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# Error threshold estimates for surface code with loss of qubits

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We estimate optimal thresholds for surface codes in the presence of loss via an analytical method developed in statistical physics. The optimal threshold for the surface code is closely related to a special critical point in a finite-dimensional spin glass, which is disordered magnetic material. We compare our estimations to the heuristic numerical results reported in earlier studies. Further application of our method to the depolarizing channel, a natural generalization of the noise model, unveils its wider robustness even with loss of qubits.

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*Introduction*. Against corruption by environmental noise as well as imperfection in implementation, the state of qubits describing quantum information cannot be stable and must be recovered by elaborate procedures, such as quantum error correction [1,2]. Quantum error corrections usually work on the computational error on qubits, which do not go out of the basis for computations. Therefore errors come from losses of the physical resource, qubits, which can degrade the performance of the error correction. However, if one can detect and identify the locations of losses, a modified scheme can recover the original information. Stace and Barret have suggested an error-correcting code, a family of Kitaev's surface codes [3], which is robust against both the computational errors and losses by a modified scheme based on the location of the lost qubits [4].

In the present paper, we estimate precise values of the error thresholds for the modified error correcting code against both of the computational errors and loss of qubits by use of a systematic theory developed in statistical physics. The key of our analysis is hidden in the disordered magnetic system, spin glasses. Several spin glass models have a special symmetry with exact solvable subspace known as the Nishimori line [5,6]. The critical point in this subspace, called the multicritical point, corresponds to the optimal error threshold in the surface code [3]. A combination of the duality with the real-space renormalization technique, which is often used to identify the singular points in statistical mechanical models, can derive precise estimations for the optimal error thresholds [7,8] and systematically approach the exact solutions [9,10]. By using this method, we fill in the blanks on the analytical study of the optimal error thresholds on several surface codes with loss of qubits. The results reported in this paper provide upper bounds against error rates for any error-correcting schemes. They serve as important benchmarks against which any constructive error-correcting procedure as recently proposed in Ref. [11] can be compared.

*Surface code and spin glass*. Let us consider qubits set on each edge $(ij)$ of the square lattice embedded on a torus (genus 1). We define the star operator $X_s = \otimes_{(ij)\in s} X_{(ij)}$ for each site $s$, and plaquette operator $Z_p = \otimes_{(ij)\in p} Z_{(ij)}$ for each plaquette $p$ (site on the dual lattice), where $X$ and $Z$ are Pauli matrices. The product consists of four edges adjacent to each site or plaquette. The stabilizer group is given by the simultaneous eigenstates with the positive eigenvalues for these operators $X_s$ and $Z_p$. Since the star and plaquette operators consist of unit loops on the dual and original square lattices, any contractible loop by $X_s$ and $Z_p$ products on each lattice acts trivially on the code space. On the other hand, any noncontractible loops on the lattice can map the code space to itself in a nontrivial manner. If we set $L \times L$ lattice on a torus, we have $2L^2$ qubits and $2(L^2-1)$ stabilizers. The remaining degrees of freedom of 2 implies existence of two noncontractible loops, winding around the hole of the torus $L_v$ and winding around the body of the torus $L_t$, and ones $L_v^*$ and $L_t^*$ on the dual lattice. These loops can be written in terms of the products of operators as $\bar{Z}_v = \prod_{(ij)\in L_v} Z_{(ij)}$, $\bar{X}_v = \prod_{(ij)\in L_v^*} X_{(ij)}$, $\bar{Z}_t$, and $\bar{X}_t$, which are called logical operators. The logical operators can form Pauli algebra of two effective qubits encoded in the topological degrees of freedom on the torus as $[\bar{Z}_v, \bar{Z}_t] = [\bar{X}_v, \bar{X}_t] = 0$, and $\bar{X}_t \bar{Z}_v = (-1)^{\delta_{tv}} \bar{Z}_v \bar{X}_t$. The combinations of noncontractible loops yield $2^4 = 16$ different homology classes for the original and dual square lattices on a single torus. We need to distinguish them for protecting the information from corruption.

To evaluate the performance of the error-correcting code, let us define a noise model where each qubit independently gets errors as

$$\rho \to p_I \rho + (p_X X\rho X + p_Y Y\rho Y + p_Z Z\rho Z). \qquad (1)$$

Although, if we employ the following analytical method, we can estimate precise values of the error thresholds for "any" cases of $p_I, p_X, p_Y$, and $p_Z$, we restrict ourselves to two cases: $p_X = p_Z = p$, $p_Y = p^2$, and $p_I = (1-p)^2$ (uncorrelated case), and $p_X = p_Y = p_Z = p/3$ and $p_I = 1-p$ (depolarizing channel case) for simplicity, where $0 \leqslant p \leqslant 1$. The error $Y_{(ij)}$ can be regarded as a multiple error $X_{(ij)}$ and $Z_{(ij)}$. The errors $Z_{(ij)}$ and $X_{(ij)}$ can be described as chains $E$ and $E^*$ on the original and dual lattices. The endpoints of the error chains $\partial E$ and $\partial E^*$ can be detected by applications of star and plaquette operators due to anticommutation of adjacent errors with operators. From the knowledge of endpoints $\partial E$ and $\partial E^*$ without the homology class of the error chains, error syndrome, we infer the most likely homology class of error chains, while considering any reasonable choices. Since $E' = E + C$ and $E'^* = E^* + C^*$, where $C$ and $C^*$ are the contractible loops on both of the lattices, are in an equivalent class with the error chains, the probability for the homology class $\bar{E}$ and $\bar{E}^*$ of the

 

error chains can be written as [3]

$$P(\bar{E},\bar{E}^*|\partial E,\partial E^*) = P(\bar{E},\bar{E}^*) \Big/ \sum_i P_{D_i}(\bar{E},\bar{E}^*), \quad (2)$$

where $P(\bar{E},\bar{E}^*) \propto \sum_{C,C^*} \prod_{\langle ij\rangle} \exp(K\tau_{ij}^E \tau_{ij}^C + K\tau_{ij}^{E^*}\tau_{ij}^{C^*})$ for the uncorrelated case. The summation is taken over all the possibilities of $C$ and $C^*$, and the product is over all the edges. The parameter $K$ stands for the importance of or preference to choose the inferred error chain. The quantity in the denominator $P_{D_i}(\bar{E},\bar{E}^*)$ denotes the probability with the different homology class specified by the logical operators $D_i$ ($i = 1,2,\ldots,2^4$). We here use $\tau_{ij}^E$ to represent the inferred error chains, which takes $\pm 1$ [$\tau_{ij}^E < 0$, when $(ij) \in E$], and also for $E^*$, $C$, and $C^*$. The loop constraints $\prod_{(ij)} \tau_{ij}^C = 1$ and $\prod_{(ij)} \tau_{ij}^{C^*} = 1$ allow us to use another expression by the Ising variables $\tau_{(ij)}^C = \sigma_i\sigma_j$ and $\tau_{(ij)}^{C^*} = \sigma_i^*\sigma_j^*$ for each lattice on the torus. By using these expressions, we can find that $P(\bar{E},\bar{E}^*)$ is written as the square of the partition function of the $\pm J$ Ising model

$$P(\bar{E},\bar{E}^*) \propto \sum_{\sigma,\sigma^*} \prod_{\langle ij\rangle} e^{K(\tau_{ij}^E\sigma_i\sigma_j + \tau_{ij}^{E^*}\sigma_i^*\sigma_j^*)}, \quad (3)$$

where $\tau_{ij}^E$ and $\tau_{ij}^{E^*}$ are the signs of the quenched random couplings in the context of spin glasses. When we set $K = K_{\rm ind}$, where $\exp(2K_{\rm ind}) = (1-p)/p$ (Nishimori line), the inference of the error chains is an optimal recovery procedure to identify the most likely homology class [5]. Each of the quenched random couplings follows the distribution function of the error chains $P(E,E^*) = \prod_{\langle ij\rangle} P(\tau_{ij}^E)P(\tau_{ij}^{E^*})$ for the uncorrelated case, where

$$P\left(\tau_{ij}^E\right) = (1-p)\delta_{\tau_{ij}^E,1} + p\delta_{\tau_{ij}^E,-1}. \quad (4)$$

Similarly, we can evaluate the probability $P(\bar{E},\bar{E}^*)$ for the homology class of the error chains $\bar{E}$ and $\bar{E}^*$ for the depolarizing channel case as

$$P(\bar{E},\bar{E}^*) \propto \prod_{\langle ij\rangle} e^{K\tau_{ij}^{\bar{E}} + K\tau_{ij}^{\bar{E}^*} + K\tau_{ij}^{\bar{E}}\tau_{ij}^{\bar{E}^*}}, \quad (5)$$

where we set the parameter $K = K_{\rm dep}$ as on the Nishimori line $\exp(4K_{\rm dep}) = 3(1-p)/p$. This is written in terms of the partition function of the eight-vertex model with quenched random interaction [12]

$$Z_{\rm dep} = \sum_{\sigma,\sigma^*} \prod_{\langle ij\rangle} e^{K(\tau_{ij}^E\sigma_i\sigma_j + \tau_{ij}^{E^*}\sigma_i^*\sigma_j^* + \tau_{ij}^E\tau_{ij}^{E^*}\sigma_i\sigma_j\sigma_i^*\sigma_j^*)}, \quad (6)$$

where $\tau_{ij}^E$ and $\tau_{ij}^{E^*}$ follow the distribution function through $P(E,E^*) = \prod_{\langle ij\rangle} P_{\rm dep}(\tau_{ij}^E,\tau_{ij}^{E^*})$ as $P_{\rm dep}(1,1) = 1-p$ while $P_{\rm dep}(1,-1) = P_{\rm dep}(-1,1) = P_{\rm dep}(-1,-1) = p/3$. We emphasize that, if we tune the probability function appropriately, we can apply our analysis as shown below to the inhomogeneous case with $p_X \neq p_Y \neq p_Z$.

In the context of statistical physics, $D_i$ represents the domain wall. In the low-temperature region implying a small $p$, the order of the degrees of freedom suppresses the fluctuation of the domain wall. The cost for the free energy difference due to the domain wall diverges as $\sum_{E,E^*} P(E,E^*)P(\bar{E},\bar{E}^*|\partial E,\partial E^*) \to 1$ for $L \to \infty$. This means that we can infer the equivalent class with the original error chains. On the other hand, in the high-temperature region, the cost vanishes and $\sum_{E,E^*} P(E,E^*)P(\bar{E},\bar{E}^*|\partial E,\partial E^*) \to 1/16$. This implies that the failure of the recovery occurs at the critical point. Therefore the location of the critical point on the Nishimori line, the multicritical point, identifies the optimal error threshold.

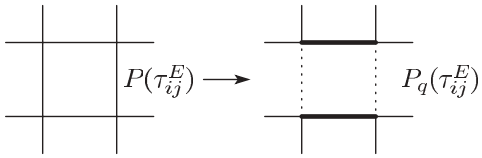


FIG. 1. Reconstruction of the damaged lattice by use of the weight-zero and irregular weight. The dashed line denotes the lack of qubits and weight-zero edge. The bold line expresses the irregular weight edges after the reconstruction. In this case, $n = 2$.

*Loss of qubits and bond dilution.* Loss of qubits on the lattice implies the modification of the stabilizers as well as the logical operators. However, we can reform a complete set of stabilizers even on the damaged lattice due to a loss of qubits following the proposed scheme in Ref. [4]. The effect of lost qubits appears in the pattern of error chains $E$ and $E^*$, and their weight for the probability, which degrades the performance of the error-correcting code. To infer the most likely homology class based on the knowledge of the error chains on the damaged lattice, we reconstruct the original lattice by assigning weight-zero edges on the lost qubits and irregular weight edges $p'$ adjacent to the lost qubits as $1-2p'^n$, where $n$ is the number of the shared qubits in adjacent edges as in Fig. 1. The weight-zero edges imply that we need to consider a diluted version of the original spin glass system as in Eqs. (3) and (6). It can be achieved by a simple modification of the distribution function for $\tau_{ij}^E$ and $\tau_{ij}^{E^*}$ into, for the uncorrelated case, $P^q(\tau_{ij}^E) = (1-q)P(\tau_{ij}^E) + q\delta_{\tau_{ij}^E,0}$ and $P^q(\tau_{ij}^{E^*})$, where $q$ denotes the ratio of loss of qubits. Similarly, for the depolarizing channel case, $P_{\rm dep}^q(\tau_{ij}^E,\tau_{ij}^{E^*}) = (1-q)P_{\rm dep}(\tau_{ij}^E,\tau_{ij}^{E^*}) + q\delta_{\tau_{ij}^E\tau_{ij}^{E^*},0}$. In addition, we have to take into account the effects of irregular weight edges $p'$ adjacent to the lost qubits as carefully discussed in Ref. [13]. The effect can be described by highly correlated distribution function depending on the pattern of the lost qubits, although we omit its detailed expression.

*Duality analysis for spin glasses.* Analyses to clarify the critical phenomena in finite-dimensional spin glasses are intractable in general. However, a recent development in spin glass theory enables us to estimate the precise value of the special critical point on the Nishimori line, which corresponds to the optimal error threshold [7–10]. The method as shown below is based on the duality, which can identify the location of the critical point especially on two-dimensional spin systems [14]. Let us first review the simple pure Ising model case. The duality is a symmetry argument that considers the low- and high-temperature expansions of the partition function $Z = \sum_{\sigma_i} \prod_{\langle ij\rangle} \exp(K\sigma_i\sigma_j)$. The painful calculation of both expansions can be replaced by a simple manipulation with the binary Fourier transformation for the local part of the Boltzmann factor, namely, the edge Boltzmann factor

$x_0 = \exp(K)$ and $x_1 = \exp(-K)$ [14]. The low-temperature expansion can be expressed by $x_0$ and $x_1$. On the other hand, the high-temperature expansion is given by the binary Fourier transformation $x_0^* = (x_0 + x_1)/\sqrt{2}$ and $x_1^* = (x_0 - x_1)/\sqrt{2}$. We use this fact and find a double expression of the partition function as

$$\{x_0(K)\}^{2L^2} z(u_1(K)) = \{x_0^*(K)\}^{2L^2} z(u_1^*(K)), \tag{7}$$

where $z$ is the normalized partition function $z(u_1) = Z/\{x_0(K)\}^{2L^2}$ and $z(u_1^*) = Z/\{x_0^*(K)\}^{2L^2}$. We here define $u_1(K) = x_1(K)/x_0(K) = \exp(-2K)$ and $u_1^*(K) = x_1^*(K)/x_0^*(K) = \tanh K$. The well-known duality relation $\exp(-2K^*) = \tanh K$ is given by rewriting $u_1^*(K)$ by $u_1(K^*)$, which implies a transformation of the temperature. Then the principal Boltzmann factors $x_0(K)$ and $x_0^*(K)$ with edge spins parallel holds $x_0(K_c) = x_0^*(K_c)$ at the critical point $\exp(-2K_c) = \tanh K_c$.

We employ the replica method, which is often used in theoretical studies on spin glasses, in order to generalize the duality analysis to spin glasses [7,8]. Let us consider the duality for the replicated partition function as $[Z_{\rm ind}^n]$ and $[Z_{\rm dep}^n]$ simply $[Z^n]$, where $[\ldots]$ is the configurational average for the quenched randomness according to the distribution functions. The multiple $(2^n)$ Fourier transformation again leads us to the double expression of the replicated partition function as

$$\begin{aligned}&\{x_0(q,K)\}^{2L^2} z(u_1(q,K), u_2(q,K), \ldots) \\ &\quad = \{x_0^*(q,K)\}^{2L^2} z(u_1^*(q,K), u_2^*(q,K), \ldots),\end{aligned} \tag{8}$$

where the subscript of $u_k$ and $u_k^*$ stands for the number of antiparallel pairs among $n$ replicas on each edge. Unfortunately we cannot replace $u_k^*(q,K)$ by $u_k(q^*,K^*)$ as the pure case, since the replicated partition function is multivariable. Nevertheless we can estimate the precise location of the critical point even for spin glasses by considering a wider range of the local part of the Boltzmann factor given after the summation of the internal spins. For instance, in the case on the square lattice, we define the cluster Boltzmann factor $x_k^{\rm cl}$, where the subscript $k$ denotes the configuration of the edge (white-colored) spins as in Fig. 2. We set the equation to lead the location of the critical point as, inspired by the case without quenched randomness,

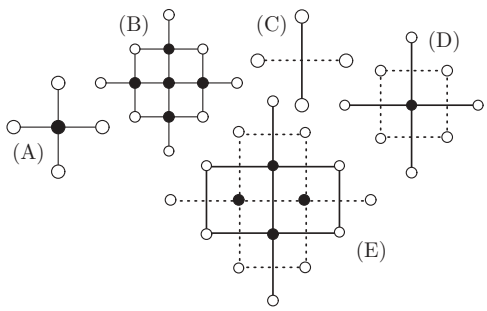


FIG. 2. Clusters for the uncorrelated and depolarizing channel cases. The cluster Boltzmann factor is defined without omitting many-body interactions generated after the decimation of the black spins on the original lattice. The dotted line denotes the dual lattice for $\sigma_i^*$ and $\tau_{(ij)}^{E^*}$.

TABLE I. Comparison of the approximations by the single bond and the clusters A and B for the uncorrelated case and by a heuristic method [4]. We add the improved result given by Ref. [13] to the above list, $p_c^{(0)} = 0.1065$.

| $q$ | $p_c$ | $p_c$ (A) | $p_c$ (B) | $p_c^0$ [4] |
|---|---|---|---|---|
| 0.00 | 0.11003 | 0.10928 | 0.10918 | 0.10486 |
| 0.10 | 0.09240 | 0.09196 | 0.09189 | 0.08816 |
| 0.20 | 0.07245 | 0.07235 | 0.07233 | 0.06997 |
| 0.30 | 0.04984 | 0.05004 | 0.05009 | 0.04836 |
| 0.40 | 0.02462 | 0.02492 | 0.02500 | 0.02561 |
| 0.45 | 0.01155 | 0.01174 | 0.01179 | 0.00757 |

$x_0(K) = x_0^*(K)$ [7–10],

$$x_0^{\rm cl}(q,K) = x_0^{\rm cl*}(q,K). \tag{9}$$

The equality even without the use of the cluster can give the precise solutions of the critical point for the multicritical point of $\pm J$ Ising model $q = 0$ as $p_c = 0.1100$ [7,8]. Although the above method is not exact, if we increase the size of the used cluster, we can systematically approach the exact solution for the critical points of the $\pm J$ Ising model in the higher temperature region than the Nishimori line [9,10].

*Results.* In the present study, we consider two clusters as A and B as well as a single edge for the uncorrelated case and a single crossing edge C and two clusters D and E for the depolarizing case as in Fig. 2. We show in Table I several estimations given by Eq. (9) for the uncorrelated case. Although, for the uncorrelated case of the B cluster, we have considered the highly correlated distribution function by taking into account the effect of the irregular weight, the results have not been changed from those obtained by a simple distribution function $P_q(E)$, which is the same as one for the bond-diluted spin glass. As discussed in Ref. [13], the high correlation between the loss of qubits is found to emerge as a finite-size effect in the numerical investigation (for $q \geqslant 0.45$). Such a complicated effect does not spoil our analysis. All the results for any $q$ do not show drastic changes that depend on the size of the used cluster. This means that our analyses are correct enough to capture accurate locations of the optimal error threshold. The optimal error thresholds indicate the upper bounds for error threshold by any heuristic methods. As shown in Fig. 3, we compare our results with

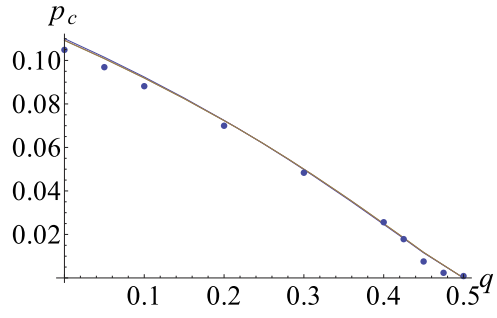


FIG. 3. (Color online) Results for the uncorrelated case. The dots stand for numerical data given in Ref. [4], where the error bars are suppressed in this scale. The curves are almost overlapped in this scale and express our results by the duality.

TABLE II. Results for the depolarizing channel case.

| $q$ | $p_c$ (C) | $p_c$ (D) | $p_c$ (E) | $p_c$ [11] |
|---|---|---|---|---|
| 0.00 | 0.18929 | 0.18886 | 0.18852 | 0.164 |
| 0.10 | 0.16025 | 0.15985 | 0.15960 | |
| 0.20 | 0.12690 | 0.12656 | 0.12641 | |
| 0.30 | 0.08844 | 0.08819 | 0.08815 | |
| 0.40 | 0.04454 | 0.04440 | 0.04443 | |
| 0.45 | 0.02121 | 0.02114 | 0.02117 | |

the inference by use of the matching algorithm, namely, the ground state as $K \to \infty$ [4], in which we denote the error thresholds as $p_c^0$. We confirm that the heuristic matching algorithm of inference gives $p_c \approx p_c^0$, presumably $p_c = p_c^0$ for large $q$. We also give several results for the depolarizing channel case in Table II. Similar to the case without loss of qubits ($q = 0$) as reported in Ref. [12], the depolarizing channel is more resilient than the uncorrelated case even with loss of qubits. For comparison, let us take an earlier study on an error recovery procedure for the depolarizing channel in Ref. [11]. Our result implies that there is still possibility to improve the performance of such a constructive procedure. All the obtained values are almost stable in the third digits. In a practical sense, our estimations for error thresholds serve as the reference values.

*Conclusion*. We have estimated the error thresholds for the surface code with loss of qubits via a finite-dimensional spin glass theory for both the uncorrelated and depolarizing channel cases, and shown the greater resilience of the depolarizing channel even with the loss of qubits.

In relation to spin glass, the comparison between the error thresholds $p_c^0$ by a suboptimal method corresponding to the inference in the ground state [4] and the optimal ones $p_c$ shows a fascinating feature of the phase boundary of the $\pm J$ Ising model as $p_c \approx p_c^0$ [5]. Future study will be desirable to solve the remaining problem in the realm of spin glasses: $p_c = p_c^0$ or not.

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