Critical slowing down exponents in structural glasses: Random orthogonal and related models

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(Received 23 June 2012; published 13 August 2012)

An important prediction of mode-coupling theory is the relationship between the power-law decay exponents in the β regime and the consequent definition of the so-called exponent parameter λ. In the context of a certain class of mean-field glass models with quenched disorder, the physical meaning of λ has recently been understood, yielding a method to compute it exactly in a static framework. In this paper we exploit this new technique to compute the critical slowing down exponents for such models including, as special cases, the Sherrington-Kirkpatrick model, the p-spin model, and the random orthogonal model.

DOI: 10.1103/PhysRevB.86.064204 PACS number(s): 64.70.Q−, 64.60.Ht, 75.10.Nr

I. INTRODUCTION AND FRAMEWORK

It is well known that mean-field spin-glass models have a low-temperature phase in which the replica symmetry is broken, with a breaking pattern that depends on the specific model. The models displaying a static discontinuous transition which are consistently described by a finite number of breakings are characterized by critical slowing down and a dynamical transition at a temperature higher than the static one.

They share some relevant properties of structural glasses;[1–4] more specifically, the dynamical equations are exactly equivalent to those predicted by mode-coupling theory (MCT) above the mode-coupling temperature T_{mc} where ergodicity breaking occurs. The time autocorrelation function in the high-temperature phase displays a fast decay to a plateau (β regime) and then a second relaxation to equilibrium (α regime). Approaching the dynamical transition temperature (called T_d in the spin-glass context) the length of the plateau grows progressively until it diverges exactly at T_d, where the system remains stuck forever in one of the most excited metastable states in a complex free energy landscape. According to MCT the approach to the plateau and the decay from it are both characterized by a power-law behavior, respectively

$$C(t) \simeq q_d + ct^{-a}, \tag{1}$$

$$C(t) \simeq q_d - c' t^{b}, \tag{2}$$

where q_d is the height of the plateau. The length of the plateau (the α-relaxation time τ_α) diverges as

$$\tau_\alpha \propto (T - T_d)^{-\gamma}. \tag{3}$$

The three exponents satisfy the following relations that are exact in the framework of MCT (see for example Ref. 5):

$$\frac{\Gamma^2(1-a)}{\Gamma(1-2a)} = \frac{\Gamma^2(1+b)}{\Gamma(1+2b)} = \lambda, \quad \gamma = \frac{1}{2a} + \frac{1}{2b}, \tag{4}$$

where Γ is the standard Euler Gamma function and λ is the so-called exponent parameter.

This relation between the exponents has been proven to be robust under higher order corrections to standard MCT.[6] The exponent parameter λ and, consequently, the exponents a and b have been computed exactly only for the spherical p-spin model[7] since the dynamical equations are particularly simple and correspond to the so-called schematic MCT models. In the general case, instead, an approximate value of λ can be obtained analytically within MCT. Sources of errors in its expression are the approximations for the vertices describing mode coupling and the numerical uncertainties in the computation of the static structure factor of the liquid. On the other hand there is the possibility to fit experimental or simulation results to get the exponents a and b. The estimated exponents display a good agreement with relationship (4) although, as expected, with a value of λ which is slightly different from the approximate analytical expression obtained in MCT. Therefore it is usually assumed that the relationship (4) is exact while λ is treated as a tunable parameter (see Refs. 8,9 for example).

In the case of continuous transitions, there is no dynamic arrest preceding the static transition, the time correlation function does not display the two-step relaxation, and consequently, no exponent b is defined. At the thermodynamic transition, for long times, the correlation decays to the equilibrium value q_{EA} with a power law of the kind $C(t) \simeq q_{EA} + ct^{-a}$. The equilibrium order parameter q_{EA} is zero at the transition in the absence of magnetic field.

It has been recently pointed out[10,11] that there exists a connection between the exponent parameter and the static Gibbs free energy, which allows one to compute λ in a completely thermodynamic framework, even in cases which go beyond schematic MCT. In the following we will briefly summarize the method.

Given a fully connected model it is possible to compute the Gibbs free energy $\Gamma(Q)$ as a function of the order parameter that, in the case of a spin-glass transition, is the well known overlap matrix Q. The value of the order parameter can be determined through a saddle point calculation and $\Gamma(Q)$ can then be expanded around this solution. For our "dynamic" purposes, the expansion has to be performed around a replica symmetric saddle point solution $Q_{ab}^{SP} = q^{SP}$. This gives rise to eight different kinds of third-order terms, but only two of them will be relevant, namely,

$$w_1 \mathrm{Tr}(\delta Q^3) = w_1 \sum_{a,b,c} \delta Q_{ab}\delta Q_{bc}\delta Q_{ca} \tag{5}$$

and

$$w_2 \sum_{a,b} \delta Q_{ab}^3. \tag{6}$$

In the case of discontinuous transitions it can be shown[10,11] that the following relation holds at the dynamical transition, giving the connection between the dynamical exponents a and b and the static coefficients, namely,

$$\lambda = \frac{w_2(T_d)}{w_1(T_d)}, \tag{7}$$

where λ is given in Eq. (4); the expansion of the Gibbs free energy has to be performed around the value of the overlap yielding the height of the plateau at the dynamical transition, q_d.

Since the coefficients have to be computed at the dynamical transition, where quantities at infinite time do *not* relax to their equilibrium (thermodynamic) value but remain stuck at their value inside the most excited metastable states, the averages should then be computed *inside* a single state. This corresponds to taking a 1-RSB (replica symmetry breaking) ansatz with breaking parameter $m \to 1$ or, equivalently, a RS ansatz with the number of replicas $n \to 1$.[12–15]

In this paper we will use the second strategy which is technically much simpler than the first one; therefore we cannot treat the case of a dynamical transition in the presence of a magnetic field, since the mutual overlap (q_0) between states is nonzero and the 1-RSB ($m \to 1$)/RS ($n \to 1$) equivalence does not hold. On the other hand, for continuous transitions a relation between the exponent a and the two coefficients w_1 and w_2 analogous to Eq. (7) holds at the static point:

$$\lambda = \frac{w_2(T_s)}{w_1(T_s)}. \tag{8}$$

In this case, since the continuous static transition coincides with the dynamical one (e.g., in the Sherrington-Kirkpatrick model), the dynamical quantities at infinite time relax to their static value[16] and the averages can be computed in a replica symmetric ansatz taking finally the limit $n \to 0$. For this reason, if the transition is continuous, the RS ansatz will be sufficient to treat the case in the presence of a magnetic field.

In order to compute the two coefficients w_1 and w_2 we must compute the Gibbs free energy as a function of the overlap and then expand it to third-order around the RS saddle point value q. In fully connected models, we introduce a replicated external field ε and the free energy reads

$$f(\varepsilon) = -\frac{1}{\beta n N} \ln \int dQ \exp N\left(\mathcal{S}[Q] + \mathrm{Tr}\,\varepsilon Q\right) \tag{9}$$

which, for $N \to \infty$, can be evaluated at the saddle point

$$f(\varepsilon) = -\frac{1}{\beta n} \max_Q \left(\mathcal{S}[Q] + \mathrm{Tr}\,\varepsilon Q\right), \tag{10}$$

where we compute the maximum of the argument of the exponential function. Notice that the equation above defines $f(\varepsilon)$ as the *anti-Legendre-transform* ($\overline{\mathscr{L}}$) of the effective action

$$f(\varepsilon) = \overline{\mathscr{L}}(\mathcal{S}[Q]) \tag{11}$$

and, by definition, the Gibbs free energy $\Gamma(Q)$ is the *Legendre transform* ($\mathscr{L}$) of $f(\varepsilon)$, yielding

$$\Gamma(Q) \equiv \mathscr{L}(f(\varepsilon)) = \mathscr{L}(\overline{\mathscr{L}}(\mathcal{S}[Q])) = \mathcal{S}[Q]. \tag{12}$$

Therefore, in fully connected models, the functional form of the Gibbs free energy is equal to the one of the effective action and we can then directly expand the latter. The general form of the third-order term in the free energy is

$$\mathcal{S}^{(3)} = \sum_{(ab)(cd)(ef)} W_{ab,cd,ef}\, \delta Q_{ab} \delta Q_{cd} \delta Q_{ef} \tag{13}$$

with

$$W_{ab,cd,ef} = \frac{\partial^3 \mathcal{S}(Q)}{\partial Q_{ab} \partial Q_{cd} \partial Q_{ef}}. \tag{14}$$

Since $a \neq b$, $c \neq d$, and $e \neq f$ and the coefficients W are computed in the RS ansatz, they eventually can be expressed as linear combinations of eight independent coefficients $w_{1,\dots,8}$. On top of this, restricting the variations to the replicon subspace (R),[30] only two coefficients yield relevant information approaching the dynamic arrest.[10,11,17] We will denote them as w_1 and w_2:

$$\begin{aligned} \mathcal{S}_R^{(3)} &= \sum_{(ab)(cd)(ef)} W_{ab,cd,ef}\, \delta Q_{ab}^R \delta Q_{cd}^R \delta Q_{ef}^R \\ &= w_1 \mathrm{Tr}(\delta Q^R)^3 + w_2 \sum_{ab} \left(\delta Q_{ab}^R\right)^3, \end{aligned} \tag{15}$$

which follows quite straightforwardly from Eq. (13) applying the replicon constraint to the variations.

In this paper we apply this technique to study the critical slowing down of a general model of mean-field Ising spin glass which includes, as particular cases, the Sherrington-Kirkpatrick (SK) model, the p-spin model, and the random orthogonal model (ROM). The outline of the paper is the following: In Sec. II we introduce the general model; in Sec. III we give the details of the computation of the parameter exponent λ for the general case and briefly present the result for the SK model and p-spin model. In Sec. IV we compute λ for the ROM model and compare our exact result with numerical simulations. Finally, in Sec. V we give our conclusions and remarks.

II. THE GENERAL MODEL

In this section we will consider a class of mean-field models with Hamiltonian

$$\mathcal{H} = -\sum_{i<j} J_{ij}\sigma_i\sigma_j - \sum_p \sqrt{\frac{R_p}{p!}} \sum_{i_1<\dots<i_p} K^p_{i_1,\dots,i_p} \sigma_{i_1}\cdots\sigma_{i_p}, \tag{16}$$

where σ_i are N Ising spins. The 2-body interaction matrix is constructed in the following way:[18]

$$J = \mathcal{O}^T \Xi \mathcal{O}, \tag{17}$$

where $\mathcal{O}$ is a random $O(N)$ matrix chosen with the invariant Haar measure,[19] which is a uniform measure on the group of orthogonal matrices.[31] On the other hand, Ξ is a diagonal matrix with elements independently chosen from a distribution $\rho(\xi)$. In order to ensure the existence of the thermodynamic limit, the support of $\rho(\xi)$ must be finite and independent of N. The p-body interactions K^p are independent, identically distributed (i.i.d.) Gaussian variables with variance

$$\frac{p!}{N^{p-1}}, \tag{18}$$

and the parameters R_p define a function $R(x)$ such that

$$R_p = \frac{d^p R}{dx^p}(x)\Big|_{x=0}. \tag{19}$$

As shown in Refs. 18,20 and 21 for this class of mean-field spin glasses, the general form of the replicated free energy is

$$-n\beta f = \max_{Q,\Lambda} \mathcal{S}[Q,\Lambda] \tag{20}$$

with

$$\mathcal{S}[Q,\Lambda] = \frac{1}{2}\mathrm{Tr}\, G(\beta Q) + \frac{\beta^2}{2}\sum_{ab} R(Q_{ab}) - \frac{1}{2}\mathrm{Tr}\, Q\Lambda + \ln\left[\mathrm{Tr}_{\sigma_a} \exp\left(\frac{1}{2}\sum_{a,b}\Lambda_{ab}\,\sigma_a\sigma_b\right)\right], \tag{21}$$

where Λ is an auxiliary field and $G : M_{n\times n} \to M_{n\times n}$ is a (in general rather complicated) function in the space of $n\times n$ matrices, formally defined through its power series around zero. The particular form of G depends on the choice of the eigenvalue distribution $\rho(\xi)$. In the following we will consider mainly two cases: Wigner law and bimodal.

For later convenience we define at this point

$$G_k(x) = \frac{d^k G(x)}{dx^k}, \quad R_k(x) = \frac{d^k R(x)}{dx^k}. \tag{22}$$

Given the effective action (21), the saddle point equations in Λ and Q respectively read

$$Q_{ab} = \langle\langle \sigma_a\sigma_b\rangle\rangle, \quad \Lambda_{ab} = \beta[G_1(\beta Q)]_{ab} + \beta^2 R_1(Q_{ab}), \tag{23}$$

where the average $\langle\langle\cdot\rangle\rangle$ is computed with the measure

$$\mathcal{W}(\Lambda,\sigma) = \frac{e^{(1/2)\sum_{a,b}\Lambda_{a,b}\sigma_a\sigma_b}}{\mathrm{Tr}_\sigma\, e^{(1/2)\sum_{a,b}\Lambda_{a,b}\sigma_a\sigma_b}}. \tag{24}$$

In the replica symmetric ansatz ($Q_{ab}=q$, $\Lambda_{ab}=\hat\lambda$ for $a\neq b$ and $Q_{aa}=q_d$, $\Lambda_{aa}=\hat\lambda_d$), Eq. (23) becomes

$$q = \langle m^2\rangle,$$
$$\hat\lambda = \frac{\beta}{n}[G_1(\beta(1+(n-1)q)) - G_1(\beta(1-q))] + \beta^2 R_1(q), \tag{25}$$

where $m=\tanh(z)$ and the average $\langle\cdot\rangle$ is computed with the measure

$$\mu(\hat\lambda) = e^{-(z^2/2\hat\lambda)}\cosh^n(z)\frac{e^{-n\hat\lambda/2}}{(2\pi\hat\lambda)^{1/2}}. \tag{26}$$

In the next section we study in detail the (dynamical) critical behavior of this class of models and we show how to compute the critical slowing down exponents. One of our main results, derived in detail in the next section, will be a closed formula for the exponent parameter:

$$\lambda = \frac{w_2}{w_1} = \frac{R_3(q) + 2\beta^4\mathcal{D}(\beta,q)^3\,\mathcal{C}_2}{\beta G_3(\beta(1-q)) + 2\beta^4\mathcal{D}(\beta,q)^3\,\mathcal{C}_1}, \tag{27}$$

where

$$\mathcal{D}(\beta,q) \equiv G_2(\beta(1-q)) + R_2(q) \tag{28}$$

and

$$\mathcal{C}_1 \equiv \langle(1-m^2)^3\rangle, \quad \mathcal{C}_2 \equiv 2\langle m^2(1-m^2)^2\rangle. \tag{29}$$

III. COMPUTATION OF THE MCT EXPONENTS

As explained in the Introduction, in order to compute the parameter exponent λ we have to expand the effective action to third order in Q and then restrict the variations to the replicon subspace, obtaining straightforwardly the two coefficients w_1 and w_2. In the present case, the effective action contains the auxiliary field Λ which will be eliminated making use of the saddle point equation (23). The expansion of Eq. (21) to third order gives

$$\begin{aligned}\delta\mathcal{S}[Q,\Lambda] \simeq\ & \frac{1}{2}\mathrm{Tr}\left[\beta G_1(\beta Q^{SP})\delta Q + \frac{1}{2}\beta^2 G_2(\beta Q^{SP})\delta Q\delta Q + \frac{1}{3!}\beta^3 G_3(\beta Q^{SP})\delta Q\delta Q\delta Q\right]\\ &+ \frac{\beta^2}{2}\sum_{ab}\left[R_1\big(Q^{SP}_{ab}\big)\delta Q_{ab} + \frac{1}{2}R_2\big(Q^{SP}_{ab}\big)\delta Q^2_{ab} + \frac{1}{3!}R_3\big(Q^{SP}_{ab}\big)\delta Q^3_{ab}\right]\\ &- \frac{1}{2}\mathrm{Tr}[Q^{SP}\delta\Lambda + \Lambda^{SP}\delta Q + \delta\Lambda\delta Q] + \frac{1}{2}\sum_{ab}\langle\langle\sigma^a\sigma^b\rangle\rangle\delta\Lambda_{ab}\\ &+ \frac{1}{2\times 4}\sum_{ab,cd}\langle\langle\sigma^a\sigma^b\sigma^c\sigma^d\rangle\rangle_C\delta\Lambda_{ab}\delta\Lambda_{cd} + \frac{1}{3!\times 8}\sum_{ab,cd,ef}\langle\langle\sigma^a\sigma^b\sigma^c\sigma^d\sigma^e\sigma^f\rangle\rangle_C\delta\Lambda_{ab}\delta\Lambda_{cd}d\Lambda_{ef}.\end{aligned} \tag{30}$$

A comment is needed for the first line of Eq. (30): The "scalar like" Taylor expansion of a matrix functional $f(M)$, $f : M_{n\times n} \to M_{n\times n}$ around some M_0 (different from the null matrix), is correct only if $[M_0,\delta M]=0$.

In the present case Q^{SP} is replica symmetric while δQ is, in principle, simply symmetric. The commutation condition for a RS matrix with a symmetric matrix reads

$$\sum_c \delta Q_{cb} = \sum_c \delta Q_{ac} \quad \forall a,b, \tag{31}$$

which is satisfied in any subspace orthogonal to the anomalous one (see Ref. 17), i.e., both in the longitudinal and in the replicon sector.[32] Equating to zero the first order of Eq. (30) we obtain the saddle point equations (25).

Considering that the variations δQ and $\delta\Lambda$ are in the replicon subspace the second order term simplifies as follows:

$$[\beta^2(\tilde{g}_2 - g_2) + \beta^2 r_2] \sum_{ab} \delta Q_{ab}^2 - 2\sum_{ab} \delta\Lambda_{ab}\delta Q_{ab} + \langle(1-m^2)^2\rangle \sum_{ab} \delta\Lambda_{ab}^2, \tag{32}$$

where here and in the following formulas we define the four constants (two diagonal and two off-diagonal):

$$\tilde{g}_k = \frac{(n-1)G_k(\beta(1-q)) + G_k(\beta(1+(n-1)q))}{n},$$
$$g_k = \frac{G_k(\beta(1+(n-1)q)) - G_k(\beta(1-q))}{n}, \tag{33}$$
$$\tilde{r}_k = R_k(1), \quad r_k = R_k(q),$$

and G_k and R_k are defined in Eq. (22). For the system to be critical, the replicon eigenvalue must vanish and, consequently, the Hessian determinant must be zero. Imposing this condition we get the following equality:

$$\langle(1-m^2)^2\rangle = \frac{1}{[\beta^2(\tilde{g}_2 - g_2) + \beta^2 r_2]}. \tag{34}$$

Equation (34), together with Eq. (25), leads to the criticality condition, which locates the dynamical or static transition point depending on the value ($n = 1,0$) of the replica number.

Now we want to eliminate the auxiliary field. The Λ saddle point equation (23) reads (up to second order)

$$\delta Q_{ab} = \frac{1}{2}\sum_{cd} \langle\langle\sigma^a\sigma^b\sigma^c\sigma^d\rangle\rangle_C \delta\Lambda_{cd} + \frac{1}{4}\sum_{cd,ef} \langle\langle\sigma^a\sigma^b\sigma^c\sigma^d\sigma^e\sigma^f\rangle\rangle_C \delta\Lambda_{cd} d\Lambda_{ef}. \tag{35}$$

Exploiting the property of the replicon subspace and the criticality condition we can write the variation in the following way:

$$\delta Q_{ab} = \langle(1-m^2)^2\rangle\delta\Lambda_{ab} + \langle(1-m^2)^3\rangle \sum_c \delta\Lambda_{ac}\delta\Lambda_{cb} + 2\langle m^2(1-m^2)^2\rangle\delta\Lambda_{ab}^2. \tag{36}$$

Inverting the equation we obtain

$$\delta\Lambda_{ab} = [\beta^2(\tilde{g}_2 - g_2) + \beta^2 r_2]\delta Q_{ab} - [\beta^2(\tilde{g}_2 - g_2) + \beta^2 r_2]^3 \times \left[\langle(1-m^2)^3\rangle \sum_c \delta Q_{ac}\delta Q_{cb} + 2\langle m^2(1-m^2)^2\rangle \delta Q_{ab}^2\right]. \tag{37}$$

Now we recall that

$$\mathcal{C}_1 \equiv \langle(1-m^2)^3\rangle, \quad \mathcal{C}_2 \equiv 2\langle m^2(1-m^2)^2\rangle, \tag{38}$$

and plug the constraint (37) into (30), obtaining three different contributions to the third order in δQ, namely

$$-\frac{1}{2}\mathrm{Tr}[\delta\Lambda\delta Q] \to \frac{1}{2}[\beta^2(\tilde{g}_2 - g_2) + \beta^2 r_2]^3 \times \left[\mathcal{C}_1 \delta Q_{ab} \sum_c \delta Q_{bc}\delta Q_{ca} + \mathcal{C}_2 \delta Q_{ab}^3\right], \tag{39}$$

$$\frac{1}{2\times 4}\sum_{ab,cd} \langle\langle\sigma^a\sigma^b\sigma^c\sigma^d\rangle\rangle_C \delta\Lambda_{ab}\delta\Lambda_{cd} \to -\frac{1}{2}[\beta^2(\tilde{g}_2 - g_2) + \beta^2 r_2]^3 \times \left[\mathcal{C}_1 \delta Q_{ab} \sum_c \delta Q_{bc}\delta Q_{ca} + \mathcal{C}_2 \delta Q_{ab}^3\right], \tag{40}$$

$$\frac{1}{3!\times 8}\sum_{ab,cd,ef} \langle\langle\sigma^a\sigma^b\sigma^c\sigma^d\sigma^e\sigma^f\rangle\rangle_C \delta\Lambda_{ab}\delta\Lambda_{cd} d\Lambda_{ef} \to \frac{1}{3!}[\beta^2(\tilde{g}_2 - g_2) + \beta^2 r_2]^3 \times \left[\mathcal{C}_1 \sum_{abc} \delta Q_{ab}\delta Q_{bc}\delta Q_{ca} + \mathcal{C}_2 \sum_{ab} \delta Q_{ab}^3\right]. \tag{41}$$

Summing all the third-order contributions in Eq. (30), we eventually find

$$w_1 = \frac{1}{2}\frac{\beta^3}{3!}(\tilde{g}_3 - g_3) + \frac{1}{3!}[\beta^2(\tilde{g}_2 - g_2) + \beta^2 r_2]^3 \mathcal{C}_1,$$
$$w_2 = \frac{1}{2}\frac{\beta^2}{3!} r_3 + \frac{1}{3!}[\beta^2(\tilde{g}_2 - g_2) + \beta^2 r_2]^3 \mathcal{C}_2. \tag{42}$$

Substituting Eq. (33) one immediately obtains a general expression for the exponent parameter, which is the main result of this paper, cf. Eq. (27):

$$\lambda = \frac{w_2}{w_1} = \frac{R_3(q) + 2\beta^4 \mathcal{D}(\beta,q)^3\, \mathcal{C}_2}{\beta G_3(\beta(1-q)) + 2\beta^4 \mathcal{D}(\beta,q)^3\, \mathcal{C}_1},$$

where $\mathcal{D}(\beta,q)$ is defined in Eq. (28). Notice that the expression (27) is completely general and holds for every model belonging to this class, while the details of the model enter in the specific form of the functions G and R.

A. SK model on the dAT line

The Sherrington-Kirkpatrick model[22] is described by the Hamiltonian

$$\mathcal{H} = -\frac{1}{2}\sum_{ij} J_{ij}\sigma_i\sigma_j - H\sum_i \sigma_i, \tag{43}$$

where the couplings are i.i.d. random variables distributed according to a Gaussian with zero mean and variance $1/N$. This model belongs to the class defined above, with

$$R(x) = \frac{x^2}{2}, \quad G(x) = 0,$$

or, conversely, $R = 0$ and $G = x^2/2$ (see Ref. 22), except for the presence of the magnetic field term. We will see in a while that this affects the result in a very simple way.

It is well known that in the SK model there exists a line of instability of the replica symmetric solution in the β-H plane, the de Almeida–Thouless (dAT) line,[23] where the so-called replicon eigenvalue of the stability matrix vanishes. In this section we want to compute the decay exponent of the time correlation function along this line. In order to get the result we first have to find solutions simultaneously satisfying saddle point and dAT equations, respectively

$$q = \int d\mu(z)\tanh^2(\beta\sqrt{q}z + \beta H), \quad (44)$$

$$1 = \beta^2 \int d\mu(z)\,\mathrm{sech}^4(\beta\sqrt{q}z + \beta H),$$
$$d\mu(z) = \frac{1}{\sqrt{2\pi}} e^{-(z^2 2)}\,dz. \quad (45)$$

In the case of continuous transitions, the presence of the magnetic field only modifies the definition of the parameter m in Eq. (27) which becomes

$$m = \tanh(z + \beta H) \quad (46)$$

without changing the formal expression for the coefficients. As $R(x) = \frac{1}{2}x^2$, implying $R_2(x) = 1$ and $R_3(x) = 0$, the expression for the exponent parameter in the SK model in a field reads

$$\lambda = \frac{\mathcal{C}_2}{\mathcal{C}_1} \equiv \frac{2\langle m^2(1-m^2)^2\rangle}{\langle(1-m^2)^3\rangle}. \quad (47)$$

Our result exactly coincides with the one obtained by Sompolinsky and Zippelius[16] in a purely dynamical framework.

B. Multi-P-spin Ising model

Starting from a Hamiltonian of the kind of Eq. (16) without the first term leads to a generalized version of the p-spin model,[24] in which many multibody interaction terms are considered, depending on the actual form of the function R. As shown in Refs. 25–27, in these models, the thermodynamic properties, the critical dynamics, and replica symmetry breaking structure depend on the relative strength of the coupling terms (the coefficients of the expansion of R). Indeed, in order to treat a particular case, before applying our technique, one should understand the behavior of the corresponding model.

The simple p-spin Ising model is characterized by $R(x) = ax^p$, where an $a \neq 1$ affects only the variance of the couplings distribution and indeed rescales the temperature. For $p > 2$, in the absence of any external magnetic field, the model displays a dynamical transition at T_d, then at a lower temperature a static discontinuous transition from the paramagnetic RS phase to the 1-RSB spin glass and, at a further lower temperature, a second transition to a full-RSB spin glass.[24] Focusing on the first transition, we can compute the critical dynamic exponents in the present general framework while a specific analysis was presented by some of us in Ref. 28. In order to recover the same model, we have to set $R(x) = x^p/2$, which reduces Eq. (25)

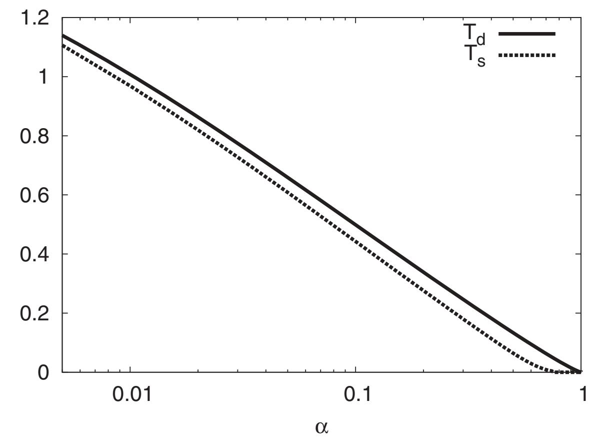

FIG. 1. The static and dynamical critical temperature as a function of the parameter α. They both diverge for $\alpha \to 0$ and are zero at $\alpha = 1$.

and (34) to

$$q = \langle m^2\rangle, \quad \hat{\lambda} = \frac{p\beta^2}{2} q^{p-1},$$
$$1 = \frac{p(p-1)\beta^2 q^{p-2}}{2}\langle(1-m^2)^2\rangle, \quad (48)$$

and Eq. (27) to

$$\lambda = \frac{2\langle m^2(1-m^2)^2\rangle + \frac{2(p-2)q^{3-2p}}{\beta^4 p^2(p-1)^2}}{\langle(1-m^2)^3\rangle}, \quad (49)$$

as was found in Ref. 28.

IV. RANDOM ORTHOGONAL MODEL

The random orthogonal model (ROM)[18,20] is obtained with the choice $R = 0$ and the following eigenvalue distribution for the matrix Ξ in Eq. (17):

$$\rho(\xi) = \alpha\,\delta(\xi - 1) + (1-\alpha)\,\delta(\xi + 1). \quad (50)$$

This model displays a glassy transition regardless of the value of the tunable parameter α. The case with $\alpha = 1/2$ has been

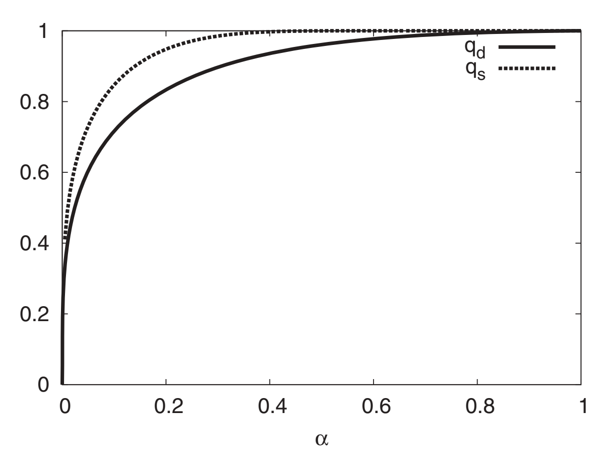

FIG. 2. The static and dynamical critical overlap as a function of the parameter α.

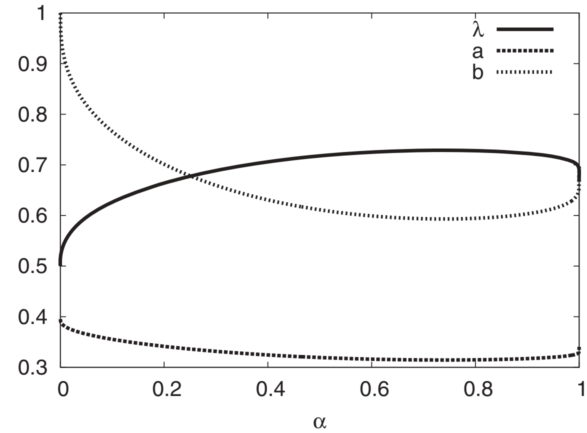

FIG. 3. Solid line: the exponent parameter λ. Dashed line: the exponent a. Dot-dashed line: the exponent b.

extensively studied in Ref. 20 while the general case was treated in Ref. 18.

It has been shown that as a consequence of the choice of the eigenvalue distribution (50), the function G appearing in the effective action reads[33]

$$\begin{aligned}2G(x) = (2\alpha - 1)\ln\{\psi(x,\alpha) + 2x + 2\alpha - 1\} \\ - \ln\{\psi(x,\alpha) + 1 + 2(2\alpha - 1)x\} \\ - (2\alpha - 1)\ln(2\alpha) - 1 + \ln(2) + \psi(x,\alpha),\end{aligned} \quad (51)$$

with

$$\psi(x,\alpha) = [1 + 4x(2\alpha - 1 + x)]^{1/2}.$$

Using Eqs. (23) we can determine the transition temperature T_d and the dynamical overlap q_d which are shown in Figs. 1 and 2 and coincide with those found in Ref. 18.

Once the critical point is obtained as a function of α, using formula (27) specialized to the ROM case, we obtain the value of the exponent parameter $\lambda(\alpha)$ and of the critical slowing down exponents $a(\alpha)$ and $b(\alpha)$ which are shown in Fig. 3. We find numerically that for $\alpha \to 1$ the exponent parameter goes to $\lambda = \frac{2}{3}$ as in the Ising p-spin model for $p \to \infty$, while for $\alpha \to 0$ we find $\lambda = \frac{1}{2}$ as in the Ising p-spin model for $p \to 2$ (see Table I). In particular, for $\alpha = 13/32$ we have $b = 0.628$. We now use this value to compare with numerical simulations.

A. Comparison with Monte Carlo data

There are recent numerical simulations by Sarlat *et al.*[29] on the fully connected ROM which give an estimate for the MCT exponents a, b, and γ. They choose $\alpha = \frac{13}{32} \simeq 0.4$ in order to have higher transition temperatures and a good separation between the static and dynamical critical temperature.

Their direct estimate of the exponent b is 0.62, while their direct estimate of γ is 2.1 which, through the exact MCT relations given in Eq. (4) that we recall,

$$\frac{\Gamma^2(1+b)}{\Gamma(1+2b)} = \frac{\Gamma^2(1-a)}{\Gamma(1-2a)} = \lambda, \quad \gamma = \frac{1}{2a} + \frac{1}{2b},$$

yields $b_{MCT} \simeq 0.75$. Our exact computation yields instead $b_{th} \simeq 0.628$ (see Table I), which suggests that the best estimate of the exponent b in Ref. 29 is the direct one, which is very close to the actual value.

TABLE I. The exponent parameter λ for special values of α.

α	λ	to compare with
1	2/3	$p \to \infty$ Ising[28]
13/132	0.7076	Num. Sim.[29]
0	1/2	$p \to 2$ Ising[3,28]

V. SUMMARY AND CONCLUSIONS

In the present work we have introduced a general fully connected model for Ising spins, which combines an orthogonal two-body interaction with a set of p-body interactions. Exploiting a technique that has been recently introduced,[10,11] based on the equivalence between statics and long-time dynamics, we have been able to find an analytic expression for the exponent parameter λ, in a purely static framework. As particular cases of the general model we have studied the Sherrington-Kirkpatrick model along the de Almeida–Thouless line, the p-spin model, and the random orthogonal model. For the SK model we find the same result found by Sompolinsky and Zippelius in Ref. 16. For the p-spin model we recover, as a by-product of the general model, the results given in detail in Ref. 28.

We have studied the critical behavior of the parametric class of random orthogonal models at arbitrary values of the constant $\alpha \in [0,1]$, which determines the distribution of the eigenvalues of the interaction matrix. The exponent parameter and the two MCT exponents have been determined analytically for any α and in particular we have looked at $\alpha = 13/32$ in order to make a comparison with existing numerical simulations.[29] Our exact result is in very good agreement with the one obtained in the Monte Carlo study, through the numerical estimate of the exponent b (late β regime). On the other hand, a direct estimate of the exponent γ gives a result that is quite far from what we found here. Assuming that in Ref. 29 a proper interpolation was performed, this might suggest that, for the present model, the strong finite-size corrections affect the value of γ much more than b.

Numerical interpolations at criticality are very sensitive for glassy models and the corresponding estimates can strongly suffer from this drawback. Our analytic computation allows one to overcome this difficulty.

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[30] The subspace where $\sum_b \delta Q^R_{ab} = 0$.

[31] Note that the orthogonalization of matrices with independent uniformly distributed random entries does not lead to uniformly distributed orthogonal matrices.

[32] Any element of the anomalous subspace can be described by a one-index field, i.e., by a vector ψ^α restricted to the condition $\sum_\alpha \psi^\alpha = 0$. A generic anomalous field can then be written as $\psi_A^{\alpha\beta} = 1/2(\psi^\alpha + \psi^\beta)$. Any element of the longitudinal subspace can be described by a scalar $\psi^{\alpha\beta} = \psi$.

[33] Note that the corresponding formula in Ref. 18 contains a typing mistake: ln(2) is taken with the negative sign, which would give a negative entropy at infinite temperature.