81 (2010) p.064415.
[16] E. Marinari, G. Parisi and J.J. Ruiz-Lorenzo, Phys. Rev. B 58 (1998) p.14852.
[17] H.G. Ballesteros, A. Cruz, L.A. Fernandez, V. Martin-Mayor, J. Pech, J.J. Ruiz-Lorenzo, A. Tarancon, P. Tellez, C.L. Ullod and C. Ungil, Phys. Rev. B 62 (2000) p.14237.
[18] T. Jörg, Phys. Rev. B 73 (2006) p.224431.
[19] H.G. Katzgraber, M. Korner and A.P. Young, Phys. Rev. B 73 (2006) p.224432.
[20] M. Hasenbusch, A. Pelissetto and E. Vicari, J. Stat. Mech. (2008) p.L02001.
[21] K. Hukushima and K. Nemoto, J. Phys. Soc. Jpn. 65 (1996) p.1604.
[22] S. Caracciolo, R.G. Edwards, A. Pelissetto and A.D. Sokal, Nucl. Phys. B 403 (1993) p.475.
[23] P. Contucci, C. Giardina, C. Giberti, G. Parisi and C. Vernia, Phys. Rev. Lett. 99 (2007) p.057206.