4

Duality of the Two-dimensional Ising Model

Being dual is in the nature of things.

Elias Canetti

In this chapter we will begin our study of the Ising model on the two-dimensional lattice. In two dimensions the model has a phase transition, with critical exponents that have different values from those obtained in the mean field approximation. For this reason, it provides an important example of critical phenomena. As we will see in great detail in this chapter and in the next, among all exactly solved models of statistical mechanics, the two-dimensional Ising model is not only the one that has been most studied but it is also the model that has given a series of deep mathematical and physical results. Many solutions of the model stand out for the ingenious methods used, such as the theory of determinants, combinatorial approaches, Grassmann variables, or elliptic functions. Many results have deeply influenced the understanding of critical phenomena and have strongly stimulated new fields of research. Ideas that have matured within the study of the two-dimensional Ising model, such as the duality between its high- and low-temperature phases, have been readily generalized to other systems of statistical mechanics and have also found important and fundamental applications in other important areas such as, for instance, quantum field theory. Equally fundamental is the discovery that in the vicinity of the critical point, the dynamics of the model can be described from the relativistic Dirac equation for Majorana fermions.

This chapter is devoted to the study of some properties of the model that can be established by means of elementary considerations. We will discuss, in particular, the argument by Peierls that permits us to show the existence of a phase transition in the model. We will also present the duality relation that links the expressions of the partition functions in the low- and high-temperature phase of a square lattice, and the partition functions of the triangle and hexagonal-lattices. In the last case, it is necessary to make use of an identity, known as the *star–triangle equation*, that will be useful later on to study the commutativity properties of the transfer matrix. At the end of the chapter, we will also discuss the general formulation of the duality transformations for lattice statistical models.

4.1 Peierls's Argument

In 1936 R. Peierls published an article with the title *On the Model of Ising for the Ferromagnetism* in which he proved that the Ising model in two or higher dimensions has a low-temperature region in which the spontaneous magnetization is different from zero. Since at high temperature the system is disordered, it follows that there must exist a critical value of the temperature at which a phase transition takes place.

Peierls's argument starts with the initial observation that to each configuration of spins there corresponds a set of closed lines that separate the regions in which the spins assume values $+1$ from those in which they assume values -1, as shown in Fig. 4.1. If it is possible to prove that at sufficiently low temperatures the mean value of the regions enclosed by the closed lines is only a small fraction of the total volume of the system, one has proved that the majority of the spins is prevalently in the state in which there is a spontaneous magnetization.

There are several versions of the original argument given by Peierls. The simplest generalizes the argument already used in the one-dimensional case (see Chapter 2, Section 2.1) and concerns the stability of the state with a spontaneous magnetization. Let's consider the two-dimensional Ising model at low temperatures and suppose that it is in the state of minimal energy in which all the spins have values $+1$. The thermal fluctuations create domains in which there are spin flips, such as the domain in Fig. 4.1. The creation of such domains clearly destabilizes the original ordered state. There is an energetic cost to the creation of the domain shown in Fig. 4.1, given by

$$\Delta E = 2J\,L, \tag{4.1.1}$$

where L is the total length of the curve. There are, however, many ways of creating a closed curve of a given total perimeter L. In fact, the domain in which the spins are flipped can be placed everywhere in the lattice and moreover can assume different shapes. To estimate the number of such configurations, imagine that the closed line is created by a random motion on the lattice of total number of steps equal to L. If we

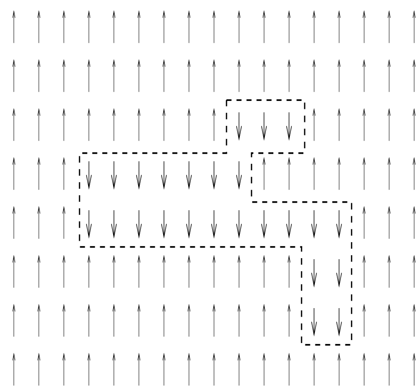

Fig. 4.1 *Closed lines that enclose a region with a flipped value of the spin.*

assume that at each step of this motion there are only two possibilities,[1] we have 2^L ways of drawing a closed curve of length L. The corresponding variation of the entropy is given by

$$\Delta S = k \ln(2^L). \tag{4.1.2}$$

Hence the total variation of the free energy associated to the creation of such a domain is

$$\begin{aligned} \Delta F = \Delta E - T\Delta S &= 2JL - kT\ln(2^L) \\ &= L(2J - kT\ln 2). \end{aligned} \tag{4.1.3}$$

Therefore the system is stable with respect to the creation of such domains of arbitrary length L (i.e. $\Delta F \geq 0$) if

$$T \leq T_c = \frac{2J}{k\ln 2} = 2.885\,\frac{J}{k}. \tag{4.1.4}$$

Note that such an estimate is surprisingly close to the exact value of the critical temperature $T_c = 2.269...J/k$ that we will determine in the next section.

4.2 Duality Relation in Square Lattices

Peierls's argument shows that the two-dimensional Ising model has two different phases: the high-temperature phase in which the system is disordered and the low-temperature phase in which the system is ordered, with a non-zero spontaneous magnetization. The exact value of the critical temperature at which the phase transition happens was first determined by H.A. Kramers and G.H. Wannier by using a duality relation between the high- and the low-temperature partition functions.[2] The self-duality of the two-dimensional Ising model on a square lattice is one of its most important properties, with far-reaching consequences on its dynamics. To prove it, we need to study the series expansions of the high/low-temperature phase of the model. We will see that these expansions have an elegant geometrical interpretation in terms of a counting problem of the polygons that can be drawn on a lattice. In the next section we will consider the square lattice and, in later sections, the triangle and hexagonal lattices.

4.2.1 High-temperature Series Expansion

Consider a square lattice $\mathcal{L}$ with M horizontal links and M vertical links. In the thermodynamical limit $M \to \infty$, M coincides with the total number N of the lattice sites. In the following we will consider a Hamiltonian with different coupling constants,

[1] On a square lattice, starting from a given site, one can move in four different directions. However, taking four instead of two as possible directions of the motion gives an upper estimate of the entropy, since it does not take into account that the final curve is a closed contour.

[2] The self-duality of the model that we are going to discuss only holds in the absence of an external magnetic field.

along the horizontal and vertical directions. Let J and J' be these coupling constants, respectively. For the partition function of the model at zero magnetic field we have

$$Z_N = \sum_{\{\sigma\}} \exp\left[K \sum_{(i,j)} \sigma_i \sigma_j + L \sum_{(i,k)} \sigma_i \sigma_k\right], \tag{4.2.1}$$

where the first sum is on the spins along the horizontal links and the second sum along the vertical links, with

$$K = \beta J; \quad L = \beta J'.$$

By using the identity

$$\exp\left[x \sigma_i \sigma_l\right] = \cosh x \left(1 + \sigma_i \sigma_l \tanh x\right), \tag{4.2.2}$$

the partion function can be written as

$$Z_N = (\cosh K \cosh L)^M \sum_{\{\sigma\}} \prod_{(i,j)} (1 + v \sigma_i \sigma_j) \prod_{(i,k)} (1 + w \sigma_i \sigma_k), \tag{4.2.3}$$

with

$$v = \tanh K; \quad w = \tanh L.$$

Both parameters v and w are always less than 1 for all values of the temperature, except for $T = 0$ when their value is $v = w = 1$. In particular, they are small parameters in the high-temperature phase and it is natural to look for a series expansion of the partition function near $T = \infty$.

If we expand the two products in (4.2.3), we have 2^{2M} terms, since there are $2M$ factors (one for each segment), and each of them has two terms. We can set up a graphical representation for this expansion associating a line drawn on the horizontal link (i, j) to the factor $v \sigma_i \sigma_j$ and a line on the vertical link (i, k) to the factor $w \sigma_i \sigma_k$. No line is drawn if there is instead the factor 1. Repeating this operation for the 2^{2M} terms, we can establish a correspondence between these terms and a graphical configuration on the lattice $\mathcal{L}$. The generic expression of these terms is

$$v^r w^s \sigma_1^{n_1} \sigma_2^{n_2} \sigma_3^{n_3} \ldots$$

where r is the total number of horizontal lines, s the total number of vertical lines, while n_i is the number of lines where i is the final site. It is now necessary to sum over all spins of the lattice in order to obtain the partition function. Since each spin σ_i assumes values ± 1, we have a null sum unless all $n_1, n_2, \ldots, n_N$ are even numbers and, in this case, the result is $2^N v^r w^s$. Based on these considerations, the partition function can be expressed as

$$Z_N = 2^N (\cosh K \cosh L)^M \sum_P v^r w^s, \tag{4.2.4}$$

where the sum is over all the line configurations on $\mathcal{L}$ with an even number of lines at each site, i.e. *all closed polygonal lines P of the lattice $\mathcal{L}$*. Therefore, apart from a prefactor, the partition function is given by the geometrical quantity

$$\Phi(v, w) = \sum_P v^r w^s. \tag{4.2.5}$$

It is easy to compute the first terms of this function. The first term is equal to 1 and corresponds to the case in which there are no polygons on the lattice. The second term corresponds to the smallest closed polygon on the lattice $\mathcal{L}$, i.e. a square with unit length, as shown in Fig. 4.2. The number of such squares is equal to N, since they can be placed on any of the N sites of the lattice. Each of them has a weight $(vw)^2$, hence the second term of the sum (4.2.5) is equal to $N(vw)^2$. The next closed polygonal curve is a rectangle of six sides: there are two kinds of them, as shown in Fig. 4.3, each with a degeneracy equal to N, and width v^4w^2 for the first and v^2w^4 for the second.

Using the first terms, the function $\Phi(v, w)$ is given by

$$\Phi(v, w) = 1 + N(vw)^2 + N(v^4w^2 + v^2w^4) + \cdots \tag{4.2.6}$$

The computation of the next terms becomes rapidly more involved although it can be clearly performed in a systematic way: presently, the first 40 terms of such a series are known. For our purposes it is not necessary to introduce all these terms, since the duality properties can be established just by exploiting the geometrical nature of the sum (4.2.5).

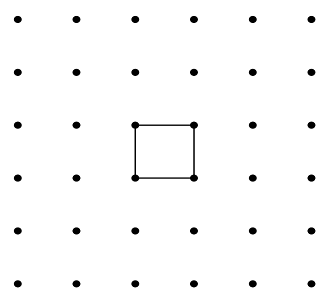

Fig. 4.2 *Second term of the high-temperature expansion.*

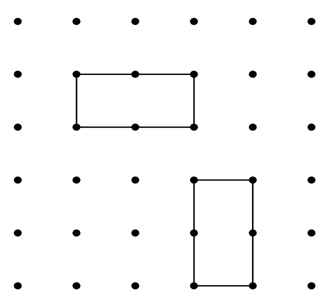

Fig. 4.3 *Third term of the high-temperature expansion.*

4.2.2 Low-temperature Series Expansion

In the low-temperature phase, according to Peierls's argument, the spins tend to align one with another. The series expansion of the partition function in this phase can be obtained as follows. For a given configuration of the spins, let r and s be the numbers of vertical and horizontal links in which the two adjacent spins are antiparallel. Since M is the total number of vertical links as well as of the horizontal ones, we have $(M - r)$ vertical links and $(M - s)$ horizontal links in which the adjacent spins are parallel. The contribution to the partition function of such a configuration is

$$\exp\left[K(M - 2s) + L(M - 2r)\right].$$

Besides a constant, this expression depends only on the number of links in which the spins are antiparallel. These segments will be called *antiparallel links.*

It is now convenient to introduce the concept of a *dual lattice.* This notion, which is familiar in crystallography, has already been met in the discussion of the four-color problem (see Appendix C of Chapter 2). For any planar lattice $\mathcal{L}$, we can define another lattice $\mathcal{L}_D$ that is obtained by placing its sites at the center of the original lattice $\mathcal{L}$ and joining pairwise those relative to adjacent faces, i.e. those sharing a common segment. It is easy to see that the dual lattice of a square lattice is also a square lattice, simply displaced by a half-lattice space with respect to the original one (see Fig. 4.4), while the dual lattice of a triangular lattice is a hexagonal one and vice versa.

Given the geometrical relation between the dual and the original lattices, it is easy to see that the spins can be equivalently regarded as defined on the sites of the original lattice $\mathcal{L}$ or at the center of the faces of the dual lattice $\mathcal{L}_D$. This allows us to introduce a useful graphical formalism. Given a configuration, we can associate to its antiparallel links a set of lines of the dual lattice by the following rule: if two next neighbor spins are antiparallel, then draw a line along the segment of $\mathcal{L}_D$ that crosses them, draw no line if they are parallel. By applying this rule, on the dual lattice $\mathcal{L}_D$ there will be r horizontal lines and s vertical lines. However, it is easy to see that there should always

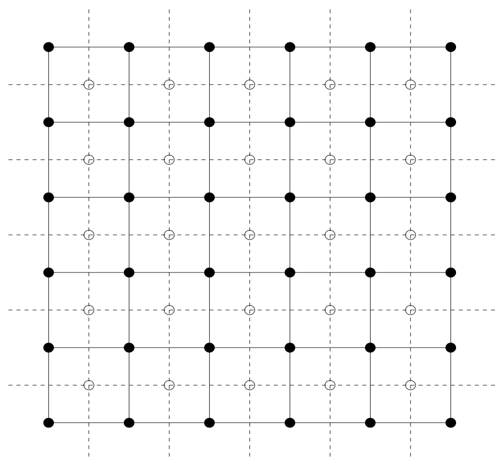

Fig. 4.4 *Dual square lattices.*

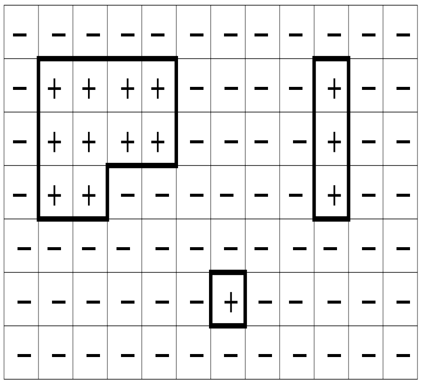

Fig. 4.5 *Polygons that separate the domains with spins* $+1$ *and* -1.

be an even number of lines passing through each site, since there is an even number of next successive changes among the adjacent faces. The drawn lines must therefore form closed polygons on the dual lattice $\mathcal{L}_D$, as illustrated in Fig. 4.5.

It is evident that the closed polygons that have been obtained in this way are nothing else that the perimeters of the different magnetic domains where, inside them, all spins are aligned in the same direction. Since for any given set of polygons there are two corresponding configurations (one obtained from the other by flipping all the spins), the partition function can be written as

$$Z_N = 2\exp[M(K+L)] \sum_{\tilde{P}} \exp[-(2Lr + 2Ks)], \tag{4.2.7}$$

where the sum is over all closed polygons $\tilde{P}$ on the dual lattice $\mathcal{L}_D$. This is the low-temperature expansion, because when $T \to 0$, both K and L are quite large and the dominant terms are given by small values of r and s. Therefore, also in this case the partition function is expressed by a geometrical quantity

$$\tilde{\Phi}\left(e^{-2L}, e^{-2K}\right) = \sum_{\tilde{P}} \exp[-(2Lr + 2Ks)]. \tag{4.2.8}$$

Consider the first terms of this series. The first term is equal to 1 and corresponds to the situation in which all spins assume the same value. The second term corresponds to the configuration in which there is only one spin flip: in this case there are two horizontal antiparallel links and two vertical antiparallel links that altogether form a square. The degeneracy of this term is equal to N, since the spin that has been flipped can be placed on any of the N sites of the lattice. The next term is given by the rectangle with six segments that can be elongated either horizontally or vertically: these rectangles correspond to next neighbor spins that are antiparallel to all other spins of the lattice. Taking into account the degeneracy N and the orientation of the rectangle, the contribution of this term to the partition function is $N(e^{-4L-8K} + e^{-8L-4K})$. With these first terms, the function $\tilde{\Phi}(e^{-2L}, e^{-2K})$ is expressed by

$$\tilde{\Phi}\left(e^{-2L}, e^{-2K}\right) = 1 + Ne^{-4L-4K} + N(e^{-4L-8K} + e^{-8L-4K}) + \cdots \tag{4.2.9}$$

From what was said above, it should now be clear that all terms of the function $\tilde{\Phi}$ have the same origin as those of the function Φ.

4.2.3 Self-duality

In the last two sections we have shown that the partition function of the two-dimensional Ising model on a square lattice can be expressed in two different series expansions, one that holds in the high-temperature phase, the other in the low-temperature phase, given in eqns (4.2.4) and (4.2.7), respectively. The final expressions involve a function that has a common geometric nature, i.e. a sum over all the polygonal configurations that can be drawn on the original lattice and its dual. For finite lattices, $\mathcal{L}$ and $\mathcal{L}_D$ differ only at the boundary. In the thermodynamical limit this difference disappears and the two expressions can be obtained one from the other simply by a change of variables. For $N \to \infty$ one has $M/N = 1$: substituting K and L in eqn (4.2.5) with $\tilde{K}$ and $\tilde{L}$ given by

$$\tanh \tilde{K} = e^{-2L}; \quad \tanh \tilde{L} = e^{-2K}, \tag{4.2.10}$$

and comparing with eqn (4.2.8), we have in fact

$$\tilde{\Phi}\left(e^{-2\tilde{K}}, e^{-2\tilde{L}}\right) = \Phi(v, w). \tag{4.2.11}$$

This implies the following identity for the partition function

$$\frac{Z_N[K, L]}{2^N (\cosh K \cosh L)^N} = \frac{Z_N[\tilde{K}, \tilde{L}]}{2 \exp[N(\tilde{K} + \tilde{L})]}. \tag{4.2.12}$$

Equation (4.2.10) can be expressed in a more symmetrical form:

$$\sinh 2\tilde{K} \, \sinh 2L = 1; \quad \sinh 2\tilde{L} \, \sinh 2K = 1. \tag{4.2.13}$$

Analogously, eqn (4.2.12) can be written as

$$\frac{Z_N[K, L]}{(\sinh 2K \sinh 2L)^{N/4}} = \frac{Z_N[\tilde{K}, \tilde{L}]}{(\sinh 2\tilde{K} \sinh 2\tilde{L})^{N/4}}. \tag{4.2.14}$$

These equations show the existence of a symmetry of the two-dimensional Ising model and establish the mapping between high- and low-temperature phases of the model. Large values of K and L are equivalent to small values of $\tilde{K}$ and $\tilde{L}$, and vice versa large values of $\tilde{K}$ and $\tilde{L}$ correspond to small values of K and L. It must be stressed that this correspondence between the two phases can also be useful from a computational point of view.

We can now identify the critical point. Let's consider first the isotropic case, i.e. $K = L$ and, correspondingly, $\tilde{K} = \tilde{L}$. At the critical point the partition function presents a divergence: assuming that this happens at the value K_c, the same should happen also at $\tilde{K} = K_c$ thanks to eqn (4.2.14). These two values can be different but, making the further hypothesis that there is only one critical point – a hypothesis that

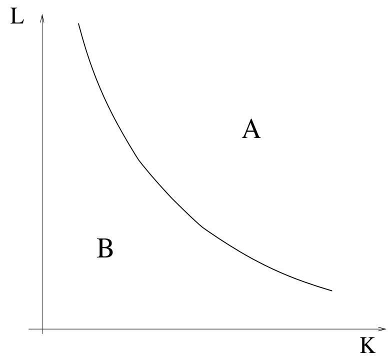

Fig. 4.6 *Critical curve.*

is fully justified from the physical point of view – these two values must coincide and the critical point is thus identified by the condition

$$\sinh 2K_c \,=\, 1; \qquad T_c^{square} \,=\, 2.26922...J. \tag{4.2.15}$$

The arguments presented above were given originally by Kramers and Wannier.

Let us consider now the general case in which there are two coupling constants. Note that combining eqn (4.2.13), we have

$$\sinh 2K \sinh 2L \,=\, \frac{1}{\sinh 2\tilde{K} \sinh 2\tilde{L}}. \tag{4.2.16}$$

This equation implies that, under the mapping $(K, L) \to (\tilde{K}, \tilde{L})$, the region A in Fig. 4.6 is transformed into the region B and vice versa, leaving invariant the points along the curve

$$\sinh 2K \sinh 2L \,=\, 1. \tag{4.2.17}$$

If there is a line of fixed points in A, there should be another line of fixed points also in B. Assuming that there is only one line of fixed points, this is expressed by eqn (4.2.17). Therefore this is the condition that ensures the criticality of the Ising model with different coupling constants along the horizontal and vertical directions. This equation plays an important role both in the solution proposed by Baxter for the Ising model and in the discussion of its hamiltonian limit.

4.3 Duality Relation between Hexagonal and Triangular Lattices

The duality transformation of the square lattice can be generalized to other lattices. In this section we discuss the mapping between the low- and high-temperature phases of the Ising model defined on the triangular and hexagonal lattices shown in Fig. 4.7.

Let us introduce the coupling constants K_i and L_i $(i = 1, 2, 3)$ relative to the triangle and hexagonal lattices, respectively, as shown in Fig. 4.8. In the absence of a

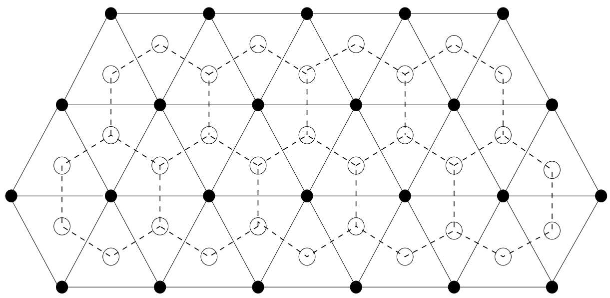

Fig. 4.7 *Dual lattices: hexagonal and triangular lattices.*

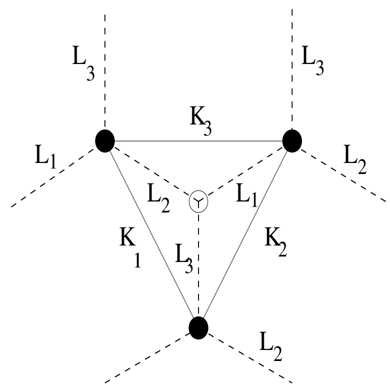

Fig. 4.8 *Coupling constants on the triangular and hexagonal lattices.*

magnetic field, the partition function of the hexagonal lattice is given by

$$Z_N^{\mathrm{H}}(\mathcal{L}) = \sum_{\{\sigma\}} \exp\left[\mathcal{L}_1 \sum \sigma_l \sigma_i + \mathcal{L}_2 \sum \sigma_l \sigma_j + \mathcal{L}_3 \sum \sigma_l \sigma_k\right], \qquad (4.3.1)$$

with $\mathcal{L}_i = L_i/kT$. In the exponential term, the sums refer to all next neighbor pairs of spins along the three different directions of the hexagonal lattice. Similarly, in the absence of the magnetic field, we can write the partition function on the triangular lattice as

$$Z_N^{\mathrm{T}}(\mathcal{K}) = \sum_{\{\sigma\}} \exp\left[\mathcal{K}_1 \sum \sigma_l \sigma_i + \mathcal{K}_2 \sum \sigma_l \sigma_j + \mathcal{K}_3 \sum \sigma_l \sigma_k\right], \qquad (4.3.2)$$

with $\mathcal{K}_i = K_i/kT$ and the sums in the exponentials on all next neighbor pairs of spins in the three different directions of the triangular lattice.

Let's consider the high-temperature expansion of the partition function on the triangular lattice. Put $v_i = \tanh \mathcal{K}_i$, we have

$$Z_N^{\mathrm{T}}(\mathcal{K}) = (2\cosh \mathcal{K}_1 \cosh \mathcal{K}_2 \cosh \mathcal{K}_3) \sum_P v_1^{r_1} v_2^{r_2} v_3^{r_3}, \qquad (4.3.3)$$

where the sum is over all closed polygons on the triangular lattice, with the number of sides equal to r_i $(i = 1, 2, 3)$ along the three different directions.

Consider now the low temperature expansion of the partition function on the hexagonal lattice. This is obtained by drawing the lines corresponding to the antiparallel links on the dual lattice. Since the triangular lattice of N sites is the dual of the hexagonal lattice with $2N$ sites, in this case we have[3]

$$Z_{2N}^{\mathrm{H}}(\mathcal{L}) \;=\; e^{[N(\mathcal{L}_1+\mathcal{L}_2+\mathcal{L}_3)]} \sum_P \exp[-2\mathcal{L}_1 r_1 + \mathcal{L}_2 r_2 + \mathcal{L}_3 r_3], \tag{4.3.4}$$

where the sum is over the closed polygons of the triangular lattice with the number of sides r_i $(i = 1, 2, 3)$ along the three directions.

Since in both expressions there is the same geometrical function given by the sum over polygons drawn on the triangular lattice, imposing

$$\tanh \mathcal{K}_i^* \;=\; \exp[-2\mathcal{L}_i], \quad i = 1, 2, 3 \tag{4.3.5}$$

the two partition functions are related as

$$Z_{2N}^{\mathrm{H}}(\mathcal{L}) \;=\; (2a_1a_2a_3)^{N/2}\, Z_N^{\mathrm{T}}(\mathcal{K}^*), \tag{4.3.6}$$

where

$$a_i \;=\; \sinh 2\mathcal{L}_i = 1/\sinh 2\mathcal{K}_i^*, \quad i = 1, 2, 3.$$

The relation (4.3.5) can be written in a more symmetrical way as

$$\sinh 2\mathcal{L}_i \, \sinh 2\mathcal{K}_i^* \;=\; 1. \tag{4.3.7}$$

As in the square lattice, the duality relation (4.3.7) implies that when one of the coupling constant is small, the other is large and vice versa. However, the duality relation alone cannot determine in this case the critical temperature of the two lattices, since they are not self-dual. Fortunately, there exists a further important identity between the coupling constants of the two lattices that permits us to identify the singular points of the free energies of both models. This identity is the *star–triangle identity* and, because of its importance, it is worth a detailed discussion.

4.4 Star–Triangle Identity

The star–triangle identity plays an important role in the two-dimensional Ising model. In addition to the exact determination of the critical temperature for triangular and hexagonal lattices, this identity also enables us to establish the commutativity of the transfer matrix of the model for special values of the coupling constants. This aspect will be crucial for the exact solution of the model discussed in Chapter 6.

To prove such an identity, first observe that the sites of the hexagonal lattice split into two classes, i.e. the hexagonal lattice is bipartite. The sites of type A interact only with those of type B and vice versa, while there is no direct interaction between sites of the same type (see Fig. 4.9). The generic term that enters the sum in the partition

[3]For large N, the number of links along each of the three directions is equal to N.

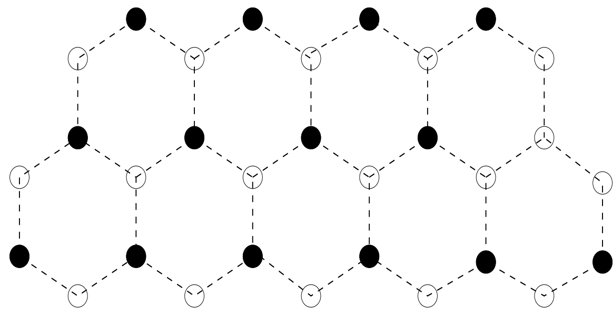

Fig. 4.9 *Bipartition of the hexagonal lattice: site of type A (black sites) and type B (white sites).*

function (4.3.1) can be written as

$$\prod_b W(\sigma_b; \sigma_i, \sigma_j, \sigma_k), \tag{4.4.1}$$

where the product is over all sites of type B and the above quantity is expressed by the Boltzmann weight

$$W(\sigma_b; \sigma_i, \sigma_j, \sigma_k) \,=\, \exp\left[\sigma_b(\mathcal{L}_1\sigma_i + \mathcal{L}_2\sigma_j + \mathcal{L}_3\sigma_k\right]. \tag{4.4.2}$$

Since each spin of type B appears only once in (4.4.1), it is simple to sum on them in the expression of the partition function, with the result

$$Z_N^{\mathrm{E}}(\mathcal{L}) \,=\, \sum_{\sigma_a} \prod_{i,j,k} w(\sigma_i, \sigma_j, \sigma_k), \tag{4.4.3}$$

where

$$w(\sigma_i, \sigma_j, \sigma_k) \,=\, \sum_{\sigma_b = \pm 1} W(\sigma_b; \sigma_i, \sigma_j, \sigma_k) \,=\, 2\,\cosh(\mathcal{L}_i\sigma_i + \mathcal{L}_2\sigma_j + \mathcal{L}_3\sigma_k). \tag{4.4.4}$$

The value of each spin is ± 1 and using the identity

$$\cosh[\mathcal{L}\sigma] \,=\, \cosh L, \qquad \sinh[\mathcal{L}\sigma] \,=\, \sigma\,\sinh L$$

we have

$$\begin{aligned} w(\sigma_i, \sigma_j, \sigma_k) = c_1c_2c_3 + +\sigma_j\sigma_k c_1 s_2 s_3 \\ + \sigma_i\sigma_j s_1 s_2 c_3 \sigma_i\sigma_k s_1 c_2 s_3, \end{aligned} \tag{4.4.5}$$

where we have defined

$$c_i \,\equiv\, \cosh\mathcal{L}_i, \qquad s_i \,\equiv\, \sinh\mathcal{L}_i.$$

It is important to note that the quantity $w(\sigma_i, \sigma_j, \sigma_k)$ can be written in such a way to be proportional to the Boltzmann factor of the triangular lattice! This means that there should exist some parameters $\mathcal{K}_i$ and a constant $\mathcal{D}$ such that

$$w(\sigma_i, \sigma_j, \sigma_k) \;=\; \mathcal{D}\, \exp\left[\mathcal{K}_1\sigma_j\sigma_k + \mathcal{K}_2\sigma_i\sigma_k + \mathcal{K}_3\sigma_j\sigma_k\right]. \tag{4.4.6}$$

These parameters can be determined by expanding the exponential as

$$\exp[x\sigma_a\sigma_b] \;=\; \cosh\, x + \sigma_a\sigma_b \sinh\, x,$$

and comparing with eqn (4.4.5). Doing so, we obtain the important result that the products $\sinh 2\mathcal{L}_i\, \sinh 2\mathcal{K}_i$ are all equal

$$\sinh 2\mathcal{L}_1\, \sinh 2\mathcal{K}_1 \;=\; \sinh 2\mathcal{L}_2\, \sinh 2\mathcal{K}_2 \;=\; \sinh 2\mathcal{L}_3\, \sinh 2\mathcal{K}_3 \;\equiv\; h^{-1} \tag{4.4.7}$$

with the constant h equal to

$$h \;=\; \frac{(1-v_1^2)(1-v_2^2)(1-v_3^2)}{4\left[(1+v_1v_2v_3)(v_1+v2v_3)(v_2+v1v_3)(v_3+v1v_2)\right]^{1/2}}, \tag{4.4.8}$$

where $v_i = \tanh \mathcal{K}_i$, while the constant $\mathcal{D}$ is expressed by

$$\mathcal{D}^2 \;=\; 2h\, \sinh 2\mathcal{L}_1 \sinh 2\mathcal{L}_2 \sinh 2\mathcal{L}_3.$$

The identity (4.4.6) admits a natural graphical interpretation: as shown in Fig. 4.8, summing over the spin of type B at the center of the hexagonal lattice (the one at the center of the star), a direct interaction is generated between the spins of type A placed at the vertices of a triangle. In this way one can switch between the Boltzmann factor of the star of the hexagonal lattice and the Boltzmann factor of the triangular lattice.

4.5 Critical Temperature of Ising Model in Triangle and Hexagonal Lattices

By using the star–triangle identity, it is now easy to determine the critical temperatures of the Ising model on triangular and hexagonal lattices. In fact, substituting the identity (4.4.6) in (4.4.3), the consequent expression is precisely the partition function of the Ising model on a triangular lattice made of $N/2$. Hence, rescaling $N \to 2N$, one has

$$Z_{2N}^{\mathrm{H}}(\mathcal{L}) \;=\; \mathcal{D}^N\, Z_N^{\mathrm{T}}(\mathcal{K}). \tag{4.5.1}$$

Using this equation, together with the duality relation (4.3.6), we obtain a relation that involves the partiton function alone of the triangular lattice

$$Z_N^{\mathrm{T}}(\mathcal{K}) \;=\; h^{-N/2}\, Z_N^{\mathrm{T}}(\mathcal{K}^*), \tag{4.5.2}$$

with

$$\sinh 2\mathcal{K}_i^* \;=\; h\, \sinh 2\mathcal{K}_i, \quad i = 1, 2, 3, \tag{4.5.3}$$

and h given in (4.4.8). Thanks to (4.5.3), there is a one-to-one correspondance between the point $(\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3)$ (relative to the high-temperature phase of the model) and the point $(\mathcal{K}_1^*, \mathcal{K}_2^*, \mathcal{K}_3^*)$ (relative to the low-temperature phase). If, in the space of the coupling constants, there is a line of fixed points under this mapping, this clearly corresponds to the value $h = 1$. For equal couplings ($\mathcal{K}_1 = \mathcal{K}_2 = \mathcal{K}_3 \equiv \mathcal{K}$), from (4.4.8) we have the equation

$$\frac{(1-v^2)^3}{4\left[(1+v^3)v^3(1+v)^3\right]^{1/2}} \;=\; 1, \tag{4.5.4}$$

with $v = \tanh \mathcal{K}$. Taking the square of both terms of this equation and simplifying the expression, one arrives at

$$(1+v)^4(1+v^2)^3(v^2-4v+1) \;=\; 0.$$

The only solution that also satisfies (4.5.4) and has a physical meaning is given by

$$v_c \;=\; 2-\sqrt{3}.$$

This root determines the critical temperature of the homogeneous triangular lattice

$$\tanh \frac{K}{kT_c} \;=\; 2-\sqrt{3},$$

or, equivalently

$$\sinh \frac{2K}{kT_c} \;=\; \frac{1}{\sqrt{3}}. \tag{4.5.5}$$

Numerically

$$T_c^{tr} \;=\; 3.64166...K. \tag{4.5.6}$$

Using eqn (4.3.7) we can obtain the critical temperature of the Ising model on a homogeneous hexagonal lattice

$$\sinh \frac{2L}{kT_c} \;=\; \sqrt{3}. \tag{4.5.7}$$

Its numerical value is given by

$$T_c^{hex} \;=\; 1.51883...L. \tag{4.5.8}$$

It is interesting to compare the value of the critical temperatures (4.5.6) and (4.5.8) with the critical temperature of the square lattice $T_c^{square} = 2.26922J$, given by eqn (4.2.15). At a given coupling constant, the triangular lattice is the one with the higher critical temperature, followed by the square lattice, and then the hexagonal lattice. The reason is simple: the triangular lattice has the higher coordination number, $z = 6$, the hexagonal lattice has the lower coordination number, $z = 3$, while the square lattice is in between the two, with $z = 4$. The higher number of interactions among the spins of the triangular lattice implies that such a system tends to magnetize at higher temperatures than those of the other lattices.