Two-step relaxation next to dynamic arrest in mean-field glasses: Spherical and Ising p-spin model

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Critical slowing down dynamics of supercooled glass-forming liquids is usually understood at the mean-field level in the framework of mode coupling theory, providing a two-time relaxation scenario and power-law behaviors of the time correlation function at dynamic criticality. In this work we derive critical slowing down exponents of spin-glass models undergoing discontinuous transitions by computing their Gibbs free energy and connecting the dynamic behavior to static "in-state" properties. Both the spherical and Ising versions are considered and, in the simpler spherical case, a generalization to arbitrary schematic mode coupling kernels is presented. Comparison with dynamic results available in literature is performed. Analytical predictions for the Ising case are provided for any p.

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I. INTRODUCTION

The slowing down of the dynamics of supercooled glass-forming liquids corresponds to a nontrivial underlying thermodynamic landscape. An unusual time behavior of the density correlation function emerges, with respect to the exponential decay: a separation between fast (β) and slow (α) relaxation modes takes place and the correlation function develops a plateau approaching a dynamic arrest transition.[1–3]

Theoretical advances have been made studying such systems with the so-called mode coupling theory (MCT),[3–8] a mean-field theoretical description of many-particle systems, able to identify the separation of two relaxation processes and a *dynamical transition temperature* T_d at which ergodicity breaks down with the system undergoing a structural arrest. In this framework, for $T \to T_d$ the time spent at the plateau increases and diverges at T_d. At criticality a power-law behavior arises for the correlation close to the plateau and the exponents governing the approach to ($\sim t^{-a}$) and departing from ($\sim t^b$) of the plateau are related as

$$\frac{\Gamma^2(1-a)}{\Gamma(1-2a)} = \frac{\Gamma^2(1+b)}{\Gamma(1+2b)} \equiv \lambda, \tag{1}$$

where λ is called the *exponent parameter*, a model dependent quantity (functional of the static structure factor[9]), that is usually treated like a tunable parameter.

Since the works of Kirkpatrick, Thirumalai, and Wolynes,[10–15] the behavior of glass-forming liquids and structural glasses has been linked with dynamics and thermodynamics of a certain class of mean-field spin-glass (SG) models, sometimes called *mean-field* glasses or *discontinuous* spin glasses. These include the p-spin models,[16–18] displaying a dynamic transition at which ergodicity breaks down, with a behavior of the correlation function close or identical to the one predicted by MCT. Below this dynamic transition the Boltzmann measure splits in many well defined metastable states. At a temperature T_s glassy states of lowest free energy become thermodynamically stable inducing a phase transition.[11,17,19] The static transition is governed by an entropy crisis and plays the role of the hypothesized Kauzmann transition, related to the vanishing of the configurational entropy of the liquid.[1,20,21] Below this point, named random first-order transition, the model develops a spin-glass phase with one step *replica symmetry breaking* (1RSB). Besides the conjectured thermodynamic analogy between discontinuous spin glasses and structural glasses, cf. *mosaic theory*,[15,22,23] the former can as well be exploited for studying critical slowing down dynamics and MCT properties at the dynamic arrest transition.

In this paper we do not address the study of thermodynamics. We, rather, focus on the dynamic transition, applying a recently introduced relationship between critical slowing down exponents and thermodynamic "in-state" quantities.[24,25]

The rest of the paper is organized as follows. In Sec. II we present and discuss the static-dynamic relationship and the analytic procedure leading to the determination of the critical slowing down exponents. In Sec. III, using spherical p-spin models whose dynamics down to T_d is equivalent to the one of schematic MCT's, we derive the critical exponents both in the classical MCT way and with the static general method, analytically verifying the correctness of such a static-dynamic relationship. In Sec. IV we determine the critical slowing down exponents at the dynamic transition of the Ising p spin.

II. STATIC COMPUTATION OF DYNAMIC SLOWING DOWN EXPONENTS

In Ref. 24 the following relationship has been proposed between static observables and the so-called exponent parameter λ, cf. Eq. (1):

$$\lambda = \frac{w_2}{w_1}, \tag{2}$$

where coefficients w_1, w_2 are obtained from the expansion at the third order of the appropriate Gibbs free-energy $\Gamma[q]$ function of the order parameters, i.e., the replica overlap matrix elements $q_{\alpha\beta}$. The thermodynamic potential Γ is the Legendre transform of the free-energy functional as a function of replica "pinning" fields.[26–28] It coincides with the Franz-Parisi potential,[29] by means of which the dynamic transition is identified as the spinodal point of an excited local minimum at a nonzero overlap value equal to the plateau value of the correlation function. Critical slowing down exponents can, indeed, be obtained from the coefficients of the expansion of Γ around the replica symmetric (RS) saddle point q^{RS} yielding that local minimum.

For $T \gtrsim T_d$ the Gibbs potential can be expanded in powers of overlap fluctuations around the dynamic solution, $\delta q_{\alpha\beta} \equiv Q_{\alpha\beta} - Q^{RS}_{\alpha\beta}$:

$$\delta\Gamma[\delta q_{\alpha\beta}] \sim \sum_{(\alpha\beta),(\gamma\delta)} M_{\alpha\beta,\gamma\delta}\delta q_{\alpha\beta}\delta q_{\gamma\delta} + \sum_{(\alpha\beta)(\gamma\delta)(\varepsilon\varphi)} W_{\alpha\beta\gamma\delta\varepsilon\varphi}\delta q_{\alpha\beta}\delta q_{\gamma\delta}\delta q_{\varepsilon\varphi}, \quad (3)$$

where $\delta\Gamma[\delta q_{\alpha\beta}] = \Gamma[Q_{\alpha\beta}] - \Gamma[Q^{RS}_{\alpha\beta}]$ and the first-order term is absent at the saddle point. Here and below, brackets in sums over replica indices mean that only distinct replicas must be considered.

The dynamic transition is associated to the vanishing of the lowest eigenvalue of the mass term, the *replicon*. This defines a critical direction in the replica space, along which the relaxation is nonexponential. Projecting the expansion of $\Gamma >$ along the replicon direction $\delta^{(r)}q_{ab}$ (see next section for the details), one obtains

$$\delta\Gamma[\delta^{(r)}q_{\alpha\beta}] \sim w_1 \mathrm{Tr}(\delta^{(r)}q)^3 + w_2 \sum_{(\alpha\beta)}(\delta^{(r)}q_{\alpha\beta})^3. \quad (4)$$

Summarizing, in order to obtain the MCT exponents for mean-field discontinuous SG models one can proceed with the following protocol:

(1) Compute the averaged replicated action $\Gamma[Q]$.

(2) Compute the expansion around the dynamical RS solution up to the third order, cf. Eq. (4).

(3) Solve the system equation for the saddle point and the vanishing of the replicon, both in the $n \to 1$ limit which allows us to work on the dynamic metastable state of the Gibbs potential.

(4) Evaluate the coefficients w_1, w_2 of the third order along the replicon direction.

(5) Compute λ by means of Eq. (2) and a and b with Eq. (1).

III. SPHERICAL *P*-SPIN MODEL

In this section we focus on the spherical version of the fully connected p-spin model. Its relevance in the context of mean-field glasses is due to the equivalence of its equations of motion with those provided by the schematic MCT for undercooled liquids, the framework where the exponent parameter λ and its relation to critical slowing down exponents were first introduced.

The static approach to the study of the dynamic properties in the spherical model(s) is, therefore, a crucial verification of the validity of Eq. (2). The Hamiltonian of the model reads

$$\mathcal{H} = -\sum_{i_1<\ldots<i_p} J_{i_1\ldots i_p}\sigma_{i_1}\ldots\sigma_{i_p} - h\sum_i \sigma_i, \quad (5)$$

where the couplings J are Gaussian independent identically distributed variables with

$$P(J) = \sqrt{\frac{N^{p-1}}{\pi p!}}\exp\left(-\frac{N^{p-1}}{p!}J^2\right) \quad (6)$$

and the spins σ are real value variables subject to a global constraint:

$$\sum_i \sigma_i^2 = N. \quad (7)$$

Due to the spherical constraint this model is analytically solvable in all details needed for our scope. This allows us to show how to compute third-order coefficients and dynamic exponents a and b step by step.

Through a saddle-point calculation, it is possible to determine the replicated partition function:[17]

$$\overline{Z^n} = e^{S(\infty)}\int_{|q|>0}\prod_{\alpha<\beta}\sqrt{\frac{N}{2\pi}}dq_{\alpha\beta}\exp\{-N\Gamma[Q]\}$$

$$\Gamma[Q] = -\frac{\mu}{2p}\sum_{\alpha\beta}Q^p_{\alpha\beta} - \frac{(\beta h)^2}{2}\sum_{\alpha\beta}Q_{\alpha\beta} - \frac{1}{2}\ln|Q| + \frac{(\beta h)^4}{2}\left(\sum_{\alpha\beta}Q_{ab}\right)^2, \quad (8)$$

where the overbar means average over disorder, cf. Eq. (6), $\mu = p\beta^2/2$, $Q_{\alpha\beta} = 1/N\sum_i \sigma_i^\alpha\sigma_i^\beta$, and

$$e^{S(\infty)} = e^{N[1+\ln(2\pi)]/2}\pi^{-1/2}\left[1 + O\left(\frac{1}{N}\right)\right]. \quad (9)$$

Differentiating Eq. (8) with respect to $Q_{\alpha\beta}$ one obtains the saddle-point condition:

$$\mu Q^{p-1}_{\alpha\beta} + (\beta h)^2 + (Q^{-1})_{\alpha\beta} = 0, \quad \forall\alpha\neq\beta, \quad (10)$$

where the term proportional to h^4 is absent because it is irrelevant in both the $n \to 0,1$ limits.

For our purposes we also need the third-order Taylor expansion of Γ in the overlap fluctuations around the RS solution,

$$\delta q_{\alpha\beta} = Q_{\alpha\beta} - Q^{RS}_{\alpha\beta}, \quad Q^{RS}_{\alpha\beta} = (1-q)\delta^\beta_\alpha + q,$$

where δ^β_α is the Kronecker δ. It yields

$$2\delta\Gamma[\delta q] \simeq \frac{1}{2!}\sum_{(\alpha\beta)(\gamma\delta)}\Gamma''_{\alpha\beta\gamma\delta}\delta q_{\alpha\beta}\delta q_{\gamma\delta} + \frac{1}{3!}\sum_{(\alpha\beta)(\gamma\delta)(\varepsilon\varphi)}\Gamma'''_{\alpha\beta\gamma\delta\varepsilon\varphi}\delta q_{\alpha\beta}\delta q_{\gamma\delta}\delta q_{\varepsilon\varphi}, \quad (11)$$

where $\delta\Gamma[\delta q] = \Gamma[Q] - \Gamma[Q^{RS}]$. The first order is absent at the saddle point and

$$\Gamma''_{\alpha\beta\gamma\delta} \equiv \frac{\partial^2\Gamma}{\partial Q_{\alpha\beta}\partial Q_{\gamma\delta}} = -(p-1)\mu Q^{p-2}_{\alpha\beta}\delta^\gamma_\alpha\delta^\delta_\beta + (Q^{-1})_{\alpha\gamma}(Q^{-1})_{\delta\beta} + (\beta h)^4 \quad (12)$$

$$\Gamma'''_{\alpha\beta\gamma\delta\varepsilon\varphi} = \frac{\partial^3\Gamma}{\partial Q_{\alpha\beta}\partial Q_{\gamma\delta}\partial Q_{\varepsilon\varphi}} = -(p-1)(p-2)\mu Q^{p-3}_{\alpha\beta}\delta^\gamma_\alpha\delta^\delta_\beta\delta^\varepsilon_\alpha\delta^\varphi_\beta - 2(Q^{-1})_{\alpha\varepsilon}(Q^{-1})_{\varphi\gamma}(Q^{-1})_{\delta\beta}. \quad (13)$$

In order to study the critical dynamic behavior we work within a RS Ansatz with $n \to 1$,[26–28,30] and restrict our analysis to the replicon subspace, defined by the conditions

$$\sum_\alpha \delta q_{\alpha\beta} = \sum_\beta \delta q_{\alpha\beta} = 0. \tag{14}$$

The second condition is a consequence of the first one as far as $q_{\alpha\beta}$ is a symmetric matrix. The vanishing of the eigenvalue of the Hessian in this subspace, the so-called *replicon* eigenvalue, yields the criticality condition.

We initially perform the analysis to the case without external magnetic field ($h = 0$). Imposing the saddle-point condition, cf. Eq. (10), at the dynamic transition point (RS with $n \to 1$) we have the saddle-point equation

$$\mu q^{p-1} = \frac{q}{(1-q)}. \tag{15}$$

Imposing the vanishing of the replicon we furthermore find

$$(p-1)\mu q^{p-2} = \frac{1}{(1-q)^2}, \tag{16}$$

leading to

$$q_d = \frac{p-2}{p-1}, \quad \mu_d = \frac{(p-1)^{p-1}}{(p-2)^{p-2}} \tag{17}$$

for the value of the overlap at the dynamic transition and the inverse dynamic temperature. The dynamic transition point is also the point at which the nontrivial dynamic saddle-point solution appears ($q = q_d \neq 0$): Eqs. (15) and (16) are therefore *not* independent and indeed Eq. (16) can be obtained also as the derivative of Eq. (15).

Considering fluctuations exclusively in the replicon subspace (r), one can considerably simplify the expansion of Γ, cf. Eq. (11), as

$$2\delta\Gamma[\delta q^{(r)}] \simeq -(p-1)(p-2)\mu q^{p-3} \sum_{(\alpha\beta)} \left(\delta q^{(r)}_{\alpha\beta}\right)^3 - \frac{2}{(1-q)^3} \sum_{(\alpha\beta\gamma)} \delta q^{(r)}_{\alpha\beta}\delta q^{(r)}_{\beta\gamma}\delta q^{(r)}_{\gamma\alpha}. \tag{18}$$

In this case the tensorial form of Eq. (13) is so simple that we can straightforwardly compute the values of the cumulants w_1 and w_2, yielding

$$\frac{w_2}{w_1} = \frac{(p-1)(p-2)}{2}\mu q^{p-3}(1-q)^3, \tag{19}$$

which, imposing Eq. (16), reduces to

$$\frac{w_2}{w_1} = \frac{(p-2)(1-q)}{2q}. \tag{20}$$

Using the value of q_d at the transition, cf. Eq. (17), one obtains for λ the p-independent value

$$\lambda = \left.\frac{w_2}{w_1}\right|_d = \frac{1}{2}, \tag{21}$$

which coincides with the result reported in Refs. 18 and 31.

A. Case of uniform magnetic field

The presence of a magnetic field term ($h \neq 0$) can change the nature of the transition. As shown in Refs. 17 and 18 for values of the field $h > h_{tr} = \sqrt{p^{2-p}(p-2)^p/2}$ the transition becomes continuous: no plateau occurs and, at the transition, the long-time limit of the correlation function does not jump discontinuously, though the relaxation behavior in time is still a power law in the β regime. The exponent a (sometimes called ν) is the only one defined. We first study this continuous transition and then we move to the discontinuous one.

The value of a can be computed as for the discontinuous transition case, except for the fact that now one has to work in the $n \to 0$ limit.[24,25] This implies that the RS expression of Eq. (10) becomes

$$\mu q^{p-1} + (\beta h)^2 = \frac{q}{(1-q)^2} \tag{22}$$

rather than Eq. (15). Equations (16) and (19), instead, do not change. From Ref. 17 we know that the transition line is parametrically defined as

$$T^2 = \frac{p(p-1)}{2}(1-q)^2 q^{p-2}, \tag{23}$$

$$h^2 = \frac{p(p-2)}{2} q^{p-1} \tag{24}$$

with $1 - 2/p < q < 1$. Given a value of the field h, one can thus straightforwardly compute the corresponding values of q and T from Eqs. (23) and (24) and obtain λ from Eq. (20).

For the discontinuous transition in a field $h < h_{tr}$, two nonzero overlap values are relevant: the plateau value q_1 and the long-time limit $q_0 = C(\infty)$. Comparing with the results of Ref. 17 we observe that the exponent parameter λ is still given by the ratio w_2/w_1, cf. Eq. (20), now evaluated on $q = q_1$, solution of the following equations for the dynamic critical values of $T_d(h)$, $q_0(h)$, and $q_1(h)$:

$$\frac{1}{(1-q_1)^2} = \mu(p-1)q_1^{p-2}, \quad \text{cf. Eq. (16)},$$

and

$$\frac{1}{1-q_1} - \frac{1}{1-q_0} = \mu\left(q_1^{p-1} - q_0^{p-1}\right),$$
$$\mu q_0^{p-1} = \frac{q_0}{(1-q_0)^2} - (\beta h)^2,$$

which are the 1RSB saddle-point equations [cf. Eq. (10)].

B. Generalization to arbitrary schematic MCT models

Comparing the dynamics of the p-spin spherical model[18] with the MCT differential equation for the correlation function,[6,8] one can notice that the function μq^{p-1} ($\mu\phi^{p-1}$ in the MCT notation), derivative of the first term of the action (8), plays the role of the MCT memory kernel.[8,11,18] One can generalize this argument, through schematic MCT, to a generic polynomial kernel,[3]

$$\Lambda(t) = \mathcal{F}[\{v\},\phi(t)] = \sum_p v_p \phi(t)^{p-1}. \tag{25}$$

First, it is convenient to introduce the so-called modified Laplace transform:

$$\mathcal{LT}[G(t)](z) \equiv i\int_0^\infty e^{izt}G(t)dt, \quad \mathrm{Im}\, z > 0. \tag{26}$$

In order to lighten the notation, in the following $G(z)$ will denote the Laplace transform of the function $G(t)$.

Studying the critical relaxation of the correlation function, we are interested in describing the power-law approach to the plateau,

$$\phi(t) \simeq q_d + (\tau_0/t)^a, \tag{27}$$

where q_d is the plateau value (also called the nonergodicity parameter) and τ_0 is the characteristic microscopic time, smaller than any time at which nonexponential slowing down occurs. We are interested also in the Von Schweidler law for the departure from the plateau,

$$\phi(t) \simeq q_d - (t/\tau)^b, \tag{28}$$

where τ is the relaxation time to equilibrium, that goes to ∞ as $(T - T_d)^{-\gamma}$ as $T \to T_d$.

Deviations of the correlation $\phi(t)$ from the plateau are thus small both in the so-called β regime (approach to plateau) and in leaving the plateau when the system starts relaxing to equilibrium. We can indeed assume that

$$|G(t)| \ll 1, \qquad G(t) \equiv \phi(t) - q_d. \tag{29}$$

The correlation function and its Laplace transform,

$$\phi(z) \simeq -\frac{q_d}{z} + G(z), \tag{30}$$

have to satisfy the MCT dynamical equation and its Laplace transform:

$$\tau_0 \partial_t \phi(t) + \phi(t) = -\int_0^t du \Lambda(t-u)\partial_u \phi(u) \tag{31}$$

$$\frac{\phi(z)}{1 + z\phi(z)} = i\tau_0 + \Lambda(z). \tag{32}$$

Plugging Eq. (30) into the left-hand side of Eq. (32) and expanding up to the second order, one obtains ($t \gg \tau_0$)

$$z\frac{\phi(z)}{1+z\phi(z)} \simeq \frac{-q_d + zG(z)}{1-q_d}\left[1 - \frac{zG(z)}{1-q_d} + \frac{z^2G(z)^2}{(1-q_d)^2}\right]. \tag{33}$$

The Laplace transform of the expansion of the memory kernel reads

$$z\Lambda(z) \simeq -\mathcal{F}(\{v\},q_d) + \mathcal{F}'(\{v\},q_d)G(z) + \frac{1}{2}\mathcal{F}''(\{v\},q_d)\mathcal{LT}[G(t)^2](z). \tag{34}$$

One has to compare Eqs. (33) and (34) order by order in $G(t)$. The zeroth and first order are the standard long-time MCT equation and its derivative, cf. Eqs. (15) and (16),

$$\mathcal{F}(\{v\},q_d) = \frac{q_d}{1-q_d}, \tag{35}$$

$$\mathcal{F}'(\{v\},q_d) = \frac{1}{(1-q_d)^2}. \tag{36}$$

The third order yields

$$zG(z)^2 + \lambda z \mathcal{LT}[G(t)^2](z) = 0 \tag{37}$$

with

$$\lambda \equiv \tfrac{1}{2}\mathcal{F}''(\{v\},q_d)(1-q_d)^3. \tag{38}$$

Assuming a power-law solution $G(t) \sim (\tau_0/t)^a$, else $\sim (t/\tau)^b$, one gets back Eq. (1) with an exponent parameter λ coinciding with Eq. (38).

This result can also be obtained studying a model whose action is a slight generalization of the simple p-spin action, cf. Eq. (8),

$$\Gamma[Q] = -\frac{1}{2}\sum_{\alpha\beta} A(Q_{\alpha\beta}) - \frac{1}{2}\ln|Q|, \tag{39}$$

such that $\mathcal{F}[\{v\},x] = A'(x)$. For polynomial function $A(x)$ this action describes a model with a Hamiltonian composed by a sum of p-spin interaction terms, like Eq. (5).[32–35] From the action Eq. (39) one can easily derive Eqs. (35) and (36), with a kernel given by Eq. (25), yielding critical temperature (identical to the mode coupling temperature) and critical plateau value of the correlation.

Expanding Eq. (39) to third order, cf. Eq. (11), yields

$$\lambda = \frac{w_2}{w_1} = \frac{1}{2}\Lambda''(q_d)(1-q_d)^3 \tag{40}$$

coinciding with Eq. (38) and verifying the method proposed for the schematic MCT models. This is the simplest case, where equations of motion can be solved and analytic results are available. We now move to consider a more difficult case for which λ cannot be computed directly solving the dynamic equations and Eq. (2) remains the only way, known so far, to determine the critical slowing down exponents.

IV. ISING *p*-SPIN MODEL

In this section we focus on the Ising version of the fully connected p-spin model. The Hamiltonian of the model reads

$$\mathcal{H} = -\sum_{i_1<\cdots<i_p} J_{i_1\ldots i_p}\sigma_{i_1}\ldots\sigma_{i_p}, \tag{41}$$

where the couplings J are Gaussian distributed again with Eq. (6) and $\sigma_i = \pm 1$. It is well known[16] that this model displays a random first-order transition at $T = T_s$ to a glassy 1RSB stable phase preceded by a distinct dynamic transition at $T = T_d$ and, at a lower temperature, a continuous thermodynamic transition to a spin-glass full RSB phase. For our purposes, we only focus on the dynamic transition at T_d.

Averaging over disorder and introducing the overlap matrix through an auxiliary matrix Λ_{ab} yields the replicated action:

$$\Gamma[Q,\Lambda] = \frac{\beta^2}{4}\sum_{(\alpha\beta)} Q^p_{\alpha\beta} - \frac{1}{2}\sum_{(\alpha\beta)}\Lambda_{\alpha\beta}Q_{\alpha\beta} + \ln\mathrm{Tr}_{\{\sigma\}}\mathcal{W}[\Lambda;\sigma], \tag{42}$$

$$\mathcal{W}[\Lambda;\sigma] = \exp\left(\frac{1}{2}\sum_{(\alpha\beta)}\Lambda_{\alpha\beta}\sigma_\alpha\sigma_\beta\right) \tag{43}$$

that has to be evaluated through a saddle-point calculation. The derivative with respect to $Q_{\alpha\beta}$ yields

$$\frac{p\beta^2}{2} Q_{\alpha\beta}^{p-1} = \Lambda_{\alpha\beta}, \quad \alpha \neq \beta \tag{44}$$

leading to

$$\Gamma[Q] = -\frac{(p-1)\beta^2}{4} \sum_{(\alpha\beta)} Q_{\alpha\beta}^p + \ln \mathrm{Tr}_\sigma \mathcal{W}[Q;\sigma],$$
$$\mathcal{W}[Q;\sigma] = \exp\left(\frac{p\beta^2}{4} \sum_{(\alpha\beta)} Q_{\alpha\beta}^{p-1} \sigma_\alpha \sigma_\beta\right). \tag{45}$$

This concludes the first step of our protocol. The first-order derivative reads

$$\Gamma'_{\alpha\beta} = \frac{\partial \Gamma}{\partial Q_{\alpha\beta}} = \theta Q_{\alpha\beta}^{p-2}(\langle \sigma_\alpha \sigma_\beta \rangle - Q_{\alpha\beta}), \tag{46}$$

where $\theta = p(p-1)\beta^2/4$ and $\langle \ldots \rangle$ means average over the weight $\mathcal{W}[Q]$, cf. Eq. (45). The vanishing of this equation yields the saddle-point condition that, solved in the RS Ansatz with $n \to 1$, yields

$$q = \langle \hat{m} \rangle^2 = \mathcal{N}^{-1} \int dz \mathcal{W}[z] \tanh(z)^2, \tag{47}$$

$$\hat{m} = \tanh(z),$$
$$\mathcal{W}[z] = \exp\left(-\frac{z^2}{p\beta^2 q^{p-1}}\right) \cosh(z), \tag{48}$$
$$\mathcal{N} = \int dz \mathcal{W}[z] = \sqrt{\pi p} \beta q^{(p-1)/2} \exp\left(\frac{p\beta^2 q^{p-1}}{4}\right).$$

The second-order derivative reads

$$\Gamma''_{\alpha\beta,\gamma\delta} = \theta(p-2) Q_{\alpha\beta}^{p-3} \delta_\alpha^\gamma \delta_\beta^\delta (\langle \sigma_\alpha \sigma_\beta \rangle - Q_{\alpha\beta}) + \theta Q_{\alpha\beta}^{p-2} \left(\theta Q_{\alpha\beta}^{p-2} \langle \sigma_\alpha \sigma_\beta \sigma_\gamma \sigma_\delta \rangle_c - \delta_\alpha^\gamma \delta_\beta^\delta\right), \tag{49}$$

where the presence of connected averages $\langle \ldots \rangle_c$ is a consequence of the direct derivative of the term $\ln \mathrm{Tr}_\sigma \mathcal{W}[Q;\sigma]$ (cf. the Appendix). In order to impose criticality, $\Gamma''_{\alpha\beta,\gamma\delta}$, evaluated at the saddle-point condition and projected onto the replicon subspace, should vanish. The first part of Eq. (49) is proportional to $\Gamma'_{\alpha\beta}$, cf. Eq. (46), and it does not contribute. The vanishing of the second part, in the RS Ansatz, reads

$$2\theta q^{p-2}(1 - 2q + r) - 1 = 0, \tag{50}$$

as detailed in the Appendix. Here $r = \langle \hat{m} \rangle^4$. This allows us to rewrite Eq. (50) as

$$\tfrac{1}{2} = \theta q^{p-2} \langle (1 - \hat{m}^2)^2 \rangle = \theta q^{p-2} \langle \mathrm{sech}^4(x) \rangle. \tag{51}$$

Once the saddle point and the vanishing of the second-order derivative [Eq. (49)] are imposed, the third-order derivative reads

$$\frac{Q_{\alpha\beta}^{6-3p}}{\theta^3} \Gamma'''_{\alpha\beta,\gamma\delta,\varepsilon\varphi} = \frac{(p-2) Q_{\alpha\beta}^{3-2p}}{\theta^2} \delta_\alpha^\gamma \delta_\beta^\delta \delta_\alpha^\varepsilon \delta_\beta^\varphi + \langle \sigma_\alpha \sigma_\beta \sigma_\gamma \sigma_\delta \sigma_\varepsilon \sigma_\varphi \rangle_c \tag{52}$$

TABLE I. Dynamic exponents in the Ising p-spin model.

p	T_d	q_d	λ	a	b
$\to 2$	1	0	1/2	0.395	1
2.05	0.916	0.051	0.556	0.379	0.892
2.2	0.808	0.198	0.652	0.346	0.768
2.5	0.724	0.428	0.719	0.320	0.609
3	0.682	0.643	0.743	0.308	0.570
4	0.678	0.815	0.746	0.307	0.565
5	0.700	0.881	0.743	0.308	0.570
6	0.727	0.915	0.739	0.310	0.576
7	0.756	0.935	0.736	0.311	0.581
8	0.784	0.948	0.733	0.313	0.586
9	0.812	0.957	0.731	0.314	0.589
$\to \infty$	$\sqrt{\frac{p}{4 \ln p}}$	1	2/3	0.340	0.700

and this allows us to write the coefficients of the expansion, cf. Eq. (4), as

$$\frac{q_d^{6-3p}}{8\theta^3} w_1 = 1 - 3q_d + 3r_d - u_d = \langle (1 - \hat{m}^2)^3 \rangle, \tag{53}$$

$$\frac{q_d^{6-3p}}{8\theta^3} w_2 = 2(q_d - 2r_d + u_d) + \Delta = 2\langle \hat{m}^2 (1 - \hat{m}^2)^2 \rangle + \Delta, \tag{54}$$

$$\Delta = \frac{2(p-2) q_d^{3-2p}}{\beta_d^4 p^2 (p-1)^2}, \tag{55}$$

where $u = \langle \hat{m} \rangle^6$. As it happens in the Sherrington-Kirkpatrick (SK) model,[24,36] the term Δ vanishes if $p = 2$ and it can be considered as the correction to w_2 due to the multibody interaction. With this results one can obtain the numerical values of the coefficients and the exponent a reported in Table I.

A. $p \to 2$ limit

The interest in the behavior of the model for p close to 2 is due to its relation with the SK model. Carrying on an expansion for small $\epsilon = p - 2$, one can expect and, actually, self-consistently verify that the finite jump of the overlap at the transition is of order ϵ. One can consequently expand the action for small $Q_{\alpha\beta}$, still considering the transition as discontinuous.

From Eq. (44) one obtains

$$\frac{1}{\beta^2} \Lambda_{\alpha\beta}^{1-\varepsilon} = Q_{\alpha\beta}. \tag{56}$$

Putting this result in the action Γ, Eq. (42), and expanding for small $\Lambda_{\alpha\beta}$ leads to

$$\Gamma[\Lambda] = -\frac{1}{4\beta^2} \sum_{\alpha\beta} \Lambda_{\alpha\beta}^{2-\varepsilon} + \frac{1}{4} \sum_{\alpha\beta} \Lambda_{\alpha\beta}^2 + \frac{1}{6} \mathrm{Tr} \Lambda^3 + O(\Lambda^4). \tag{57}$$

The first three derivatives read

$$2\Gamma'_{\alpha\beta} = -\frac{2-\varepsilon}{2\beta^2}\Lambda_{\alpha\beta}^{1-\varepsilon} + \Lambda_{\alpha\beta} + (\Lambda^2)_{\alpha\beta}, \tag{58}$$

$$2\Gamma''_{\alpha\beta,\gamma\delta} = -\frac{(2-\varepsilon)(1-\varepsilon)}{2\beta^2}\Lambda_{\alpha\beta}^{-\varepsilon}\delta_\alpha^\gamma\delta_\beta^\delta + \delta_\alpha^\gamma\delta_\beta^\delta + \delta_\alpha^\gamma\Lambda_{\delta\beta} + \Lambda_{\alpha\gamma}\delta_\beta^\delta, \tag{59}$$

$$2\Gamma'''_{\alpha\beta,\gamma\delta,\varepsilon\varphi} = \frac{(2-\varepsilon)(1-\varepsilon)\varepsilon}{2\beta^2}\Lambda_{\alpha\beta}^{-1-\varepsilon}\delta_\alpha^\gamma\delta_\beta^\delta\delta_\alpha^\varepsilon\delta_\beta^\varphi + \delta_\alpha^\gamma\delta_\delta^\varepsilon\delta_\beta^\varphi + \delta_\alpha^\varepsilon\delta_\gamma^\varphi\delta_\beta^\delta. \tag{60}$$

The first two derivatives yield the criticality conditions. In the RS Ansatz, in the $n \to 1$ limit, these reduce to the equations

$$\frac{1}{\beta^2}\hat{\lambda}^{1-\varepsilon} = \hat{\lambda} - \hat{\lambda}^2, \tag{61}$$

$$\frac{1-\varepsilon}{\beta^2}\hat{\lambda}^{-\varepsilon} = 1 - 2\hat{\lambda}, \tag{62}$$

for β and $\hat{\lambda}$, where $\hat{\lambda}$ is the off-diagonal part of $\Lambda_{\alpha\beta}$. At the dynamic transition, the system equation is solved by

$$\hat{\lambda}_d \simeq \varepsilon \tag{63}$$

$$\beta_d \simeq 1 - \frac{1}{2}\varepsilon\ln\varepsilon. \tag{64}$$

Evaluating the third-order derivative, Eq. (60), on β_d and $\hat{\lambda}_d$ leads to

$$\lambda = \left.\frac{w_2}{w_1}\right|_d = \frac{1}{2}. \tag{65}$$

This result agrees with the $a = 0.395$ proposed by Kirkpatrick and Thirumalai[12] studying the dynamics of a soft-spin version of the model in the $p \to 2$ limit.

We note that the behavior of the parameter λ is discontinuous as a function of p at $p = 2$. Indeed in the SK model, i.e., precisely at $p = 2$, we have $\lambda = w_2 = 0$, while as soon as $\epsilon > 0$ we have $\lambda = 1/2$. This happens because the coefficient w_2 is proportional to the third derivative of $q^{2+\epsilon}$ which is singular at $q = 0$ as soon as ϵ is different from zero.

B. $p \to \infty$ limit

The behavior of the p-spin model for large p has been previously studied, due to its relation to the random energy model (REM).[37–39] In this section we indeed present the calculation of $\lambda = w_2/w_1$ in the limit $p \to \infty$.

As in the $p \to 2$ case, it is more convenient to work with the auxiliary variable $\hat{\lambda}$, the RS off-diagonal element of the matrix $\Lambda_{\alpha\beta}$, which is related to q by Eq. (44). Furthermore, in order to keep q finite, one should expect and consistently verify that $\hat{\lambda}$ diverges in the large p limit. From Eqs. (44) and (47) one obtains

$$q(\hat{\lambda}) = \left(\frac{2\hat{\lambda}}{p\beta}\right)^{1/(p-1)} \simeq 1 - e^{-\lambda/2}\sqrt{\frac{\pi}{2\hat{\lambda}}}, \quad \hat{\lambda} \gg 1, \tag{66}$$

where, in the last expression, only the leading term, for $\hat{\lambda} \gg 1$, has been retained. Differentiating this equation and keeping the leading order,

$$\frac{2q(\hat{\lambda})}{p\hat{\lambda}} = e^{-\hat{\lambda}/2}\sqrt{\frac{\pi}{2\hat{\lambda}}}, \tag{67}$$

one recovers the set of equations for the criticality condition. Solving Eqs. (66) and (67) for $\hat{\lambda}$ and for q yields

$$\hat{\lambda}_d \simeq 2\ln p, \tag{68}$$

$$T_d \simeq \sqrt{\frac{p}{4\ln p}}. \tag{69}$$

As $p \to \infty$ the critical dynamic temperature diverges and the dynamic overlap $q_d \to 1$, cf. Eq. (66).

We now move to the computation of the exponent parameter λ. Equations (53) and (54) and their ratio λ, in the large $\hat{\lambda}$ limit, reduce to

$$w_1 = \frac{3\pi}{8}e^{-\hat{\lambda}/2}\sqrt{\frac{1}{2\pi\hat{\lambda}}}, \tag{70}$$

$$w_2 = \frac{\pi}{4}e^{-\hat{\lambda}/2}\sqrt{\frac{1}{2\pi\hat{\lambda}}} + \Delta, \tag{71}$$

$$\lambda = \frac{w_2}{w_1} = \frac{2}{3}\left(1 + \Delta e^{\hat{\lambda}/2}\sqrt{\frac{32\hat{\lambda}}{\pi}}\right). \tag{72}$$

Evaluating the last term one obtains

$$\Delta e^{\hat{\lambda}/2}\sqrt{\frac{32\hat{\lambda}}{\pi}} \simeq \sqrt{\frac{1}{2\pi\ln^3 p}}\left(1 - \frac{1}{p}\sqrt{\frac{\pi}{4\ln p}}\right)^{-2p}, \tag{73}$$

which goes to zero as $p \to \infty$.

Our result is, indeed, $\lambda \to 2/3$, corresponding to an exponent $a = 0.340$.

C. Addition of ferromagnetic couplings

It is possible to generalize the previous results allowing the couplings J to have a nonzero mean:

$$P(J) = \sqrt{\frac{N^{p-1}}{\pi p!}}\exp\left[-\frac{N^{p-1}}{p!}\left(J - \frac{p!J_0}{N^{p-1}}\right)^2\right], \tag{74}$$

To treat the case $J_0 \neq 0$ one has to introduce a nonzero magnetization for the system and the Gibbs effective action, cf. Eq (45), becomes

$$\Gamma[Q,\Lambda,m,x] = \frac{\beta^2}{4}\sum_{ab}Q_{ab}^p + \beta J_0\sum_a m_a^p - \frac{1}{2}\sum_{ab}Q_{ab}\Lambda_{ab} - \sum_a m_a x_a + \ln\mathrm{Tr}_{\{\sigma\}}\mathcal{W}[\Lambda,x;\sigma],$$

$$\mathcal{W}[\Lambda,x;\sigma] \equiv \exp\left(\frac{1}{2}\sum_{ab}\Lambda_{ab}\sigma_a\sigma_b + \sum_a x_a\sigma_a\right), \tag{75}$$

where the fields x_a play for the magnetization m_a the same role of Λ for the overlap. Through a saddle-point calculation one arrives at two coupled equations:

$$q = \langle \hat{m}^2 \rangle = \mathcal{N}^{-1} \int dz \mathcal{W}[z] \tanh(z)^2, \tag{76}$$

$$m = \langle \hat{m} \rangle = \mathcal{N}^{-1} \int dz \mathcal{W}[z] \tanh(z), \tag{77}$$

$$\mathcal{W}[z] = \exp\left(-\frac{(z - p\beta J_0 m^{p-1})^2}{p\beta^2 q^{p-1}}\right), \tag{78}$$

$$\mathcal{N} = \int dz \mathcal{W}[z] = \sqrt{\pi p}\beta q^{(p-1)/2}. \tag{79}$$

The solution with $m = 0$, the SG phase, is always present but, for J_0 large enough, a ferromagnetic (FM) RS solution appears discontinuously with $m = q > 0$ and turns out to be the stable one.

Since in the action (75) J_0 couples only to the magnetization m, the SG phase, where $m = 0$, is not affected by the presence of a nonzero mean of the couplings. As a result, along the whole dynamic SG transition the physics does not change and the values of the exponents a and b are constant.

The paramagnetic/ferromagnetic (PM/FM) transition is a usual thermodynamic first-order transition with a ferromagnetic spinodal line.[40] Two relevant point are, indeed, present: the tricritical point between the SG, FM, and PM phases, and the intersection between the dynamic transition line and the FM spinodal line. It has been shown[40,41] that both the relevant points belong to the Nishimori line (NL)

$$J_0^{NL}(T) = \frac{1}{2T}, \tag{80}$$

a line in the J_0,T phase diagram. This fact is due to the following property of systems on the NL:[41]

$$\lim_{n\to 0} \Gamma\big(\beta, J_0^{NL}(\beta), q, m\big)\big|_{m=q} = \lim_{n\to 1} \Gamma(\beta, 0, q, 0). \tag{81}$$

Equation (81) corresponds to the statement that the static Gibbs free-energy potential along the NL is equal to the dynamic Gibbs free energy along the $J_0 = 0$ axis. This means that, in order to obtain the ratio w_2/w_1 at the dynamical transition with $n \to 1$, one can, simplifying the numerical calculation, evaluate those coefficients along the NL, at the spinodal point and working with $n \to 0$. Furthermore, since on the NL the melting process is equivalent to a glassy transition,[40] one can argue that the exponent calculated through our procedure controls this melting process too.

V. CONCLUSIONS

In the present work we have applied a method[24] to compute the slowing down exponents on the Ising and spherical versions of the frustrated p-spin model with Gaussian interaction. This method allows us to derive the mode coupling exponents of the critical dynamics by means of an analytical static-driven computation. These exponents govern the power-law approach to $[C(t) \simeq q_d + (t/\tau_0)^{-a}]$ and departure from $[C(t) \simeq q_d - (t/\tau)^b]$ the plateau value at $C(t) = q_d$.

For the spherical case, the method is exactly equivalent to standard schematic dynamic approaches. We verified this, reproducing the analytic results obtained by means of a Langevin description of the dynamics in Ref. 18 and reproducing, in full generality, all schematic MCT exponents.[3]

For the Ising version we presented our computation and the exact values of both exponents for any value of p. One can, and it is usually done, approximate discrete by soft spins in order to construct a dynamic equation. In that case, though, the final computation for $p > 2$ differs: the discrete case depends on the value of p, whereas the soft one does not. Our results agree with the computation performed in Ref. 12 on a soft-spin approximation in the $p \to 2$ limit. In this limit we found a discontinuity with the Sherrington-Kirkpatrick model ($p = 2$), with a finite jump of the exponent parameter λ from 0 to 1/2. For the sake of completeness, also the $p \to \infty$ limit has been characterized. The study of the Ising p-spin model eventually includes the case of a nonzero mean of the couplings ($J_0 \neq 0$): up to a critical value, where a ferromagnetic transition takes place, the behavior of the model does not change with J_0, validating our estimates for the exponents along the whole SG transition line.

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APPENDIX: COMPUTATION OF THE CUMULANTS IN THE REPLICON SUBSPACE

In this appendix we present the calculation of the connected cumulants of four and six replicas projected in the replicon subspace. At first it should be noticed that in the RS Ansatz four replica index quantities could take only three different values, depending on how many replica indices are repeated. For

$$\begin{aligned} C^{(4)}_{(\alpha\beta)(\gamma\delta)} &= \langle \sigma_\alpha\sigma_\beta\sigma_\gamma\sigma_\delta \rangle_c \\ &= \langle \sigma_\alpha\sigma_\beta\sigma_\gamma\sigma_\delta \rangle - \langle \sigma_\alpha\sigma_\beta \rangle \langle \sigma_\gamma\sigma_\delta \rangle \end{aligned} \tag{A1}$$

we have

$$C^{(4)}_{(\alpha\beta)(\alpha\beta)} = 1 - q^2, \tag{A2}$$

$$C^{(4)}_{(\alpha\beta)(\alpha\delta)} = q - q^2, \tag{A3}$$

$$C^{(4)}_{(\alpha\beta)(\gamma\delta)} = r - q^2, \tag{A4}$$

where here α, β, γ, δ are considered different. This allows us to express the tensorial form of $C^{(4)}$:

$$\begin{aligned} C^{(4)}_{(\alpha\beta)(\gamma\delta)} = &(1 - 2q + r)\big(\delta^\gamma_\alpha\delta^\delta_\beta + \delta^\delta_\alpha\delta^\gamma_\beta\big) \\ &+ (q - r)\big(\delta^\gamma_\alpha + \delta^\delta_b + \delta^\delta_\alpha + \delta^\gamma_b\big) \\ &+ (r - q^2). \end{aligned} \tag{A5}$$

Summing over all replica indices the $C^{(4)}$ times the fluctuations in the replicon subspace [Eq. (14)], the only term that does not vanish is $(1-2q+r)$, with a factor 2 due to the exchange of α with β:

$$\sum_{(\alpha\beta),(\gamma\delta)} C^{(4)}_{(\alpha\beta)(\gamma\delta)}\delta q^{(r)}_{\alpha\beta}\delta q^{(r)}_{\gamma\delta} = 2(1-2q+r)\sum_{(\alpha\beta)}\left(\delta q^{(r)}_{\alpha\beta}\right)^2. \quad \text{(A6)}$$

Six replica index quantities, like

$$\begin{aligned} C^{(6)}_{(\alpha\beta)(\gamma\delta)(\varepsilon\varphi)} &= \langle\sigma_\alpha\sigma_\beta\sigma_\gamma\sigma_\delta\sigma_\varepsilon\sigma_\varphi\rangle_c \\ &= \langle\sigma_\alpha\sigma_\beta\sigma_\gamma\sigma_\delta\sigma_\varepsilon\sigma_\varphi\rangle - \langle\sigma_\alpha\sigma_\beta\sigma_\gamma\sigma_\delta\rangle\langle\sigma_\varepsilon\sigma_\varphi\rangle \\ &\quad - \langle\sigma_\gamma\sigma_\delta\sigma_\varepsilon\sigma_\varphi\rangle\langle\sigma_\alpha\sigma_\beta\rangle - \langle\sigma_\alpha\sigma_\beta\sigma_\varepsilon\sigma_\varphi\rangle\langle\sigma_\gamma\sigma_\delta\rangle \\ &\quad + 2\langle\sigma_\alpha\sigma_\beta\rangle\langle\sigma_\gamma\sigma_\delta\rangle\langle\sigma_\varepsilon\sigma_\varphi\rangle, \end{aligned} \quad \text{(A7)}$$

can take eight different values. The computation follows as in the previous case[42] leading to

$$\begin{aligned} &\sum_{(\alpha\beta),(\gamma\delta)(\varepsilon\varphi)} C^{(6)}_{(\alpha\beta)(\gamma\delta)(\varepsilon\varphi)}\delta q^{(r)}_{\alpha\beta}\delta q^{(r)}_{\gamma\delta}\delta q^{(r)}_{\varepsilon\varphi} \\ &\quad = 8(1-3q+3r-u)\sum_{(\alpha\beta\gamma)}\delta q^{(r)}_{\alpha\beta}\delta q^{(r)}_{\beta\gamma}\delta q^{(r)}_{\gamma\alpha} \\ &\qquad + 16(q-2r+u)\sum_{(\alpha\beta)}\left(\delta q^{(r)}_{\alpha\beta}\right)^3, \end{aligned} \quad \text{(A8)}$$

which concludes the calculation.

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