Statistical Field Theory and Applications: An Introduction for (and by) Amateurs.

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by

Denis BERNARD

LPT-ENS & CNRS

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Chapter 3 was written in part by Jesper Jacobsen

Comments are welcome

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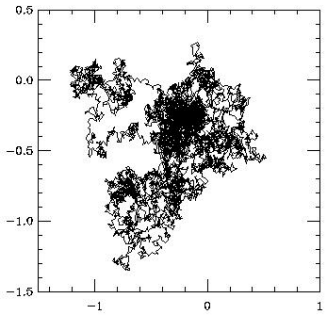
1 Introduction: what are we aiming at describing ?

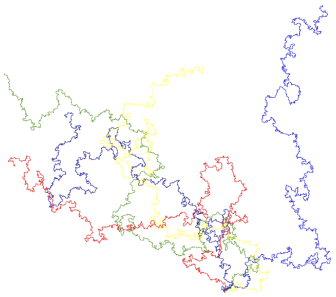
Statistical Field Theory is an important subject in theoretical physics, with a wide range of interdisciplinary applications ranging from statistical physics or condensed matter physics to higher energy physics or random geometry, which has undergone many progresses in the recent years. It is of course intimately linked to Quantum Field Theory.

Statistical Field Theory aims at dealing with the behavior of systems (classical or quantum) with a large –actually infinitely large– number of interacting degrees of freedom. These systems have very interesting and peculiar behaviors: they have different phases with different characters, they manifest phase transitions, their behaviors are dominated by collective modes and/or refined geometrical patterns, etc. Their understanding and analysis make contact with very elegant mathematical structures (say probability theory, representation theory, geometry) and with remarkable concepts, notably the renormalization group which is nowadays a cornerstone of Physics and its ramification.

Statistical Field Theory aims at an understanding of those behaviors on the basis of a few physical principles. This is particularly true for its application to critical phase transitions through their universality property. These are characterized by sharp transitions in the physical properties of statistical systems controlled by external parameters. They are induced by collective phenomena which involve large fluctuations over long distances without scale separation. Statistical field theory provides tools to deal with many nested degrees of freedom, with large fluctuations, over a cascade of scales.



A sample of a Brownian curve in 2D.



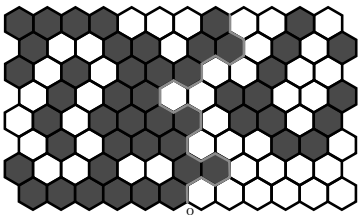
Samples of 2D self-avoiding walks, alias polymers.

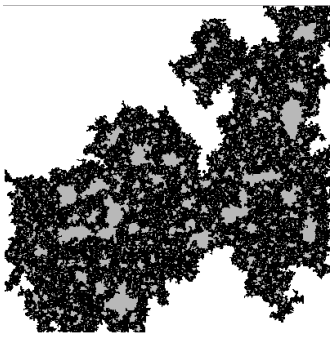
Understanding random patterns is at the core of the comprehension of many physical phenomena or mathematical structures, and the Brownian motion is a historical example of such structures. Although the relevant geometries can be as simple as gentle curved or surfaces, many relevant patterns are however not well-described by an integer dimension but have a fractal character, at least over some length scale. For instance, the singular behavior of second order phase transitions are compatible with fractal surfaces separating phases at the critical point. In general, the deeper our understanding of these geometries, the more complete our predictions can be. See Figure.

Polymers provide simple examples of such fractal geometry going beyond the simple model of Brownian motions. These can be modeled by self avoiding random walks which may be viewed as the paths drawn by a random walker on a square lattice constrained not to visit twice any site

of the lattice. This constraint mimics the self-repulsion of the polymers. They undergo a phase transition when the temperature is varied: at low temperature the polymers is compactly curled up while at large temperature it extends macroscopically and resembles a fluctuating smooth curve. At the critical point, the polymers possess macroscopic fractal shapes which nowadays can be described using statistical field theory tools. See Figure.

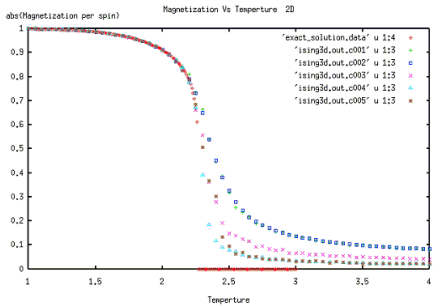
Percolation is another geometrical example manifesting phase transition and generating random fractal structures. Imagine that we randomly color the cells of a honeycomb lattice black or white. The rule is that each cell has a probability p of being white and $1-p$ of being black. If p is small, most of the cells will be black, with a few small islands of white cells. As p is increased, these white clusters grow larger until a critical value p_c at which there is a non-zero probability that one of these clusters spans the whole domain, no matter how large it is. Percolation is important as a model for random inhomogeneous systems, for instance if the black clusters represent untapped oil pockets it is much easier to extract the oil if they percolate. See Figure.



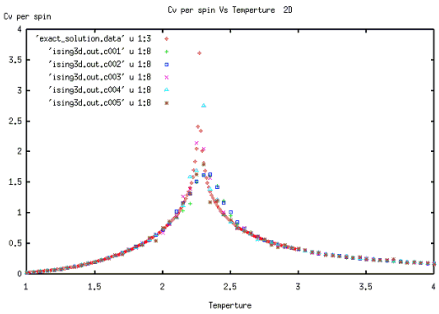


A small sample of definition (left).
A critical percolating clusters (right).

Critical phase transitions (2nd order phase transitions) of statistical systems are the main actors in statistical field theory. There is a large variety of physical systems exhibiting second order phase transitions. Standard examples are the para-to-ferro magnetic transition in magnetic materials, the superfluidity transition in quantum fluids, the superconductivity transition of certain metallic materials at low temperature, etc. Second order phase transitions have a universality property in the sense that the types of phase transitions fall into a relatively small number of categories, known as universality classes, which all behave similarly. Close to the transition there are singularities in the thermodynamical functions and, in parallel, large anomalous fluctuations and power law behavior on the correlation functions. These singular behaviors, characterized by scaling exponents of thermodynamical functions, are understood within statistical field theory. The size of the fluctuations is characterized by a length, the correlation length ξ, which is the typical extension of the domain over which the degrees of freedom are correlated. At the critical point, fluctuations are all over the scale and as a consequence this length diverges. On one hand, this has dramatic consequences because all degrees of freedom are then coupled, at any scale, making the analysis difficult (to say the least), but one the other hand, it renders a continuous description with statistical field theory possible and it is at the origin of the universality property.

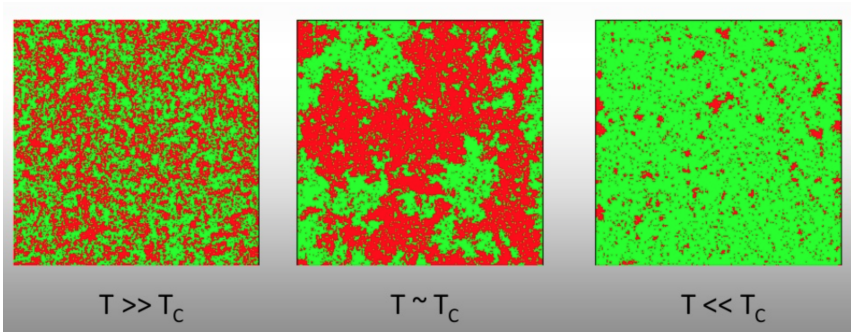


Magnetization versus temperature in the 2D Ising model.



Heat capacity versus temperature in the 2D Ising model.

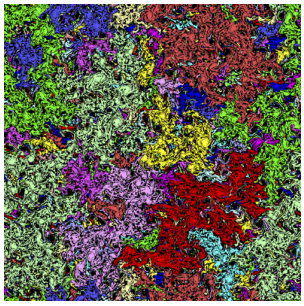
The archetypical statistical model of phase transition is the Ising model for magnetic transition. The Ising degrees of freedom are simple spin variables s_i, whose values are either $+$ or $-$, defined on the sites i of a lattice. The statistical weight of a given spin configuration is proportional to the Boltzmann weight $e^{-E[\{s\}]/T}$, with T the temperature and with interaction energy $E[\{s\}] = -J\sum_{i,j} s_i s_j$ where the sum is restricted to the neighbour spins on the lattice. For $J > 0$, the configuration with aligned spins are the most probable at low temperature. There is a phase transition at a certain critical temperature T_c. At $T > T_c$, spins have a tendency not to be aligned and the mean magnetization vanishes, while at $T < T_c$ spins get ordered, they form large clusters, and there is a non zero spontaneous magnetization. At $T = T_c$, fluctuations are large, clusters of identical spins form scale invariant fractal patterns. See Figure.



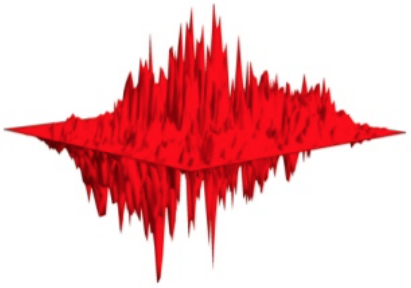
Ising configurations

Statistical Field Theory is also applicable to statistical systems out-of-equilibrium involving a large number of coupled degrees of freedom fluctuating over large range of scales. Here, fluid turbulence or turbulent transports are the main —yet unsolved— classical examples. Turbulence in fluids is one of those natural phenomena for which we think we understand the underlying physics but are nevertheless unable to make analytic precise predictions. Turbulence occurs when a piece of fluid with low viscosity is stirred at large length scales. There is then a continual cascade of energy from large to small scales where the energy is dissipated. This cascade of energy may be described statistically in terms of random velocity fluctuations, with a power-law spectrum of their Fourier components. Those fluctuations are all over the scales over which

the cascade takes place and, as a consequence, are potentially describable with statistical field theory methods. In two dimensions the fluid vorticity clusters form random mutli-scale patterns bearing similarities with clusters of critical statistical transitions. See Figure.



Vorticity clusters in 2D turbulence



A sample of the 2D massless discrete Gaussian free field

As its name suggests, statistical field theory has grown from the merging of field theory, classical or quantum, and statistical physics. As in classical field theory, statistical field theory deals with the continuum of degrees of freedom encompassed in field variables. Those degrees of freedom, which for instance encode for the large distance behaviour of relevant fluctuations close to a critical phase transition, are characterized by their geometrical properties and symmetries. Concretely the fields take values in manifolds specified by those properties. Because it deals with the fluctuations of these degrees of freedom, statistical field theory gives statistical weights to any of the field configurations. At equilibrium these weights are specified according to the Boltzmann rules for statistical physics. Naively —and formally—, statistical field theory may be viewed as an attempt to define probability measures on the spaces of field configurations —which are indeed very infinite dimensional spaces. The construction of these measures is of course guided by statistical physics but also by a few other general principles, say local property inherited from the local nature of the underlying degrees of freedom and their interaction, symmetry reflecting global or internal geometrical properties depending on the systems, etc. The renormalization group, which provides both a conceptual framework and efficient tools to deal with these infinitely many coupled degrees of freedom, is a key principle underlying the construction and the use of statistical field theory. Its impact on Physics —in general not only in statistical or quantum field theory— cannot be underestimated. See Figure.

By its very nature, Statistical Field Theory borrows tools from statistics or probability theory, say Gaussian variables or fields. It also uses techniques from representation theory, in particular because of its relation with Quantum Field Theory. Geometry, group theory and dynamics have been historical cross points between mathematics and theoretical physics, as recently illustrated by the impact of geometry or symmetry algebra on gauge theory, on the standard model of particle physics, or on critical Conformal Field Theory. The recent evolution advocates for a deeper role of probability theory in this mathematics-and-physics interaction, especially in statistical field theory.

A non-exhaustive list of references:

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2 Brownian motions, random paths and stochastic processes

The aim of this chapter is to introduce simple but fundamental random objects and to get acquainted with their path integral representations. Both —the objects and the path integrals— are baby examples of structures involved in statistical field theory. Illustrating such structures in the case of stochastic processes yields tools which find applications in field theory but also in other scientific domains. We are going to start with random walks and random paths defined over discrete lattices and then take the limit to continuous spaces. This will allow us to introduce the important notion of scaling limits that we will be encountered many times in the following.

2.1 Random walks and random paths

We start by discussing simple examples of random paths and random curves. The simplest is of course that of random walks which we will later use to define the Brownian motion, one of the basic objects random geometry deals with. We will introduce a few other examples whose statistics are coded into sums over paths.

• *Random walks: basics.*

For simplicity let us (first) consider random walks on $\mathbb{Z}$ (generalizations are easy). Let $\Lambda_a \equiv a\mathbb{Z}$ be the line with edges of size a. We consider symmetric walks, with equal probability to move to the left and to the right. Let q_0 be the initial walker's position and Q_n be its position after n step. The rules are $Q_n \to Q_{n+1} = Q_n + a\epsilon_{n+1}$ with $\epsilon_{n+1} = \pm$ with probability $\frac{1}{2}$. Hence

$$Q_n = q_0 + a\sum_{k=0}^{n-1} \epsilon_{k+1},$$

with ϵ_k independent and identically distributed random variables (i.i.d.) with $\mathbb{P}[\epsilon_k = \pm] = \frac{1}{2}$. The mean and the covariance of the position Q_n are easy to compute (set $q_0 = 0$ for simplicity):

$$\mathbb{E}[Q_n] = 0, \quad \mathbb{E}[Q_n^2] = a^2\,\mathbb{E}[\sum_{k,l}\epsilon_k\epsilon_l] = a^2\,\mathbb{E}[\sum_k \epsilon_k^2] = a^2 n.$$

This last equation tells us that the typical displacement scales as the square of the number of steps: $\ell \simeq \sqrt{n}$. We can compute other expectations, for instance the two point functions:

$$\mathbb{E}[Q_nQ_m] = a^2\,\mathbb{E}[\sum_{k=1}^{n}\sum_{l=1}^{m}\epsilon_k\epsilon_l] = a^2\,\mathbb{E}[\sum_{k=1}^{\min(n,m)}\epsilon_k^2] = a^2\min(n,m).$$

Note that Q_n is not a Gaussian variable but a sum of binomial variables.

We can of course consider more general random walks, say asymmetric walks with $\mathbb{P}[\epsilon_k = +] = p$ and $\mathbb{P}[\epsilon_k = -] = q$ with $p + q = 1$. Then (for $q_0 = 0$) the mean is $\mathbb{E}[Q_n] = a(p-q)n$ and the variance $\mathbb{E}[Q_n^2] - \mathbb{E}[Q_n]^2 = a^2\,4pq\,n$, so that there is a drift. We can also imagine random walks in inhomogeneous (potential) landscape (i.e. with the probability to move to the left or to the right depending on the position).

• *Interlude: sum over paths.*

Expectations on random walks are tautologically represented as (discrete) path sums or integrals. A walk ω say of length N can be coded with its series of steps $\omega = (\epsilon_1, \cdots, \epsilon_N)$

or by its successive positions $\omega = (q_0, q_1, \cdots, q_N)$ with $q_k - q_{k-1} = \epsilon_k$. Observables may for instance be the positions at different times whose expectations are the multipoint functions $\mathbb{E}[Q_{k_1} \cdots Q_{k_p}]$. Let f be such a (general) observable, i.e. a function defined over the space of walks: $f : \omega \to f(\omega)$. Its expectation reads (as usual in probability theory, or in statistical physics dealing with Boltzmann sum)

$$\mathbb{E}[f] = \sum_{\omega:\text{path}} p_\omega \, f(\omega),$$

where p_ω is the probability of occurence of this walk. For random walks $p_\omega = p^{\#\text{right}} q^{\#\text{left}}$. This is a sum over configuration (with Boltzmann weight proportional to p_ω) or a sum of discrete paths (with a weight assign to each path).

Of course we are free to change the measure on path p_ω (if it yields interesting enough models). For instance we can choose to weight the path according to their length, say $p_\omega \propto z^{|\omega|}$ with $z > 0$ and $|\omega|$ the number of the steps of the walk ω. This ensemble may be called that of free random paths, and we will discuss it more in details in a little while. We can either consider the canonical or grand-canonical ensembles, depending whether we fix the length of the curves or not. For $z = 1/2$ (in $D = 1$) this ensemble of walks is that of standard symmetric random walks with $p = q = 1/2$. This alternative formulation offers possible extensions. For instance, we can also look at curves with free ends or connecting to preselected end points. We can then look at the continuous (scaling) limit of those models, etc. We can also restrict the set of paths that are sampled. For instance, we may demand that the paths are not self-touching, and hence also not self-intersecting. Such paths are called self avoiding walks and we make them random by equipping them with the probability measure $p_\omega \propto z^{|\omega|}$. Self avoiding walks are sometimes used as a model for polymer physics. We leave all of these as exercises and we shall concentrate ourselves on the scaling limit of the random walks and on the Brownian motion.

Path integral for Brownian motion, for stochastic processes, for quantum mechanics and/or for field theories will simply be 'formal' extension/definition to continuous paths (which are maps from some interval to some manifolds) or to higher dimensional analog maps from some 'base manifold' to some 'target manifold'.

2.2 Scaling limits

We give here two presentations of the Brownian motion: via a continuous limit of random walks or via its characteristic properties. We also discuss a scaling limit of the free paths defined above which will turn out to be related to the Gaussian free field of statistical field theory. The notion of scaling limits is very important: it encompasses the passage from a discrete model to a continuous model.

• *Brownian motion: basics.*

We may define the Brownian motion from a scaling limit of random walks. We want to take the continuous limit with the lattice spacing going to zero, i.e. $a \to 0$. Since $\mathbb{E}[Q_n^2] = a^2 n$ we see that we have to simultaneously take the limit $n \to \infty$ keep the product $a^2 n$ fixed in order to get a non trivial result. Let $t = a^2 n$ be called the time. Then the Brownian motion is defined by:

$$W_t := \lim_{\substack{a \to 0, \, n \to \infty \\ a^2 n = t}} Q_n = \lim_{a \to 0} a \sum_{k=1}^{[t/a^2]} \epsilon_k,$$

with $[t/a^2]$ the integer part of t/a^2. This is a scaling limit (with scaling dimensions assign to physical quantities). From the previous computation for random walks we have

$$\mathbb{E}[W_t] = 0, \quad \mathbb{E}[W_t W_s] = \min(t, s).$$

In particular $\mathbb{E}[(W_t - W_s)^2] = |t - s|$. The central limit theorem also implies that W_t is a Gaussian variable. Indeed, a direct application of this theorem (see the exercise Section) gives:

$$\mathbb{E}\big[e^{izW_t}\big] = e^{-t\frac{z^2}{2}}.$$

The Brownian motion possesses a few remarkable properties, some of which are directly inherited from those of random walks:
— Markov property, in the sense that the future of a Brownian trajectory after a time s is independent of the past but only dependent of the present position W_s;
— Identically distributed increments, in the sense that the increments $W_{t+s} - W_s$ and $W_t - W_0$, for any t and s, have identical distributions. They are also independent from the past. These two properties are inherited from similar properties which hold true for random walks;
— Translation invariance, in the sense that the increments of W_t and $W_t + a$ have the same law;
— Dilatation invariance, in the sense that W_t and $\lambda^{-1} W_{\lambda^2 t}$ have the same law (notice the change in time).

It is worth knowing that these properties, together with continuity of the curves $t \to W_t$ are characteristic of the Brownian motion. A Brownian path is almost surely not differentiable.

The generalization to higher dimensions in $\mathbb{R}^D$ is simple: one takes D copies of independent Brownian motions in correspondance with each of the D-orthogonal direction:

$$\vec{W}_t = (W_t^1, \cdots, W_t^D),$$

with W_t^i, $i = 1, \cdots, D$, independent identical distributed (i.i.d.) Brownian motions. The D-dimension Brownian motion possesses all the properties of the 1D Brownian motion (translation and dilatation invariance, Markov properties, i.i.d. of the increments) plus rotation invariance $\vec{W}_t \to R \cdot \vec{W}_t$ with $R \in SO(D)$ in dimension $D \geq 2$.

• *Scaling limit of asymmetric random walks*

We now look how to take the continuum limit of asymmetric random walks with different probability to step to the left or to the right. That is: asymmetric walks with $\mathbb{P}[\epsilon_k = +] = p$ and $\mathbb{P}[\epsilon_k = -] = q$. Since $\mathbb{E}[Q_n] = a(p - q)n$, the continuous limit under the Brownian scaling $a^2 n = t$ exists only if the probability p and q depend appropriately on the mesh a. We should have $(p - q) = a\nu$ with ν fixed, or equivalently $p = \frac{1}{2}(1 + \nu a)$ and $q = \frac{1}{2}(1 - \nu a)$. Then,

$$\mathbb{E}[Q_n] = a(p - q)n \simeq \nu\, t, \quad \mathbb{E}[Q_n^2] - \mathbb{E}[Q_n]^2 = a^2\, 4pq\, n \simeq (\nu t)^2 + t.$$

As in the symmetric case, we can thus define the limit of the asymmmetric walks by

$$X_t = \lim_{\substack{a \to 0,\, n \to \infty \\ a^2 n = t}} Q_n, \quad \text{with} \quad p = \frac{1}{2}(1 + \nu a),\ q = \frac{1}{2}(1 - \nu a).$$

By the central limit theorem again, X_t is a Gaussian process with mean μt and covariance t:

$$\mathbb{E}\big[e^{izX_t}\big] = e^{-t\frac{z^2}{2} + i\nu t}.$$

There is a constant drift, with velocity ν, in the sense that it satisfies

$$dX_t := X_{t+dt} - X_t = \nu\, dt + dW_t,$$

with W_t a Brownian motion. This type of noisy evolution equation is an example of stochastic differential equations.

Notice that we could have chosen to make the probabilities to jump to the left or to the right inhomogeneous, so that the drift ν would also be inhomogeneous, position space dependent, say $\nu = a(x)$. Then we would have got a more general stochastic differential equation of the form $dX_t = a(X_t)\, dt + dW_t$. We will come to it in a little while.

The noticeable point about this limit is that we have to let the parameters p and q of the discretized microscopic model to approach the 'critical' value $\frac{1}{2}$ appropriately when $a \to 0$ for the continuous limit to exist. This is a simple instance of what is called the scaling limit. This is a typical procedure for extracting non-trivial continuous field theory from discrete version thereof.

• *Scaling limit of free random paths*

We now describe the scaling of free paths and its connexion with what you will later recognise as Gaussian free field. It also provides a very simple statistical model of random geometrical structures, namely curves.

Recall that free path were defined above as the statistical ensemble of curves weighted according to their lengths. Let us be a bit more precise. Let Λ be a D dimensional square lattice with mesh size a: $\Lambda_a = (a\mathbb{Z})^D$. We let $\mathbf{e}_j$, $j = 1, \cdots, D$, be a basis of orthonormal vectors in $\mathbb{R}^D$, so that points $x \in \Lambda_a$ are $x = a\sum_j n_j \mathbf{e}_j$ with n_j integers. We shall deal with paths Γ starting at the origin 0 and ending at fixed point x, i.e. paths $\Gamma = (x_0, \cdots, x_i, \cdots, x_{|\Gamma|})$ with $x_0 = 0$ and $x_{|\Gamma|} = x$. The successive points in Γ have to be lattice neighbour so that the segment $[x_i, x_{i+1}]$ from an edge of the lattice.

The statistical ensemble of free paths is defined as the set of random paths drawn on Λ_a whose statistics is specified by assigning a Boltzmann weight $w_\Gamma := \mu^{|\Gamma|}$ to each path Γ, with μ a real number ($\mu > 0$) and $|\Gamma|$ be the number of bonds of the path Γ, so that $a|\Gamma|$ is its length. The parameter μ is often called the fugacity. The probability of a given path Γ from 0 to x is thus $\mu^{|\Gamma|}/Z(x)$ with $Z(x)$ the partition function conditioned on paths from 0 to x: $Z(x) = \sum_{\Gamma:\, 0\to x} \mu^{|\Gamma|}$.

As usual in statistical physics, the partition function is a generating function for the number of configuration of a given 'energy' —here of a given length. Thus, by construction,

$$Z(x) = \sum_{\Gamma:\, 0\to x} \mu^{|\Gamma|} = \sum_{N\geq 0} \mu^N\, W_N^{\text{free}}(x) = \delta_{x;0} + \sum_{N>0} \mu^N\, W_N^{\text{free}}(x),$$

with $W_N^{\text{free}}(x)$ the number of paths from 0 to x with N bonds. We note that $W_{N=0}^{\text{free}}(x) = \delta_{x;0}$ because a path of zero length does not escape from its original position.

We are now aiming at understanding the properties of this statistical ensemble of curves and their limit in continuous space obtained by sending the mesh of the lattice to zero.

Because paths arriving at a given point have to visit one of its neighbour points before reaching its destination (if it has more than one bond) the partition function $Z(x)$ has to satisfy the following difference equation:

$$Z(x) = \delta_{x;0} + \mu \sum_{j=1}^{D} \big(Z(x + a\mathbf{e}_j) + Z(x - a\mathbf{e}_j)\big).$$

This is a linear equation, involving the discrete Laplacian on Λ_a. As a consequence, it can be solved by Fourier transform giving

$$Z(x) = \int_{\mathrm{BZ}} \frac{d^D k}{(2\pi/a)^D} \frac{e^{ik\cdot x}}{1 - 2\mu \sum_j \cos(ak \cdot \mathbf{e}_j)}.$$

with BZ the Brillouin zone of the square lattice: BZ:$= [-\frac{\pi}{a}, \frac{\pi}{a}]^D$. This explicit expression makes clear that $Z(x)$ converges if the fugacity is smaller than a critical value: i.e. it converges for $|\mu| < \mu_c$. This value actually depends on which lattice the model is defined but $\mu_c = 1/2D$ is for Λ_a the square lattice. The partition function $Z(x)$ has a simple pole singularity near the critical fugacity, i.e. $Z(x) \simeq \frac{\text{const.}}{\mu_c - \mu}$. This singular behaviour codes for the asymptotic number of curves from 0 to x with a large number N of bonds: $W_N^{\text{free}}(x) \simeq \text{const.}\, \mu_c^{-N}$ at large N.

So good so far for the discrete model. Let us now look at its continuous limit $a \to 0$. The first naive attempt would be to take this limit keeping all other parameters fixed, say μ, x fixed. Then, the above equation for $Z(x)$ implies that $(1 - 2D\mu)Z(x) \simeq \delta_{x;0}$ as $a \to 0$, and hence that $Z(x)$ is trivial in this limit in the sense that all geometrical objects it codes for have collapse around the origin.

That is: we learn that, if we keep all parameters fixed, the naive continuous limit do not describe extended structures in space. We have to adjust the parameters (here the fugacity μ while taking the continuous limit). This is called the 'scaling limit'. Letting $\mu \to \mu_c$ put more and more weight on long paths and gives a chance to be get a meaningful continuous limit of extended geometrical objects.

Let us thus look at the continuous limit $a \to 0$ but adjusting $\mu \to \mu_c$ as a function of a. Expanding the discrete equation for Z in power of a yields (recall that $\mu_c^{-1} = 2D$)

$$(\mu^{-1} - \mu_c^{-1})Z(x) = \mu^{-1}\delta_{x;0} + a^2\,(\Delta_x Z)(x) + \cdots,$$

with Δ_x the Laplacian in $\mathbb{R}^D$. Thus, we see that to get a non trivial limit we have to let $\mu^{-1} - \mu_c^{-1} = a^2 m^2$ as $a \to 0$ with the parameter m fixed (m has the dimension of the inverse of a length). Let us then define the scaling limit of $Z(x)$ by

$$G(x) := \frac{1}{2D} \lim_{a\to 0}\; a^{2-D}\, Z(x),$$

with the limit understood as the scaling limit with $\mu^{-1} - \mu_c^{-1} = a^2 m^2$. By construction, it is non-trivial and it satisfies

$$\big(-\Delta_x + m^2\big)G(x) = \delta(x),$$

where we use the fact that $a^{-D}\delta_{x;0} \to \delta(x)$, the Dirac measure centred at the origin, as $a \to 0$. Notice that we had to incorporate an extra factor a^{-D} in $G(x)$ to get a meaningful continuous limit, so that $G(x)$ has the dimension of the $(2-D)$-th power of a length —we shall say that its scaling dimension is $D-2$—, whereas $Z(x)$ was initially dimensionless. The scaling function $G(x)$ is thus the Green function of the massive Laplacian in $\mathbb{R}^D$. Its explicit expression

$$G(x) = \int \frac{d^D k}{(2\pi)^D} \frac{e^{ik\cdot x}}{m^2 + k^2},$$

can either be found by taking the limit of the explicit formula for $Z(x)$ or by solving the differential equation it satisfies. As we shall see in a few Chapters, $G(x)$ is the two point function for a Gaussian free field.

Let us insist on the notion of scaling limit: to get a meaningful continuous limit we had to let the parameter μ of the model to approach its critical value μ_c in a specific way as a function of the lattice cutoff.

2.3 Brownian motions and path integrals

We define and compute the Brownian transition kernel. This will then be used to give a path integral representation of the Brownian motion. This type of path integrals is a baby exemple of those we will encounter later when dealing with random fields.

• *Brownian motion: its transition kernel.*

We now characterize and determine the transition kernel $\mathbb{P}_t(x_0, x)\, dx$ which is the probability for a Brownian motion, started at position x_0, to be at point x, up dx, at time t. We will get the standard result that

$$\mathbb{P}_t[x_0; x]\, dx = \frac{dx}{\sqrt{2\pi t}} \exp\Big(- \frac{(x_0 - x)^2}{2t}\Big). \tag{1}$$

Note that this is the solution of the heat equation $\big[\partial_t - \frac{1}{2}\partial^2_{x_0}\big]\, \mathbb{P}_t[x_0; x] = 0$ with initial condition $\mathbb{P}_{t=0}[x_0; x] = \delta(x_0 - x)$. The Markov property implies that it satisfies the convolution property

$$\mathbb{P}_t[x_0; x] = \int dy\, \mathbb{P}_t[x_0; y]\mathbb{P}_t[y; x].$$

The Brownian process is indeed Gaussian with i.i.d. increments.

More precisely, let us compute the probability $\mathbb{P}_t[x_0; \Omega]$ for a Brownian motion started at x_0 to be in the interval Ω at time t, i.e. the probability that $W_t \in \Omega$. Let us start with the discretizing model and look for the probability $\mathbb{P}_n[x_0; \Omega]$ for a random walker to be in Ω after n step staring at x_0. Since the walker would have done a first step either to the left or to the right with probability $1/2$, we have:

$$\mathbb{P}_{n+1}[x_0; \Omega] = \frac{1}{2}\big(\mathbb{P}_n[x_0 + a; \Omega] + \mathbb{P}_n[x_0 - a; \Omega]\big),$$

In the continuous limit $a \to 0$, $n \to \infty$, with $t = a^2 n$, we get:

$$\begin{aligned} \big(\mathbb{P}_{n+1}[x_0; \Omega] - \mathbb{P}_n[x_0; \Omega]\big) &\simeq a^2 \partial_t\, \mathbb{P}_t[x_0; \Omega] \\ \big(\mathbb{P}_n[x_0 + a; \Omega] + \mathbb{P}_n[x_0 - a; \Omega] - 2\mathbb{P}_n[x_0; \Omega]\big) &\simeq a^2\, \partial^2_{x_0}\, \mathbb{P}_t[x_0; \Omega] \end{aligned}$$

The probability $\mathbb{P}_t[x; \Omega]$ is therefore solution of the heat equation:

$$\partial_t \mathbb{P}_t[x_0; \Omega] = \frac{1}{2}\partial^2_{x_0}\, \mathbb{P}_t[x_0; \Omega].$$

If Ω is small and centered around a point x we may write $\mathbb{P}_t[x; \Omega]$ as a probability density $\mathbb{P}_t[x_0; x]\, dx$, and for a domain ω of arbitrary size we have:

$$\mathbb{P}_t[x_0; \Omega] = \int_\Omega dx\, \mathbb{P}_t[x_0; x]$$

By linearity, the probability density is also solution of the heat equation with initial condition $\mathbb{P}_{t=0}[x_0; x] = \delta(x_0 - x)$.

• *The path integral representation of the Brownian motion.*

Brownian motion are random curves. We aim at representing their probability measure as an integral sum over paths. Consider the Brownian motion on the time interval $[0,t]$ starting at initial point $x_i = x_0$. We are going to show that its transition kernel from x_i to final point x_f admits the following path integral representation

$$\mathbb{P}_t(x_i,x_f) = \int_{\substack{x(0)=x_i\\ x(t)=x_f}} [Dx]\, e^{-\frac{1}{2}\int_0^t ds\, \dot{x}_s^2}, \tag{2}$$

where the symbols $\int [Dx]...$ "means" the integral sum over continuous paths.

This may sounds a bizarre definition because the integral over continuous paths does not seem to be well-defined (at least we didn't defined it) and furthermore the derivative $\dot{x}_s$ does not exist for the Brownian motion so that the meaning of the integrals $\int dx\, \dot{x}^2$ in unclear (it depends on which curves/paths the measure $[Dx]$ is supported). Path integrals are almost, if not always, not well defined mathematically except in few cases. The Brownian motion is such exceptional case: the Brownian path integral is "tautologically" defined by:

$$\int [Dx]\, e^{-\frac{1}{2}\int_0^t ds\, \dot{x}_s^2}\, (\cdots) := \mathbb{E}[\,(\cdots)\,],$$

with (here) $\mathbb{E}[\cdots]$ denoting the Brownian expectations. Alternatively, we may formally write: $[Dx]\, e^{-\frac{1}{2}\int_0^t ds\, \dot{x}_s^2} = d\mathbb{P}$, with $\mathbb{P}$ the Brownian measure which is a mathematically well defined notion.

The construction of the path integral goes by discretizing the time interval $[0,t]$ and taking a limit when the mesh of this discretization goes to zero (hence the number of intermediate points becomes infinite). Consider a Brownian curve on interval $[0,t]$ and look at the transition kernel $\mathbb{P}_t(x_i,x_f)$. We use a trick consisting in introducing successive cuts and intermediate points. Let us first introduce a single intermediate point t', $0<t'<t$. Thanks to the convolution property of the transition kernel we may write (See Figure)

$$\mathbb{P}_t(x_i,x_f) = \int dx'\, \mathbb{P}_{t'}(x_i,x')\mathbb{P}_{t-t'}(x',x_f).$$

By iterating and dividing the interval $[0,t]$ in N sub-intervals $[k\delta,(k+1)\delta]$, for $k=0,\cdots,N-1$, of equal length $\delta := t/N$, we write

$$\mathbb{P}_t(x_i,x_f) = \int [\prod_{k=1}^{N-1} dx_k]\, \mathbb{P}_\delta(x_i,x_1)\cdots\mathbb{P}_\delta(x_{N-1},x_f).$$

Using $\mathbb{P}_\delta(x,y) = \frac{1}{\sqrt{2\pi\delta}}\, e^{-\frac{(x-y)^2}{2\delta}}$, this can equivalently be written as (with $x_0 = x_i$ and $x_N = x_f$)

$$\mathbb{P}_t(x_i,x_f) = \int [\prod_{k=1}^{N-1} \frac{dx_k}{\sqrt{2\pi\delta}}]\, e^{-S^{(N)}},$$

where the function $S^{(N)}$, call the discretized action, is $S^{(N)} = \frac{1}{2\delta}\sum_{k=0}^{N-1}(x_{k+1}-x_k)^2$.

Up to this stage the formula is exact. We have simply sampled the Brownian curve at a large, but finite, number of points. The path integral formulation comes from taking the limit $N \to \infty$ and thus $\delta = t/N \to 0$. By definition the path integral representation is set to be

$$\int [Dx]\, e^{-S[x]} := \lim_{N\to\infty} \int \big[\prod_{k=1}^{N-1} \frac{dx_k}{\sqrt{2\pi\delta}} \big]\, e^{-S^{(N)}}.$$

Now the discretized action can alternatively be written as $S^{(N)} = \delta \sum_{k=0}^{N-1} \big(\frac{x_{k+1}-x_k}{\delta}\big)^2$. The terms $(\frac{x_{k+1}-x_k}{\delta})^2$ are the discretized version of the velocity $\dot{x}^2$, and the formal limit of $S^{(N)}$ is:

$$S^{(N)} = \delta \sum_{k=0}^{N-1} \big(\frac{x_{k+1}-x_k}{\delta}\big)^2 \longrightarrow S[x] := \frac{1}{2}\int_0^t ds\, \dot{x}_s^2.$$

Denoting by $[Dx]$ the formal limit of $[\prod_{k=0}^{N-1} \frac{dx_k}{\sqrt{2\pi\delta}}]$ when the number of sampled points increase, we get (by definition) the path representation of the transition kernel

$$\mathbb{P}_t(x_i, x_f) = \int [Dx]\, e^{-S[x]}, \quad S[x] = \frac{1}{2}\int_0^t ds\, \dot{x}_s^2,$$

with the boundary condition $x(0) = x_i$ and $x(t) = x_f$.

2.4 The 2D Brownian motion

The 2D Brownian motion is the simple two component generalization of the 1D Brownian motion. Besides the basic properties and symmetries (translation, rotation, dilatation invariances) it possesses an extra remarkable property: it is conformally invariant.

The conformal invariance of 2D Brownian trajectories has first been understood by P. Lévy. This invariance concerns the set of traces formed by these trajectories, i.e. their graphs, it does not concern the time parametrized trajectories. These traces are not invariant realisation by realisation, independently, but their statistical ensemble is conformal invariant. This invariance manifests itself in the following way. Let us imagine sampling a Brownian motion in the unit disk started from the origin and stopped the first instance it touches the boundary of the disk. By conformal transformation, these curves can be transported in another planar domain having a topology identical to that of the disk, for example a rectangle. One then obtains a statistical ensemble of curves inscribed in this new planar domain. Conformal invariance asserts that this set of curves possesses a statistical distribution identical to that of the Brownian motion started from a point in the new domain (the image of the origin by the conformal map) and stopped as soon as it touches the boundary of this new domain. That is: we cannot distinguish the Brownian curves transported by conformal transformations to the Brownian curves themselves.

• *2D conformal transformations.*

Conformal transformations in 2D are geometric transformations that preserve angles. Let $(x, y) \in \mathbb{R}^2$ be the coordinates of a point in the plane and $z = x + iy$ its complex coordinate. Conformal transformations are locally holomorphic transformations:

$$z \to w = f(z)$$

Let $z_1 = z_0 + \delta z_1$ and $z_2 = z_0 + \delta z_2$ two neighbour points of z_0. The two small vectors δz_1 and δz_2 joining z_0 to z_1 or z_2 are transformed into two vectors δw_1 and δw_2 joining $w_0 = f(z_0)$ to $w_1 = f(z_1)$ or $w_2 = f(z_2)$. To first order, we have:

$$\delta w_1 = f'(z_0)\,\delta z_1, \quad \delta w_2 = f'(z_0)\,\delta z_2.$$

The angle between the vectors δw_1 and δw_2 is therefore identical to that between the vectors δz_1 and δz_2. Each of these vectors has been rotated, by an angle equal to the argument of $f'(z_0)$, and has been dilated proportionally to the modulus of $f'(z_0)$.

• *Conformal invariance of the 2D Brownian motion.*

Lévy's argument consists in promoting the global invariance by rotation and dilatation into a local invariance (hence, the conformal invariance) using the locality properties of the Brownian motion. This principle has a domain of applicability that extends beyond the study of Brownian curves. Let us consider a Brownian curve stopped at time T. Let us divide the time intervalle $[0, T]$ in a large number $\mathcal{N}$ of intervalles $[t_i, t_{i+1}]$ with $0 = t_0 < t_1 < \cdots < t_{\mathcal{N}} = T$, and decompose the trajectory as the sum of all its increments between these successive times:

$$\mathbf{X}_t - \mathbf{X}_0 = (\mathbf{X}_{t_1} - \mathbf{X}_{t_0}) + (\mathbf{X}_{t_2} - \mathbf{X}_{t_1}) + \cdots + (\mathbf{X}_t - \mathbf{X}_{t_{j-1}}),$$

for $t \in [t_{j-1}, t_j]$. All the increments $\mathbf{X}_{t_{j+1}} - \mathbf{X}_{t_j}$ are statistically independent and distributed identically to $\mathbf{X}_{t_{j+1}-t_j}$. This decomposition means that Brownian curves can be reconstructed by concatenating its increments.

Let us now transform each of these increments by a rotation R_j and by a dilatation with scale factor λ_j, which may vary from one increment to the next. We get:

$$(\mathbf{X}_{t_{j+1}} - \mathbf{X}_{t_j}) \to \lambda_j\, R_j \cdot (\mathbf{X}_{t_{j+1}} - \mathbf{X}_{t_j}) \equiv_{\text{in law}} \mathbf{X}_{\lambda_j^2 (t_{j+1} - t_j)}$$

where in the last equivalence we used the global invariance of the Brownian motion. Thus, after concatenation of the rotated and dilated increments, we obtain curves with the same statistic as the Brownian curves but with a different temporal parameterization (since the time increments $t_{j+1} - t_j$ have been transformed into $\lambda_j^2 (t_{j+1} - t_j)$). Arbitrary conformal transformations can be applied to the Brownian curves by naively taking a limit where the above discretization is increasingly fine.

This argument, which can be made rigorous, indicates that the image of a Brownian curve by a conformal transformation is yet another Brownian curve up to time reparameterization. The conformal invariance of Brownian curves is at the origin of many of their peculiar properties. For instance, it has recently been proved that the exterior perimeter of the graph a Brownian trajectory is a fractal curve of dimension $4/3$.

To conclude, let us note that this property is based on the following principle: "Global invariance under dilatations and rotations plus locality implies invariance under local dilations and rotations, i.e. under conformal transformations". This is a principle that find applications in many physical systems.

2.5 Brownian motions and stochastic differential equations

Stochastic differential equations provide a framework to model and analyse the effects of noise or random external forces on dynamical systems. In the physics literature they are often represented

as Langevin-type equations of the form

$$\dot{X}_t = a(X_t) + \epsilon\, \xi_t,$$

with $a(x)$ some function (or vector field) driving the dynamical variable X in absence of noise and ξ_t representing the noise with ϵ a parameter coding for the amplitude of that noise. Depending on the physical problem various assumption can be made about the noise statistics, e.g, it can be correlated or uncorrelated in time.

- *Stochastic differential equations: discrete and continuous.*

The simplest case —but nevertheless quite rich— amounts to assume that ξ_t is a white-in-time Gaussian noise with zero mean and covariance $\mathbb{E}[\xi_t\, \xi_s] = \delta(t-s)$. The δ-correlation codes for the absence of memory so that ξ_t is a highly fluctuating quantity. Since the process ξ_t is ill-defined mathematically, we shall write the above equation is a slightly different form:

$$dX_t = a(X_t)dt + \epsilon\, dB_t, \tag{3}$$

with $dX_t = X_{t+dt} - X_t$ and B_t a (normalized) Brownian motion with zero mean and covariance $\mathbb{E}[B_t B_s] = \min(t,s)$. In particular, $dB_t = B_{t+dt} - B_t$ and $\mathbb{E}[dB_t^2] = dt$. The parameter ϵ has been introduced by convenience to scale the amplitude of the noise. Since ξ_t can be viewed as the time derivative of the Brownian motion —which, mathematically speaking, almost surely does not exists— these are two (equivalent) ways of writing the same equation. Of course one can also consider more general SDEs, say with more variables. A proper definition of solution of this equation is the integrated version of the SDE in the form $X_t = \int_0^t a(X_s)ds + \epsilon\, B_t$.

The process X_t defined by the SDEs $dX_t = a(X_t)dt + \epsilon\, dB_t$ may be viewed as the scaling limit of asymmetric random walks but space dependent with probabilities to move to the left or right, so that they induce a space dependent drift. Concretely one consider the scaling limit of asymmetric random walks, defined on the lattice $h\mathbb{Z}$ with mesh size h, and probability $p(x)$ (resp. $q(x)$ to move to the right (resp. to the left) scaling with the lattice mesh as $p(x) = \frac{1}{2}(1 + a(x)h)$ and $p(x) = \frac{1}{2}(1 - a(x)h)$ as $h \to 0$.

We may also look at more general SDEs of the form $dX_t = a(X_t)dt + b(X_t)dB_t$ with $b(x)$ a non-constant function. Dealing with them however requires some care as the irregularity of the Brownian motion as function of t demands to make precise what is meant by the product $b(X_t)dB_t$. This will be briefly discussed below.

A discretized version of this equation is (with $\delta := \delta t$ the elementary time step):

$$X_{n+1} - X_n = \delta\, a(X_n) + \epsilon\, \xi_{n+1}. \tag{4}$$

The ξ_k are Gaussian i.i.d. with zero mean, $\mathbb{E}[\xi_k] = 0$ and covariance $\mathbb{E}[\xi_k \xi_l] = \delta_{k;l}\, \delta$. It is important to note that ξ_k are typically of order $\sqrt{\delta}$. The integrated version is

$$X_n - X_0 = \delta \sum_{k=0}^{n-1} a(X_k) + \epsilon \sum_{k=1}^{n} \xi_k.$$

Recall that $B_n = \sum_{k=1}^n \xi_k$ is a discretized version of the Brownian motion, and we may view ξ_n as discrete time derivative, $\xi_{n+1} = B_{n+1} - B_n$. The form of this discretization, in which we have chosen to sample $a(X)$ at the starting point of the discretized intervals, is called the Itô convention.

• *Transition kernels and the Fokker-Planck equation.*

We are interested in the transition kernel $dx\,\mathbb{P}_t(x_0,x)$ or $dx\,\mathbb{P}_t(x_0\to x)$ which is the probability to be at position x up to dx at time t starting from point x_0. It is such that

$$\mathbb{E}_{x_0}[F(X_t)] = \int dx\,\mathbb{P}_t(x_0,x)\,F(x),$$

for any test function F. Another (formal) way to define it uses Dirac delta function: $\mathbb{P}_t(x_0,x) = \mathbb{E}_{x_0}[\delta(X_t - x)]$. By the Markov property the kernel satisfies the composition property:

$$\mathbb{P}_{t+s}(x_0,x) = \int dy\,\mathbb{P}_t(x_0,y)\mathbb{P}_s(y,x).$$

It defines a semi-group. The transition kernel transports probability distributions (by duality). If $dx_0\,Q_0(x_0)$ is the initial distribution, then its transport at time t is

$$Q_t(x) = \int dx_0\,Q_0(x_0)\,\mathbb{P}_t(x_0,x).$$

Similar formulas apply in the discrete version: $\mathbb{E}_{x_0}[F(X_n)] = \int dx\,\mathbb{P}_n(x_0,x)\,F(x)$, etc.

We are know going to prove that the transition kernel is solution of a partial differential equation (similar to the Schrödinger equation).

$$\partial_t\mathbb{P}_t(x_0,x) = \mathcal{H}\cdot\mathbb{P}_t(x_0,x), \quad \mathcal{H} := \partial_x\big(\frac{\epsilon^2}{2}\partial_x - a(x)\big). \tag{5}$$

This is called the Fokker-Planck equation. Notice its interpretation as a conservation law for probabilities: $\partial_t\mathbb{P}_t(x_0,x) = \partial_x\mathbb{J}_t(x)$, with probability current $\mathbb{J}_t(x) := (\frac{\epsilon^2}{2}\partial_x - a(x))\mathbb{P}_t(x_0,x)$. We can write $\mathbb{P}_t = e^{t\mathcal{H}}$. By construction $\mathcal{H}$ is a non positive operator. This operator actually fully determine the process: its data is equivalent to that of the process. At large time $\mathbb{P}_t$ converges toward the stationary measure if it exists. The approach to stationary is governed by the spectrum of $\mathcal{H}$. Note the analogy with quantum mechanics.

The invariant measure $\mathbb{P}_{\rm inv}(x)$ should satisfies $\mathcal{H}\cdot\mathbb{P}_{\rm inv} = 0$. A solution with vanishing probability current should satisfies $(\frac{\epsilon^2}{2}\partial_x - a(x))\mathbb{P}_{\rm inv}(x) = 0$. Let us set $a(x) = -U'(x)$ (this is always possible in 1D, with one variable x, but not in higher dimension). The function U is called the potential. Then, if normalizable, the invariant measure (with zero probability flux) is:

$$\mathbb{P}_{\rm inv}(x) = Z^{-1}\,e^{-2U(x)/\epsilon^2},$$

with the normalization constant Z such that $\int dx\,\mathbb{P}_{\rm inv}(x) = 1$ (if normalizable). This is Boltzmann distribution for a potential $U(x)$ and temperature $\epsilon^2/2$, so we can set $\epsilon = \sqrt{2k_BT}$.

Let us now derive the differential equation satisfied by the kernel. We compute the conditioned expectation $\mathbb{E}[F(X_{n+1})|\mathcal{F}_n]$, conditioned on the knowledge of the process up to step n. We have to compute $\mathbb{E}[F(X_n + \delta A_n + \epsilon\,\xi_{n+1}|\mathcal{F}_n]$ with $A_n = a(X_n)$. Conditioning on $\mathcal{F}_n$ means that $X_n,\ A_n$ are fixed (non random) number in this expectation, so that the expectation is only about the gaussian variable ξ_{n+1}. We compute by expanding in power of δ (recall that ξ_k are

typically of order $\sqrt{\delta}$):

$$\begin{aligned} & \mathbb{E}[F(X_n + \delta A_n + \epsilon\,\xi_{n+1})|\mathcal{F}_n] \\ = \;& F(X_n) + \delta\, A_n\, F'(X_n) + \epsilon\, F'(X_n)\mathbb{E}[\xi_{n+1}] + \frac{\epsilon^2}{2}\, F''(X_n)\mathbb{E}[\xi_{n+1}^2] + \mathcal{O}(\delta^{3/2}), \\ = \;& F(X_n) + \delta\,\big(A_n\, F'(X_n) + \frac{\epsilon^2}{2}\, F''(X_n)\big) + \cdots \end{aligned}$$

The r.h.s. depends only on data up to step n (i.e. it is $\mathcal{F}_n$-measurable) as it should be. As a consequence

$$\mathbb{E}[F(X_{n+1}] = \mathbb{E}[F(X_n) + \delta\,\big(a(X_n)\, F'(X_n) + \frac{\epsilon^2}{2}\, F''(X_n)\big)] + \cdots$$

Up to an integration by part, this is equivalent to (recall that δ is the time step δt)

$$\mathbb{P}_{n+1}(x_0, x) - \mathbb{P}_n(x_0, x) = \delta\,\big(\,\frac{\epsilon^2}{2}\partial_x^2 - \partial_x\, a(x))\mathbb{P}_n(x_0, x)\big) + \mathcal{O}(\delta^2).$$

Hence we get the Fokker-Planck equation in the continuum limit.

We could also have derived a differential equation for $\mathbb{P}_t(x_0, x)$ but acting the initial point x_0. This equation is the dual of the one acting x, that it $\partial_t \mathbb{P}_t(x_0, x) = \big(\frac{\epsilon^2}{2}\partial_{x_0} + a(x_0)\big)\partial_{x_0}\mathbb{P}_t(x_0, x)$. As for the Brownian motion, it can be derived by looking at what happened after one step but taking into account that the probabilities to move to the right or to the left of space dependent with the appropriate scaling as the mesh of the lattice goes to zero.

• <u>*Itô versus Stratonovich.*</u>

We can of course consider more general SDEs —and those naturally occur in physical problems. Instead of Sees of the form $dX_t = a(X_t)dt + \epsilon\, dB_t$ we can more generally look at SDEs

$$dX_t = a(X_t)dt + b(X_t)dB_t,$$

with $a(x)$ and $b(x)$ smooth function and with B_t a Brownian motion, $dB_t^2 = dt$. A proper definition of solution of this equation is in the integrated version:

$$X_t = \int_0^t a(X_s)ds + \int_0^t b(X_s)dB_s.$$

We have to be careful how we define the integrals, in particular the stochastic integral. Let us defined them by discretization. Different conventions differ from where we sample the function in the Riemann sum. Two important conventions are the so-called Itô or Stratonovich convention, with $\xi_{k+1} = (B_{k+1} - B_k)$:

$$\begin{aligned} \text{Ito} \quad &: \quad X_n = \delta \sum_{k=0}^{n-1} a(X_k) + \sum_{k=0} b(X_k)\xi_{k+1}, \\ \text{Stratonovich} \quad &: \quad X_n = \delta \sum_{k=0}^{n-1} \frac{1}{2}(a(X_{k+1} + a(X_k)) + \sum_{k=0} \frac{1}{2}(b(X_{k+1}) + b(X_k))\xi_{k+1}, \end{aligned}$$

in the large N limit with $\delta = 1/N$ and $n\delta = t$. With the Itô convention we sample the function at the beginning of the sub-intervals, with the Stratonovich convention with sample it with the mean of its value at both ends of the sub-intervals. Of course there are intermediate alternative choices in which we weight differently the sums. The important property of Itô convention is that $\xi_{k+1} = B_{k+1} - B_k$ is independent of $b(X_k)$ and all $b(X_l)$, $l \leq k$.

Itô and Stratonovich convention for stochastic integrals are related by

$$\int b(B_s)\, dB_s|_{\text{Ito}} = \int b(B_s)\, dB_s|_{\text{Strato}} - \frac{1}{2}\int_0^T b'(B_s)\, ds.$$

This equation is simply checked in the case $b(B) = B$. The simplest way to check it is to verify it for $\int B_t dB_t$. One has $\int_0^T B_t dB_t|_{\text{Ito}} = \frac{1}{2}B_T^2 - \frac{1}{2}T$ and $\int_0^T B_t dB_t|_{\text{Strato}} = \frac{1}{2}B_T^2$. Here is a more general but naive proof. Let $\mathfrak{I}$ be the difference between the Itô minus Stratonovich integrals $\int b(B_s)dB_s$. We have:

$$\begin{aligned}
\mathfrak{I} &= -\frac{1}{2}\sum_k (b(B_k) - b(B_{k-1}))\,(B_k - B_{k-1}), \\
&\simeq -\frac{1}{2}\sum_k b'(\hat{B}_{k-1})\,(B_k - B_{k-1})^2 \simeq -\frac{1}{2}\sum_k b'(\hat{B}_{k-1})\,(\delta t)_k = -\frac{1}{2}\int b'(B_s)\, ds.
\end{aligned}$$

The Itô calculus is very efficient "computationally" because (i) it is simply based on Taylor expansions in dB_t and dt with the rule $dB_t^2 = dt$, and (ii) in product of the form $b(B_t)dB_t$ the random quantities $b(B_t)$ and dB_t are independent. In particular, for any regular enough function F we have

$$dF(B_t) =_{\text{Ito}} F'(B_t)dB_t + \frac{1}{2}F''(B_t)dt.$$

In the Stratonovich convention the formula will be $dF(B_t) =_{\text{Strato}} F'(B_t)dB_t$ as with Leibniz rules but the quantities $F'(B_t)$ and dB_t are there not independent.

2.6 Path integral representation for SDEs

The path integral representation (also called the MSR representation) of a SDE of one variable of the form $dX_t = a(X_t)dt + \epsilon\, dB_t$ with $dB_t^2 = dt$ is specified by the action (in the Itô convention):

$$S[x] = \int_0^t ds\, \frac{(\dot{x}_s - a(x_s))^2}{2\epsilon^2} \tag{6}$$

Here $\dot{x}_s := \partial_s x_s$. The action codes for the Boltzmann weight e^{-S} in the path integral measure. This means that expectation of function $F(X_t)$ testing the process can be written as path integral. For instance the probability $\mathbb{P}_t(x_0, x)$ to go from x_0 to x can be written as

$$\mathbb{P}_t(x_0, x) = \int_{\substack{x(0)=x_0 \\ x(t)=x}} [Dx]\, e^{-S[x]} \tag{7}$$

We shall discuss some property of this representation and use it in the following section to derive exact —and non perturbative— results in the small noise limit.

We are going to present two derivations: the first using a discretization of the SDE, the second directly using the continuous formulation.

• *An approach by discretization.*

With the Itô convention, the discrete form of the SDE reads

$$X_{n+1} - X_n = \delta\, a(X_n) + \epsilon\, \xi_{n+1},$$

with δ the small time interval, $\delta = dt$, and ξ_k's Gaussian i.i.d. with covariance $\mathbb{E}[\xi_k^2] = \delta$, with initial data $X_0 = x_0$. Imagine we aim at computing the expectation $\mathbb{E}[F(X_1, \cdots, X_N)]$ testing the process at different times. The expectation is with respect to the Gaussian noise ξ_k so that it corresponds to the measure (up to a proportionality coefficient):

$$\prod_{k=0}^{N-1} d\xi_k\, e^{-\xi_{k+1}^2/2\delta}$$

Now, given the ξ_k's we have to recursively solve the difference equations $x_{k+1} - x_k = \delta\, a(x_k) + \epsilon\, \xi_{k+1}$ for $k = 0$ up to $N-1$. These are recursive equations (in the sense that given the n first x_k's we can solve for x_{n+1}). At this point there are two possible routes: either we integrate directly on the ξ_k's, or we implement these constraints by introducing series of δ-functions in the integrals. We choose the second way because it has a wider application (in more general context). So let us enforce these constraints by inserting the following series of integrated δ-functions:

$$[Dx] \prod_{k=0}^{N-1} \delta\big(x_{k+1} - x_k - \delta\, a(x_k) - \epsilon\, \xi_{k+1}\big).$$

Because the constraints are recursively linear in x_{k+1}, their implementations do not involve any Jacobian. This would not have been true if we would have chosen another discretization form (i.e. it is only true for the Itô prescription). We can then represent these delta-functions via their Fourrier transforms: Fourier integral:

$$\delta(x_{k+1} - x_k - \delta\, a(x_k) - \epsilon\, \xi_{k+1}) = \int \frac{d\phi_{k+1}}{2\pi}\, e^{-i\phi_{k+1}(x_{k+1} - x_k - \delta\, a(x_k) - \epsilon\, \xi_{k+1})}.$$

Gathering, we get the measure $[\prod_k d\xi_k d\phi_k dx_k]\, e^{-S}$ with action

$$S = \frac{1}{2\delta} \sum_{k=0}^{N-1} \xi_{k+1}^2 + i \sum_{k=0}^{N-1} \phi_{k+1}(x_{k+1} - x_k - \delta a_k - \epsilon\, \xi_{k+1}).$$

Doing successively the Gaussian integrals over the ξ's first and the ϕ's yields:

$$\begin{aligned} S &\equiv \frac{\delta\, \epsilon^2}{2} \sum_{k=0}^{N-1} \phi_{k+1}^2 + i \sum_{k=0}^{N-1} \phi_{k+1}(x_{k+1} - x_k - \delta a_k) \\ &\equiv \sum_{k=0}^{N-1} \frac{(x_{k+1} - x_k - \delta a_k)^2}{2\delta\, \epsilon^2}, \end{aligned}$$

By convention notation $\equiv$ means "equivalent up to Gaussian integration". The integrations are over $\xi_1, \cdots, \xi_N$, and $\phi_1, \cdots, \phi_N$ but $x_1, \cdots, x_{N-1}$ with x_0 and $x_N = x_f$ fixed. The last form can be expanded into

$$S = \frac{1}{\epsilon^2}\Big(\sum_{k=0}^{N-1} \frac{(x_{k+1} - x_k)^2}{2\delta} - \sum_{k=0}^{N-1} a_k\, (x_{k+1} - x_k) + \frac{\delta}{2} \sum_{k=0}^{N-1} a_k^2 \Big)$$

The first term is the Gaussian Brownian measure $\frac{1}{2}\int ds\,\dot{x}_s^2$, the second is the Itô integral $\int a_s\,dx_s$ and the third is the Riemann integral $\frac{1}{2}\int ds\,a_s^2$. Taking the large N limit we get the action:

$$\begin{aligned} S[x] &= \frac{1}{\epsilon^2}\big(\frac{1}{2}\int ds\,\dot{x}_s^2 - \int a_s\,dx_s + \frac{1}{2}\int ds\,a_s^2\big) \\ &= \int_0^T ds\,\frac{(\dot{x}_s - a(x_s))^2}{2\epsilon^2} \end{aligned}$$

Note that the formula for the action depends on the fact that we use the Itô convention. It will also be different if we were dealing with a SDE of the form $dX_t = a(X_t)dt + b(X_t)dB_t$. The construction is slightly different if we use the Stratonovich convention.

• *A continuous description.*

Here we do the same computation in the 'physicist's way' directly in the continuous formulation. This will give us the opportunity to become acquainted with formal manipulations of path-integrals. We start again from the SDE

$$dX_t = a(X_t)dt + \epsilon\,dW_t,$$

with W_t a normalized Brownian motion with $dW_t^2 = dt$. Let us code this equation into $\mathfrak{F}(x_t) := \dot{x}_t - a(x_t) - \epsilon\dot{w}_t$. We aim at integrating over $\dot{w}_t$, with the Brownian measure $[Dw]\,e^{-\frac{1}{2}\int ds\,\dot{w}_s^2}$, with the contraint that x_t is the solution (supposed to be unique) of the equation $\mathfrak{F}(x_t) = 0$. We have to enforce this constraint by inserting in the path-integral a multi-variable δ-function $\prod_t \delta(x_t - x_t^*)$ with x_t^* the solution of the differential equation. Recall the properties of Dirac delta function of single variable is such that $\delta(f(x))|f'(x)|dx = \delta(x-x^*)dx$ for any function $f(x)$ with single zero at x^*. The extension of this formula to the infinite dimensional case corresponding to the curve x_t reads

$$[Dx]\times\prod_s \delta\big(x_s - x_s^*\big) = [Dx]\times\prod_s \delta(\mathfrak{F}(x_s))\times\Big|\mathrm{Det}\big[\frac{\delta\mathfrak{F}}{\delta x}\big]\Big|,$$

where $\mathrm{Det}\big[\frac{\delta\mathfrak{F}}{\delta x}\big]$ is the infinite dimensional (functional) determinant of the functional operator with entries $\big[\frac{\delta\mathfrak{F}(x_s)}{\delta x_{s'}}\big]$. This operator is $[\partial_t - a'(x_t)]$ viewed as acting on functions of the variable t. Thus the path-integral measure we start with reads

$$[Dw][Dx]\,e^{-\frac{1}{2}\int ds\,\dot{w}_s^2}\times\prod_s \delta\big(\dot{x}_s - a(x_s) - \epsilon\dot{w}_s\big)\,\big|\mathrm{Det}[\partial_t - a'(x_t)]\big|.$$

As before, we can represent the Dirac delta function using Fourier transform. The only delicate point is that now this is continuous product of δ-function so that the conjugated variable, which we denote ϕ_s, as to be a function of the time variable, so that

$$\prod_s \delta\big(\dot{x}_s - a(x_s) - \epsilon\dot{w}_s\big) = \int [D\phi]\,e^{i\int ds\,\phi_s(\dot{x}_s - a(x_s) - \epsilon\dot{w}_s)}.$$

The field ϕ_s plays the role of Lagrange multipliers imposing the constraint. Gathering all the bits, we can write the path-integral measure as

$$[Dw][Dx][D\phi]\,e^{-\frac{1}{2}\int ds\dot{w}_s^2}\,e^{i\int ds\,\phi_s(\dot{x}_s - a(x_s) - \epsilon\dot{w}_s)}\,\big|\mathrm{Det}[\partial_t - a'(x_t)]\big|\,.$$

We now have to deal with the determinant (the Jacobian). We know from the discrete formula we previously discuss that this Jacobian is 1, at least with the Itô convention (this will be proved in more detail in the exercise section)[1] :

$$\frac{\mathrm{Det}[\partial_t - a'(x_t)]}{\mathrm{Det}[\partial_t]} = 1, \quad \text{(Ito convention)}.$$

We give an alternative naive argument (if not proof) working directly in the continuum. This determinant is the Jacobian for the change of variable from the function x_t to the function z_t, solution of $\dot{z}_t = \dot{x}_t - a(x_t) - \epsilon\, w_t$. With the Itô convention, which is a strict forward in time convention, the function z_t as to be thought as $z_t = x_t - \int_0^{t^-} ds\, (a(x_s) + \epsilon\, w_s)$. Hence, with this convention, the Jacobian is one because it is the determinant of the 'matrix' $[\frac{\delta z_t}{\delta x_s}]$ which is (in this convention) strictly triangular with 1 on the diagonal:

Hence, with this convention, we got the measure $[Dw][Dx][D\phi]\, e^{-S[x]}$ with action:

$$S[x] = \frac{1}{2}\int ds\, \dot{w}_s^2 + i \int ds\, \phi_s(\dot{x}_s - a(x_s) - \epsilon \dot{w}_s).$$

The term $\dot{w}_s^2$ represents the Brownian Gaussian measure, and ϕ_s plays the role of Lagrange multiplier enforcing the SDE. We can now do the Gaussian integrals successively the Gaussian integrals over w_s and then ϕ_s to obtain:

$$\begin{aligned} S[x] &\equiv \frac{\epsilon^2}{2}\int ds\, \dot{\phi}_s^2 + i \int ds\, \phi_s(\dot{x}_s - a(x_s)) \\ &\equiv \frac{1}{2\epsilon^2}\int ds\, (\dot{x}_s - a(x_s))^2 \end{aligned}$$

By convention notation $\equiv$ means "equivalent up to Gaussian integration". Note that this Jacobian, and hence this convention, do not matter in the small nose limit $\epsilon \to 0$.

• *Path integral perturbation and the Girsanov theorem.*

The path integral representation has a simple interpretation in terms of Radon-Nikodym derivatives (i.e. changes of probability measures), martingales and the Girsanov theorem. This provides a formal way to make it rigorous mathematically.

Recall the action (with $\epsilon = 1$ for simplicity)

$$S[x] = \frac{1}{2}\int_0^t ds\, (\dot{x}_s - a(x_s))^2.$$

The weight e^{-S} factorizes into the product of two terms: $e^{-\frac{1}{2}\int ds\, \dot{x}_s^2}$ and $e^{\int_0^T ds\, \dot{x}_s\, a(x_s) - \frac{1}{2}\int_0^T ds\, a(x_s)^2}$. Even though there is clearly a question of convention ('Itô versus Stratonovich') let us write formally $\int_0^T ds\, \dot{x}_s\, a(x_s) = \int_0^T a(x_s)\, dx_s$. Then we have:

$$[Dx]\, e^{-S[x]} = [Dx]\, e^{-\frac{1}{2}\int_0^T ds\, \dot{x}_s^2} \times e^{\int_0^T a(x_s)\, dx_s - \frac{1}{2}\int_0^T ds\, a(x_s)^2}.$$

[1] Actually, we have

$$\mathrm{Det}[\partial_t - a'(x_t)] = \mathrm{Det}[\partial_s]\, \mathrm{Det}\big[\delta(t-s) - \theta(t-s)a'(x_s)\big] = \mathrm{Det}[\partial_s]\, e^{-\theta(0)\int ds\, a'(x_s)}.$$

As usual, there is an ambiguity in defining $\theta(0)$, which is linked to ordering factors. The different conventions for stochastic calculus, say Itô versus Stratonovich, correspond to different convention for $\theta(0)$. Itô convention corresponds to $\theta(0) = 0$, Stratonovich convention to $\theta(0) = 1/2$.

The first term is the Brownian measure and the second codes for the change of measure, weighting differently the trajectories of the process. Let M_T be this extra weight:

$$M_T = e^{\int_0^T a(x_s)\, dx_s - \frac{1}{2}\int_0^T ds\, a(x_s)^2}.$$

We can then write the measures as

$$[Dx]\, e^{-S[x]} = [Dx]\, e^{-\frac{1}{2}\int_0^T ds\, \dot{x}_s^2} \times M_T.$$

Hence we may interpret the path integral representation in terms of the Brownian measure as :

$$\int [Dx]\, e^{-S[x]} \cdots = \mathbb{E}_o\big[\, M_T \cdots \,\big],$$

with $\mathbb{E}_o[\cdots]$ the Brownian measure for X and M_t the exponential martingale defined above. The process $t \to M_t$ (defined as an Itô integral) is known to be a martingale under the Brownian measure, solution of the stochastic differential equation $dM_t = a(X_t)M_t\, dX_t$ for $dX_t^2 = dt$. Alternatively we can write this relation as $d\mathbb{P} = d\mathbb{P}_o \times M_T$, with $\mathbb{P}_o$ the Brownian measure and $\mathbb{P}$ that for the process associated to the SDE $dX_t = a(X_t)dt + dW_t$. This representation of the measure of the process X_t is known as the Girsanov theorem.

• <u>*Itô versus Stratonovich.*</u>

Here is a comment on the echo of the Itô and Stratonovich conventions on the path integral representation of SDEs. Although they yield naively different representations, these conventions are of course compatible. Let us again consider the (simple) SDE $dX_t = a(X_t)dt + dW_t$ with $dW_t^2 = dt$. The action of path integral representation depends on the convention we are using because, as mentioned above and proved in an exercice below, the determinant Jacobian is convention dependent. One has:

$$\mathrm{Det}[\partial_t - a'(x_t)] = \mathrm{Det}[\partial_t] \times e^{-\alpha \int ds\, a'(x_s)},$$

with $\alpha = \frac{1}{2}$ with the Stratonovich convention and $\alpha = 0$ with the Itô convention. As a consequence the path integral actions are respectively:

$$\begin{aligned} S_{\mathrm{Ito}} &= \frac{1}{2}\int_0^T ds\, (\dot{x}_s^2 - a(x_s))^2, \\ S_{\mathrm{Strato}} &= \frac{1}{2}\int_0^T ds\, (\dot{x}_s^2 - a(x_s))^2 + \frac{1}{2}\int_0^T ds\, a'(x_s). \end{aligned}$$

The two representations are of course compatible because when expanding the square $\int ds(\dot{x}_s^2 - a(x_s))^2$ we have to view the cross term $\int ds\dot{x}_s\, a(x_s)$ as the stochastic integral $\int dx_s\, a(x_s)$ which is convention dependent by definition. Compatibility is recovered because Itô and Stratonovich integrals are related as follows (see above): $\int_0^T a(x_s)\, dx_s|_{\mathrm{Ito}} = \int_0^T a(x_s)\, dx_s|_{\mathrm{Strato}} - \frac{1}{2}\int_0^T a'(x_s)\, ds$. Each convention has his own avantages: Ito calculus encodes directly for the independence of the increment of the Brownian motion but chain rules and changes of variables are simpler in the Stratonovich convention.

2.7 The small noise limit and the Arrhenius law

The path integral representation may sound a bit formal –actually, it is quite formal. Except in a few cases, Gaussian integrals for instance, it cannot be computed. It is nevertheless adapted to formal manipulations which allow to extract general properties that the measure should satisfy —or it expected to satisfy— if it can be defined. It is also computable when there is a small parameter which can be used to approximate the integral via a saddle point. In the context of quantum mechanics this approximation is called the semi-classical approximation, in the context of field theory it is called instantons for historical reasons. This saddle point approximation allows us to derive results which are exact in the small noise limit. For instance, the presence of noise may allow the dynamical systems to follows paths, say to through an energy barrier, which would be forbidden in absence of noise. The small noise saddle point approximation provides a way to derive exact results for the probability of the occurence of these forbidden moves even though these probabilities are non perturbative in ϵ (they are asymptotically zero as $\epsilon \to 0$ and do not admit a power series expansion in ϵ).

• *Small noise and saddle points.*

We want to study the small noise limit (i.e. the limit $\epsilon \to 0$). In this case the action $S[x] = \frac{1}{2\epsilon^2} \int ds\, (\dot{x}_s - a(x_s))^2 + \alpha \int ds\, a'(x_s)$ is independent of the convention (i.e. independent on α). So we take

$$S[x] = \frac{1}{2\epsilon^2} \int_0^T ds\, (\dot{x}_s - a(x_s))^2 =: \frac{1}{\epsilon^2} \mathcal{A}.$$

The path integral, with measure $[Dx]\, e^{-S}$ is dominated in the limit $\epsilon \to 0$ by the saddle points (actually the minimum) of S. We may write $S = \mathcal{A}/\epsilon^2$ to make the ϵ dependence explicit. Let us call "classical" the trajectory minimizing the action $\mathcal{A}$. These trajectories are solutions of the Euler-Lagrange equations of motion derived from this action. These classical solutions are also/sometimes called "instantons". Naively, we may write

$$\int [Dx] e^{-S[x]} \, [...] \asymp_{\epsilon \to 0} \, e^{-\epsilon^{-2} \mathcal{A}_{\text{classical}}} \, [...]_{\text{classical}}. \tag{8}$$

A more precise formulation of this equation consists in taking the logarithm of both sides (and this is what the symbol $\asymp$ means). There could be a sum over all classical solutions in case there are many (but the one with the minimal action contributes the most).

A more precise formulation proved rigorously (Freidlin-Wentzell theory) is the following. Let $\hat{x}_s$ be a given smooth path from x_i to x_f on the time interval $[0, T]$. Then the probability that the stochastic path x_s is (arbitrary) close the given path is such that

$$\lim_{\delta \to 0} \lim_{\epsilon \to 0} \epsilon^2 \log \mathbb{P}\Big[\sup_{0 \le s \le T} |x_s - \hat{x}_s| < \delta \Big] = -\frac{1}{2} \int_0^T ds \, \big(\dot{\hat{x}}_s - a(\hat{x}_s) \big)^2$$

Note that this action (and hence minus the logarithm of the probability) is always positive. It is zero (and hence maximal) on the noiseless trajectories $\dot{\hat{x}}_s = a(\hat{x}_s)$. Given two points x_i and x_f, such noiseless trajectories do not always exist. This probability is maximal on the classical trajectory (minimizing the classical action). The formula codes more information because it also measures how fast the probability decreases for trajectories which deviate from the classical ones.

• *Saddle points and classical instantons.*

Instantons are solutions of the Euler-Lagrange (classical) equation (these are not the noiseless equations of motion, they are more general):

$$\ddot{x} = a(x)a'(x) = -\partial_x V_{\text{eff}}(x), \quad \text{with } V_{\text{eff}} = -\frac{1}{2}a^2(x).$$

This corresponds to the motion of a classical particle in a potential V_{eff}. The 'energy' is conserved $\mathfrak{e} = \frac{1}{2}\dot{x}^2 - \frac{1}{2}a^2(x) = \frac{1}{2}\dot{x}^2 + V_{\text{eff}}$. The momentum is $p = \dot{x} - a(x)$ and the hamiltonian is $\mathfrak{h}(p,x) = \frac{1}{2}\dot{x}^2 - \frac{1}{2}a^2(x) = \frac{p^2}{2} + pa(x)$. The action $\mathcal{A}(T,x)$, evaluated along the classical trajectory and view as a function of the time T and final position x satisfies the hamilton-Jacobi equation

$$\partial_T \mathcal{A} = \mathfrak{h}(p = \partial_x \mathcal{A}, x) = \frac{1}{2}(\partial_x \mathcal{A})^2 + a(x)\partial_x \mathcal{A}.$$

We can also write the equation of motion keeping track of the auxiliary field (the Lagrange multiplier). Then $S = i\int_0^T ds\, \phi_s(\dot{x}_s - a(x_s)) + \frac{\epsilon^2}{2}\int_0^T ds\, \phi_s^2$. To factorize ϵ we choose to redefine $\phi_s = -i\epsilon^{-2}\xi_s$). Then $\epsilon^2\, S = \int_0^T ds\, \xi_s(\dot{x}_s - a(x_s)) - \frac{1}{2}\int_0^T ds\, \xi_s^2$. The (classical) equations of motion are $\xi_s = \dot{x}_s - a(x_s)$ and $\dot{\xi}_s = -a'(x_s)\xi_s$. They are clearly equivalent to the equations above.

Notice that noiseless trajectories, solutions of $\dot{x} = a(x)$ are always solutions (because they satisfy $\ddot{x} = a(x)a'(x)$). They are zero 'energy' solution $\mathfrak{e} = 0$. These are not the only zero 'energy' trajectories. The other set are those with reversed/opposite velocity, i.e. $\dot{x} = -a(x)$. The former are sometime called the 'downhill trajectories', the later the 'uphill' trajectoires (the names comes from looking that these trajectories in the landscape specified by the potential $u(x)$ such that $a(x) = -U'(x)$).

$$\begin{aligned} \text{downhill} \quad &: \quad \dot{x}_t = a(x_t),\ p_t = 0,\ \mathfrak{e} = 0,\ \mathcal{A} = 0 \\ \text{uphill} \quad &: \quad \dot{x}_t = -a(x_t),\ p_t \neq 0,\ \mathfrak{e} = 0,\ \mathcal{A} = 2\big(U(x_f) - U(x_i)\big). \end{aligned}$$

We have given the value of the action on these trajectories. The uphill trajectories exist only if $U(x_f) \geq U(x_i)$, because $\mathcal{A} \geq 0$ (hence they are effectively uphill; they are induced by the noise). The downhill trajectories exist only if $U(x_f) \leq U(x_i)$.

• *Application to the Arrhenius law.*

Let $a(x) = -U'(x)$ and take the potential U to be a double well. **Figure**. The stationary measure is $P(x)dx = Z^{-1}\, e^{-2U(x)/\epsilon^2} dx$; it is very localized around the minimas when $\epsilon \to 0$ (which means that a population of particle in such landscape, if stationary w.r.t. to this process, will mostly be concentrated around the minima). We may identify ϵ^2 with the temperature (recall $\epsilon^2 = 2k_B T$). We aim at evaluating the probability that thermal noise/fluctuations induce transitions of particle from one minima to the next. So we want to evaluate the probability that there is transition from x_i (close to one minima) to x_f (close to another minima; take $x_i < x_f$ to fix the setup), that we may define by[2]

$$\mathbb{P}[x_i \to x_f] := \text{Sup}_T\ \mathbb{P}_T[x_i \to x_f].$$

[2]It is clear that if x_i and x_f are separated by an energy barrier the maximum of $\mathbb{P}_T[x_i \to x_f]$ is attained asymptotically when T goes to infinity, and then $\mathbb{P}_T[x_i \to x_f]$, as a function of x_f, is asymptotically proportional to the invariant measure $e^{-2U(x_f)/\epsilon^2}$.

In the small noise limit $\mathbb{P}_T[x_i \to x_f] \simeq_{\epsilon\to 0} e^{-\epsilon^{-2}\mathcal{A}(T;x_i\to x_f)}$. Maximizing over the time T to reach the final position amounts to extremize the action $\mathcal{A}$ over T, hence to impose $\partial_T \mathcal{A} = 0$. By the Hamilton-Jacobi equation, this corresponds to zero 'energy' solution. Hence the transition probability is

$$\mathbb{P}[x_i \to x_f] \asymp_{\epsilon\to 0} e^{-\epsilon^{-2}\mathcal{A}(x_i\to x_f;\mathfrak{e}=0)}.$$

Now which zero energy trajectory we have to chose depends on the profil of the potential $U(x)$ (recall that we set $a(x) = -U'(x)$). When the potential goes down we take the noiseless (classical) downhill trajectories with $\mathcal{A} = 0$, when the potential goes up we have to take the "anti-noiseless" uphill trajectoires with $\mathcal{A} = 2(\Delta U)$. Hence,

$$\mathbb{P}[x_i \to x_f] \asymp_{\epsilon\to 0} e^{-2(\Delta U)_{\rm tot}/\epsilon^2} = e^{-(\Delta U)_{\rm tot}/k_B T},$$

with $(\Delta U)_{\rm tot}$ the sum of the potential differences along uphill trajectories. Recall that $\epsilon^2/2$ is identified with the temperature. This is the Arrhenius law/formula. We can actually do a bit better and compute the pre-factor (but we leave this as an exercise...).

2.8 Exercises

• *Exercise 2.8.1: Random variables and generating functions.*

Let X be a real random variable. Let its characteristic function (also called generating function) be defined by

$$\Phi(z) := \mathbb{E}[\, e^{izX}\,].$$

It is always well defined for real z (why?), but its properties under analytical continuation depend on the distribution. Let $W(z)$ be defined as $W(z) := \log \Phi(z)$ or $\Phi(z) = e^{W(z)}$.
(i) Expand Φ and W in power of z and identify the first few Taylor coefficients.
(ii) Suppose that X is a Poisson variable taking integer values with $\mathbb{P}[X = n] = \frac{\lambda^n}{n!}\, e^{-\lambda}$. What are its means, its covariance and its generating function?
(iii) Suppose that X is a Gaussian variable with probability distribution density

$$\mathbb{P}[\, X \in [x, x + dx]\,] = \frac{dx}{\sqrt{2\pi\sigma}}\, e^{-x^2/2\sigma}.$$

Verify that $\mathbb{P}$ is correctly normalized and compute its characteristic function.

• *Exercise 2.8.2: Random Gaussian vectors.*

Let $\vec{X}$ be a N-dimensional Gaussian random vector with real coordinates X^i, $i = 1, \cdots, N$. By definition it probability distribution is

$$\mathbb{P}(X)\, d^N X = d^N X\, \big(\frac{\det[G]}{(2\pi)^N}\big)^{\frac{1}{2}}\, \exp\big(-\frac{1}{2}\langle X|G|X\rangle\big),$$

with $\langle X, G, X\rangle = \sum_{ij} X^i G_{ij} X^j$, where the real symmetric form G_{ij} is supposed to be non degenerate. Denote by $\widehat{G}$ its inverse: $G_{ij}\widehat{G}^{jk} = \delta^i_k$.
(i) Verify that this distribution is normalized, that is:

$$\int \frac{d^N X}{(2\pi)^{N/2}}\, e^{-\frac{1}{2}\langle X|G|X\rangle} = \big(\det[G]\,\big)^{-\frac{1}{2}}.$$

(ii) Verify that $\mathbb{E}[X^i] = 0$ and $\mathbb{E}[X^i X^j] = \widehat{G}^{ij}$.
(iii) Show the generating function $\mathbb{E}[e^{i\langle U|X\rangle}]$ with $\langle U|X\rangle = \sum_i U_i X^i$ is

$$\mathbb{E}[e^{i\langle U|X\rangle}] = e^{-\frac{1}{2}\langle U|\widehat{G}|U\rangle}.$$

Notice that U and X belong to dual spaces.

• *Exercise 2.8.3: The law of large number and the central limit theorem.*

The aim of this exercise is to prove (a simplified version of) the central limit theorem, which we used above. Let ϵ_k, $k = 1, \cdots, n$, be independent identically distributed (i.i.d.) variables. To simplify matter, let us suppose that $\epsilon = \pm$ with probability $\frac{1}{2}$. This theorem says that the sum $\hat{S}_n = \frac{1}{\sqrt{n}} \sum_k \epsilon_k$, is a Gaussian variable in the limit $n \to \infty$ (the more precise statement is that it converges in law).
(i) Prove that

$$\mathbb{E}[e^{iz\,\hat{S}_n}] = \big[\cos(\frac{z}{\sqrt{n}})\big]^n \;\xrightarrow[n\to\infty]{}\; e^{-\frac{z^2}{2}},$$

and conclude.
Hint: Recall the Taylor expansion $\cos(\frac{z}{\sqrt{n}}) = 1 - \frac{z^2}{2n} + o(\frac{1}{n})$ and use $\lim_{n\to\infty}[1 - \frac{y}{n} + o(\frac{1}{n})]^n = e^{-y}$ (which can be proved by taking the logarithm).

• *Exercise 2.8.4: Free random paths.*

The aim of this exercise is to complete the study of free paths presented in the main text. See the main text for a precise definition of the statistical ensemble of free paths. Recall the definition of $Z(x)$ as the partition function of free paths from 0 to x:

$$Z(x) = \delta_{x;0} + \mu \sum_{j=1}^{D} \big(Z(x + a\mathfrak{e}_j) + Z(x - a\mathfrak{e}_j)\big).$$

(i) Compute the Fourier transform of $Z(x)$ and prove the explicit expression of $Z(x)$ as an integral over the Brouillon zone given in the text.
(ii) Let $\Delta^{\text{dis.}}$ be the discrete Laplacian and write $\Delta^{\text{dis.}} = \Theta - 2D\,\mathbb{I}$ with Θ the lattice adjacency matrix and $\mathbb{I}$ the identity matrix. We view Θ as acting on functions via $(\Theta \cdot f)(x) = \sum_{j=1}^{D} \big(f(x + a\mathfrak{e}_j) + f(x - a\mathfrak{e}_j)\big)$. Show that:

$$Z(x) = \langle 0| \frac{1}{\mathbb{I} - \mu\,\Theta} |x\rangle,$$

with $|x\rangle$ the δ-function peaked at x, i.e. $\langle y|x\rangle = \delta_{y;x}$. Deduce from this formula that $Z(x)$ converges for $|\mu| < \mu_c$ with $\mu_c = 1/2D$.
(iii) Let $Z(x) = \sum_N \mu^N W_N^{\text{free}}(x)$ for $|\mu| < \mu_c$. Give an expression of $W_N^{\text{free}}(x)$ as matrix elements of powers of the matrix Θ and give a geometrical interpretation of this formula.
(iv) Prove the formula for the Green function $G(x)$ given in the main text.

• *Exercise 2.8.5: Computation of a path integral Jacobian determinants.*

The aim of this exercise is to compute the determinant $\text{Det}[\partial_t - A(t)]$ of the linear map acting functions $f(t)$ as follows $f(t) \to (Jf)(t) = f'(t) - A(t)f(t)$ with $A(t)$ a given function. Instead of computing directly this determinant we factorize the derivation operator and we write $\text{Det}[\partial_t - A(t)] := \text{Det}[\partial_t] \times \text{Det}[1 - K]$. The operator K is defined by integration as follows:

$$K : f(t) \to (Kf)(t) = \int_0^t ds A(s) f(s),$$

for any function f defined on the finite interval $[0, T]$. The aim of this exercise is thus to compute the determinant $\text{Det}[1 - K]$ and to prove that

$$\text{Det}[1 - K] = e^{-\alpha \int_0^T ds\, A(s)},$$

with α a parameter depending on the regularization procedure ($\alpha = 0$ for Itô and $\alpha = \frac{1}{2}$ for Stratonovich conventions). This illustrates possible strategy to define and compute functional –infinite dimensional– determinants.

To define the determinant $\text{Det}[1 - K]$ we need to discretize it by representing the integral of any function by a Riemann sum. Let us divide the interval $[0, T]$ in N sub-interval $[n\delta, (n+1)\delta]$ with $n = 0, \cdots N - 1$ and $\delta = T/N$. We will then take the limit $N \to 0$. To simplify notation

we denote $f_n := f(n\delta)$. There are many possible discretizations but we shall only consider two of them (which correspond to the Itô and Stratonovich conventions):

$$\text{Ito} \quad : \quad \int_0^t f(t)dt := \lim_{N\to\infty} \delta \sum_{k=0}^{n-1} f_k,$$

$$\text{Stratonovich} \quad : \quad \int_0^t f(t)dt := \lim_{N\to\infty} \delta \sum_{k=0}^{n-1} \frac{1}{2}(f_k + f_{k+1}).$$

(i) Write the regularized action of the operator K on function f by writing the expression of $(Kf)_n$.
(ii) Show that the operator $1-K$ is lower triangular and determine the diagonal entries (which are convention dependent).
(iii) Deduce, by taking the large N limit, the formula for the determinant:

$$\text{Ito} \quad : \quad \text{Det}[1-K] = 1,$$

$$\text{Stratonovich} \quad : \quad \text{Det}[1-K] = e^{-\frac{1}{2}\int_0^T ds\, A(s)}.$$

More general discretization are defined by sampling differently the Riemann sum as follows: $\int_0^t f(t)dt = \lim_{N\to\infty} \delta \sum_{k=0}^{n-1}((1-\alpha)f_k + \alpha f_{k+1})$. Following the same strategy as above, it is then clear that $\text{Det}[1-K] = e^{-\alpha\int_0^T ds\, A(s)}$.

• *Exercise 2.8.6: Levy's construction of the Brownian motion.*

The path integral representation is actually closely related to an older (!) construction of the Brownian motion due to P. Levy. The aim of this exercise is to present the main point of Levy's approach which constructs the Brownian paths by recursive dichotomy.

We aim at constructing the Brownian curves on the time interval $[0,T]$ starting point x_0. The construction is recursive.
(a) First, pick the end point x_T with the Gaussian probability density $\frac{dx_T}{\sqrt{2\pi T}}\, e^{-(x_T-x_0)^2/2T}$ and draw (provisionally) a straight line from x_0 to x_T.
(b) Second, construct the intermediate middle point $x_{T/2}$ at time $T/2$ by picking it randomly from the Gaussian distribution centered around the middle of the segment joining x_0 to x_T, and with the appropriate covariance to be determined. Then, draw (provisionally) two straight lines from x_0 to $x_{T/2}$ and from $x_{T/2}$ to x_T.
(ic) Next, iterate by picking the intermediate points at times $T/4$ and $3T/4$, respectively, from the Gaussian distribution centered around the middle point of the two segments drawn between x_0 and $x_{T/2}$ and between $x_{T/2}$ and x_T, respectively, and with the appropriate covariance. Then draw (provisionally) all four segments joining the successive points x_0, $x_{T/4}$, $x_{T/2}$, $x_{3T/4}$ and $x_{T/2}$.
(d) Iterate ad infinitum...

Show that this construction yields curves sampled with the Brownian measure.
Hint: This construction works thanks to the relation

$$\frac{(x_i-x)^2}{2(t/2)} + \frac{(x-x_f)^2}{2(t/2)} = \frac{(x_i-x_f)^2}{2t} + \frac{(x-\frac{x_i+x_f}{2})^2}{2(t/4)}$$

• *Exercise 2.8.7: The over-damped limit of the noisy Newtonian particle.*

Consider Newton's equation for a particle of mass m subject to a friction and random forcing (white noise in time). That is, consider the SDEs:

$$dX_t = \frac{P_t}{m}\,dt, \quad dP_t = -\gamma\,dX_t + dB_t,$$

with X_t the position and P_t the momentum. We are interested in the limit $m \to 0$ (or equivalently γ large). Let us set $m = \epsilon^2$ to match the Brownian scaling. Then show that:
(i) the process γX_t^ϵ converges to a Brownian motion B_t;
(ii) $Y_t^\epsilon := \epsilon\,\dot{X}_t^\epsilon$ converges to a finite random variable with Gaussian distribution.

That is: Introducing the mass, or ϵ, is a way to regularize the Brownian curves in the sense that X_t^ϵ admits a time derivative contrary to the Brownian motion. But quantities such as Y_t^ϵ, which are naively expected to vanish in the limit $\epsilon \to 0$, actually do not disappear because the smallness of ϵ is compensated by the irregularities in $\dot{X}_t^\epsilon$ as $\epsilon \to 0$. For instance $\mathbb{E}[\frac{1}{2}m\dot{X}_t^2]$ is finite in the limit $m \to 0$. Such phenomena –the existence of naively zero but nevertheless finite quantities due to the emergence of irregular structures in absence of regularizing– is commun phenomena in statistical field theory, (sometimes) called 'anomaly'.

• *Exercise 2.8.8: SDEs with 'multiplicative' noise.*

Generalize the results described above for a more general SDE of the form

$$dX_t = a(X_t)dt + b(X_t)dB_t$$

with $a(x)$ and $b(x)$ smooth non constant functions. To deal with the small noise limit one may introduce a small parametr ϵ by rescaling $b(x)$ via $b(x) \to \epsilon\,b(x)$.
(i) Prove that the Fokker-Planck operator for SDEs reads

$$\mathcal{H} = \partial_x\big(\frac{1}{2}\partial_x\,b^2(x) - a(x)\big)$$

(ii) Verify that the invariant measure (if normalizable) is

$$\mathbb{P}_{\rm inv}(x)\,dx = b^{-2}(x)\,e^{-2s(x)}\,dx, \quad s(x) := -\int^x dy\frac{a(y)}{b^2(y)}.$$

What is the invariant measure if the later is not normalizable?
What is then the physical interpretation of this new measure?
(iii) Show that the action of the path integral representation of these SDEs is

$$S = \frac{1}{2}\int_0^T ds\,\frac{(\dot{x}_s - a(x_s))^2}{b^2(x_s)}.$$

in the small noise limit $\epsilon \ll 1$. Verify (by going back to the discret formulation) that this way of writing the action is still valid away from the small noise limit provided that one carefully defined the integrals.

• *Exercise 2.8.9: Multivariable SDEs*

Generalize all these results for multivariable SDEs of the form $dX^i = a^i(X)\,dt + b^i_j(X)\,dB^j_t$ where B^j are Brownian motions with covariance $\mathbb{E}[B^i_t B^j_s] = \delta^{ij}\min(t,s)$.

3 Statistical lattice models

In this Chapter we present basic examples of lattice models of statistical mechanics and the 2D Ising model with more details. This allows us to introduce the important concept of transfer matrix which is based on algebraic rewriting of the Boltzmann sums. The correspondences between statistical lattice models and lattice formulations of field theories is spelled out.

3.1 Examples of statistical lattice models

By a statistical lattice model is meant a model of statistical mechanics defined on a graph $G = (V, E)$ with vertices V and edges E. Most of the time, G is a regular lattice embedded in D dimensions, such as a square or a (hyper)cubic lattice. In some situations it is of interest to study models on more general classes of graphs, for instance the set of all planar triangulations. The statistical sum then carries over both the statistical degrees of freedom and the graphs themselves, weighted by the number of vertices, and it provides a discretisation of 2D quantum gravity.

The statistical degrees of freedom are typically discrete variables, called *spins* and denoted S_i. In most situations the spins are defined on the vertices V, so that $i \in V$, and the interaction takes place along the edges E, via the definition of some (dimensionless) energy functional, called *Hamiltonian* and denoted $\mathcal{H} = E/(k_B T)$, that depends only on pairs of neighbouring spins. The properties of the system can be studied via the correlation functions of spins, by which is meant the mean (expectation) values of functions of spins at selected vertices, such as $\langle S_i \rangle$, $\langle S_i S_j \rangle$, etc.

Although the interaction in such models is short-ranged, they can have a very rich behaviour at long distances. When $D \geq 2$ there typically exists a critical temperature T_c so that the system is ordered for $T < T_c$ and disordered at $T > T_c$. In the low-temperature phase, the majority of the spins takes a particular value, meaning that we have spontaneous symmetry breaking with a non-zero value of a suitably defined order parameter. The transition at T_c can be first-order (with latent heat), or second (or maybe higher) order. In the latter case, the system is said to be critical, and it will display statistical fluctuations at all length scales, meaning more precisely that its correlation functions transform covariantly under a change of the length scale. Concretely, the two-point function of spins will decay with distance as a power law (sometimes involving also a logarithmic factor), $\langle S_i S_j \rangle \propto |i - j|^{-\alpha}$, defining some critical exponent α. For $T \neq T_c$, it would instead decay as an exponential, $\langle S_i S_j \rangle \propto \mathrm{e}^{-|i-j|/\xi}$, defining a characteristic length ξ, called *correlation length*, beyond which the effective interaction becomes negligible.

• *Ising model*

The simplest —and most well-studied— model of this type is the Ising model with $S_i = \pm 1$ and

$$\mathcal{H} = -K \sum_{(ij)\in E} S_i S_j - B \sum_{i \in V} S_i \,. \tag{9}$$

Here $K = J/(k_B T)$ is the (dimensionless) coupling constant, and B the corresponding magnetic field. The Ising model can be solved exactly —by which we mean that the partition function and correlation functions can be computed analytically— in $D = 1$, and also for $B = 0$ on regular lattices in $D = 2$, for any value of the temperature T. The exact solutions in $D = 2$ with $B \neq 0$, and in $D > 2$ with $B = 0$ seem beyond reach, despite considerable effort; but in both cases we have nevertheless a good understanding of the long distance behaviour from field

theory. The Ising model has a $\mathbb{Z}_2$ symmetry, $S_i \to -S_i$, which is explicitly broken for $B \neq 0$, and spontaneously broken for $B = 0$ and $T < T_c$.

Below we present the exact solution of the $D = 2$ zero-field Ising model.

The properties of statistical models depend subtly on which correlation functions one is interested in. In the case of the 2D Ising model, it is also of interest to study the behaviour of *domain walls*, which are the boundaries between regions with $S_i = 1$ and $S_i = -1$. To be precise, they are closed curves on the dual lattice G^*. One may then ask questions like:
— What is the fractal dimension of a domain wall, or of a region of spins surrounded by one domain wall?
— What is the decay of the probability that two (or more) domain walls extend from the neighbourhood of a point i to the neighbourhood of a distant point j?
This type of questions have highly non-trivial answers even in the case $T = \infty$. So for $T = \infty$, the 2D Ising model behaves trivially in terms of the spins (with $\xi = 0$), but non-trivially in terms of the non-locally defined domain walls. This gives rise to an active research field known as *random geometry*.

• *Potts model*

The Ising model is a special case of the Q-state Potts model. Let $Q \in \mathbb{N}$ and define, for each $i \in V$, a spin $S_i = 1, 2, \ldots, Q$. The Hamiltonian is

$$\mathcal{H} = -K \sum_{(i,j)\in E} \delta(S_i, S_j)\,, \tag{10}$$

where the Kronecker delta function is defined as

$$\delta(S_i, S_j) = \begin{cases} 1 & \text{if } S_i = S_j \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

Note that is we had defined the $Q = 2$ model with $S_i = \pm 1$, then $2\delta(S_i, S_j) = S_i S_j + 1$, so this is equivalent to the Ising model with $K_{\text{Potts}} = 2K_{\text{Ising}}$.

The Potts model has a second order phase transition also for $Q = 3$ and $Q = 4$. But more interestingly, it is possible to reformulate it geometrically for *real* values of Q. To see this, notice first that by (11) we have the identity

$$\mathrm{e}^{K\delta(S_i,S_j)} = 1 + v\delta(S_i, S_j)\,, \tag{12}$$

where we have defined $v = \mathrm{e}^K - 1$. Now, it is obvious that for any edge-dependent factors h_e one has

$$\prod_{e\in E}(1 + h_e) = \sum_{E'\subseteq E} \prod_{e\in E'} h_e\,. \tag{13}$$

where the subset E' is defined as the set of edges for which we have taken the term h_e in the development of the product $\prod_{e\in E}$. In particular, taking $h_e = v\delta(\sigma_i, \sigma_j)$ we obtain for the partition function

$$Z = \sum_S \mathrm{e}^{-\mathcal{H}} = \sum_{E'\subseteq E} v^{|E'|} \sum_S \prod_{(ij)\in E'} \delta(S_i, S_j) = \sum_{E'\subseteq E} v^{|E'|} Q^{k(E')}\,, \tag{14}$$

where $k(E')$ is the number of connected components in the graph $G' = (V, E')$, i.e. the graph obtained from G by removing the edges in $E \setminus E'$. Those connected components are called

clusters, and (14) is the Fortuin-Kasteleyn *cluster representation* of the Potts model partition function. The sum over spins S in the original definition of Z has now been replaced by a sum over edge subsets, and Q appears as a parameter in (14) and no longer as a summation limit. Therefore it makes sense to take any real $Q > 0$.

• *Bond percolation*

For $Q = 1$ the Potts model is seemingly trivial, with partition function $Z = (1 + v)^{|E|}$. Instead of setting $Q = 1$ brutally, one can however consider taking the *limit* $Q \to 1$. This leads to the important special case of *bond percolation*.

Let $p \in [0, 1]$ and set $v = p/(1 - p)$. We then consider the rescaled partition function

$$\widetilde{Z}(Q) \equiv (1-p)^{|E|} Z = \sum_{E' \subseteq E} p^{|E'|} (1-p)^{|E|-|E'|} Q^{k(E')} \,. \tag{15}$$

We have of course $\widetilde{Z}(1) = 1$. But formally, what is written here is that each edge is present in E' (i.e., percolating) with probability p and absent (i.e., non percolating) with probability $1 - p$. Appropriate correlation functions and derivatives of $\widetilde{Z}(Q)$ in the limit $Q \to 1$ furnish valuable information about the geometry of the percolation clusters. For instance

$$\lim_{Q \to 1} Q \frac{\mathrm{d}\widetilde{Z}(Q)}{\mathrm{d}Q} = \langle k(E') \rangle$$

gives the average number of clusters.

Bond percolation is an extreme example of random geometry. The local occupation numbers of edges are completely uncorrelated, and all corresponding correlation functions are trivial. However, non-trivial questions about the fractal geometry of the clusters can be asked in analogy with the above discussion of Ising domain walls.

3.2 Transfer matrices

The transfer matrix $\mathcal{T}$ is a linear operator that builds the partition function Z of a D-dimensional statistical lattice model with short-range interactions, by relating the states of two adjacent $(D - 1)$ dimensional slices. With suitable modification $\mathcal{T}$ can be used to study correlation functions as well.

The basic strategy is as follows. Divide the lattice into two parts, called *past* and *future*, by intersecting it with a $(D-1)$ dimensional (hyper)plane, called a *time slice*. The vertices contained in the time slice belong to the *present*. Let $\vec{S}_t$ denote the states of all spins corresponding to vertices in the present, and let $Z_t(\vec{S}_t)$ be the partition function of the past part of the system, conditioned by the state $\vec{S}_t$ of the present vertices. We interpret t as a *time* parameter, and we consider building up the whole lattice by letting the time evolve in discrete steps.

Consider now another time slice at time $t+1$. We suppose that it differs from the one at time t by having had the present shifted a bit into the future, in such a way that vertices belonging to the present at time t and at time $t+1$ are either identical or nearest neighbours. Knowing $\vec{S}_t$ and $\vec{S}_{t+1}$ it is then possible to infer the part $w(t, t+1)$ of the Boltzmann weights that describe the interaction between the spins in $\vec{S}_t$ and those in $\vec{S}_{t+1}$. We can then write

$$Z_{t+1}(\vec{S}_{t+1}) = \sum_{\vec{S}_t} w(t, t+1) Z_t(\vec{S}_t) \,, \tag{16}$$

where $w(t, t+1)$ are the matrix elements of the linear operator $\mathcal{T}_{t+1}$:

$$w(t, t+1) = \langle \vec{S}_{t+1} | \mathcal{T}_{t+1} | \vec{S}_t \rangle \, .$$

Eq. (16) is just the component form of $Z_{t+1} = \mathcal{T}_{t+1} Z_t$, where Z_t and Z_{t+1} are vectors and $\mathcal{T}$ is a matrix, whence the name transfer matrix. Iterating this relation we obtain finally the complete partition function

$$Z = \sum_{\vec{S}_T} Z_T = \sum_{\vec{S}_0, \vec{S}_T} \langle \vec{S}_T | \mathcal{T}_T \cdots \mathcal{T}_3 \mathcal{T}_2 \mathcal{T}_1 | \vec{S}_0 \rangle \, , \tag{17}$$

where $\vec{S}_0$ and $\vec{S}_T$ describe the possible initial and final states. When writing this, care must be taken so that the interaction along each edge of the lattice is represented by one and only one transfer matrix, since otherwise some double-counting would occur.

For a regular lattice, we can choose always to shift the time slices in the same way, in the direction orthogonal to the time slice, so that both the state space $\vec{S}_t$ and the transfer matrix $\mathcal{T}_t$ are independent of t. If we further assume periodic boundary conditions in the direction perpendicular to the time slice, the initial and final spins can be identified, viz. $\vec{S}_T = \vec{S}_0$, and (17) reduces to

$$Z = \mathrm{Tr}\big(\mathcal{T}^T \big) . \tag{18}$$

In most situations $\mathcal{T}$ is diagonalisable, and since Boltzmann weights are strictly positive the Perron-Frobenius theorem guarantees that the largest (dominant) eigenvalue Λ_0 is non-degenerate. We have then

$$Z \simeq (\Lambda_0)^T \, . \tag{19}$$

for T large. The behaviour of the partition function in the thermodynamic limit (large system size) is thus determined by the largest transfer matrix eigenvalue.

• *Example: 1D Ising model*

The Hamiltonian is

$$\mathcal{H} = -K \sum_{t=1}^{T} S_t S_{t+1} - \frac{B}{2} \sum_{t=1}^{T} (S_t + S_{t+1}) \, , \tag{20}$$

where each time slice consists of a single spin S_t. Note that we have rewritten the interaction with the magnetic field B in a convenient way that makes S_t and S_{t+1} play symmetric roles. Since $S_t = \pm 1$ takes two values, $\mathcal{T}$ is just a 2×2 matrix:

$$\mathcal{T} = \begin{pmatrix} \mathrm{e}^{K+B} & \mathrm{e}^{-K} \\ \mathrm{e}^{-K} & \mathrm{e}^{K-B} \end{pmatrix} .$$

• *Example: Dimer-monomer mixture in 1D*

Consider a 1D lattice with T sites and free boundary conditions. We wish to cover the lattice with two different kinds of objects —dimers that are small rods covering two adjacent sites, and monomers that are points covering just one site— in such a way that each lattice site is covered by precisely one object. The Boltzmann weights for dimers and monomers are w_d and w_m respectively.

To deal with this situation, we let S_t describe the states of the edges rather than the vertices. More precisely, $S_t = 0$ or 1, depending on whether the edge just to the left of the site t is empty or occupied by a dimer. Then

$$\mathcal{T} = \begin{bmatrix} w_m & 1 \\ w_d & 0 \end{bmatrix} ,$$

and the initial and final state vectors are

$$|S_0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} , \qquad \langle S_T| = \begin{bmatrix} 1 & 0 \end{bmatrix} ,$$

In (17) there is no sum over S_0 and S_T, since the boundary states have been fixed in this way, by requiring that no dimer can stick out of the system. So $Z = \langle S_T|\mathcal{T}^T|S_0\rangle$. We leave as an exercise to show that

$$Z = \sum_{n=0}^{\lfloor T/2 \rfloor} \binom{T-n}{T-2n} (w_d)^n (w_m)^{T-2n}$$

by using the transfer matrix formalism. Can you provide also a direct combinatorial argument?

• *Further remarks*

In $D = 2$, each time slice describes the set of spins along a (say) horizontal line intersecting the lattice. Therefore, for a lattice of size $N_h \times N_v$, the transfer matrix of the Ising model has dimension 2^{N_h}. To describe an infinite lattice, we must take the $N_h \to \infty$ limit, and dealing with this is the crux of solving such a model exactly. We shall see it done below.

The transfer matrix formalism is quite malleable and can deal with a variety of situations, provided that the concepts of time slices and state spaces are carefully rethought. Here are a few possibilities:

- Interactions of longer (but finite range): Make the time slices thicker (i.e. consist of several adjacent layers).
- Continuous degrees of freedom: In each matrix product, replace the summation over the discrete degrees of freedom by an integral.
- Inhomogeneities: Replace $\mathcal{T}$ by a time-dependent $\mathcal{T}_t$.
- Random graphs: Sum over both the statistical degrees of freedom and the graphs themselves. Use a time slice whose size varies as it is swept over the graph.

3.3 The 2D Ising model

We now consider the zero-field Ising model on a 2D square lattice. The Hamiltonian is

$$\mathcal{H} = -\sum_{x=1}^{N_x} \sum_{y=1}^{N_y} \left[K_x \, S_{x,y} S_{x+1,y} + K_y \, S_{x,y} S_{x,y+1} \right] , \tag{21}$$

where we have chosen different coupling constants, K_x and K_y, in the horizontal and vertical directions. We assume periodic boundary conditions in both directions.

• *Existence of a phase transition*

We first present an argument due to Peierls that the 2D Ising model has a phase transition at some non-zero temperature T. We assume here $K_x = K_y \equiv K = J/T$, where J is the physical (temperature independent) coupling constant.

The ground state in which all spins are aligned has energy $E_0 = 2JN_xN_y$. An excitation consisting of a droplet of overturned spins within the ordered ground state has energy $E_1 = E_0 + 2J\ell$, where ℓ is the perimeter of the droplet. The number of possible shapes of the droplet equals the number of closed walks on the square lattice of length ℓ, with self-intersections at the vertices being allowed. Viewing such walks as an exploration process, one has at least 1 and at most 3 possibilities for each step (the walk cannot backtrace). So the number of walks is of order c^ℓ, with $1 < c < 3$ (the precise value of c is known but does not matter for this argument).

The contribution to Z of such droplet configurations is $c^\ell \mathrm{e}^{-E_1/T}$, so the free energy is

$$F = E - TS = E_0 + (2J - T\log c)\ell\,.$$

Now, if $T < T_\mathrm{c} \equiv \frac{2J}{\log c}$ we have $F > 0$ for any ℓ, so droplets will be exponentially suppressed. Conversely, if $T > T_\mathrm{c}$ we have $F < 0$ for large enough ℓ, so droplets will proliferate. Therefore we expect a phase transition at $T_\mathrm{c} > 0$.

By contrast, in 1D a droplet corresponds to just two units of domain wall, so the excitation energy is constant. Therefore we will have $T_\mathrm{c} = 0$ in that case, as is easily confirmed by solving the model exactly.

• *High-temperature expansion*

The Boltzmann weight of a configuration of Ising spins S reads

$$W[S] = \prod_{x,y} \exp(K_x\, S_{x,y}S_{x+1,y}) \exp(K_y\, S_{x,y}S_{x,y+1})\,.$$

Since the product of two spins $S_iS_j = \pm 1$, we have the identity

$$\exp(KS_iS_j) = \cosh K + \sinh K\ S_iS_j = \cosh K \times (1 + w\ S_iS_j)\ , \tag{22}$$

where $w = \tanh K$. If we drop the overall multiplicative factor, we get:

$$W[S] \propto \prod_{x,y} (1 + w_x S_{x,y}S_{x+1,y})(1 + w_y S_{x,y}S_{x,y+1})\,,$$

with

$$w_x = \tanh K_x\,, \qquad w_y = \tanh K_y\,.$$

This product can be expanded graphically, associating the term 1 with an empty edge, and the term wS_iS_j with an occupied edge. Thus

$$Z = (\cosh K_x)^{|E_x|}(\cosh K_y)^{|E_y|} \sum_{\{S\}} \sum_{A\subseteq E} \left[w_x^{|E_x(A)|} w_y^{|E_y(A)|} \prod_{(ij)\in A} S_iS_j \right]\,,$$

where $|E_x(A)|$ and $|E_y(A)|$ are the number of horizontal and vertical edges contained in the edge subset A. To get a non-zero contribution, every factor S_i should occur as an even power, so a non-zero contribution is obtained only if A is a set of closed polygons:

$$Z = (\cosh K_1)^{|E_x|}(\cosh K_2)^{|E_y|}\ 2^{|V|} \sum_{A \text{ polygons}} w_x^{|E_x(A)|} w_y^{|E_y(A)|}\,. \tag{23}$$

This exact rewriting of Z is called a *high-temperature expansion*, since $w = \tanh K \ll 1$ when $K \ll 1$.

• *Low-temperature expansion*

We can also expand Z around a fully ordered state. The excitations are then droplets of spins of the opposite sign, which are bordered by domain walls that live on the dual graph G^*, again a square lattice. These domain walls are again polygon configurations:

$$Z = 2\mathrm{e}^{K_x|E_x|+K_y|E_y|} \sum_{A^* \subseteq E^*} (w_x^*)^{|E_y(A^*)|}(w_y^*)^{|E_x(A^*)|}\,,$$

where

$$w_x^* = e^{-2K_x}\,, \qquad w_y^* = e^{-2K_y}\,.$$

Notice how w_x^* is now conjugate to $|E_y(A^*)|$, since a horizontal edge in E is dual to a vertical edge in E^*, and vice versa. This is called a *low-temperature expansion*.

• *Duality*

It follows that if the high-temperature rewriting of Z has a singularity at some parameters (w_x, w_y), then the low-temperature rewriting of Z must have the same singularity at the dual parameters (w_y^*, w_x^*). This is called a *duality transformation*. Since the square lattice is self-dual, the two rewritings are in terms of the same polygon expansion, so we have related singularities of Z at two sets of parameter values.

It might of course be that Z really has a pair of distinct singularities (critical points), but if we suppose —as seems more likely— that there is a *unique* critical point, then it must occur where

$$w_x = w_y^*\,, \qquad w_y = w_x^*\,.$$

It is easy to show the involution property $(w_x^*)^* = w_x$, so there is actually only one relation:

$$\tanh K_x = \mathrm{e}^{-2K_y}\,. \tag{24}$$

This fixes the selfdual manifold along which the Ising model is critical.

The critical properties (critical exponents) are identical all along the self-dual variety (24). Indeed, in the continuum limit we can just scale the x and y directions with opposite scale factors, so that the system becomes isotropic. This is known as anisotropic scaling. The advantage is that we may then solve the Ising model by going to the completely anisotropic limit, where $K_x \ll 1$ and $K_y \gg 1$, while maintaining the relation (24). In this limit the dynamics of the system simplifies considerably, since all interactions are close to the identity in the transfer matrix formalism (where we transfer along the y-direction). We now give the details of this solution.

• *Transfer matrix and Hamiltonian*

A spin configuration along a horizontal row of the lattice is denoted $\vec{S}_1 = \{S_{1,y}, S_{2,y}, \ldots, S_{N_x,y}\}$. The transfer matrix $\mathcal{T}$ transfers this into another configuration $\vec{S}_2$ in which y has been replaced by $y+1$: $\vec{S}_2 = \{S_{1,y+1}, S_{2,y+1}, \ldots, S_{N_x,y+1}\}$. Explicitly this reads

$$\langle \vec{S}_2 | \mathcal{T} | \vec{S}_1 \rangle = \prod_{x=1}^{N_x} \exp\left(K_x\, S_{x,y} S_{x+1,y} + K_y\, S_{x,y} S_{x,y+1}\right) .$$

We can separate the horizontal and vertical interactions by writing $\mathcal{T} = \mathcal{T}_v \mathcal{T}_h$, with

$$\begin{aligned} \langle \widetilde{\vec{S}}_1 | \mathcal{T}_h | \vec{S}_1 \rangle &= \prod_{x=1}^{N_h} \exp\left(K_x S_{x,y} S_{x+1,y}\right) \delta(\widetilde{\vec{S}}_1, \vec{S}_1) , \\ \langle \vec{S}_2 | \mathcal{T}_v | \widetilde{\vec{S}}_1 \rangle &= \prod_{x=1}^{N_h} \exp\left(K_y \widetilde{S}_{x,y} S_{x,y+1}\right) . \end{aligned}$$

The individual factors in each product commute, but $\mathcal{T}_h$ does not commute with $\mathcal{T}_v$. However, in the completely anisotropic limit ($K_x \ll 1$ and $K_y \gg 1$) both matrices are close to the identity, up to an unimportant overall factor in $\mathcal{T}_v$. Indeed we can write

$$\mathcal{T}_h = \mathrm{e}^{-\mathcal{H}_h} , \qquad \mathcal{T}_v = \mathrm{e}^{K_x N_x} \mathrm{e}^{-\mathcal{H}_v} ,$$

where the matrices $\mathcal{H}_h, \mathcal{H}_v \ll 1$. In particular we have

$$\mathrm{e}^{-\mathcal{H}_h} \mathrm{e}^{-\mathcal{H}_v} \approx (1 - \mathcal{H}_h)(1 - \mathcal{H}_v) \approx 1 - \mathcal{H}_h - \mathcal{H}_v \approx \mathrm{e}^{-\mathcal{H}_h - \mathcal{H}_v} .$$

To leading order, $\mathcal{T}_h$ reads

$$\langle \widetilde{\vec{S}}_1 | \mathcal{T}_h | \vec{S}_1 \rangle = \delta(\widetilde{\vec{S}}_1, \vec{S}_1) \left(1 + K_x \sum_{x=1}^{N_x} S_{x,y} S_{x+1,y}\right)$$

and in terms of the Pauli matrices

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

we have

$$\mathcal{T}_h = 1 + K_x \sum_{x=1}^{N_x} \tau_x^3 \tau_{x+1}^3 .$$

Here the subscript of the Pauli matrices indicate on which tensorand they act. Re-exponentiating we get the horizontal part of the Hamiltonian:

$$\mathcal{H}_h = -K_x \sum_{x=1}^{N_x} \tau_x^3 \tau_{x+1}^3 .$$

Let us now examine one factor in $\mathcal{T}_v$, acting at position x:

$$\left[\mathrm{e}^{K_y \widetilde{S}_{x,y} S_{x,y+1}}\right]_{\widetilde{S}_{x,y}}^{S_{x,y+1}} = \mathrm{e}^{K_y} \begin{pmatrix} 1 & \mathrm{e}^{-2K_y} \\ \mathrm{e}^{-2K_y} & 1 \end{pmatrix} = \mathrm{e}^{K_y} \left(1 + \mathrm{e}^{-2K_y} \tau^1\right) \approx \mathrm{e}^{K_y} \exp\left(\mathrm{e}^{-2K_y} \tau^1\right) .$$

Combining this, we arrive at

$$\mathcal{H}_v = -\mathrm{e}^{-2K_y} \sum_{x=1}^{N_x} \tau_x^1 .$$

Summarising our achievements this far, if we write the transfer matrix as

$$\mathcal{T} = \mathrm{e}^{K_y N_x} \mathrm{e}^{-H} ,$$

and introduce the parameters

$$\gamma = \mathrm{e}^{-2K_y} , \qquad \beta = K_x ,$$

then the corresponding 1D quantum Hamiltonian is

$$\mathcal{H} = -\gamma \sum_{x=1}^{N_x} \tau_x^1 - \beta \sum_{x=1}^{N_x} \tau_x^3 \tau_{x+1}^3 , \tag{25}$$

This is known as the *1D transverse field Ising spin chain*. By the argument of anisotropic rescaling, its critical behaviour is identical to that of the original 2D Ising model.

• *Ordered phase*

When $\gamma \ll \beta$, there are two degenerate ground states of $\mathcal{H}$ with energy $E_0 = -\beta N_x$:

$$|\Uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \qquad |\Downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

They correspond of course to the two ordered states in the 2D formulation of the Ising model.

An elementary excitation is obtained by reversing one of the N_x spins, and the corresponding energy is $E_1 = -\beta N_x + 2\beta$. Recalling (25) the corresponding eigenvalues of $\mathcal{T}$ are then

$$\lambda_0 = \mathrm{e}^{K_y N_x} \mathrm{e}^{-E_0} , \qquad \lambda_1 = \mathrm{e}^{K_y N_x} \mathrm{e}^{-E_1} ,$$

so the correlation length is finite:

$$\xi = \left(\log \frac{\lambda_0}{\lambda_1}\right) = \frac{1}{E_1 - E_0} = \frac{1}{2\beta} .$$

• *Disordered phase*

When $\gamma \gg \beta$, we look instead for the dominant eigenvector of τ^1. The ground state of $\mathcal{H}$ is then

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

with energy $E_0 = -\gamma N_x$. This corresponds to the completely disordered state in the 2D formulation: a equally weighted superposition of all possible states.

An elementary excitation is obtained by replacing one of the factors in the tensor product by the other eigenvector of τ^1, which is $(1, -1)$ with eigenvalue -1. The energy is then $E_1 = -\gamma N_x + 2\gamma$, and we have again a finite correlation length:

$$\xi = \frac{1}{E_1 - E_0} = \frac{1}{2\gamma} \,.$$

Below we shall show that the exact expression for ξ, valid for any values of γ and β is

$$\xi = \frac{1}{2|\gamma - \beta|} \,.$$

Notice that this diverges for $\gamma = \beta$, so this must correspond to T_c. This matches indeed the selfduality criterion (24) in the completely anisotropic limit $K_x \ll 1$:

$$K_x = \mathrm{e}^{-2K_y} \,,$$

in agreement with (25).

• *Jordan-Wigner transformation*

We rotate the Pauli matrices by means of a unitary transformation to obtain the equivalent Hamiltonian

$$\mathcal{H} = -\gamma \sum_{x=1}^{N_x} \tau_x^3 - \beta \sum_{x=1}^{N_x} \tau_x^1 \tau_{x+1}^1 \,.$$

Introduce the raising and lowering operators $\tau_x^{\pm} = \frac{1}{2}(\tau_x^1 \pm i\tau_x^2)$. They satisfy the anticommuting relations

$$(\tau_x^+)^2 = (\tau_x^-)^2 = 0 \,, \quad \tau_x^+ \tau_x^- + \tau_x^- \tau_x^+ = 1 \,,$$

which makes the problem start looking fermionic. However, these $\tau^{\pm}$ commute on two different sites, so a trick is needed to obtain the required anticommutativity also in this case.

This is accomplished by the Jordan-Wigner transformation:

$$a_x = \exp\Big(i\pi \sum_{y=1}^{x} S_y^- S_y^+\Big) S_x^+ \,, \qquad a_x^\dagger = \exp\Big(-i\pi \sum_{y=1}^{x} S_y^- S_y^+\Big) S_y^- \,,$$

It is straightforward to show that we now have the required fermionic relation

$$a_x^\dagger a_y + a_y a_x^\dagger = \delta_{x,y} \,.$$

The Hamiltonian is then

$$\mathcal{H} = \gamma \sum_{x=1}^{N_x} \Big(a_x^\dagger a_x - a_x a_x^\dagger\Big) - \beta \sum_{x=1}^{N_x} \Big(a_x^\dagger - a_x\Big)\Big(a_{x+1}^\dagger + a_{x+1}\Big) \,. \tag{26}$$

It is quadratic in the fermion operators.

• *Diagonalisation of $\mathcal{H}$*

By means of a discrete Fourier transformation

$$a_x = \frac{1}{\sqrt{N_x}} \sum_k \mathrm{e}^{ikx} a_k \,, \qquad a_x^\dagger = \frac{1}{\sqrt{N_x}} \sum_k \mathrm{e}^{-ikx} a_k^\dagger$$

this becomes

$$\mathcal{H} = \gamma \sum_k \left(a_k^\dagger a_k - a_k a_k^\dagger \right) - \beta \sum_k \left(a_k^\dagger - a_{-k} \right) \left(a_{-k}^\dagger + a_k \right) \mathrm{e}^{ik} \,,$$

where the sum is over the Brillouin zone, $k \in [-\pi, \pi]$, in steps of $\frac{2\pi}{N_x}$. Note that this couples only the wave numbers k and $-k$, so we can write

$$\mathcal{H} = \sum_{k \geq 0} \mathcal{H}_k \,,$$

where $\mathcal{H}_k$ reads in matrix notation

$$\mathcal{H}_k = 2 \begin{pmatrix} a_k^\dagger & a_{-k} \end{pmatrix} \begin{pmatrix} \gamma - \beta \cos k & -i\beta \sin k \\ i\beta \sin k & -\gamma + \beta \cos k \end{pmatrix} \begin{pmatrix} a_k \\ a_{-k}^\dagger \end{pmatrix} .$$

This latter form can be diagonalised by successively applying the relabelling $a_{-k}^\dagger = i\widetilde{a}_{-k}^\dagger$ and $a_{-k} = -i\widetilde{a}_{-k}$, and the Bogoliubov transformation (orthogonal rotation)

$$\begin{pmatrix} a_k \\ \widetilde{a}_{-k}^\dagger \end{pmatrix} = \begin{pmatrix} \cos\phi_k & \sin\phi_k \\ -\sin\phi_k & \cos\phi_k \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix} .$$

Omitting the details, the result is

$$\mathcal{H} = \sum_{k>0} h_k \left(c_k^\dagger c_k - c_{-k} c_{-k}^\dagger \right) \,, \tag{27}$$

where

$$h_k = \left[(\gamma - \beta)^2 + 4\gamma\beta \sin^2(k/2) \right]^{1/2} \,. \tag{28}$$

The ground state of (27) is obtained by making only the minus term in the parenthesis contribute, viz., by leaving all the fermionic modes empty. An elementary excitation corresponds to populating just one fermionic mode q. The excitation energy is

$$E_1(q) - E_0 = 2h_q \,.$$

The lowest-lying excitation corresponds to $q \to 0$, so we get the correlation length

$$\xi = \frac{1}{E_1(q \to 0) - E_0} = \frac{1}{2|\gamma - \beta|} \tag{29}$$

as already announced.

If we define the "mass" $m = |\gamma - \beta|/\sqrt{\gamma\beta}$, then the excitation energy has the relativistic form

$$(E_1(q) - E_0)^2 \propto m^2 + q^2 \,, \tag{30}$$

and the mass vanishes at T_c. In other word, at the critical point we have a massless field theory. According to (27) it is a theory of free fermions, but of a particular type. Namely, the modes with positive (resp. negative) momentum k have an energy that increases when k becomes more positive (resp. negative), so they move to the right (resp. left) in Fourier space. Therefore they are often called *right-movers* (resp. *left-movers*). Such particles are called *Majorana fermions*.

By contrast, the standard Dirac fermions move left or right for any k, independently of the sign. (There is one more type of free fermions known as symplectic fermions.)

3.4 Lattice field theory

[... To be completed...]

3.5 Partition functions and path integrals

The aim of the two following sections is to point out analogies and relations between models of statistical physics and of quantum mechanics. These relations are already present, more or less explicitly, in the transfer matrix formulation of lattice statistical models —which is grounded on time evolution of time slice 'spin configuration states'.

We first start by describing a path integral representation of the partition function of a quantum system at finite temperature. We assume the reader is familiar enough —or at least know or heard about it— with Feynman path integral representation of Quantum Mechanics. This will be done by implementing an analytic continuation in the time variable known as the "Wick rotation". We then extend this construction to field theory.

• *Feynmann path integrals*

In quantum mechanics the evolution operator $U(t)$ is solution of the Schrödinger equation (we shall later put $\hbar = 1$)

$$i\hbar\partial_t U(t) = H(t)U(t),$$

with $H(t)$ the system hamiltonian, so that the system state $|\psi(t)\rangle$ at time t is $U(t)|\psi_0\rangle$, with $|\psi_0\rangle$ its initial state. For time independent hamiltonian H, it is simply $U(t) = e^{-itH}$. It may be explicitly written if one knows the spectral decomposition of the hamiltonian. Feynman path integral is an alternative representation of this evolution operator as some over all possible system histories. Let q be the dynamical variables of the system and $|q\rangle$ are the eigen-position states. Then, the path integral representation of the matrix elements of the evolution operator reads:

$$\langle q'|U(t)|q\rangle = \int_{\substack{q(0)=q \\ q(t)=q'}} [Dq]\, e^{\frac{i}{\hbar}S[q]} \tag{31}$$

with $S(q)$ the classical action evaluated along the trajectories $s \to q_s$. The integral sum is formally over all paths starting at q and ending at q'. This is similar to the path integral representation of the Brownian motion discussed in a previous chapter except for the important complex extra factor 'i' inherent to quantum mechanics. See any Quantum Mechanics text book for more details.

• *The Euclidean evolution operator and Wick's rotation.*

Wick rotation is based on the fact that the evolution operator $U(t)$ is analytic in the lower half complex plane $\Im\mathrm{m}\, t < 0$. Indeed, if ψ_α form an ortho-normalized eigen-basis of H (which is supposed to be time independent) with eigenvalue E_α, we have $U(t) = \sum_\alpha |\psi_\alpha\rangle\, e^{-\frac{i}{\hbar}tE_\alpha}\, \langle\psi_\alpha|$. Its matrix elements between position eigenstates $|q\rangle$ and $\langle q'|$ are

$$\langle q'|U(t)|q\rangle = \sum_\alpha \psi_\alpha(q')\, e^{-\frac{i}{\hbar}tE_\alpha}\, \overline{\psi_\alpha(q)},$$

with $\psi_\alpha(q) = \langle q|\psi_\alpha\rangle$. All these matrix elements are indeed analytic for $\Im\mathrm{m}\, t < 0$ if the hamiltonian is bounded from below (because the series is absolutely convergent for $\Im\mathrm{m}\, t < 0$). This analyticity property is valid provided its hamiltonian is bounded from below.

Thanks to this property, we can analytically continue the time evolution operator from real time to complex time $t \to -i\tau$ with τ real and called the "Euclidean time". This analytic continuation is called the Wick rotation. Let U_E be this analytic continuation. It satisfies the differential equation

$$-\partial_\tau U_E = H\, U_E,$$

where the hamiltonian H can be time dependent or not. For H time independent, $U_E = e^{-\tau H}$ (because $-it = -\tau$). For a particle in a potential $V(q)$, this is a kind of heat kernel, solution of a generalized heat equation of the form

$$\partial_\tau U_E = -\big(-\frac{\Delta_q}{2m} + V(q)\big)U_E.$$

The Euclidean evolution operator also admits a path integral representation

$$\langle q'|U_E(\tau)|q\rangle = \int_{\substack{q_{(s=0)}=q \\ q_{(s=t)}=q'}} [Dq]\, e^{-S_E[q]}, \tag{32}$$

with Euclidean action S_E defined as (here the integration variable s is again the Euclidean time)

$$S_E[q] = \int_0^\tau ds\, \big[\frac{1}{2}\dot{q}_s^2 + V(q)\big].$$

Notice the difference of signs between the real time action (defined by a Lagrangian) and the Euclidean action (defined after analytic continuation). These differences can be justified/derived in different ways: either using the previous construction based on a discretization of the path integral —as for the path integral of quantum mechanics or for the Brownian motion—, or directly by noticing that under the Wick rotation $\dot{q}^2 \to -\dot{q}^2$ and $ids \to ds$ so that $i\int ds[\dot{q}^2 - V(q)] \to -\int ds[\dot{q}^2 + V(q)]$, or equivalently $iS[q] \to -S_E[q]$.

For a free particle, in absence of external potential, U_E is simply the heat kernel and the Euclidean path integral with action $S_E[q] = \frac{1}{2}\int ds\, \dot{q}_s^2$ is that of the Brownian motion. Hence the Euclidean path integral for a particle in a potential V can be written as Brownian expectation (and hence it is better defined its the real time analogue). Namely,

$$\int [Dq]\, e^{-S_E[q]} = \mathbb{E}\big[\, e^{-\int ds\, V(X_s)}\,\big],$$

where $\mathbb{E}[\cdots]$ the expectation over the Brownian motion X_s. This representation of the Euclidean evolution operator as a Brownian expectation is called the Feynman-Kac representation. Note that it is slightly different form the representation of SDEs with drift which we previously discussed.

• *Path integral representation of partition functions.*

The euclidean evolution operator is $U_E = e^{-\tau H}$. It may be compared with the density matrix of a system with hamiltonian H at temperature $k_B T = 1/\beta$ which reads:

$$\rho_{\text{Gibbs}} = Z^{-1}\, e^{-\beta H}.$$

Here $Z = \text{Tr}(e^{-\beta H})$ is the partition function, a key quantity of statistical physics at equilibrium (recall that $F = -k_B T \log Z$ is the free energy). Computing the partition function amounts the trace of the Euclidean evolution operator with $\tau = \beta = 1/k_B T$, that is

$$Z = \text{Tr}(e^{-\beta H}) = \int dq\, \langle q|U_E(\tau = \beta)|q\rangle.$$

This of course can be represented by a path integral with periodic boundary condition, with period β:

$$Z = \int_{q_{(\beta)}=q_{(0)}} [Dq]\, e^{-S_E(q)}. \tag{33}$$

Periodicity is a way to implement the trace sum. This is an important formula, because it generalizes to any quantum systems and in particular to quantum field theory.

We can similarly write path integral representations of thermal expectations of operators. If A is an observable of the quantum system with hamiltonian H, its expectation at temperature $1/\beta$ is defined to be

$$\langle A\rangle_\beta = \frac{\text{Tr}(A\, e^{-\beta H})}{\text{Tr}(e^{-\beta H})}.$$

Its path integral representation is thus given by the ratio

$$\langle A\rangle_\beta = \frac{\int_{q_{(\beta)}=q_{(0)}} [Dq]\, A(\tau_0)\, e^{-S_E(q)}}{\int_{q_{(\beta)}=q_{(0)}} [Dq]\, e^{-S_E(q)}}.$$

For instance, if the observable A is a function $A(q)$ of the operator q, its insertion amounts to insert $A(q_{(\tau_0)})$ in the path integral. Again we have periodic boundary conditions to close the trace. The time τ_0 at which we insert the operator A does not matter (because of the periodicity we can always split $e^{-\beta H}$ is two pieces of move them around the trace).

Note that physical quantities, such as expectations or correlations, are ratio of path integrals. Taking this ratio improves some of the problems of the (any way ill-defined) path integral measure because the undefined normalization cancels between the numerator and the denominator.

• *Application to field theory.*

It is a small step to go (formally) from high dimensional hamiltonian systems to field theory. To make it concrete one may discretized space on a lattice with vertices at points $\mathbf{x}_k$. The dynamical variables are then countable variables $\phi(\mathbf{x}_k)$ so that one has the correspondance

$$\{\mathbf{q}\} \leftrightarrow \{\phi(\mathbf{x}_k)\}.$$

One may also take (formally) the continuous limit and view the value of the field $\phi(\mathbf{x})$ at space point $\mathbf{x}$ as the dynamical variables. They don't form a countable set but we nevertheless apply the previous formal manipulations.

A (classical) field theory comes equipped with an action $S[\phi]$, depending on the field configurations, which determine the field dynamics. By hypothesis, it can be written in terms a Lagrangian density $\mathcal{L}(\phi, \dot{\phi})$ as

$$S[\phi] = \int dt d\mathbf{x}\, \mathcal{L}(\phi, \dot{\phi}).$$

The simplest example is that of free massive field for which the action is:

$$S[\phi] = \int dt d\mathbf{x}\, \mathcal{L} = \int dt d\mathbf{x}\, \big(\frac{1}{2}(\dot{\phi})^2 - \frac{1}{2}(\partial_{\mathbf{x}}\phi)^2 - V(\phi)\big)$$

where $V(\phi) = \frac{1}{2}m^2\phi^2$ with m the mass of the field.

This classical field theory may (formally) be quantized using the previous path integral formalism. Upon quantization, a basis of the Hilbert space of the quantum field theory is (formally) made of eigen-field configuration states $|\{\phi(\mathbf{x})\}\rangle$. These are the analogue of the eigen-position states $|\mathbf{q}\rangle$. To specify the quantum dynamical we may give ourselves two configurations at two different times (at two different time slices), $\{\phi_0(\mathbf{x})\}$ at time t_0 and $\{\phi_1(\mathbf{x})\}$ at time t_1, and postulate that the quantum amplitude is (formally) defined by the path integral

$$\langle\{\phi_1(\mathbf{x})\}|U(t_1,t_0)|\{\phi_0(\mathbf{x})\}\rangle = \int_{\substack{\phi(\mathbf{x},t_0)=\phi_0(\mathbf{x})\\ \phi(\mathbf{x},t_1)=\phi_1(\mathbf{x})}} [D\phi]\, e^{(i/\hbar)\, S[\phi]},$$

with the boundary condition specified by the two field configurations. The integral is over 'path' in the space of field configurations.

We can of course Wick's rotate this path integral, as we did for quantum mechanics. This yields a Euclidean path integral over field configurations in one dimension higher (we set $\hbar = 1$):

$$\int [D\phi]\, e^{-S_E[\phi]},$$

with Euclidean action $S_E[\phi] = \int dx \big(\frac{1}{2}(\nabla\phi)^2 + V(\phi)\big)$ where ∇ is the gradient derivative along all directions (space and Euclidean time). Here we use the notation $x = (\tau, \mathbf{x})$, with τ the Euclidean time, which effectively span a space of one dimension higher than that of $\mathbf{x}$. In particular the partition function $Z(\beta) = \mathrm{Tr}(e^{-\beta H})$ of a quantum field at temperature $1/\beta$ is given by the Euclidean path integral with periodic condition along the Euclidean time:

$$Z(\beta) = \int_{\phi(\beta,\mathbf{x})=\phi(0,\mathbf{x})} [D\phi]\, e^{-S_E(\phi)}.$$

To make these formal definitions workable is what quantum and statistical field theory is about.

It is worth having in mind a geometrical picture related to these path integral representations. If the quantum field theory is defined on the infinite line the path integral representation of its partition function takes place over a cylinder of circumference β and infinite height (the long axis of the cylinder is the infinite line on which the quantum system is defined, the periodic circle is associated with the finite temperature and the periodicity comes form taking the traces). If the quantum field theory is defined on a finite interval, then the path integral takes place on

a cylinder of finite length. If the quantum system is periodic with period R (i.e. it is defined on a circle of radius R), the path integral takes place on a torus. Generalizations to higher dimensional quantum systems is clear. If the field theory is defined over a manifold $\mathcal{M}$, then its partition function path integral takes values on $\mathcal{M} \times S^1$ with S^1 the circle of period β. It may be useful to have these geometrical pictures in mind. See Figure.

3.6 The classical/quantum correspondance

The previous constructions provide instances of a correspondance between the physics of quantum systems and that of classical systems but in one dimension higher —a correspondance which was already transparent in the transfer matrix formulation of lattice statistical models. The aim this section is to spell out this correspondance and to establish the dictionary between these two setups

• *D-classical/$(D-1)$-quantum correspondance.*

By interpreting the Euclidean path integral as the partition function of a classical system whose Boltzmann weights e^{-S_E} we get a correspondance between the physics of quantum system in dimension d at finite temperature and that of classical system in $D = d + 1$ dimension.

Consider a lattice statistical model defined on hyper-cubic lattice in dimension D with spin variables $S_{i_0,i_1,\cdots,i_{D-1}}$ on each point of the lattice with $(i_0, i_1, \cdots, i_{D-1})$ the coordinates the vertices. We can choose one of the direction, say the first one corresponding to the index i_0, and declare it to be the Euclidean time direction. Accordingly we can slice the D-dimensional hyper-cube as $(D-1)$ hyper-surfaces equally spaced along the Euclidean time direction. The set of the spin variables on these $(D-1)$ hyper-surfaces may be viewed as the dynamical variables of a quantum systems. Collecting all values of those variables on all slices along the chosen Euclidean time direction gives a Euclidean time trajectories. The energy of the configuration of all spins variables on all hyper-surfaces slices (i.e. on the original hyper-cube) is then identified with the Euclidean action of these trajectories. Hence, if this identification applies, the partition sum of the classical statistical model is identified with the (discretized) path integral of these trajectories.

This mapping is particularly explicit in the transfer matrix formulation of lattice statistical models. The transfer matrix $\mathcal{T}$ is identified with the Euclidean evolution over a lattice distance δ, that is we have the identification $\mathcal{T} = e^{-\delta H}$ with H the hamiltonian. For a statistical model in dimension D, it acts on Hilbert space defined over an hypersurface of dimension $d = D - 1$ which is identified with the quantum Hilbert space.

• *Thermal expectations, energy gap and correlation length.*

Through the previous correspondance we associate a D-dimensional classical statistical systems with partition function Z with a $(D-1)$ quantum systems with hamiltonian H. The quantum system is thought to be defined on the hyper-surface orthogonal to one of the direction classical system which, by convention is called the Euclidean time direction. This correspondance is such that if the classical system is made periodic, then

$$Z(L) = \mathrm{Tr}(e^{-LH}),$$

where the trace is over the quantum system Hilbert space.

Accordingly we have the correspondance:

$D-dim.$ "*classical*" *system*		$(D-1)-dim.$ "*quantum*" *system*
lattice spacing	$\leftrightarrow$	discretization
configuration energy βE_c	$\leftrightarrow$	Euclidean action S_E
periodic system L	$\leftrightarrow$	finite temperature β
inverse correlation length ξ^{-1}	$\leftrightarrow$	energy gap $\Delta E = E_1 - E_0$

The first three statement are clear and were already discussed at length. The last statement about the correlation length needs an explanation. It requires computing one and two point correlation functions.

Let us first look at the expectation of some observable A. We first (and provisionally) makes the Euclidean time direction periodic of period L. We will later take the large L limit. Following the above strategy we may represent the expectation values as

$$\langle A\rangle_L = \frac{\mathrm{Tr}\big(e^{-LH}\,A\big)}{\mathrm{Tr}\big(e^{-LH}\big)},$$

where $Z(L) = \mathrm{Tr}\big(e^{-LH}\big)$ is the partition function with H the hamiltonian coding for the evolution along the Euclidean time. By decomposing on the eigen-state of H, we write

$$\langle A\rangle_L = \frac{1}{Z(L)}\sum_n e^{-LE_n} A_{nn},$$

where E_n are the eigen-values of H and A_{nn} the matrix elements of A in the eigen-basis. The partition function is $Z(L) = \sum_n e^{-LE_n}$. In the large L limit, the minimum energy E_0 dominates (assuming that there is a gap). This projects the sum on the ground state $|0\rangle$. The vacuum energy compensate (as it should be) in the ratio, and we get

$$\langle A\rangle_L \simeq_{L\to\infty} A_{00}.$$

A formula which may be summarized as: taking the infinite size limit projects on the ground state.

Let us now pick B and A two operators and consider their two point functions at different sites at position τ_1 and τ_2 along the Euclidean time direction. Let $\ell = \tau_2 - \tau_1 > 0$ be their distance. We may then represent this correlation function as follows:

$$\langle B(\ell)A(0)\rangle_L = \frac{\mathrm{Tr}\big(e^{-(L-\ell)H}\,B\,e^{-\ell H}\,A\big)}{\mathrm{Tr}\big(e^{-LH}\big)},$$

with $\ell = \tau_2 - \tau_1 > 0$ the distance between the two insertion points. We want to know how this correlation function behavior at large distance but small compare to the system size, i.e. $1 \ll \ell \ll L$. Decomposing the trace on the hamiltonian eigenstates gives

$$\langle B(\ell)A(0)\rangle_L = \frac{1}{Z}\sum_{n,m} e^{-(L-\ell)E_n} B_{nm}\, e^{-\ell E_m} A_{mn}.$$

Imposing $1 \ll \ell \ll L$ amounts to take the limits $L \to \infty$ then $\ell \to \infty$ is this order. Again the eigen-state with minimum energy dominates in the large L limit, and

$$\langle B(\ell)A(0)\rangle_{L=\infty} = \sum_m B_{0m}\, e^{-\ell(E_m - E_0)} A_{m0}.$$

The dominating contribution in this last sum again comes from the vacuum energy with the terms $B_{00}A_{00} = \langle B\rangle\langle A\rangle$. Let us defined the connected correlation function by

$$\langle B(\ell)A(0)\rangle_L^{\text{conn.}} := \langle B(\ell)A(0)\rangle_L - \langle B(\ell)\rangle_L \langle A(0)\rangle_L.$$

In the large L limit we have $\langle B(\ell)A(0)\rangle_{L=\infty}^{\text{conn.}} = \sum_{m\neq 0} B_{0m}\, e^{-\ell(E_m - E_0)} A_{m0}$. Hence, the dominating contribution at large distance $\ell \gg 1$ comes the first excited state and we have

$$\langle B(\ell)A(0)\rangle_{L=\infty}^{\text{conn.}} \simeq \text{const.} e^{-\ell/\xi}$$

with

$$\xi^{-1} = E_1 - E_0.$$

By definition, ξ is then the correlation length. Notice that the decay is exponentially fast only if the gap is not vanishing (in the infinite size limit).

3.7 Exercises

• *Exercise 3.7.1: Fermionic representation of the 2D Ising model*

The aim of this exercise is to complete the study of the 2D Ising model presented in the main text. Recall the definition of the 2D Ising model given in the text.
(i) Prove –or argue for– the expression for the Ising transfer matrix.
(ii) Recall the Jordan-Wigner transformations given in the main text which construct fermionic operators in terms of Pauli matrices via

$$a_x = e^{i\pi \sum_{y=1}^{x} \tau_y^- \tau_y^+} \tau_x^+, \quad a_x^\dagger = e^{-i\pi \sum_{y=1}^{x} \tau_y^- \tau_y^+} \tau_x^-.$$

Show that we may alternatively write

$$a_x = \Big(\prod_{y=1}^{x-1} \tau_y^z\Big) \tau_x^+, \quad a_x^\dagger = \Big(\prod_{y=1}^{x-1} \tau_y^z\Big) \tau_x^-.$$

Verify that they satisfy the canonical fermionic relation $a_x^\dagger a_y + a_y a_x^\dagger = \delta_{x,y}$.
(iii) Complete the proof of the diagonalisation of the Ising hamiltonian and its spectrum. Proof that, after an appropriate Bogoliubov transformation on the fermion operators, the Ising hamiltonian (25) can be written in the form (27) given in the main text, which we recall here,

$$\mathcal{H} = \sum_{k>0} h_k \left(c_k^\dagger c_k - c_{-k} c_{-k}^\dagger\right),$$

with single particle spectrum $h_k = \left[(\gamma-\beta)^2 + 4\gamma\beta \sin^2(k/2)\right]^{1/2}$.

• *Exercise 3.7.2: Spin operators, disorder operators and parafermions.*

The aim of this exercise –and the following two– is to study some simple consequences of group symmetry in lattice statistical models.

Let us consider a lattice statistical model on a two dimensional square lattice $\Lambda := a^2\mathbb{Z}^2$ with spin variables s on each vertex of the lattice. These variables take discrete or continuous values, depending on the models. We consider neighbour spin interactions with a local hamiltonian $H(s,s')$ so that the Boltzmann weight of any given configuration $[c]$ is

$$W([c]) := \prod_{[i,j]=\text{edge}} w_{[i,j]}, \quad w_{[i,j]} = e^{-H(s_i,s_j)},$$

where, by convention, $[i,j]$ denotes the edge connecting the vertices i and j. Let $Z := \sum_{[c]} W([c])$ be the partition function.

Let us suppose that a group G is acting the spin variables. We denote by R the corresponding representation. Furthermore we assume that the interaction is invariant under this group action so that, by hypothesis,

$$H(R(g)\cdot s, R(g)\cdot s') = H(s,s'), \quad \forall g \in G.$$

(i) *Transfer matrix:* Define and construct the transfer matrix for these models.

(ii) *Spin operators:* Spin observables, which we denote $\sigma(i)$, are defined as the local insertions of the spin variables at the lattice site i. That is: $\sigma(i)$ is the function which to any configuration associate the variable s_i.
Write the expectations of the spin observables $\langle \sigma(i_1) \cdots \sigma(i_N) \rangle$ as a sum over configurations weighted by their Boltzmann weights.
Write the same correlation functions in terms of the transfer matrix.
(iii) *Disorder operators:* Disorder observables are defined on the dual lattice and are indexed by group elements. Let Γ be a closed anti-clockwise oriented contour on the square lattice $\tilde{\Lambda}$ dual to Λ –the vertices of $\tilde{\Lambda}$ are the center of the faces of Λ. Let ℓ denote an oriented edge of Γ. It crosses an edge of Λ and we denote by ℓ^- and ℓ^+ the vertices of this edge with ℓ^- inside the loop Γ. The disorder observable $\mu_g(\Gamma)$ for $g \in G$ is defined as

$$\mu_\Gamma(g) := \exp\big(\sum_{\ell \in \Gamma} (H(s_{\ell^-}, s_{\ell^+}) - H(s_{\ell^-}, R(g) s_{\ell^+})\big),$$

Inserting $\mu_\Gamma(g)$ in the Boltzmann sum amounts to introduce a defect by replacing the hamiltonian $H(s_{\ell^-}, s_{\ell^+})$ by its rotated version $H(s_{\ell^-}, R(g) s_{\ell^+})$ on all edges crossed by Γ.
Write the expectations of disorder observables in terms of the transfer matrix.

• *Exercise 3.7.3: Symmetries, conservation laws and lattice Ward identities*

The aim of this exercise is to understand some of the consequences of the presence of symmetries. The relations we shall obtain are the lattice analogue of the so-called Ward identities valid in field theory.

We consider the same two dimensional lattice model as in previous exercise. We recall that we assume the Bolztmann weight to be invariant under a symmetry group G in the sense that

$$H(R(g) \cdot s, R(g) \cdot s') = H(s, s'), \quad \forall g \in G.$$

(i) Let i_k be points on the lattice Λ and Γ a contour as in previous exercise.. Show that the group invariance implies that

$$\langle \mu_\Gamma(g) \prod_k \sigma(i_k) \rangle = \prod_{i_k \text{ inside } \Gamma} R_{i_k}(g) \cdot \langle \prod_k \sigma(i_k) \rangle,$$

where $R_{i_k}(g)$ denote the group representation R acting on the spins at site i_k.
(ii) Show that $\mu_g(\Gamma)$ is invariant under any smooth continuous deformation of Γ as long as the deformation does not cross points of spin insertions (it is homotopically invariant).

We now look at the consequences of these relations for infinitesimal transformations. Suppose that G is a Lie group and Lie(G) its Lie algebra. Let us give a name to small variations of H by defining $\partial_X H$. For $g = 1 + \epsilon\, X + \cdots$ with $X \in \text{Lie}(G)$, we set

$$H(s, R(g)s') - H(s, s') =: \epsilon\, \partial_X H(s, s') + \cdots .$$

For $\ell = [\ell^-, \ell^+]$ an oriented edge of Γ as in previous exercise and $X \in \text{Lie}(G)$, we let

$$*J_\ell^X := \partial_X H(s_{\ell^-}, s_{\ell^+}),$$

They are specific observables, called *currents*, whose correlation functions are defined as usual via insertion into the Bolztmann sums.

(i) Show that the following equality holds:

$$\langle \sum_{\ell\in\Gamma} *J_\ell^X \cdot \prod_i \sigma(i)\rangle = \langle (\sum_{i_k \text{ inside } \Gamma} R_{i_k}(X)) \cdot \prod_i \sigma(i)\rangle,$$

if some spin observables are inserted inside Γ.
(ii) Deduce that, if there is no observables inserted inside Γ, then the following equality holds inside any expectation values:

$$\sum_{\ell\in\Gamma} *J_\ell^X = 0,$$

That is: The second of these two equations is a conservation law (i.e. it is the analogue of the fact that $\int *J = 0$ if $*J$ is a closed form, or equivalently, if J is a conserved current), the first tells about the consequences of this conservation law when insertion of observables are taken into account. It is analogous to the Gauss law in electrodynamics. They are called Ward identities in field theory.

• *Exercise 3.7.4: Para-fermionic operators and braiding relations*

The aim of this exercise is to develop further the notion of spin and disorder operator using the transfer matrix approach and the corresponding operator formalism.

We consider the same 2D group invariant lattice model as in previous exercises. The sites of the 2D square lattice are labeled (n, m) with n and m integers. Let $\mathcal{T}$ be the transfer matrix for a one-step evolution. It acts on the Hilbert space $\mathcal{H} = \otimes_{(n,0)} V_{(n,0)}$, product of the local Hilbert spaces along an horizontal line ($m = 0$). Let n_∞ the most left point on this line.

Spin operators: Define the spin operator $\hat{\sigma}(n)$ by spin multiplication on site $(n, 0)$. Let $\hat{\sigma}(n, m) := \mathcal{T}^{-m}\hat{\sigma}(n)\mathcal{T}^m$ be its translation by m vertical step.

Disorder operators: Define the disorder operator $\hat{\mu}_g(n)$ by the insertion of $R(g)$ on all sites strictly on the left of the site $(n, 0)$, i.e. on all site $(k, 0)$ with $n_\infty \leq k < n$. Let $\hat{\mu}_g(n, m) = \mathcal{T}^{-m}\hat{\mu}_g(n)\mathcal{T}^m$ be its translation by m vertical step.
(i) Show the generalized commutation relations, called braiding relations, for $g \in G$:

$$\hat{\sigma}(n_1)\,\hat{\mu}_g(n_2) = \begin{cases} \hat{\mu}_g(n_2)\,\hat{\sigma}(n_1) & \text{, if } n_1 \geq n_2 \\ \hat{\mu}_g(n_2)\,(R(g)\cdot\hat{\sigma}(n_1)) & \text{, if } n_1 < n_2 \end{cases}$$

(ii) Show that, for $g, h \in G$:

$$\hat{\mu}_g(n_1)\,\hat{\mu}_h(n_2) = \hat{\mu}_h(n_2)\,\hat{\mu}_{h^{-1}gh}(n_1) \quad \text{, if } n_1 < n_2.$$

Para-fermions: Let $\psi_g^R(n)$ be the product of operators $\hat{\psi}_g^R(n) = \hat{\sigma}(n)\hat{\mu}_g(n)$.
(ii) Compare this definition with the Jordan-Wigner transformation.
(iv) Show that for all $g, h \in G$:

$$\hat{\psi}_g^R(n_1)\,\hat{\psi}_h^R(n_2) = \hat{\psi}_h^R(n_2)\,(R(g)\cdot\hat{\psi}_{h^{-1}gh}^R(n_1)) \quad \text{, if } n_1 < n_2.$$

(v) Determine what are these commutation relations when $G = \mathbb{Z}_2$. Compare with fermionic commutation relations. Extend these questions to $G = \mathbb{Z}_N$.

- *Lattice gauge theory.*

  [...To be completed...]

- *Self-avoiding walks and the $O(n)$-models*

  [...To be completed...]

4 Critical systems and mean field theory.

The aim of this chapter is to describe basics facts about second order phase transitions and critical systems. These include scaling exponents, coding for the singular behavior of thermodynamic quantities at the transition, and the important notion of universality. We explain the Landau theory which is based on mean field approximations as well as its domain of validity. First steps on how to go beyond mean field theory, away from its domain of validity, are discussed.

4.1 Critical systems: phenomenology

Critical systems refer to second order phase transitions. Classical thermodynamics classifies phase transitions according to the behavior of the thermodynamical functions at the transition. (See standard textbook for basic facts about phases transitions). For first order phase transition the first derivative of the free energy is discontinuous at the phase transition point. For second order phase transition, the first order derivative is continuous there is a discontinuity in a higher order derivative of the free energy.

• *Second order phase transitions.*

Second order phase transitions are continuous phase transitions (no discontinuity in the response function). The fact that the transition is continuous does not mean that the behavior is not singular. Although continuous at the transition, the response functions are not a smooth functions of the external parameters (some of their derivatives develop singularities). Standard examples are:
— Para/Ferromagnet transition: Above the critical temperature T_c, the magnetization is zero, whereas below the critical temperature the magnetization becomes non zero. For a second order (and hence continuous) phase transition the magnetization vanishes at the critical temperature (in absence of external magnetic field);
— Bose condensation: Below a critical temperature, the lowest energy orbital becomes macroscopically occupied (i.e. it becomes on the order of the macroscopic number of bosons in the system);
— Superconducting or superfluid transition: Below a critical temperature, certain systems display dissipation-less flow of current or of fluid mass.
— And many more examples,...
All these transitions share a great deal of similarities and they often can be mapped to the others.

We shall use the (most standard) example of para-to-ferro transition to illustrate the phenomena. Some of the typical data of second order phase transitions are the followings:

At high temperature $T > T_c$, the system is in a disordered phase in which the local spins vary randomly and rapidly from one site to the others. For instance, in the infinite temperature Ising model ($T \gg J$) the spins at different sites are independent. The response of the system to a small magnetic field is linear: the mean magnetization $\bar{m}$ is linear in the external field and the magnetic susceptibility $\chi(T) := \partial\bar{m}(h)/\partial h|_{h=0}$ is finite. **Figure.**

At low temperature $T < T_c$, the system is ordered. The spins have tendency to be aligned, with a large probability, and they are correlated at large distances. For instance, in the zero temperature Ising model, the most probable configurations are those in which all spins point in the same direction. There is a non zero spontaneous magnetization and hence the system is in

a ferromagnetic phase. More precisely,

$$\lim_{h\to 0^+} \bar{m}(h) \neq 0.$$

in such case, on says that the symmetry is broken (the symmetry is the $\mathbb{Z}_2$ action reversing all spins). This is a key concept in the theory of second order phase transition but also in particule physics through the so-called Higgs mechanism. **Figure**

The spontaneous magnetization $\bar{m}$ (at $h = 0$) is a function of the temperature: $\bar{m} \neq 0$ at $T < T_c$ and $\bar{m} = 0$ at $T > T_c$ and the magnetization vanishes continuously at $T = T_c$. The magnetization is not a smooth function of the temperature but develops a power law behavior $\bar{m}(T) \simeq |T - T_c|^\beta$ for $T < T_c$ close to the critical temperature. **Figure.**

At the critical point $T = T_c$, the system possesses peculiar and anormal properties: the magnetization at small external field is singular, $\bar{m}(h)|_{T=T_c} \propto h^{1/\delta}$, the susceptibility diverges close to the transition, $\chi(T) \propto |T - T_c|^{-\gamma}$, etc. All these exposants β, δ, γ, etc, are non trivial.

These behavior are linked to the presence of large fluctuations at all scales so that the physics is dominate dby collective phenomena. Critical systems (at $T = T_c$) are characterized infinite correlation length. In particular there is no intrinsic characteristic length scale so that these system are invariant under dilatation.

• *Why do we care about second order phase transition?*

(i) True phase transitions (singularities in the free energy) can only occur in the thermodynamic limit (i.e., when there are an infinite number of degrees of freedim in the system, as otherwise the partition function a finite polynomial sum). Phase transitions are key examples of collective phenomena, i.e. how many objects act together to give new physics.

(ii) Phase transitions have universal properties. Phases of matter are so diverse so that many of the properties of these phases are "non-generic". However, the types of second order phase transitions turn out to fall into a relatively small number of classes known as "universality classes" which all behave similarly. Understanding the phase transitions also helps understanding the key differences between these phases.

(iii) The mathematical structure build to study phase transitions (statistical field theory and the renormalization group) turns out to be one of the most fundamental tool to understand non-trivial physics (say involving infinitely large numbers of degrees of freedom, at all scales), with applications to condensed matter and high energy physics.

4.2 The Ising mean field theory

Here we describe the mean field theory in the case of the Ising model. Although non exact the mean field theory provides a simple scheme illustrating the emergence of singular behaviors at a transition. There are different ways of deriving the mean field theory (either using a local field approximation as done below, or using a variational ansatz as done in the exercise Section).

• *The Ising model*

We consider the Ising model on a hyper-cubic lattice in dimension D, with a the mesh of the lattice a, i.e. the lattice is $a^D\mathbb{Z}^D$. On each lattice site we put a spin variable s_i, $i \in \Lambda_a$, which possibly takes two values $s_i = \pm 1$. A configuration $[s]$ is the data of all spin values on all

point of the lattice. The energy of a configuration codes for the interaction of neighbor spins. by definition, it is given by the formula:

$$E[\mathbf{s}] = -J_0 \sum_{i\sim j} s_i s_j - h \sum_i s_i$$

where the sum us over pairs of neighbor sites. The constant $J_0 > 0$ measure the strength of the interaction and h is the external magnetic field. The statistical distribution is specified by the Boltzmann rules: the probabilities for a configuration $[s]$ is proportional to $e^{-E[\mathbf{s}]/T}$ where T is the temperature. Since $J_0 > 0$, aligned configurations are favored at low temperature. The Ising model possesses a second order phase transition (at zero external field) at some critical temperature T_c.

• *Mean field theory*

The mean field theory consists in making the approximation that at each site the local spin is effectively sensible only to the"mean field" created by the neighbor spins. If z is the number of neighbor ($z = 2D$ in the case of the hyper-cubic lattice), the mean field is $h + zJ_0\bar{m}$ with $\bar{m}$ the local magnetization ($\bar{m}$ is the mean value $\langle s\rangle$ of the spin s_i). This mean field approximation consist in replacing the sum $\sum_{<ij>} s_i s_j$ by $z\bar{m}\sum_i s_i$ in the energy of a spin configuration, which is then reduced to a sum of independent terms:

$$E[\mathbf{s}] \to \bar{E}[\mathbf{s}] = -\sum_i (zJ_0\bar{m} + h)\, s_i.$$

The local mean energy is $e_i := (zJ_0\bar{m} + h)\, s_i$. Self-consistency condition demands that the mean spin (computed using the Boltzmann rules specified by $\bar{E}[\mathbf{s}]$) should be $\bar{m}$, i.e.

$$\langle s\rangle = \bar{m}, \quad \text{alias} \quad \bar{m} = \frac{e^{-\beta e(+)} - e^{-\beta e(-)}}{e^{-\beta e(+)} + e^{-\beta e(-)}} = \tanh(\beta(zJ_0\bar{m} + h)),$$

with $\beta = 1/k_B T$ the inverse temperature (in the following we take $k_B = 1$). This condition means that we equate the mean computed using a spatial averaging procedure with the thermodynamic mean. As consequence, and according to the central limit theorem, we expect this approximation to be better as the spatial dimension increases.

At zero external field (see **Figure**), the mean field equation is trivial with $\bar{m} = 0$ if $T > T_c$ but non-trivial solution with $\bar{m} \neq 0$ for $T < T_c$ (the other solution $\bar{m} = 0$ is unstable). The critical temperature is $T_c = zJ_0$. That is, we have:

$$\begin{cases} \text{if } \beta z J_0 < 1, & \bar{m} = 0, & \text{(para phase);} \\ \text{if } \beta z J_0 > 1, & \bar{m} = \pm\bar{m}_0 \neq 0, & \text{(ferro phase).} \end{cases}$$

At $h = 0$, and close to the critical point, one obtains the magnetization by simply expanding the self consistency condition in power of the magnetization. This yields

$$\frac{1}{3}(z\beta J_0\bar{m})^3 = \frac{T_c - T}{T}\,\bar{m}.$$

Hence for $T > T_c$, we recover $\bar{m} = 0$, while for $T < T_c$ we get $\bar{m}_0 \simeq (T_c - T)^{1/2}$. The phase transition is thus continuous. It is a second order phase transition.

For $h \neq 0$ but at the critical temperature, the mean field equation reads $\bar{m} = \tanh(\bar{m} + \beta_c h)$. Hence $\bar{m}_0 \simeq h^{1/3}$ since $\tanh x \simeq x - \frac{1}{3}x^3 + O(x^5)$.

Similarly, writing the mean field equation at small external field and at $T \neq T_c$ but close to the critical point yields $(\partial\bar{m}/\partial h)_{h=0} \simeq |T - T_c|^{-1}$, (for instance by evaluating the derivative of the mean field equation w.r.t to external field at $h = 0$),.

Hence, we have a second order transition with $\beta = 1/2$, $\delta = 3$, $\gamma = 1$, and one can also verifies that $\alpha = 0$ (the heat capacity is discontinuous).

4.3 Critical exponents and universality

Here we introduce the standard notation to parametrize the singular behavior of thermodynamical function at a critical point. These singular behaviors are coded into critical exponents which we list just below.

• *Critical exponents.*

— For the magnetization as function of the temperature (magnetization exponent):

$$\bar{m}(T) \simeq \begin{cases} C\,|T_c - T|^{\beta}, & \text{for } T < T_c, \\ 0, & \text{for } T > T_c. \end{cases}$$

Both the critical temperature and the amplitude C are not universal, but the exponent is.

— The magnetic susceptibility (at zero magnetic field) is defined as $\chi(T) := \partial\bar{m}(h)/\partial h|_{h=0}$. It is singular as the temperature approches T_c (it diverges):

$$\chi(T) \ = \partial\bar{m}(h)/\partial h|_{h=0} \simeq \begin{cases} \chi_+\,|T_c - T|^{-\gamma}, & \text{for } T < T_c, \\ \chi_-\,|T - T_c|^{-\gamma}, & \text{for } T > T_c. \end{cases}$$

The amplitudes $\chi_\pm$ are not universal but their ratio is. The exponent γ is universal (it could have been different on both side of the transition but this turns out not to be the case).

— The magnetisation at the critical temperature is also singular as a function of the magnetic field:

$$\bar{m}(h)|_{T=T_c} \simeq \sigma_0\, h^{1/\delta}.$$

The writing $1/\delta$, and not the reverse, is a convention.

— The heat capacity (which is the second order derivative of the free energy w.r.t. the temperature) diverges at the critical temperature (and this is often taken as a characteristic of the transition):

$$C(T) \simeq \begin{cases} A_+\,|T_c - T|^{-\alpha}, & \text{for } T > T_c, \\ A_-\,|T - T_c|^{-\alpha}, & \text{for } T < T_c. \end{cases}$$

The amplitudes $A_\pm$ are not universal but their ratio is. The exponent α is universal.

— The correlation length diverges at the transition. The correlation length codes how much two distant spins (for magnetic systems) are correlated. It could be defined via the two point connected correlation spin function, $G(i,j) = \langle S_i S_j\rangle^c = \langle S_i S_j\rangle - \langle S_i\rangle\langle S_j\rangle$, for spins at positions

x_i and x_j. At $T > T_c$, $\langle S_i \rangle = 0$, whereas $\langle S_j \rangle \neq 0$ at $T < T_c$. At distances large enough we expect an exponentiel decreases of the correlation function

$$G(i,j) \simeq \text{const. } e^{-|x_i - x_j|/\xi},$$

with ξ the (so-called) correlation length. It depends on the temperature (and on the magnetic field which we set here to zero). For $T \gg T_c$ spins are quite uncorrelated (by the very nature of the disordered phase) and get more and more correlated as T approches T_c so that ξ increases as $T \to T_c^+$ until it gets infinite as ones approaches the ferromagnetic phase (the direction of all spins in the system are correlated in the ferromagnetic phase). This divergence also occurs when approaching T_c from below. We have

$$\xi(T) \simeq \begin{cases} \xi_+ \, |T_c - T|^{-\nu}, & \text{for } T > T_c, \\ \xi_- \, |T - T_c|^{-\nu}, & \text{for } T < T_c. \end{cases}$$

Surprisingly the exponent are identical on both sides of the transition. The amplitudes $\xi_\pm$ are not universal but their ratio is.

It should be noticed that the correlation length is infinite at the critical temperature $\xi(T_c) = \infty$. This tells us that all length scales are becoming correlated as we approach the critical temperature. At the critical point there is no characteristic length and the system behavior is thus expected to be invariant under scale (global dilatation) transformation. By "usual" (folklore) argument, locality promotes this invariance to global conformal invariance.

At criticality the correlation function decreases as a power law (not exponentially)

$$G(i,j)|_{T=T_c} \simeq \frac{1}{|x_i - x_j|^{D-2+\eta}}.$$

This defines a new exponent (the scaling dimension of the spin operator).

All these exponents are in general not independent. They satisfy (so-called) scaling relations (not to be proven at this point of the lectures):

$$\begin{aligned} \alpha + 2\beta + \gamma = 2 \quad &, \quad \alpha + \beta\delta + \beta = 2, \\ \alpha + D\nu = 2 \quad &, \quad \gamma = \nu(2 - \eta). \end{aligned}$$

These relations come from assuming (or proving) that all relevant quantities, such as the (singular part) of the free energy or the correlation functions, are essentially homogeneous functions of the external parameters and of the correlation length. But a proof of them requires analyzing the RG transformations, see below.

To conclude, we present a table of the Ising critical exponents in various dimensions:

ISING	;	mean field	;	exact 2D	;	3D approx.
– – – – – – – – – – – – – – – – – – –	;	– – – –	;	– – – –	;	– – – –
magnetization : $m(T) \simeq (T_c - T)^\beta$	;	$\beta = 1/2$	;	$\beta = 1/8$	;	$\beta = 0.3264...$
critical magnetization : $m(h) \simeq h^{1/\delta}$	;	$\delta = 3$	;	$\delta = 15$	;	$\delta = 4.7898...$
magnetic susceptibility : $\chi(T) \simeq \lvert T_c - T \rvert^{-\gamma}$	;	$\gamma = 1$	;	$\gamma = 7/4$	;	$\gamma = 1.2371...$
correlation length : $\xi(T) \simeq \lvert T_c - T \rvert^{-\nu}$	;	$\nu = 1/2$	;	$\nu = 1$	;	$\nu = 0.6299...$

• *Universality*

We have already mentioned this important property. Universality means that the critical behavior near the transition does not depend on the details of the systems or of the model systems, but only on a few properties – such as symmetries, the representation of this symmetry group associated to the order parameters, etc. For instance the Ising model is the representative of the universality class for which the order parameter is a scalar —here the magnetization— and the symmetry is the $\mathbb{Z}_2$ reflexion symmetry of this scalar. The universality property is an echo the renormalization group analysis.

4.4 Landau theory

Landau theory is an effective theory to deal with second order phase transitions, very much close the Ising mean field theory, but with much wider perspectives and domains of applications. It is based on the following methodological tools:
(i) Identify a local order parameter (often called m or ϕ), which describes (macroscopically) the relevant degrees of freedom close to the transition. This parameter can be a scalar, a vector, a tensor, etc. but symmetry arguments usually helps identifying it.
(ii) Construct a local free energy functional $F[m]$ by expanding in power of m (or power of its gradient ∇m) assuming that all fluctuations are small enough. Again symmetry arguments help restricting the possible terms in this expansion. By hypothesis, the expansion coefficients depend smoothly on the external parameters (T, h, etc.).
(iii) Minimize the free energy to evaluate the thermodynamical functions and the critical exponents.

• *The Ising (mean field) universality class.*

The order parameter is $\phi = m$, the local magnetization. There is a $\mathbb{Z}_2$-symmetry. For homogeneous configurations, and homogeneous applied external field h, the free energy is proportional to the volume $F[m;T,h] = \text{Vol.}\, f[m;T,h]$ where f is the free energy per unit volume. The coupling to the external field is

$$f[m;T,h] = f_0[T,m] - hm.$$

$\mathbb{Z}_2$-symmetry demands that f_0 to be even: $f_0[T,m] = f_0[T,-m]$. Hence we expand in power of m:

$$f_0[T,m] = g_0(T) + g_2(T)\, m^2 + g_4(T)\, m^4 + \cdots .$$

We truncate the expansion at order four. For the free energy to be bounded from below we assume $g_4 > 0$ (otherwise we would have to expand to the next order hoping the next expansion coefficient is positive). The Taylor coefficient g_k are smooth functions of the temperature. The behavior of f_0 as function of m (here the variational parameter) is different whether g_2 is positive or negative. Hence the transition temperature T_c occurs when g_2 vanishes. We write ($a > 0$):

$$g_2(T) = \frac{a}{2}(T - T_c) + \cdots , \quad g_4 = \frac{g}{4!} + \cdots ,$$

where the dots refer to higher order terms in $T - T_c$.

The thermodynamical quantities are determined by minimizing the free energy with respect to m (this comes from usual arguments of thermodynamics). Writing $\partial f/\partial m = 0$ yields

$$a(T - T_c)\bar{m} + \frac{g}{3!}\bar{m}^3 = h.$$

At $h = 0$ we get

$$\bar{m}|_{h=0} \simeq \begin{cases} 0, & \text{for } T > T_c, \\ \pm\left(\frac{6a}{g}\right)^{\frac{1}{2}} (T_c - T)^{\frac{1}{2}}, & \text{for } T < T_c. \end{cases}$$

This corresponds to an exponent $\beta = 1/2$.

At non zero magnetic field $h \neq 0$, evaluating the derivative (at $h = 0$) of the above minimizing equation yields

$$\chi(T) = \partial\bar{m}/\partial h|_{h=0} \simeq \begin{cases} [a(T - T_c)]^{-1}, & \text{for } T > T_c, \\ [2a(T_c - T)]^{-1}, & \text{for } T < T_c. \end{cases}$$

This corresponds to an exponent $\gamma = 1$.

At the critical temperature, the minimizing equation is $\frac{g}{3!}\bar{m}^3 = h$. Hence,

$$\bar{m}(h)|_{T=T_c} \simeq h^{1/3}.$$

This corresponds to an exponent $\delta = 3$.

Similarly, the singular part of the heat capacity is $C \simeq -T\frac{\partial^2 f}{\partial T^2}$. For $T > T_c$, $m = 0$ and $f_0 = g_0(T)$. For $T < T_c$, the magnetization is non zero and $f_0 = g_0(T) + \frac{3a^2}{2g}(T_c - T)^2 + \cdots$. Hence, there is no divergence of the heat capacity at the transition but a jump with an extra heat capacity

$$\delta C \simeq 3a^2 T_c/g.$$

This corresponds to an exponent $\alpha = 0$. We of course recover all exponents of the Ising mean field theory.

The key point of this section is that we don't actually need any microscopic details in order to calculate exponents. All we needed to know was the symmetry of the order parameter. All we needed was a generic Taylor series expansion, and we didn't even need to know much about the value of any of the expansion coefficients. One could have ferromagnets on any shaped lattice, or any sort of microscopic physics, and still the same expansion pertains. Furthermore, we can translate this calculation to many other systems, such as ferro-electrics (where h is replaced by the externally applied electric field), if one can argue that the necessary symmetry holds.

4.5 Landau-Ginzburg theory

The Landau-Ginzburg theory is same thing as Landau theory but taking into account more carefully spatial the dependence of the magnetization. It generalizes the previous construction by taking into account inhomogeneity (in the external field or in the local magnetization). It will give us access to the correlation length. It only describes the long wave length of these inhomogeneities (not inhomogeneities at small distance). To lowest order in the derivative, the free energy reads

$$F[m; h] = \int d^D x \Big[\frac{\kappa(T)}{2} (\nabla m)^2(x) + g_2(T)\, m(x)^2 + g_4(T)\, m(x)^4 - h(x)m(x) + \cdots \Big],$$

up to the irrelevant constant (in m) term $g_0(T)$. We neglect higher order derivatives assuming that fluctuations are small and that the long wave length modes are the only relevant ones.

• *Heuristics*

One may justify the use of this functional free energy by a (naive) coarse graining approximation. Let us imagine dividing our system into small cells, large enough compared to the microscopic scale but still very small compared to lengths of interest (i.e. we want to be able to take averages within the cells and deal with them as if they were macroscopic sub-systems). We can then divide our partition function into sums over a large number of individual cells

$$Z = \sum_{[s]} e^{-\beta H[s]} = \sum_{[m]} \sum_{[s:\sum_{i\in \text{cell}} s_i = m]} e^{-\beta H[s]}.$$

The last sum is restricted such that the sum over all the spins in the cell number k is equal to the magnetization m_k. Since all cells are macroscopic sub-systems, we assign them a free energy given the values of the magnetizations in the different boxes, so that the partition function reads

$$Z \to \int [\prod_k dm_k]\, e^{-F[m_1,\cdots,m_k,\cdots]}.$$

We are defining a free energy which is a function of a magnetization, the only difference is that here the free energy is a function of the magnetization in all of the different cells. We have absorbed β into the definition of F. The sum over cells has been converted into integrals (since the cells are large enough that the averaged magnetization is essentially continuous) and then the integration over all of the small cells is essentially what we mean by a functional integral

$$Z \simeq \int [Dm]\, e^{-F[m]}.$$

where we can describe the free energy functional as an integral over a local energy functional $F[m] = \int d^D x\, f[m(x)]$. Here we have made an assumption that the free energy functional is essentially local, an assumption that is usually true if there are no long range interactions and we think of the microscopic boxes as being larger than the interaction length scale.

The mean field approximation corresponds to estimate this integral by a saddle approximation, (even though there is no small parameter) so that

$$\log Z = -F_{\min}, \quad \text{with} \quad F_{\min} = \min_{[m(x)]} F[m].$$

Notice that we have neglected all fluctuations around the "coarse grained" mean field $m(x)$ determined by a minimization problem.

• *Correlation functions*

To compute the correlation exponent requires to slightly generalize the construction of the previous section by taking into account inhomogeneity (in the external field or in the local magnetization). To lowest order in the derivative, the Landau-Ginzburg free energy reads

$$F[m;h] = \int d^D x \Big[\frac{\kappa(T)}{2} (\nabla m)^2(x) + g_2(T)\, m(x)^2 + g_4(T)\, m(x)^4 - h(x) m(x) + \cdots \Big],$$

up to the irrelevant constant term $g_0(T)$. The minimization is now with respect to $m(x)$. The minimizing condition is the differential equation

$$-\kappa(T)\, \Delta m(x) + 2 g_2(T)\, m(x) + 4 g_4(T)\, m(x)^3 = h(x).$$

Again we Taylor expand the coefficient around the critical temperature, so that

$$\kappa(T) = \kappa + \cdots, \quad g_2(T) = \frac{a}{2}(\delta T) + \cdots, \quad g_4 = \frac{g}{4!} + \cdots,$$

with $\delta T := T - T_c$, the gap to the critical point, and $a > 0$. Up to a redefinition of m, we can set $\kappa = 1$. The equation for the local magnetization then reads

$$-\Delta m(x) + a(\delta T)\, m(x) + \frac{g}{3!}\, m(x)^3 = h(x).$$

We look at the response function

$$G(x,y) := \frac{\partial m(x)}{\partial h(y)}|_{h=0},$$

which is linked to the correlation functions (coding for the fluctuations) through the fluctuation-dissipation theorem. It satisfies

$$\big(-\Delta_x + a(\delta T) + \frac{g}{2!}\, \bar{m}_0^2 \big)\, G(x,y) = \delta(x-y),$$

with m_0 the magnetization at zero external field. This is the equation for the Euclidean Green function with mass $M_{\text{eff}} = \sqrt{a(\delta T) + \frac{g}{3!}\, \bar{m}_0^2}$. This function decreases at large distances $G(r) \simeq e^{-M_{\text{eff}} r}$ with a correlation length $\xi = M_{\text{eff}}^{-1}$. See next Chapter. Recall that $m_0 = 0$ for $\delta(T) > 0$ and $\frac{g}{3!}\, \bar{m}_0^2 = -a(\delta T)$ for $\delta T < 0$. Hence,

$$\xi(T) = M_{\text{eff}}^{-1} \simeq \begin{cases} [a(T - T_c)]^{-1/2}, & \text{for } T > T_c, \\ [2a(T_c - T)]^{-1/2}, & \text{for } T < T_c. \end{cases}$$

This corresponds to the exponent $\nu = 1/2$.

At the critical point $T = T_c$, the mass M_{eff} vanishes and the correlation length ξ diverges. The Green function satisfies $-\Delta_x G(x-y) = \delta(x-y)$, so that

$$G(x)|_{T=T_c} \propto |x|^{2-D}.$$

This corresponds to the exponent $\eta = 0$.

Remark that speaking about correlation functions requires talking about the probability measure used to define and to compute them. Here we took a short cut arguing that we could use the fluctuation-dissipation theorem and the Landau-Ginzburg free energy to compute the response functions. Hence we implicitly assume that this free energy can be used as a generating functional for correlation functions (called an effective action). This requires arguing why and when the mean free approximation is exact, or not. This will be discussed below.

We can a postiori justify the truncation of the free energy expansion within the mean field approximation. The Landau-Ginzburg free energy describes the transition close the transition, i.e. for $\delta T = |T - T_c| \ll T_c$. Let us evaluate the typical order of magnitude of each term. We have seen that $m \sim (\delta T)^{1/2}$. By the minimization equation this means that $h \sim (\delta T)^{3/2}$ (more precisely that the previous description does not hold if the magnetic field is much bigger that this typical order of magnitude). We also evaluate the correlation length $\xi \sim (\delta T)^{-1/2}$. This

gives the typical order of the space variation the magnetization so that the spatial derivatives scale as $\nabla \sim \xi^{-1} \sim (\delta T)^{1/2}$. Hence, all the term is the LG free energy scale the same way, namely

$$(\nabla m)^2 \sim (\delta T)\, m^2 \sim m^4 \sim hm \sim \xi^{-4} \sim (\delta T)^2.$$

All the higher order terms, compatible with the $\mathbb{Z}_2$-symmetry, such as $(\nabla m)^4$, $(\nabla^2 m)^2$ or $m^2(\nabla m)^2$, m^6 etc. scale with higher power of (δT). They can be neglected close to the transition point. This scaling argument will be made more precise below and within the renormalization group formulation.

4.6 Upper and lower critical dimensions

This section is concerned with the domain of validity of the mean field Landau-Ginzburg description. The Landau model, as any mean field model, assumes that the fluctuations are not too important and do no play a relevant role. The upper and lower critical dimensions are the dimensional thresholds at which those fluctuations become relevant.

The fluctuations typically increase as the dimension is lowered. There are two notions, the upper and the lower critical dimension.
— If the dimension is small enough, i.e. if $D < D_i$ with D_i the lower critical dimension, the fluctuations are strong enough to destroyed any order at any temperature. That is: for $D < D_i$, there could not be any ordered phase: the fluctuations prevent them to exist.
— If the dimension is big enough, i.e. $D > D_s$, the fluctuations are small enough so that the mean field approximation is correct. That is, for $D > D_s$, the fluctuations are irrelevant and the Landau description is exact.

In the intermediate regime,

$$D_i < D < D_s,$$

the fluctuations are relevant, they do not destroy the possible existence of a phase transition. They have to be taken into account in the description of this phase transition (which is then quite non-trivial, see the following Chapters).

• *The upper critical dimension*

Let us start with the estimation of the upper critical dimension D_s. It depends on the universality class, so we shall do it for the Ising universality class (the ϕ^4 Landau theory). For the Ising class $D_s = 4$. We consider the Ising model at zero external field $h = 0$. Recall the expression for the free energy

$$F[m] = \int d^D x \Big(\frac{1}{2} (\nabla m)^2(x) + \frac{a}{2} (\delta T)\, m(x)^2 + \frac{g}{4!}\, m(x)^4 \Big),$$

with $a > 0$. The Landau theory will be correct if the typical fluctuations of the magnetization $\Delta m(x)$ are small compared to the typically magnetization m_0, i.e.

$$(\Delta m)^2 \ll m_0^2.$$

This is called the Ginzburg criteria.

Since spins are uncorrelated at distances bigger than the correlation length, the natural scale/size ℓ of such fluctuations is the correlation length: $\ell \sim \xi$. In the Landau mean field theory $\xi^2 \sim (\delta T)^{-1}$ and $m_0^2 \sim (\delta T)$. Let us estimate the free energy variation, $\Delta F = F[m_0 +$

$\Delta m] - F[m_0]$, associated to fluctuations of typical amplitude Δm. The first two terms in F scale the same ways (using that $\ell^2 \sim \xi^2 \sim (\delta T)^{-1}$) but the second scale differently with the size $\ell \sim \xi$:

$$\begin{aligned} \int d^D x \frac{1}{2} (\nabla m)^2(x) &\sim \xi^{D-2} (\Delta m)^2 \\ \int d^D x \, a(\delta T) \, m^2(x) &\sim \xi^D (\delta T) (\Delta m)^2 \sim \xi^{D-2} (\Delta m)^2, \\ \int d^D x \frac{g}{4!} \, m^4(x) &\sim \xi^D g (\Delta m)^4. \end{aligned}$$

According to the Boltzmann rules, the relative statistical weight of such fluctuations is of order $e^{-\Delta F}$. Thus the typical sizes of the maximum fluctuations are so such that ΔF is of order $O(1)$ —as, if it is much bigger it occurs with a much smaller probability.

For $D > 4$, the typical size is determined by the first terms, so that $(\Delta m)^2 \sim \xi^{2-D}$. Indeed, then $\xi^D (\Delta m)^4 \sim \xi^{4-D} \to 0$ for large ξ. Thus for $D > 4$, we have

$$(\Delta m)^2 \sim \xi^{2-D} \ll m_0^2 \sim \xi^{-2}.$$

Hence, the Landau mean field approximation is justified for $D > 4$.

For $D < 4$, the typical size is determined by the second term, so that $(\Delta m)^2 \sim \xi^{-D/2}$. Indeed, then $\xi^{D-2} (\Delta m)^2 \sim \xi^{(D-4)/2} \to 0$. Thus for $D < 4$, we have

$$(\Delta m)^2 \sim \xi^{-D/2} \gg m_0^2 \sim \xi^{-2}.$$

Hence, the Landau mean field approximation breaks down for $D < 4$. And the upper critical dimension is $D_s = 4$.

We can refine this argument a little. Consider still $D < 4$. The parameters of the model define a length scale ℓ_c by dimensional analysis via

$$\ell_c^{D-4} \simeq g.$$

Let us repeat the previous analysis but putting back the dependence on g via ℓ_c. This yields that the typical fluctuations are of size $(\Delta m)^2 \sim \xi^{-D/2} \, \ell_c^{(D-4)/2}$ for $D < 4$. Equivalently,

$$\frac{(\Delta m)^2}{m_0^2} \sim \big(\frac{\xi}{\ell_c}\big)^{(4-D)/2}.$$

Hence, $(\Delta m)^2 \ll m_0^2$ iff $\xi \ll \ell_c$. Since the correlation length has of course also to be much bigger than the microscopic length, we thus learn that for $D < 4$ the Landau theory is correct is

$$\xi_{\text{micro}} \ll \xi(T) \ll \ell_c.$$

Since the correlation length increases close to the critical point, $\xi(T) \sim (\delta T)^{-1}$, this is equivalent to saying that the Landau theory is correct if the distance to the critical temperature is big enough. In other words, the breakdown of the mean field approximation occurs very close the critical point (it only occurs as one approches the critical point).

Before closing this discussion recall that he upper critical dimension depends on the universality class, i.e. it depends on the symmetry, on the Landau expansion, etc.

• *The lower critical dimension*

Let us now discuss the lower critical dimension. A standard argument by Peierls tells that there is no ordered phase at positive temperature $T > 0$ in dimension 1. Hence the lower critical dimension is at minimum bigger than one: $D_i > 1$.

This is the lower critical dimension for universality class with discrete symmetries.

For continuous symmetry, ordered phase associated to symmetry breaking comes with massless Goldstone modes (see below). A theorem from Mermin and Wagner tells that this cannot occurs in dimension less than 2 (at least for a compact symmetry group). The argument is essentially based the IR behavior of massless Green function,

$$G_{M=0}(x) = \int \frac{d^D k}{(2\pi)^D} \frac{e^{ikx}}{k^2}.$$

It scaled like $\sim k^{D-2}$ for k small, and hence it is infrared divergent in $D < 2$. A way to regularized in $D = 2$ is to put the system in a finite volume of typical size L (see the more in-depth discussion in the chapter on free conformal field theory below), then at distances large enough

$$G(x) \simeq \log(|x|/L), \quad \text{if } D = 2,$$

The fluctuations are therefore large at large distance and the spins cannot be ordered (because if they were their two point function $G(x) = \langle S_0 \, S_x \rangle$ will asymptotically factorizes/clusters as $\langle S \rangle^2$). Thus the lower critical dimension is $D_i = 2$.

4.7 Deviation from mean field theory

Here we describe how going away from the mean field approximation leads to deal with large fluctuations encompassed in statistical field theory.

• *Naive scaling dimensions*

Let us rederive the previous results using scaling arguments –or let us make contact between this result and scaling arguments. This will be a tiny step towards reasoning used when formulation the renormalization group. We want to analyse how the theory changes when changing the size of the fluctuations. Let us write again the action as

$$F[m] = \int d^D x \Big(\frac{1}{2} (\nabla m)^2(x) + \frac{\tau}{2} \, m(x)^2 + \frac{g}{4!} \, m(x)^4 \Big).$$

Here, we set $\tau :\simeq (\delta T)$ that we view as an extra parameter similar to the coupling constant g. We want to compare this action for a profile $m(x)$ and a scaled profile $m_\lambda(x)$ in which we have scaled (scaled up if $\lambda \gg 1$) the distance (hence the size of the magnetization profile). Thus

$$m(x) \to m_\lambda(x) = \lambda^{-\Delta} \, m(x/\lambda),$$

where we used the freedom to scale the height of the profile. Δ is free parameter called the (bare) scaling dimension. We fix it by demanding that the first term in the free energy is preserved, hence

$$2\Delta = D - 2.$$

We now look how to transform the parameter g and τ as $g \to g_\lambda$ and $\tau \to \tau_\lambda$ in such way to preserve the free energy, i.e. such that

$$F[m_\lambda; g, \tau] = F[m; g_\lambda, \tau_\lambda].$$

This is may alternatively be interpreted as follows: the free energy (or the action) for configurations of size $\lambda\ell$ with parameter τ and g is identical as those of size ℓ but with parameter g_λ and τ_λ. This is a key equation, because it will generalize within the renormalisation group.

It is then a simple scaling exercise to check that

$$\tau_\lambda = \lambda^2 \, \tau, \quad g_\lambda = \lambda^{4-D} \, g.$$

Hence, the non-linear effects become more and more important at large scale (i.e. for large scale configuration) if $D < 4 = D_s$. That is: the non-linear quartic terms are relevant at large distances, at large scales, if $D < 4$.

Let us then determine the scaling dimensions of other operators, i.e. $m^{2k}(x)$ or $\nabla^{2p} m^{2k}(x)$. They for instance correspond to add terms of the form $g_k \int d^D x \, m^{2k}(x)$ in the action. Under the scaling transformation $m \to m_\lambda$, these couplings change (in order to preserve the key relation $F[m_\lambda; g_k] = F[m; g_k(\lambda)]$ as

$$g_k(\lambda) = \lambda^{D-k(D-2)} \, g_k.$$

All these couplings decrease as $\lambda \to \infty$, i.e. at large distances, for $D > 3$ (the coupling to m^6 is marginal in $D = 3$). Adding derivatives as in $\nabla^{2p} m^{2k}$ renders this decrease faster.

Thus, in $D > 3$ only the coupling to m^2 and m^4 are relevant. We can thus discard all other operators as long as we are only looking at the large distance physics.

Remark that this analysis shows that the dimension $D = 2$ is peculiar, in the sense that m has zero scaling dimension in $D = 2$. All polynomial terms scale the same way in $D = 2$ and this allow for a wide landscape of multi-critical points.

• *Beyond the Ising mean field theory*

We are going to use a (dual) reformulation of the Ising model in terms of (dual) variables close to the scalar field of the Landau theory. This reformulation goes beyond the Landau theory. We start with the Ising model on the square lattice with partition sum (we have absorbed the inverse temperature β into J and h)

$$Z = \sum_{\{s_k\}} e^{\sum_{ij} J_{ij} s_i s_j + \sum_i h_i s_i}.$$

We represent the spin-spin interaction as follows (this is called the Hubbard-Stratonovich transformation, it is simply based on Gaussian integral techniques):

$$e^{\sum_{ij} J_{ij} s_i s_j} = \text{const.} \int [\prod_k d\phi_k] \; e^{-\frac{1}{4} \sum_{ij} \phi_i J_{ij}^{-1} \phi_j + \sum_i \phi_i s_i}.$$

Here, the ϕ_k's are scalar variables at each site of the lattice. Once this identity is inserted into the partition sum we can explicitly do the sum over the spin configurations (at fixed ϕ) using

$$\sum_{[s]} e^{\sum_i (\phi_i + h_i) s_i} = \prod_i [2 \cosh(\phi_i + h_i)] = \text{const.} \; e^{\sum_i \log[\cosh(\phi_i + h_i)]}.$$

Hence, up to an irrelevant multiplicative constant (proportional to $[\det[J]|^{-1}$), the Ising partition sum reads

$$Z = \int [\prod_k d\phi_k]\ e^{-\frac{1}{4}\sum_{ij}\phi_i J^{-1}_{ij}\phi_j + \sum_i \log[\cosh(\phi_i + h_i)]},$$

Equivalently, we can change variables $\phi_i \to \sum_j J_{ij}\phi_j)$ (at the prize of aJacobian proportional to $[\det[J]|)$ so that the first quadratic term is transformed in $\frac{1}{4}\sum_{ij}\phi_i J_{ij}\phi_j$ and the second term into $\sum_i \log[\cosh(h_i + \sum_j J_{ij}\phi_j)$. The integral is convergent by construction (and the expansion we are going to do are legitimate if we do not spoil this convergence property). This Ising partition function can be thus written in the form

$$Z = \int [\prod_k d\phi_k]\ e^{-S[\phi;h]},$$

with the action

$$S[\phi;h] = -\frac{1}{4}\sum_{ij}\phi_i J_{ij}\phi_j + \sum_i \log[\cosh(h_i + \sum_j J_{ij}\phi_j)].$$

We have thus alternatively written the partition function in terms of scalar variables ϕ_k. This defines a discrete field theory, alias it defines a probability measure on configuration of the discrete field ϕ.

Let us look at what happens at $h_i = 0$. Expanding $\log[\cosh(x)] = \frac{x^2}{2} - \frac{x^4}{12} + \cdots$, the action for ϕ is then

$$S[\phi] = \frac{1}{4}\sum_{ij}\phi_i J_{ij}\phi_j - \frac{1}{2}\sum_{ijk}\phi_j J_{ij} J_{ik}\phi_k + \cdots = \frac{1}{4}\langle\phi, (J - 2J^2)\phi\rangle + \cdots$$

The dots correspond to higher order terms which would induce interactions between the various components of the field ϕ. Remember that the coup;ing constants J_{ij} couple only neighbor sites on the lattice (recall that the terms $\sum_{ij} J_{ij}s_i s_j$ in the energy), i.e. for instance for a square lattice $\hat{J}_{ij} = \frac{J_0}{2}(\delta_{i,j-1} + \delta_{i,j+1})$ with the convention that adding one to the site index amounts to move by one step on the lattice. We have $J_{ij} = \frac{J_0}{2}(\Delta_{ij} + z\mathbb{I}_{ij})$ with Δ the Laplacian on the lattice (say square lattice), $\mathbb{I}$ the identity matrix and the number z related to the number of neighbors. The action for the field ϕ can then be written as

$$S[\phi] = \text{const.}\,\langle\phi, (-\Delta + M^2)\phi\rangle + \text{higher derivatives} \times \text{higher order terms},$$

with $M^2 \propto (1 - J_0 z) \propto (T - T_c)$. We recover the quadratic part of the Landau-Ginzburg action. To be exact the action should include many more terms (actually, an infinite number of terms) coming form the expansion $\log[\cosh(h + J\cdot\phi)]$, or equivalently $\log[\cosh(h + \frac{J_0}{2}(\Delta + z)\cdot\phi)]$. However, as we have discussed above, scaling arguments tell us that only the ϕ^4 term matters, if we are aiming at describing the long distance physics. We are thus back the action

$$S[\phi] \propto \int d^D x \left[\frac{1}{2}\phi(-\Delta + M^2)\phi + \frac{g}{4!}\phi^4\right].$$

The difference with the Landau-Ginzburg mean field theory is that we are now know that we have to take into account fluctuations of the field ϕ, at least for $D < 4$, with probability measure

$$\int [D\phi]\ e^{-S[\phi]}.$$

Here the integration is over all configurations of the field ϕ. Field theory aims at making sense of such measure.

4.8 Symmetry breaking and Goldstone modes

Here we discuss why and how continuous symmetry breaking is associated with the emergence of massless modes, called Goldstone bosons.

• *$U(1)$ symmetry breaking*

Do first the standard example of $U(1)$ broken symmetry with Mexican hat potential....
[....To be completed...]

• *Goldstone modes and coset spaces.*

We present here the semi-classical (alias mean field like) argument but for a general symmetry group G broken to H. We consider situations in which the order parameter is a multi-component field that we shall denote Φ again - we denote Φ^a the field component - with action

$$S[\Phi] = \int d^D x \left[\frac{1}{2}|\nabla\Phi|^2 + V(\Phi)\right].$$

The potential V depends on all components of the field (so it is a multi-variable function). We assume that the field Φ takes values in a representation of the group G so that the group acts linearly on it: $\Phi \to g\cdot\Phi$ for $g \in G$. The action, both the kinetic and the potential terms, are supposed to be invariant under this action. In particular

$$V(g\cdot\Phi) = V(\Phi).$$

In absence of external potential, mean field solutions are minima of the potential V. Let us pick one of this minima that we shall denote φ_0: $\nabla V(\varphi_0) = 0$ or in components $\partial_a V(\varphi_0) = 0$. Let us then suppose that this so-called vacuum solution is non trivial $\varphi_0 \neq 0$ so that the symmetry G is broken, $g\cdot\varphi_0 \neq 0$ for at least some $g \in G$, and let $H \subset G$ be the subgroup that preserves φ_0 :

$$h\cdot\varphi_0 = \varphi_0, \quad h \in H \subset G.$$

H is supposed to be the maximal subgroup preserving φ_0. We aim at looking at the fluctuations around this solution. We set $\Phi(x) = \varphi_0 + \phi(x)$ and expand the action (to quadratic order):

$$S[\varphi_0+\phi] = \int d^D x \left[\frac{1}{2}\sum_a |\nabla\phi^a|^2 + \frac{1}{2}\sum_{a,b}\phi^a H_{ab}\phi^b + \cdots\right], \quad H_{ab} := \partial_a\partial_b V(\varphi_0).$$

The Hessian matrix is thus the mass matrix of the fluctuating modes ϕ. The Goldstone theorem asserts that these fluctuating modes contain massless modes, that is: the mass matrix H_{ab} possesses a number of zero eigen-values (and the eigen-vectors are the massless modes).

This properties comes from the G-invariance of the potential (with G a continuous group). This invariance reads $V(g\cdot\Phi) = V(\Phi)$. Let us apply this relation to the vacuum solution, hence $V(g\cdot\varphi_0) = V(\varphi_0)$ for $g \in G$. Take g infinitesimal $g = 1 + \varepsilon X + \cdots$, with $X \in \mathrm{Lie}(G)$ and $\varepsilon \ll 1$. Expanding the relation to second order in ε, using the fact that φ_0 is minimum, yields

$$\sum_{a,b} H_{ab}\,(X\varphi_0)^a\,(X\varphi_0)^b = 0, \quad \forall X \in \mathrm{Lie}(G).$$

However, $X\varphi_0 = 0$ for $X \in \mathrm{Lie}(H) \subset \mathrm{Lie}(G)$, because φ_0 is preserved by the subgroup H. Let us decompose $\mathrm{Lie}(G)$, as a vector space as,

$$\mathrm{Lie}(G) = \mathrm{Lie}(H) \oplus T_{G/H},$$

where $T_{G/H}$ is the complement vector space of Lie(H) in Lie(G). (It is isomorphic to the tangent space to the quotient G/H at the identity point and form a representation of H). For any $X \in T_{G/H}$, $X\varphi_0 \neq 0$ because H is supposed to be the maximal subgroup preserving φ_0.

Hence, any element of $T_{G/H}$ is associated to a massless mode. In other words, the massless zero modes of the G-invariant action broken to $H \subset G$ are in correspondance with the quotient G/H. If no other relevant physics play a role at intermediate scale, this implies that the large distance phenomena are governed by this G/H-zero modes.

4.9 Exercises

• *Exercise 4.9.1: Mean field from a variational ansatz*

The aim of this exercise is to derive the Ising mean field approximation form a variational ansatz. We consider the Ising in homogeneous external field h_i so that the configuration energy is $E[s] = -\sum_{i,j} J_{ij} s_i s_j - \sum_i h_i s_i$, with J_{ij} proportional to the lattice adjacency matrix. The Ising spins take values $s_i = \pm$. Let $Z[h]$ be its partition function. (Note that we introduce the external magnetic field with a minus sign).

As an ansatz we consider the model of independent spins in an effective inhomogeneous external field h_i^o with ansatz energy $E^o[s] = -\sum_i h_i^o s_i$, so that the ansatz Boltzmann weights are $Z_0^{-1} e^{\beta \sum_i h_i^o s_i}$ with Z_0 the ansatz partition function.
(i) Show that $Z_0 = \prod_i [2\cosh(\beta h_i^o)]$.
(ii) Using a convexity argument, show that $\mathbb{E}_0[e^{-X}] \geq e^{-\mathbb{E}_0[X]}$ for any probability measure $\mathbb{E}_0$ and measurable variable X.
(iii) Choose to be $\mathbb{E}_0$ the ansatz measure and $X = \beta(E[s] - E^o[s])$ to prove that

$$Z[h] \geq Z_0 \, e^{-\beta \mathbb{E}_0[E[s] - E^o[s]]},$$

or equivalently, $F[h] \leq F_0 - \mathbb{E}_0[E^o[s] - E[s]]$, with $F[h]$ and F_0 the Ising and ansatz free energy respectively.

The best variational ansatz is that which minimizes $F_0 - \mathbb{E}_0[E^o[s] - E[s]]$.
(iv) Compute F_0, $\mathbb{E}_0[E^o[s]]$ and $\mathbb{E}_0[E[s]]$ and show that the quantity to minimize is

$$F_0[h^o] + \sum_i h_i^o \bar{m}_i - \sum_{ij} J_{ij} \bar{m}_i \bar{m}_j - \sum_i h_i \bar{m}_i,$$

where $\bar{m}_i = -\frac{\partial F_0[h^o]}{\partial h_i^o} = \tanh(\beta h_i^o)$ is the local mean magnetization evaluated with the ansatz measure. Show that this minimization problem reduces to the Ising mean field equations.

• *Exercise 4.9.2: Thermodynamic functions and thermodynamic potentials*

The aim of this exercise is to recall a few basic fact about generating functions, thermodynamic functions and their Legendre transforms.

Let us consider a (generic) spin model and let $E[\{s\}]$ be the energy of a spin configuration $\{s\}$ with local spin s_i. We measure the energy in unit of the temperature so that the Boltzmann weights are $e^{-\beta E[\{s\}]}$. Let $Z[0] = \sum_{\{s\}} e^{-\beta E[\{s\}]}$ be the partition function.
(i) Explain why the partition function $Z[h]$ in presence an external inhomogeneous external field h is the generating function for spin correlations.
What is the expression for $Z[h]$?
Show that the generating function for this spin correlation functions can written as (with $(s,h) = \sum_i s_i h_i$)

$$\mathbb{E}[e^{(s,h)}] = \frac{Z[h]}{Z[0]},$$

(ii) Let $F[h]$ be the free energy and let $W[h] = -\big(F[h] - F[0]\big)$. Verify that

$$\log \mathbb{E}[e^{(s,h)}] = W[h].$$

(iii) Let $\Gamma(m)$ be the thermodynamic potential defined as the Legendre transform of $W[h]$. Recall that

$$\Gamma(m) = (m, h_*) - W[h_*], \quad \text{with } \frac{\partial W}{\partial h}[h_*] = m.$$

Verify that this transformation is inverted by writing

$$W[h] = (m_*, h) - \Gamma[m_*], \quad \text{with } \frac{\partial \Gamma}{\partial m}[m_*] = h.$$

• *Exercise 4.9.3: An alternative representation of the Ising partition function.*

The aim of this exercise is to explicitly do the computation leading to the representation of the Ising partition function in terms of a bosonic field. It uses a trick —representing the interaction terms via a Gaussian integral over auxiliary varaibles— which find echoes in many other problems.
(i) Prove the following representation of the Ising partition function given in the text (without looking at its derivation given there):

$$Z = \int [\prod_k d\phi_k] \; e^{-S[\phi;h]},$$

with the action

$$S[\phi; h] = -\frac{1}{4} \sum_{ij} \phi_i J_{ij} \phi_j + \sum_i \log[\cosh(h_i + \sum_j J_{ij} \phi_j)].$$

(ii) Deduce what is the representation of the Ising spin variables s_i in terms of the bosonic variables ϕ_i.

• *Exercise 4.9.4: Mean field vector models*

See the exercise booklet....

5 Statistical field theory: free theory

The aim of this chapter is to present the basic concepts of free statistical field theories. These are Gaussian theories. Basics tools, including the Wick's theorem, its graphical representation, are introduced. The connection between free statistical field theories and free quantum theories via analytic continuation is outlined.

5.1 Classical field theory: basics

Before discussing statistical field theories, let us recall a few basic facts about classical field theory. We shall only deal with scalar fields, that we generically denote ϕ. See the reference books for more details. These fields are maps from a base –or world-sheet– manifold to some target manifold. Unless otherwise specified, and to simplify matter, we take the base manifold to be the flat Euclidean space $\mathbb{R}^D$ and the target to be $\mathbb{R}^N$. That is: ϕ is a N-component map $x \in \mathbb{R}^D \to \phi(x)$ with each component ϕ^a taking real values. To each of those maps is associated a (classical) action $S[\phi]$ which is assumed to tale the form

$$S[\phi] = \int d^D x \, \mathcal{L}[\phi; \partial\phi],$$

with $\mathcal{L}[\phi; \partial\phi]$ the so-called Lagrangian density. Maps extremalizing this action are said to be solution of the classical equations of motion. These equations are the Euler-Lagrange equations which reads (for all a)

$$\partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)(x)} \Big) = \frac{\partial \mathcal{L}}{\partial \phi^a(x)}.$$

For a scalar field theory, the typical form of the action is

$$S[\phi] = \int d^D x \, \big(\frac{1}{2} (\nabla\phi)^2(x) + V(\phi(x)) \big),$$

with the first kinetic term representing curvature effects and the second term local potential effect. In this case the equations of motion are simply $-\Delta\phi + V'(\phi) = 0$ with Δ the D-dimensional Euclidean Laplacian.

Let us now recall the Noether theorem. It codes for the consequences of symmetries. Let us suppose that a classical action $S[\phi]$ is invariant under some continuous symmetries generated by the field transformations $\phi(x) \to \phi(x) + \epsilon(\delta\phi)(x) + \cdots$, for some variation $\delta\phi$, with $\epsilon \ll 1$ a small formal parameter. If the Lagrangian density is invariant under such transformation –that is $\delta\mathcal{L}[\phi; \partial\phi] = 0$–, Noether's theorem says that the following current, called the Noether current,

$$J^\mu = (\delta\phi)(x) \, \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)(x)},$$

is conserved, $\partial_\mu J^\mu = 0$, on solutions of the classical equations of motion. This conservation law can be checked directly using the Lagrange equations of motion and the invariance the Lagrangian density.

The Noether current can also be identified by looking at infinitesimal but inhomogeneous transformations $\phi(x) \to \phi(x) + \epsilon(x)(\delta\phi)(x) + \cdots$, with a position dependent small parameter

$\epsilon(x)$. If the Lagrangian density is invariant, i.e. $\delta\mathcal{L}[\phi;\partial\phi] = 0$ for constant ϵ, the variation of the action for a non constant ϵ is going to be proportional to its derivative. One can check that

$$\delta S[\phi] = \int d^D x\, \partial_\mu \epsilon(x)\, J^\mu(x),$$

with J^μ the above Noether current. Notice that the fact that the action is extremal on solutions of the equations of motion then implies that J^μ is conserved on those solutions.

This can be slightly generalized to cases in which the Lagrangian density is not invariant but its variation is a total derivative so that the action is invariant. These cases englobe translation invariance and the associated Noether current is then related to the so-called stress-tensor. See the exercise Section.

The consequences of symmetries and Noether currents in statistical field theory will be discussed in the following Chapter.

5.2 Euclidean free field theories

We aim at describing statistical sums whose configurations are the possible values of fields of the form

$$Z := \int [D\phi]\, e^{-S[\phi]},$$

where ϕ is a field $x \to \phi(x)$ defined over a base space of dimension D and taking value in some target space (say $\mathbb{R}^N$ for simplicity). If we view this integral sum as arising from a discrete lattice statistical model, the continuous description applies if the typical correlation length ξ is much bigger than the lattice space a, i.e. $\xi \gg a$.

The above statistical sum formally defines a measure on the random field configurations. The least we can ask for giving a meaning to this measure is to specify what are the correlation functions of the field at different points, that is

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle := \frac{1}{Z} \int [D\phi]\, e^{-S[\phi]}\, \phi(x_1) \cdots \phi(x_n).$$

When dealing with field theory, we shall adopt the "standard" convention to denote expectation vaues by $\langle \cdots \rangle$ instead of $\mathbb{E}[\cdots]$. If we were more precise mathematically, we would have to define more than just these n-points functions. This is actually beyond the present understanding of generic field theory (except in some special cases). Nevertheless, n-point functions of local fields is all what is needed as long as we don't look at non-local properties (say property related to extended objects or structures of field configurations).

Recall that or a scalar field theory, the typical form of the action is $S[\phi] = \int d^D x \left(\frac{1}{2}(\nabla\phi)^2 + V(\phi)\right)$, for some potential V.

We may imagine that the fields code for the shape of some kind of landscape or membrane, say imbedded in some higher dimensional manifold and parametrized whose coordinate(s) are the field components ϕ. We are then describing fluctuating shapes, a notion which may be view as part of what random geometry can be. We may also want to describe magnetic material, the coordinate will then be parametrizing the long wave length behavior of the local magnetization. There are of course many other possibilities. The field can be a scalar, a vector, etc., with multi-component, and/or take values in manifold with/without internal structures depending on the physical problem. The form of the potential of course also depend on the physical setup,

say on relevant symmetries. For instance, if we aim at describing the long distance physics of Goldstone bosons associated to the symmetry breaking of a continuous group G down to a subgroup H, the Goldstone field take values in the coset space G/H. These are then non free field theory.

• *Free field theory in continuous space.*

In this chapter we restrict ourselves to Gaussian scalar free field theory (we shall deal with interacting theory in the following Chapters). They correspond to quadratic potentials. One sets $V(\phi) = \frac{1}{2}m^2\phi^2$, so that the action becomes

$$S[\phi] = \int d^D x \, \big(\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\big). \tag{34}$$

The classical equation of motion is linear. It is the well-known Klein-Gordon equation: $-\Delta\phi + m^2\phi = 0$, so that m is identified with (or more precisely called) the mass of the field. As discussed in previous chapters, higher terms in the potential may be relevant (not all polynomial are relevant but only the first few depending on the dimension) which mean that they potentially modify the long distance behavior of the field configurations. These terms induce self-interactions, we shall analyse in the following chapter.

By integration by part (with appropriate boundary condition), the action can be written as

$$S[\phi] = \frac{1}{2}\int d^D x \, \phi(x)\big(-\Delta_x + m^2\big)\phi(x),$$

with $\Delta = \nabla^2$ the D-dimensional Laplacian. It is quadratic in the field ϕ. If we view the field ϕ as an infinite dimensional vector with component $\phi(x)$, the action is a bilinear form with matrix the positive operator $(-\Delta + m^2)$. Of primordial importance is the Green function $G(x, x')$, the inverse of this operator, defined by the differential equation

$$(-\Delta_x + m^2)G(x, x') = \delta(x, x'),$$

with $\delta(x, x')$ the Dirac distribution (viewed as the identity operator acting on the space of functions).

Since, the total action is quadratic, the field is going to be Gaussian (hence simple to describe) with covariance the Green function. That is:

$$\langle\phi(x)\rangle = 0, \quad \langle\phi(x)\phi(x')\rangle = G(x, x').$$

More will be described in a little while after a digression on Gaussian theories.

• *Lattice free field theory.*

Field theories emerging from statistical lattice models may also be defined on lattices. Let us describe their free field version. For simplicity we consider theories on a square lattice $a\mathbb{Z}^D$, with a the lattice mesh (lattice spacing). Let $\mathbf{e}_\alpha$, with $\alpha = 1, \cdots, D$, be unit vector basis in $\mathbb{R}^D$. Let $\mathbf{x} = a\mathbf{n}$ be points of the lattice with $\mathbf{n}$ integer (i.e. point on $\mathbb{Z}^D$) and $\phi_\mathbf{x}$ be the field at that point. Notice that here $\mathbf{x}$ is the dimensional position, not the dimensionless integer $\mathbf{n}$, as otherwise we would use the notation $\phi_\mathbf{n}$. The lattice action simply reads

$$S[\phi] = \frac{a^D}{2}\sum_\mathbf{x} \phi_\mathbf{x}\,[(-\Delta^{\rm dis} + m^2)\phi]_\mathbf{x},$$

with $\Delta^{\rm dis}$ the discrete analogue of the Laplacian defined through finite difference by:

$$[(-\Delta^{\rm dis}+m^2)\phi]_{\mathbf{x}} = a^{-2}\sum_{\alpha}\big(-\phi_{\mathbf{x}+a\mathbf{e}_\alpha}+(2+(am)^2)\phi_{\mathbf{x}}-\phi_{\mathbf{x}-a\mathbf{e}_\alpha}\big).$$

Clearly the discrete Laplacian is an infinite dimensional matrix $[-\Delta^{\rm dis}]_{\mathbf{x},\mathbf{x}'}$, indexed by points on the lattice, which is symmetric with $+2$ on the diagonal and -1 on the first line off the diagonal. The lattice action can alternatively be written as

$$S[\phi] = \frac{a^D}{2}\sum_{\mathbf{x},\mathbf{x}'}\phi_{\mathbf{x}}\,[(-\Delta^{\rm dis}+m^2)]_{\mathbf{x},\mathbf{x}'}\,\phi_{\mathbf{x}'},$$

from which it is clear that the theory is Gaussian.

This action is diagonalized via Fourier transformed.Let

$$\hat\phi_{\mathbf{k}} = \sum_{\mathbf{x}} e^{-i\mathbf{x}\cdot\mathbf{k}}\,\phi_{\mathbf{x}}, \quad \phi_{\mathbf{x}} = \int_{\rm BZ}\frac{d^D\mathbf{k}}{(2\pi/a)^D}\,e^{i\mathbf{x}\cdot\mathbf{k}}\,\hat\phi_{\mathbf{k}}$$

where the integration is over the Brillouin zone. For a D-dimensional square lattice of mesh size a, i.e. $a\mathbb{Z}^D$, the Brillouin zone is the hyper-cube $\mathrm{BZ}\equiv[-\frac{\pi}{a},\frac{\pi}{a}]^D$. The Laplacian is of course diagonal in the Fourier basis. The action becomes

$$S[\phi] = \frac{1}{2}\int_{\rm BZ}\frac{d^D\mathbf{k}}{(2\pi/a)^D}\,\hat\phi_{-\mathbf{k}}(-\Delta^{\rm dis}+m^2)_{\mathbf{k}}\,\hat\phi_{\mathbf{k}}.$$

In Fourier space, free field theory is thus a collection of i.i.d. Gaussian variables, indexed by the momentum $\mathbf{k}$, with mean and covariance determined by the discrete Laplacian. Notice however, that there is no zero mode if $m^2\neq 0$ but one in the massless case, and that if the model was defined on the periodic torus the integration on the momentum would be replaced by a discrete sum. See the exercise Section for further details.

5.3 Gaussian field theories

Here we summarize basic facts about Gaussian models. They all share the same structure, only the explicit expressions for their two-point functions —their covariance— differ from model to model.

• *Discrete Gaussian models.*

Let Λ be a lattice, or a domain on a lattice with a finite number of points (we may have then to take the infinite volume limit to define the thermodynamic or scaling limit). To any point $j\in\Lambda$ we associate a (scalar) variable ϕ_j, which is the value of the field at point j. We assume that this field is Gaussian (with zero mean), so that the measure is

$$\frac{1}{Z}\,[\prod_{k\in\Lambda}d\phi_k]\,e^{-S[\phi]},$$

with

$$S[\phi] = \frac{1}{2}\sum_{j,k}\phi_j M_{jk}\phi_k =: \frac{1}{2}(\phi, M\phi), \tag{35}$$

with M a real positive definite symmetric matrix, $M_{jk} = M_{kj}$. The factor Z in the above formula, which is identified as the partition sum, ensures the proper normalization of the measure –its total sum is one. The set of ϕ_k's form a (big) vector ϕ whose component are those ϕ_k's. This is a random vector, a Gaussian vector.

We need to collect basic information on multi-dimensional Gaussian variables.
— The first is about the partition sum. We have (by diagonalization and from the 1D result)

$$Z = \int [D\phi]\, e^{-\frac{1}{2}(\phi, M\phi)} = [\det\, M]^{-\frac{1}{2}}, \quad [D\phi] = \prod_{k\in\Lambda} \frac{d\phi_k}{\sqrt{2\pi}}.$$

— The second is about the two point function. We have (by change of variable and contour deformation)

$$\mathbb{E}[\phi_j\phi_k] \equiv_{\text{notation}} \langle \phi_j\phi_k\rangle = [M^{-1}]_{jk} =: G_{jk},$$

with $G = M^{-1}$ the inverse of the matrix M. The one point function vanishes: $\langle\phi_k\rangle = 0$.
— The third is about generating function (coding for higher moments and cumulants). Let u be a dual vector (i.e. a vector in the dual space) with component u^j so that $\langle u, \phi\rangle = \sum_j u^j\phi_j$. Then,

$$\mathbb{E}[e^{-(u,\phi)}] \equiv_{\text{notation}} \langle e^{-(u,\phi)}\rangle = e^{\frac{1}{2}(u,M^{-1}u)}. \tag{36}$$

The proof is done by change of variables and contour deformations; convergence of the integrals has to be discussed but there is no problem if M is positive definite. In order to avoid any convergence problem (in case one does not have a good explicit control on the measure) one usually defines the generating function with a purely imaginary vector, that is through the formula $\langle e^{i(u,\phi)}\rangle$, and look for its analytic continuation. See the Exercises section in Chapter 2.

Remark that this last expectation, as any correlation, can be viewed as the ratio of the two partition functions, one with the external field, the other without

$$\langle e^{-(u,\phi)}\rangle = \frac{Z[u]}{Z[0]},$$

with $Z[u]$ the partition with u-dependent modified action

$$Z = \int [D\phi]\, e^{-S[\phi;u]}, \quad S[\phi;u] = S[\phi] + (u,\phi).$$

The dual vector u plays the role of a source for the field ϕ.

• *Gaussian generating functions.*

The function $Z[u]$ is a generating function for multi-point correlation functions in the sense that its Taylor expansion near $u = 0$ yields the multi-point expectations:

$$\begin{aligned}
\langle\phi_k\rangle &= -\frac{\partial}{\partial u^k}\langle e^{-(u,\phi)}\rangle\big|_{u=0} = -\frac{\partial}{\partial u^k}\frac{Z[u]}{Z[0]}\Big|_{u=0},\\
\langle\phi_j\phi_k\rangle &= \frac{\partial^2}{\partial u^j\partial u^k}\langle e^{-(u,\phi)}\rangle\big|_{u=0} = \frac{\partial^2}{\partial u^k\partial u^j}\frac{Z[u]}{Z[0]}\Big|_{u=0},\\
\langle\phi_j\phi_k\phi_l\rangle &= -\frac{\partial^3}{\partial u^j\partial u^k\partial u^l}\langle e^{-(u,\phi)}\rangle|_{u=0} = -\frac{\partial^3}{\partial u^j\partial u^k\partial u^l}\frac{Z[u]}{Z[0]}\Big|_{u=0}, \quad \text{etc.}
\end{aligned}$$

It is actually useful to deal with the generating function of the simpler correlation functions, called connected correlation functions. Let $W[u]$ defined by

$$e^{-W[u]} := \langle e^{-(u,\phi)} \rangle = \frac{Z[u]}{Z[0]},$$

or alternatively

$$W[u] = -\log(Z[u]/Z[0]).$$

By construction, it is the generating function of connected correlation functions:

$$\begin{aligned} \langle \phi_k \rangle^c &:= \langle \phi_k \rangle^c = -\partial_{u^k} W[u]\big|_{u=0}, \\ \langle \phi_j \phi_k \rangle^c &:= \langle \phi_j \phi_k \rangle - \langle \phi_j \rangle \langle \phi_k \rangle = \partial_{u^j} \partial_{u^k} W[u]\big|_{u=0}, \quad \text{etc.} \end{aligned}$$

It is sometimes alternatively called the 'effective action'. The origin of the name 'connected' comes form the diagrammatic representation of the correlation functions in terms of Feynman diagrams that we shall discuss in the following chapter. They correspond to connected diagrams, those which can be split in parts without breaking one of the lines of the diagrams.

• *Wick's theorem and graph representations.*

For Gaussian theory, $W[u]$ is quadratic (cf above)

$$W[u] = \frac{1}{2}(u, M^{-1}u) = \frac{1}{2}(u, Gu).$$

Thus, the two point function is the only non vanishing connected correlation function.

All the odd correlation functions vanish and for the even correlation functions – not necessarily connected– can be written in terms of the two-point functions. Since $Z[u]/Z[0] = e^{-W[u]}$ is the generating function of the correlation function, one has,

$$\begin{aligned} \langle \phi_{j_1} \phi_{j_2} \rangle &= G(j_1, j_2), \\ \langle \phi_{j_1} \phi_{j_2} \phi_{j_3} \phi_{j_4} \rangle &= G(j_1, j_2)G(j_3, j_4) + G(j_1, j_3)G(j_2, j_4) + G(j_1, j_4)G(j_2, j_3), \quad \text{etc.} \end{aligned}$$

To simplify the notation we write $G(j, k)$ instead of G_{jk}, with $G = M^{-1}$.

The way to compute the multi-point correlation functions in terms of the two-point function for Gaussian model is called the "Wick's theorem". It allows to compute explicitly, and recursively, all correlation functions. The general formula is

$$\langle \phi_{j_1} \phi_{j_2} \cdots \phi_{j_{2n}} \rangle = \sum_{\text{pairings}} G(j_{\sigma_1}, j_{\sigma_2}) \cdots G(j_{\sigma_{2n-1}}, j_{\sigma_{2n}}) \tag{37}$$

How many pairings do we have? To enumerate them, we pick (i) the number 1 and associate to it another number, this yields $(2n-1)$ possibilities, (ii) then we pick the next smallest number and we associated to it another number left, this yields $(2n-3)$ possibilities, and so on. Altogether there are $(2n-1)(2n-3)\cdots 3 \cdots 1$ pairings, or else $\frac{(2n)!}{2^n n!}$ pairings. For $2n = 2$ this gives 1, for $2n = 4$ this gives 3, for $2n = 6$ this gives 15, etc. This number grows faster than exponentially: the number of pairing of $2n$ fields/points is $\sqrt{2}(2n/e)^n$ asymptotically (thanks to the Stirling formula $n! \sim \sqrt{2\pi n}(n/e)^n$).

One uses simple graphical representation expressing the relation between the multi-point and 2-point functions. The multi-point functions are sum of monomials made of product of the 2-point function $G(j,k)$, say $G(j_{\sigma_1}, j_{\sigma_2}) \cdots G(j_{\sigma_{2n-1}}, j_{\sigma_{2n}})$. Each graph represent of those monomial. The vertices of the graph label the field components and there is one line connecting those vertices for each insertion of the 2-point function. The construction of the graph is better explained by a few illustrative drawing than by words:

$$[...Give\ examples\ for$$
$$the\ 2\ and\ 4\ point\ and\ 6\ functions...]$$

Feynman diagrams are made of those graphs.

As it is clear form the construction a Gaussian field is specified by its two-point function. Thus, every statement made above apply to the Gaussian free field with the discrete index j replaced by the continuous position x of the fields —or its momentum k if we work in Fourier space— and with the Green function $G(x,x')$ as two-point function.

• *Back the Gaussian free field theories.*

All previous remarks apply directly to Gaussian field theories. For the lattice theories, the mapping is $\{\phi_j\} \to \{\phi_{\mathbf{x}}\}$ with $\mathbf{x} \in \Lambda$ in position space, or $\{\phi_j\} \to \{\phi_{\mathbf{k}}\}$ with $\mathbf{k} \in \mathrm{BZ}$ in Fourier space, with the covariance matrix G_{jk} mapped to the Green function

$$G(\mathbf{k},\mathbf{p}) = \frac{1}{(-\Delta^{\mathrm{dis.}} + m^2)_{\mathbf{k}}} (2\pi/a)^D \, \delta(\mathbf{k}+\mathbf{p})$$

in Fourier space.

For continuous theories defined over $\mathbb{R}^D$ the mapping is $\{\phi_j\} \to \{\phi(x)\}$ with $x \in \mathbb{R}^D$, with the covariance matrix G_{jk} mapped to the Green function $G(x,x')$ which is the kernel of the massive Laplacian, so that

$$\langle \phi(x)\phi(x') \rangle = [(-\Delta + m^2)]^{-1}(x,x') =: G(x,x').$$

All higher order correlation functions are computed using Wick's theorem.

Of course, continuous field theories over $\mathbb{R}^D$ can also be formulated in Fourier space. See below for explicit expression for the Green function in Fourier space.

Gaussian field theory can also be formulated on curve spaces (one then have to deal with the Green function of the Laplacian defined using the curved metric) or on finite domain (one then have to specify the field boundary conditions). See the exercise Section for examples.

• *Another definition of Gaussian free fields.*

One can use the previous remarks to present a variant definition (possibly more rigorous...) of Gaussian free fields via its generating function against test function. Under suitable conditions, the distribution of a random variable is specified by its characteristic function —one is the Fourier transform of the other. In particular, given a Hilbert space V with norm $||f||^2$ for $f \in V$, on can defined a random Gaussian variable ϕ in V by defining its generating function as

$$\mathbb{E}[e^{i(f,\phi)}] = e^{-\frac{1}{2}||f||^2}.$$

We can apply this to the Gaussian free field: we choose the space of function with compact support for V equipped with the norm

$$||f||^2 = \int d^D x d^D x' f(x)\, G(x,x')\, f(x'),$$

with $G(x,x')$ the Green-function. The integral is convergent (except in the massless $D=2$ case in infinite volume in which an extra finite volume regularization is needed).

The random variables are values of the form $f \to (f,\phi)$, so that the field ϕ is identified with a distribution. The duality is implemented as usual by testing the distribution against f,

$$(f,\phi) = \int d^D x\, f(x)\phi(x).$$

As generic distribution, the value of the distribution at a point does not exist (although we use the notation $\phi(x)$, as usual in physics literature). This is reflected that the 2-point function of the field at coinciding points is infinite

$$\langle \phi^2(x) \rangle = G(x,x) \to \infty.$$

But the pairings (f,ϕ) makes mathematical sense.

More mathematically rigorous definitions of bosonic free field start from this type of constructions.

5.4 Green functions

Here we collect information on Green functions and propagators.

• *Green functions in the continuum.*

The Green function is the kernel of the inverse of the operator $(-\Delta + m^2)$. Thus it is the kernel $G(x,x')$ solution of the differential equation

$$(-\Delta_x + m^2) G(x,x') = \delta(x,x').$$

In Fourier space, this reads

$$G(x,x') = G(x-x') = \int \frac{d^D k}{(2\pi)^D}\, e^{ik\cdot(x-x')}\, \hat{G}(k),$$

with (recall that $-\Delta$ acts by multiplication by k^2 in Fourier space)

$$\hat{G}(k) = \frac{1}{k^2+m^2}. \tag{38}$$

Of course G is rotation invariant. Let $r^2 = x^2$. In radial coordinates we have (here k is the norm of the momentum)

$$G(r) = \frac{S_{D-1}}{(2\pi)^{D-1}} \int \frac{dk}{2\pi}\, \frac{k^{D-1}}{k^2+m^2} \int_0^\pi d\theta\, (\sin\theta)^{D-2}\, e^{ikr\cos\theta}.$$

with $S_D = 2\pi^{D/2}/\Gamma(D/2)$ the volume of the unit sphere in $\mathbb{R}^D$. We have

$$\begin{aligned}\int d\theta\,(\sin\theta)^{2a}\,e^{ikr\cos\theta} &= \Gamma(a+\frac{1}{2})\Gamma(\frac{1}{2})\,(\frac{kr}{2})^{-a}\,J_a(kr)\\ \int dk\,k^{a+1}\frac{J_a(kr)}{k^2+m^2} &= m^a\,K_a(mr)\end{aligned}$$

Here $K_a(\cdot)$ is the modified Bessel function. This gives the formula

$$G(x) = (2\pi)^{-D/2}\,(\frac{m}{r})^{D-2}\,K_{\frac{D-2}{2}}(mr). \tag{39}$$

It is better and simpler to do the computation explicitly in dimension $D = 2$ and $D = 3$. Let us start with $D = 3$. We then have

$$\begin{aligned}G(r) &= \frac{1}{(2\pi)^2}\int\frac{k^2dk}{k^2+m^2}\int_0^\pi d\theta\,\sin\theta\,e^{ikr\cos\theta}\\ &= \frac{1}{(2\pi)^2}\int\frac{k^2dk}{k^2+m^2}\,(\frac{e^{ikr}-e^{-ikr}}{ikr}) = \frac{-i}{4\pi^2 r}\int_{-\infty}^{+\infty}dk\,\frac{k\,e^{ikr}}{k^2+m^2}.\end{aligned}$$

We can further evaluate the integral using contour integral techniques. There is simple poles in k-space at points $\pm im$. To ensure the convergence of the integrals, we deform and close the contour on the upper-half plane. By the residue theorem, we thus pick the residue at $k = im$ which yields

$$G_{D=3}(r) = \frac{1}{4\pi}\,\frac{1}{r}\,e^{-mr}.$$

In $D = 2$, the formula does not simplify so much and we have

$$G_{D=2}(r) = \int\frac{dk}{(2\pi)^2}\,\frac{k}{k^2+m^2}\int d\theta\,e^{ikr\cos\theta} = \frac{1}{2\pi}K_0(mr),$$

which cannot be written in terms of elementary function.

• *Short and large distance behaviors.*

At short distances we have $G_r(r) \propto \frac{1}{r^{D-2}}$ with a logarithmic divergence in dimension $D = 2$. More precisely

$$\begin{aligned}G_{D=2}(r) &= \frac{1}{2\pi}\log(\frac{2}{mr})+\cdots,\\ G_{D=3}(r) &= \frac{1}{4\pi r}+\cdots,\\ G_{D=4}(r) &= \frac{1}{4\pi^2 r^2}+\cdots.\end{aligned}$$

At large distances we have an exponential decrease if the mass is non vanishing

$$G_D(r) \propto \frac{e^{-mr}}{r^{D-2}},\quad r \gg m^{-1},$$

This means that the correlation length (of the field with covariance G) can be identified as the inverse of the mass: $\xi = 1/m$. This is important to remember.

At $m = 0$ (and hence at infinite correlation length), $G(x) = \int \frac{d^D k}{(2\pi)^D} \frac{e^{ik\cdot x}}{k^2}$ and

$$G_D^{m=0}(r) = \int \frac{k^{D-3}\, dk d\Omega}{(2\pi)^D} e^{ikr\cos\theta} \propto \frac{1}{r^{D-2}}.$$

Note that the massless Green function are pure power laws. There is a logarithmic divergence in dimension $D = 2$ (UV and IR divergence because $G_{D=2}(r) = \int_{|k|\leq\Lambda} \frac{dk}{k} \int \frac{d\theta}{(2\pi)^2} e^{ikr\cos\theta}$). We thus have to introduce the cut-off $\Lambda \simeq 1/a$ and we have (or can choose)

$$G_{D=2}^{m=0}(r) = \frac{1}{2\pi} \log(\Lambda r).$$

It is easy to verify that $(-\Delta_x)G(x) = \delta(x)$ in $d = 2$. The arbitrariness in choosing Λ comes form the fact that the operator $(-\Delta)$ possesses zero mode (the constant function).

• *Lattice Green function.*

On the lattice, the action is

$$S[\phi] = \frac{1}{2} \int_{\mathrm{BZ}} \frac{d^D k}{(2\pi/a)^D} \hat{\phi}_k [-\Delta^{\mathrm{dis.}} + m^2]_k \hat{\phi}_k.$$

and the Green function reads

$$G(x) = \int_{\mathrm{BZ}} \frac{d^D k}{(2\pi/a)^D} \frac{e^{ik\cdot x}}{[-\Delta^{\mathrm{dis.}} + m^2]_k},$$

where $[-\Delta^{\mathrm{dis.}} + m^2]_k = 2a^{-2} \sum_\alpha \big(\eta - \cos(ak.e_\alpha)\big)$, with $\eta = (2 + a^2 m^2)/2$, is the discrete Laplacian in Fourier space. We see that there is a natural momentum cut-off $|k| \leq \Lambda \simeq 1/a$ with a the lattice mesh size. It is also important to realize that m has the dimension of the inverse-length.

How the continuous limit is taken? First the limit is taken at x fixed. Since $x = a\mathbf{n}$ is a point on the lattice, this means we take the limit fixing the "physical" distance (not the lattice distance defined as the minimal number of step to arrive at point x). The mass is also fixed in this limit, which means that the correlation length, measured in terms of the physical distances (not the lattice distance) is kept fixed. For instance, in 1D the continuous limit consists in the limit $a \to 0$, $n \to \infty$ with $x = an$ fixed. Similarly in $[-\Delta^{\mathrm{dis.}} + m^2]_k$, we keep m fixed, that is we keep $\xi = 1/m$ the 'physical' correlation length fixed (not the lattice correlation length which would be the product am).

5.5 Products and composite operators

• *Normal ordering and composite operators.*

Product of operators at coincident points are singular. Let us for instance look at the product of the Gaussian field ϕ at two neighbour points. Let us image inserting $\phi(x)$ and $\phi(y)$, with x near y, in a correlation function containing many other insertions of ϕ at points at finite distance away from x and y, say at points ξ_k. We can compute such correlation function using the Wick's theorem. There are then two kinds of Wick's contactions: (a) those for which x and y a contracted together producing the Green function $G(x, y)$ and the other points are contracted

together, and (b) those for which x is contracted to one of the points ξ_k and y with another one. Since $G(x,y)$ is singular as $x \to y$, the Wick's contraction of the first type are singular as $x \to y$ whereas those of the second type are regular is that limit. Using $G(x,y) \simeq \text{const.}/|x-y|^{D-2}$ as $x \to y$ we can write

$$\phi(x)\phi(y) = G(x,y) + \text{reg} \simeq_{x\to y} \text{const.}/|x-y|^{D-2} + \cdots ,$$

where the 'reg' terms are regular at $x = y$.

Alternatively, we can give a name of this remaining part and write

$$\phi(x)\phi(y) = G(x,y) + : \phi(x)\phi(y) :,$$

where $: \phi(x)\phi(y) :$ is called to the 'normal order' of $\phi(x)$ and $\phi(y)$. This equation may be taken as a definition of the normal order operator. Alternatively, $: \phi(x)\phi(y) :$ may defined inside any correlation functions as $\phi(x)\phi(y)$ but without self Wick's contractions. This normal order product is now regular (because there is no self contraction) so that we can take the limit of coincident points and define the product ϕ^2 as

$$: \phi^2(x) := : \phi(x)\phi(x) := \lim_{y\to x} \big(\phi(x)\phi(y) - G(x,y)\big).$$

There are actually some freedom on the way to define this product operator as its construction only requires subtracting the singular part of the Green which is defined up to regular piece. It is a good exercise to compute the correlation functions $\langle : \phi^2(x) :: \phi^2(y) : \rangle$.

Of course this can be extended to higher point functions and higher products. For instance,

$$\begin{aligned} : \phi^2(x) : \phi(y) &= 2G(x,y) + : \phi^2(x)\phi(y) :, \\ : \phi^2(x) :: \phi^2(y) : &= 2G(x,y) + : \phi^2(x)\phi^2(y) :, \end{aligned}$$

where $: \phi^2(x)\phi(y) :$ means the insertion of the product $\phi^2(x)\phi(y)$ without self Wick's contraction, and similarly for $: \phi^2(x)\phi^2(y) :$. In particular we can now define the fourth power of ϕ as

$$: \phi^4(x) := \lim_{y\to x} \big(: \phi^2(x) :: \phi^2(y) : -2G(x,y)\big).$$

And ad-finitum, recursively.

From this analysis, we understand that field theory operators, or observables, are sensible to —or are testing— the neighbourhood of the points they are attached to.

• *Geometrical interpretation.*

We now describe a geometrical interpretation of these operators in terms of random free path. Recall the definition of random free path given in Chapter 2. There, we computed partition function $Z_{\text{path}}(0,x)$ (we call it $Z(x)$ in that Chapter) as the Boltzmann sum over free paths from the origin to the point x and we found that

$$Z_{\text{path}}(0,x) \propto \langle 0| \frac{\mathbb{I}}{\mathbb{I} - \mu\,\Theta} |x\rangle,$$

with Θ the lattice adjacency matrix and μ the fugacity. In the scaling limit ($a \to 0, \mu \to \mu_c$), it satisfies $(-\Delta + m^2) Z_{\text{path}}(0,x) = \delta(x)$. We thus have the identification

$$Z_{\text{path}}(0,x) = \langle \phi(x)\phi(0) \rangle,$$

with ϕ a massive Gaussian free field. Alternatively, by the expanding the inverse $\mathbb{I}/(\mathbb{I}-\mu\,\Theta)$ in power of μ, we may view the Gaussian correlation function as a sum over paths (because the N-th power of the adjacency matrix Θ^N codes for the sum of paths of length N). We may then interpret the field $\phi(x)$ as conditioning on a curve starting or arriving at the point x.

Similarly, given four points $x_1,\ x_2,\ x_3,\ x_4$, the partition function $Z_{\rm path}([x_1,x_2];[x_3,x_4])$ for pairs of free (possibly intersecting) paths joining x_1 to x_2 and x_3 to x_4 respectively is

$$Z_{\rm path}([x_1,x_2];[x_3,x_4]) = G(x_1,x_2)G(x_3,x_4).$$

Thus, the partition sum $Z_{\rm path}(x_1,x_2,x_3,x_4)$ over free paths joining these points pairs, independently of the set of connected pairs, is the four point functions

$$Z_{\rm path}(x_1,x_2,x_3,x_4) = \langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle.$$

As a consequence we can identify the normal operator $:\phi^2(x):$ as conditioning on two curves starting (or arriving) at point x.

Similar, the higher order normal ordered powers $:\phi^k:$ condition on k curves starting the operator insertion point:

$$:\phi^k(x): \quad = \quad \text{create/absord } k \text{ curves at } x.$$

The fact that $:\phi^2:$, the operator creating two curves at a point, has scaling dimension $D-2$ can be used to argue that the fractal dimension of free paths is $D_{\rm frac}=2$. Notice also that we may identify the diagrams representing the Wick's contraction with the free path configuration we are summing over.

5.6 Quantization of free field theories

We go back to Minkowski (real-time) quantum field theory and make connection with the Euclidean and Minkowski versions of the field theory.

• *Path integral quantization.*

Remember that the weight of the QM path integral is $e^{(i/\hbar)S}$ with S the action. We use the notation $x=(\mathbf{x},t)$ and $D=d+1$. The Euclidean action was $S[\phi]=\int d^Dx\big(\frac{1}{2}(\nabla\phi)^2+V(\phi)\big)$ with $V(\phi)=\frac{1}{2}m^2\phi^2$ for a free field. The analytic continuation to go from the Euclidean time t_E to the Minkowski time t is $t\to it_E$. The Minkoswki version of the action is

$$S[\phi]=\int dtd^d\mathbf{x}\,\mathcal{L}=\int dtd^d\mathbf{x}\,\big(\frac{1}{2}(\dot\phi)^2-\frac{1}{2}(\partial_{\mathbf{x}}\phi)^2-V(\phi)\big)=\int dtd^d\mathbf{x}\,\big(\frac{1}{2}(\nabla\phi)^2-V(\phi)\big).$$

where here the gradient square $(\nabla\phi)^2$ is defined using the Minskowky metric. The Feynman path integral reads

$$\int[D\phi]\,e^{(i/\hbar)\,S[\phi]}.$$

The dynamical variables are the field configurations (at all space points $\mathbf{x}$)

$$q\to\phi(\mathbf{x}),\quad p\to\pi(\mathbf{x}).$$

The canonical commutation relations are $\big[\phi(\mathbf{x})\,,\,\pi(\mathbf{y})\big] = i\hbar\delta(\mathbf{x}-\mathbf{y})$. They are formally represented on functional of field configurations with the field operator acting by multiplication and the conjugated momentum by derivative. We formally have a basis a state made of field configuration $|\{\phi(\mathbf{x})\}\rangle$ which are eigen-states of the field operator

$$\phi(\mathbf{x_0})|\{\phi(\mathbf{x})\}\rangle = \phi(\mathbf{x_0})|\{\phi(\mathbf{x})\}\rangle.$$

If we give ourselves two configurations at two different times (at two different time slices), $\{\phi_0(\mathbf{x})\}$ at time t_0 and $\{\phi_1(\mathbf{x})\}$ at time t_1, the quantum amplitude is (formally again) defined by the path integral with specified boundary condition:

$$\langle\{\phi_1(\mathbf{x})\}|U(t_1,t_0)|\{\phi_0(\mathbf{x})\}\rangle = \int_{\substack{\phi(\mathbf{x},t_0)=\phi_0(\mathbf{x})\\ \phi(\mathbf{x},t_1)=\phi_1(\mathbf{x})}} [D\phi]\, e^{(i/\hbar)\,S[\phi]}.$$

Of course the normalization is not very well specified, but this ill-defined normalisation cancels when computing normalized correlation functions.

Below we shall argue that taking the limit $t_0 \to -\infty$ and $t_1 \to +\infty$ this amplitude projects onto the vacuum state, that is the path integral from $-\infty$ to $+\infty$ (without specified boundary condition) is the vacuum to vacuum amplitude. The simplest (naive) argument for the limit to exist requires introducing an infinitesimal small imaginary part (or doing an analytic continuation) to the time parameters which then produce the Euclidean transfer matrix which projets on the vacuum at large time. See Chapter 3 and 4.

Unless otherwise specified, we now set $\hbar = 1$.

• *Canonical quantization.*

The canonical momentum is $\pi(\mathbf{x},t) = \dot\phi(\mathbf{x},t)$ with canonical commutation relation

$$\pi(\mathbf{x},t) = \dot\phi(\mathbf{x},t), \quad \big[\phi(\mathbf{x},t)\,,\,\pi(\mathbf{y},t)\big] = i\delta(\mathbf{x}-\mathbf{y}).$$

The action can be written in the hamiltonian form

$$S[\phi] = \int dt d^d\mathbf{x}\,\big(\pi(\mathbf{x})\dot\phi(\mathbf{x}) - \mathcal{H}\big),$$

with

$$H = \int d^d\mathbf{x}\,\mathcal{H} = \int dt d^d\mathbf{x}\,\big(\frac{1}{2}\pi(\mathbf{x})^2 + \frac{1}{2}(\partial_{\mathbf{x}}\phi(\mathbf{x}))^2 + V(\phi(\mathbf{x})\big).$$

For free field theory of non-zero mass m^2, this reads

$$H = \frac{1}{2}\int dt d^d\mathbf{x}\,\big(\pi^2 + (\partial_{\mathbf{x}}\phi)^2 + m^2\phi^2\big).$$

Let us go to Fourier space

$$\hat\phi(\mathbf{k}) = \int d^d\mathbf{x}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\phi(\mathbf{x}), \quad \phi(\mathbf{x}) = \int \frac{d^\mathbf{k}}{(2\pi)^d}\, e^{i\mathbf{k}\cdot\mathbf{x}}\hat\phi(\mathbf{k}),$$

and similarly for the momentum. The canonical commutation relation now reads $[\hat\pi(\mathbf{k}), \phi(\mathbf{k}')] = -i2\pi\delta(\mathbf{k}-\mathbf{k}')$. The hamiltonian (density) reads

$$H = \frac{1}{2}\int \frac{d^d\mathbf{k}}{(2\pi)^d}\,\big(\hat\pi(\mathbf{k})^2 + \omega_\mathbf{k}^2\,\hat\phi(\mathbf{k})^2\big), \quad \omega_\mathbf{k}^2 = \mathbf{k}^2 + m^2.$$

This hamiltonian codes for a collection of independent harmonic oscillators, indexed by the momentum $\mathbf{k}$. It is diagonalized by introducing annihiliation-creation operators $A_{\mathbf{k}}$ and $A^*_{\mathbf{k}}$ with commutation relation

$$\big[A_{\mathbf{k}}\,,\,A^*_{\mathbf{p}}\big] = (2\pi)^d\delta(\mathbf{k}-\mathbf{p}),$$

defined by

$$\hat\pi(\mathbf{k}) = \sqrt{\frac{\omega_{\mathbf{k}}}{2}}(A_{\mathbf{k}}+A^*_{\mathbf{k}}),\quad \hat\phi(\mathbf{k}) = -i\sqrt{\frac{1}{2\omega_{\mathbf{k}}}}(A_{\mathbf{k}}-A^*_{\mathbf{k}}).$$

To be finite the hamiltonian needs renormalization by normal ordering. See any book on basics quantum field theory for more detailed.

The representation of these commutations relations is done as usual on the Fock space $\mathcal{F}\simeq\otimes_{\mathbf{k}}\mathcal{F}_{\mathbf{k}}$ over the vacuum $|0\rangle$ annihilated by all $A_{\mathbf{k}}$, that is

$$A_{\mathbf{k}}|0\rangle = 0,\quad \langle 0|A^*_{\mathbf{k}} = 0,$$

for all $\mathbf{k}$. States in the Fock space are obtained by repeated action of the creation operators on the vacuum: $A^*_{\mathbf{k}_1}\cdots A^*_{\mathbf{k}_N}|0\rangle$.

Alternatively, we can solve directly the wave equation and 'quantize' them. The field ϕ satisfies the wave equation

$$\big(-\partial_t^2+\partial_{\mathbf{x}}^2+m^2\big)\phi(\mathbf{x},t)=0.$$

Hence, by Fourier transform

$$\phi(\mathbf{x},t) = \int\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\big(e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)}A_{\mathbf{k}}+e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)}A^*_{\mathbf{k}}\big)$$

The momentum $\pi(\mathbf{x},t)=\dot\phi(\mathbf{x},t)$ is

$$\pi(\mathbf{x},t) = -i\int\frac{d^d\mathbf{k}}{(2\pi)^d}\sqrt{\frac{\omega_{\mathbf{k}}}{2}}\,\big(e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)}A_{\mathbf{k}}-e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)}A^*_{\mathbf{k}}\big)$$

We impose the canonical commutation relation $\big[A_{\mathbf{k}}\,,\,A^*_{\mathbf{p}}\big]=(2\pi)^d\delta(\mathbf{k}-\mathbf{p})$. They imply the canonical commutation relation $\big[\phi(\mathbf{x},t)\,,\,\pi(\mathbf{y},t)\big]=i\delta(\mathbf{x}-\mathbf{y})$, as it should be.

We could alternatively set $A_{\mathbf{k}}=\frac{a_{\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}}}$ (if $\omega_{\mathbf{k}}$ does not vanish). The formula for field ϕ looks then a bit simpler because there is no square root:

$$\phi(\mathbf{x},t) = \int\frac{d^d\mathbf{k}}{(2\pi)^d 2\omega_{\mathbf{k}}}\big(e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)}a_{\mathbf{k}}+e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)}a^*_{\mathbf{k}}\big).$$

The operators $a_{\mathbf{k}}$ then satisfy

$$\big[a_{\mathbf{k}}\,,\,a^*_{\mathbf{p}}\big] = (2\pi)^d 2\omega_{\mathbf{k}}\,\delta(\mathbf{k}-\mathbf{p}).$$

The representation of these canonical commutations relations is as above.

• *Wick's rotation and time ordering.*

Let us compute the vacuum two point functions. We have (using only the commutation relations of the creation-annihilation operators and the fact the defining property of the vacuum)

$$\langle 0|\phi(\mathbf{x},t)\phi(\mathbf{y},s)|0\rangle\big|_{t>s} = \int\frac{d^d\mathbf{k}}{(2\pi)^d}\,\frac{1}{2\omega_{\mathbf{k}}}\,e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\,e^{-i\omega_{\mathbf{k}}(t-s)},$$

for $t > s$. We can (again) check that from this formula that the canonical commutation relations are satisfied, that is $\langle 0|[\dot{\phi}(\mathbf{x},t)\,,\,\phi(\mathbf{y},t)|0\rangle = -i\delta(\mathbf{x}-\mathbf{y})$.

We now compare this formula to the analytic continuation of the Euclidean Green function. Recall that (with τ denoting the Euclidean time)

$$G_E(\mathbf{x}-\mathbf{x}') = \int \frac{d^d\mathbf{k}dk_0}{(2\pi)^D}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')+ik_0(\tau-\tau')}\,\hat{G}_E(\mathbf{k}),$$

with

$$\hat{G}_E(\mathbf{k}) = \frac{1}{\mathbf{k}^2+k_0^2+m^2}.$$

As above, we do the integration over k_0 by contour integral. There is a pole at $k_0 = \pm i\omega_{\mathbf{k}}$. We pick one or the other depending whether $\tau > \tau'$ or the reverse (because the contour deformation is chosen such that the integral converges). Since the Euclidean Green function is symmetric under the exchange of x and x', we are free to choose either case. Suppose $\tau > \tau'$. For the integral to converge we then have to close the contour on the upper half plane and pick the pole at $+i\omega_{\mathbf{k}}$. Hence, for $\tau > \tau'$,

$$G_E(\mathbf{x}-\mathbf{x}') = \int \frac{d^d\mathbf{k}}{(2\pi)^d}\,\frac{1}{2\omega_{\mathbf{k}}}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\, e^{-\omega_{\mathbf{k}}(\tau-\tau')}.$$

If $\tau' > \tau$, we have to close the contour in the lower half plane and pick the other pole, so that we get the same result but with τ' and τ exchanged. That the general formula is with $|\tau-\tau'|$, that is

$$G_E(\mathbf{x}-\mathbf{x}') = \int \frac{d^d\mathbf{k}}{(2\pi)^d}\,\frac{1}{2\omega_{\mathbf{k}}}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\, e^{-\omega_{\mathbf{k}}|\tau-\tau'|}.$$

We now implement the analytic continuation. Remember that to go from real time quantum mechanic to Euclidean time we had to set $t = -i\tau$ with $\tau > 0$ (for convergence). That is, we analytically continue the real time to its lower complex plane. The reverse it $\tau = it$ (but we have to keep track of the imaginray part to keep track of the ordering). So let $\tau = i(t-i\epsilon)$ and $\tau' = i(s-i\epsilon')$. For $\epsilon > \epsilon'$ we have $\Re(\tau-\tau') > 0$, so the Euclidean integral produces $e^{-\omega_{\mathbf{k}}(\tau-\tau')}$. And hence the analytic continuation

$$\begin{aligned}\lim_{\substack{\epsilon,\epsilon'\to 0^+\\ \epsilon>\epsilon'}} G_E(\mathbf{x},\tau=it+\epsilon;\mathbf{y},\tau'=is+\epsilon') &= \int \frac{d^d\mathbf{k}}{(2\pi)^d}\,\frac{1}{2\omega_{\mathbf{k}}}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\, e^{-i\omega_{\mathbf{k}}(t-s)}\\ &= \langle 0|\phi(\mathbf{x},t)\phi(\mathbf{y},s)|0\rangle\big|_{t>s}.\end{aligned}$$

Thus we learn that real time correlation functions are obtained by analytic continuation of the Euclidean expectation values, and that the ordering of the operator is induced by the precise way the analytic continuation is done.

The fact that we have to add a tiny negative imaginary part to order them has a natural interpretation. Remember that when going to the unitary evolution e^{-itH} to the Euclidean one $e^{-\tau H}$ we had to set $t = -i\tau$. So adding a negative imaginary part to t amount to slightly evolve the operator forward in Euclidean time. We order the operator according to their Euclidean time (e.g. as suggested by the transfer matrix formalism).

This is summarized by the fact the analytic continuation of the Euclidean propagator gives the Feynman propagator corresponding to time ordered correlation functions. Under the wick rotation, $x_0 = \tau = it$ and $k_0 = -i\omega$, we have

$$G_E \to G_F(x,y) = \left(\frac{i}{\Delta - m^2 + i0^+} \right)_{x,y} = \langle \mathbf{T} \cdot \phi(x)\phi(y) \rangle,$$

with $\Delta = -\partial_t^2 + \partial_{\mathbf{x}}^2$ the Minkowski Laplacian and $\mathbf{T}$ the time order operator. But one has to be careful how the analytical continuation is done (cf the above discussion).

5.7 Exercises

• *Exercise 5.7.1: Translation invariance and the stress-tensor*

The aim of this exercise is to see some aspect of the relation between translation invariance and the stress-tensor. Let us consider classical scalar field theory with Lagrangian $\mathcal{L}[\phi, \partial\phi]$ and action $S[\phi] = \int d^D x\, \mathcal{L}[\phi, \partial\phi]$.
(i) Consider an infinitesimal field transformation $\phi(x) \to \phi(x) + \epsilon(\delta\phi)(x)$. Suppose that, under such transformation the Lagrangian variation is $\delta\mathcal{L}[\phi, \partial\phi] = \epsilon\partial_\mu G^\mu$ so that the action is invariant. Show that the following Noether current

$$J^\mu = (\delta\phi)\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} - G^\mu,$$

is conserved on solutions of the equations of motion.
(ii) Let us look at translations $x \to x + \varepsilon\, a$. How does a scalar field ϕ transforms under such translation? Argue that if the Lagrangian density is a scalar, then $\delta\mathcal{L} = \varepsilon\, a^\mu\partial_\mu\mathcal{L}$. Deduce that the action is then translation invariant and that associated conserved Noether current is $J^\mu_a = T^\mu_\nu a^\nu$ with

$$T^\mu_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,(\partial_\nu\phi) - \delta^\mu_\nu\,\mathcal{L}.$$

This tensor is called the stress-tensor. It is conserved: $\partial_\mu T^\mu_\nu = 0$.
(iii) Find the expression of the stress-tensor T^μ_ν for a scalar field theory with action $S[\phi] = \int d^D x\,\big(\frac{1}{2}(\nabla\phi)^2 + V(\phi)\big)$.

• *Exercise 5.7.2: Lattice scalar field and lattice Green function*

Recall that lattice scalar free theory is defined by the action

$$S[\phi] = \frac{a^D}{2}\sum_{x,x'}\phi_x\,[(-\Delta^{\rm dis} + m^2)]_{x,x'}\,\phi_{x'},$$

where ϕ_x are the value of the field at point x on the lattice and $\Delta_{\rm dis}$ discrete Laplacian on that lattice. We here consider only D-dimensional square lattice of mesh size a, i.e. $a\mathbb{Z}^D$. Let us also recall that the Fourier transforms in $a\mathbb{Z}^D$ are defined by

$$\hat\phi_k = \sum_{\mathbf{n}} e^{-ix\cdot k}\,\phi_x, \quad \phi_x = \int_{\rm BZ}\frac{d^D k}{(2\pi/a)^D}\,e^{ix\cdot k}\,\hat\phi_k$$

where the integration is over the Brillouin zone, which is the hyper-cube $\mathrm{BZ} \equiv [-\frac{\pi}{a}, \frac{\pi}{a}]^D$.
i) Verify that the Laplacian acts diagonally in the Fourier basis, with

$$(-\Delta^{\rm dis} + m^2)_k = 2a^{-2}\sum_\alpha\big(\eta - \cos(a\,k_\alpha)\big),$$

with $\eta = (2 + a^2m^2)/2$ and k_α the component of the momentum k in the direction α.
ii) Verify that in the Fourier basis the free field action reads

$$S[\phi] = \frac{1}{2}\int_{\rm BZ}\frac{d^D k}{(2\pi/a)^D}\,\hat\phi_{-k}(-\Delta^{\rm dis} + m^2)_k\,\hat\phi_k.$$

(iii) Deduce that in Fourier space, a scalar free field is thus equivalent to a collection of i.i.d. Gaussian variables, indexed by the momentum k, with mean and covariance

$$\langle \phi_k \rangle = 0, \quad \langle \phi_k \phi_p \rangle = \frac{1}{(-\Delta^{\rm dis} + m^2)_k} \, (2\pi/a)^D \, \delta(k+p).$$

• *Exercise 5.7.3: The Green function in 2D*

[... To be completed...]

• *Exercise 5.7.4: Fractal dimension of free paths.*

The fractal dimension $D_{\rm frac}$ of a set embedded in a metric space may be defined through the minimal number $\mathcal{N}_\epsilon$ of boxes of radius ϵ need to cover it by $D_{\rm frac} = \lim_{\epsilon \to 0} \log \mathcal{N}_\epsilon / \log(1/\epsilon)$.

(i) Prove that the fractal dimension of free paths is $D_{\rm frac} = 2$ using the fact that the composite operator ϕ^2, with ϕ a (massless) Gaussian free field, is the operator conditioning on two paths emerging from its insertion point.

• *Exercise 5.7.6: Two ways to compute the free energy*

The aim of this exercise is to compute the free energy, or the partition function, of a massless free boson in space dimension $d = 1$ at temperature $T = 1/\beta$. Let $D = d + 1$. Recall that the partition function is defined as $Z = {\rm Tr}(e^{-\beta H})$ where the trace is over the quantum Hilbert space with H the hamiltonian. Let us suppose that the quantum theory is define dover an interval $\mathbb{I}$ of length L. We shall be interested in the large L limit.

(i) Argue (see Chapter 3) that the partition function is given by the Euclidean path integral on the cylinder $\mathbb{I} \times \mathbb{S}_1$ with a radius β:

$$Z = \int_{\phi(\mathbf{x},\beta)=\phi(\mathbf{x},0)} [D\phi] \, e^{-S[\phi]}.$$

We shall compute the partition function by quantizing the theory along two different channels (see **Figure**):
(a) either taking the direction $\mathbb{S}_1$ as time, this Euclidean time is then period with period β;
(b) or taking the direction $\mathbb{I}$ as time, this time then runs from 0 to L with $L \to \infty$. Global rotation invariance implies that this to way of computing gives identical result. Let us check. On the way this will give us a nice relation about the Riemann ζ-function.

(ii) Explain why the first computation gives $Z = e^{-\beta L \mathcal{F}(\beta)}$, where $\mathcal{F}$ the free energy.
(iii) Explain why the second computation gives $Z = e^{-L E_0(\beta, A)}$ with $E_0(\beta) = \beta \, \mathcal{E}_0(\beta)$ where E_0 is the vacuum energy and $\mathcal{E}_0$ is the vacuum energy density (this is the Casimir effect).
(iv) Show that the free energy density of a massless boson in one dimension is:

$$\mathcal{F} = \frac{1}{\beta} \int \frac{dk}{2\pi} \log(1 - e^{-\beta |k|}) = \frac{1}{\beta^2} \int_0^\infty \frac{dx}{\pi} \log(1 - e^{-x}).$$

(v) Compute the integral to write this free energy density as

$$\mathcal{F} = -\frac{1}{\pi \beta^2} \zeta(2).$$

We have introduce the so-called *zeta*-regularisation. Let $\zeta(s) := \sum_{n>0} \frac{1}{n^s}$. This function was introduced by Euler. This series is convergent for $\Re s > 2$. It is defined by analytic continuation for other value of s via an integral representation.
(iv) Show that the vacuum energy density is $\mathcal{E}_0(\beta) = \frac{1}{\beta} \sum_n \frac{1}{2} \left| \frac{2n\pi}{\beta} \right|$. (v) This is divergent. Argue that a regularization based on analytic continuation gives

$$\mathcal{E}_0(\beta) = \frac{2\pi}{\beta^2} \zeta(-1).$$

(vi) Conclusion: A remarkable fact is that $\zeta(2) = \frac{\pi^2}{6}$ and that the analytic continuation of ζ gives $\zeta(-1) = -\frac{1}{12}$. Thus

$$\mathcal{F}(\beta) = \mathcal{E}_0(\beta) = -\frac{\pi}{6\beta^2}.$$

Actually, we could reverse the logic: physics tells us that $\zeta(-1)$ has to be equal to $-\frac{1}{12}$ because $\mathcal{E}_0$ has to be equal to $\mathcal{F}$.

- *Exercise 5.7.7: Radial quantization (at least in 2D).*

  [... To be completed...]

6 Statistical field theory: interactions.

In this Chapter we go deeper in analysing the structures and the basic tools of statistical field theories. We first write consequences of symmetries on correlation functions which are called Ward identities. We then go one by introducing generating functions of multi-point correlations functions and make the analogy with thermodynamic functions. This leads to the important concept of effective action. We describe how perturbation theory can be formulated and how it is can encoded in synthetic way in Feynman diagrams. We make the connection between the irreducible components of the Feynman diagram expansion and the effective action. Finally, we study in different ways –via loop expansion or via saddle point approximation– the $O(N)$ model at large N.

6.1 Preliminaries

For an interacting scalar theory the action is of the form $S[\phi] = S_0[\phi] + S_I[\phi]$ with $S_0[\phi]$ the free action. Standard examples are

$$S_0[\phi] = \frac{1}{2}\int d^D x[(\nabla\phi)^2 + m^2\phi^2], \quad S_I[\phi] = \int d^D x\, V(\phi),$$

with V a potential coding for self-interaction. The Boltzmann weights of field configurations then read

$$[D\phi]\, e^{-S[\phi]} = [D\phi]\, e^{-S_0[\phi]}\, e^{-S_I[\phi]}.$$

The first part is the Boltzmann measure for free Gaussian theory so that we can (formally) factorize the measure of field configuration as

$$d\mathbb{P}[\phi] = d\mathbb{P}_{\text{free}}[\phi] \times e^{-\int d^D x\, V(\phi)}.$$

This formula is however only formal for many reasons (the field ϕ is not a random function but a random distribution, the interacting and free measures may be singular, etc). Making sense of such formula is part of what statistical field theory is aiming at and the renormalization group is a key tool to reach this goal. It requires introducing a UV cut-off that we may think as coding for an underlying lattice. We shall implicitly assume below that such UV cut-off is present.

Keeping this prerequisite in mind, we can look at what constraints these measures, or the correlation functions they code for, should satisfy. This is the purpose of the Ward identities and the introduction of generating functions. Perturbation theory consists in developing the interaction terms $e^{-\int d^D x\, V(\phi)}$ is power of V and analyzing each terms of this expansion. These two techniques —symmetries and perturbative expansion— are two basic, fundamental, ways of computing —or of making sense— of correlation functions in statistical field theory. A third possible ways may be applied when there is a small parameter such that the statistical path integral can be evaluated via a saddle point approximation —in a way analogous to the WKB approximation in Quantum Mechanics or to the small noise problems we discussed in the first Chapter. This last method will be applied to the $O(N)$ model at large N at the end of this Chapter.

Field configurations are random, by construction, and a way to test their statistics is to look at expectations of functions, often called observables, of those configurations. If they depend locally on the field configuration at a point x, those observables are called either local observables,

operators or fields, indifferently. Standard (formal) observables are products of polynomials of field values at given points, say monomials $\phi^p(x)$ with p integer, but we may aim at considering any local functions of the field, say

$$F_1[\phi(x_1)] \cdots F_N[\phi(x_N)].$$

Of course these naive writings are purely formal since, on the one hand, statistical fields are singular objects (in the previous Chapter dealing with Gaussian free fields we gave them a meaning in terms of random distributions), and on the other hand, product of statistical field at coinciding points is ill-defined (as discussed in previous Chapter in the case of free Gaussian field) and we will have to give a meaning to such products. Again different strategies can be developed to attain this goal, say either using symmetries to specify the properties of fields and observables, or using perturbation expansions to recursively defined those observables, or else.

As mentioned in the previous Chapter we may look at more general theories with fields of different types (scalar, vectors, etc) taking values in different manifolds (flat or curve spaces, group manifolds or quotient spaces, etc.) with more or less simple structures. For instance, we may consider theories not over the flat space $\mathbb{R}^D$ but over a metric space $\mathfrak{M}$, equipped with a metric g. For scalar field theories the kinetic term is then going to be $\frac{1}{2}\int d^Dx\,\sqrt{|g|}\,g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$, with $|g|$ the metric determinant. More generally, we may consider scalar fields taking values in another target space $\mathfrak{E}$, equipped with a metric G, with local coordinates ϕ^A. The maps $x \to \phi^A(x)$ are then maps from the base space $\mathfrak{M}$ —sometimes called the worldsheet— to the target space $\mathfrak{E}$, and the statistical field theory codes from statistical properties of such maps. The kinetic part of the action could for instance be chosen to be $\frac{1}{2}\int d^Dx\,\sqrt{|g|}\,g^{\mu\nu}\nabla_\mu\phi^A\nabla_\nu\phi^B\,G_{AB}$, which is not a free action if the metric G is ϕ-dependent. Such models are generically called sigma-models.

Thus in general, statistical field theories and their expectations,

$$\langle\mathcal{O}_1(x_1)\cdots\rangle_{\mathfrak{M},g;\mathfrak{E},G,\cdots},$$

depends on collection of data, of geometrical nature (say, spaces, metrics, etc), or not (say, external fields, sources, etc).

Below, we shall restrict ourselves to scalar field (mostly with one component) on flat space.

6.2 Symmetries and Ward identities

In classical field theory, conservation laws are associated to symmetries –and reciprocally. If a classical action is invariant under some symmetries, there exists an associated conserved Noether current. The aim of this section is to understand how this translates in statistical field theory.

• *Symmetry and invariance*

Symmetries refer to invariances of statistical expectations or of correlations functions. There are different kinds of transformations: those which leave the space points fixed and act only the field components (for example, group symmetries, etc), and those which act on the space point (for example, translation, rotation, etc). If all the geometrical data are preserved, the transformations act on observables/operators and transform them as $\mathcal{O}(x) \to \hat{\mathcal{O}}(x)$ say. In this case invariance of the correlation functions means that

$$\langle\mathcal{O}_1(x_1)\cdots\rangle_{\mathfrak{M},g;\cdots} = \langle\hat{\mathcal{O}}_1(x_1)\cdots\rangle_{\mathfrak{M},g;\cdots}.$$

The aim of this Section is to show some of consequences of these symmetries and their relations with the (classical) symmetries of the action defining (maybe formally) the statistical field measure. These invariance relations give constraints on the correlation functions. Sometimes, the logic can be reversed and the correlation functions in a symmetric enough field theory could be defined from their symmetry properties (this is in particular the case in a large class of conformal field theory in dimension two).

• *Noether's current and invariances.*

Classically, symmetry and conservation law are tied, and if the classical action possesses some symmetries then there exists a conserved Noether current J^μ, $\partial_\mu J^\mu = 0$. The Noether current is in general defined by looking at the the variation of the action. If the field is transformed according to $\phi(x) \to \phi(x) + \epsilon\,(\delta\phi)(x))$, the variation of the action for a non constant ϵ is going to be proportional to the derivative of ϵ, so that we may write the variation of the action for a non constant transformation parameter as

$$\delta S[\phi] = -\int d^D x\,(\partial_\mu \epsilon)(x)\,J^\mu(x) = \int d^D x\,\epsilon(x)\,(\partial_\mu J^\mu)(x),$$

by integration by part. This defines the Noether current, and it is a simple exercise in classical field theory to check that it is conserved on solution of the classical equation of motion.

We now look at what are the consequences of this transformation law in the path integral formulation of statistical field theory. Those will be consequences of a change of variables in the path integral.

Let us start with the path integral $\int [D\phi]\, e^{-S[\phi]}$, and let us image implementing the change of variables $\phi(x) \to \hat\phi(x)$ with $\hat\phi(x) = \phi(x) + \epsilon(x)\,(\delta\phi)(x)$ with $\epsilon(x)$ small but position dependent. The point to understand is : how does the path integral measure varies under such transformation? They are two contributions to this variation, one from the action, the other from the change of variables.

Let us first look at the variation of the action $S[\phi] = \int d^D x\,\mathcal{L}[\phi, \partial\phi]$. As we assumed above, the variation of the action is

$$\delta S[\phi] = \int d^D x\,\epsilon(x)\,(\partial_\mu J^\mu)(x). \tag{40}$$

Here we are computing the variation of the action for any field configuration and not only for solution of the classical equations of motion (because we are going to integrate over them), so $(\partial_\mu J^\mu)$ maybe non zero (it is a fluctuating quantity). Under the change of field variables, $\phi \to \hat\phi$, the path integral measure change according

$$[D\phi] e^{-S[\phi]} = [D\hat\phi]\, e^{-S[\hat\phi]} \times \left|\mathrm{Det}\frac{[D\phi]}{[D\hat\phi]}\right| \Big[1 - \int d^D x\,\epsilon(x)\,(\partial_\mu \hat J^\mu)(x)\Big],$$

where $\left|\mathrm{Det}\frac{[D\phi]}{[D\hat\phi]}\right|$ is the Jacobian of the transformation.

We will suppose that the Jacobian of the transformation $\phi \to \hat\phi$ is trivial:

$$\text{Hypothesis :} \quad \left|\mathrm{Det}\frac{[D\phi]}{[D\hat\phi]}\right| = 1.$$

This is an important but delicate point. To be defined and evaluated, this determinant needs to be regularized (say by discretizing the theory, by defining it on a lattice, etc). This regularization amounts to discretize or regularize the path integral measure. It may happen that it is impossible to regularize this measure in such way to preserve its symmetry. In such case the Jacobian will not be trivial. This is call a (quantum) anomaly, see below.

Under this hypothesis, we get that

$$[D\phi]e^{-S[\phi]} = [D\hat{\phi}]\, e^{-S[\hat{\phi}]} \times \big[1 - \int d^D x\, \epsilon(x)\, (\partial_\mu \hat{J}^\mu)(x) + \cdots\big].$$

for (smooth enough) function $\epsilon(x)$. Integrating over ϕ (or $\hat{\phi}$) gives:

$$\langle (\partial_\mu J^\mu)(x)\rangle = 0. \tag{41}$$

That is the current is conserved in mean (away from field insertions, see below). This is an analogue of the Noether theorem. (See the Exercise section of Chapter 4 to see the lattice analogue of this construction).

• *Field insertions and Ward identities.*

Things becomes more interesting when genralizing this construction with insertions of operators, say $\mathcal{O}(y)$, or products of such operators. This operator may be thought as a local function of the field ϕ. We consider the expectation value:

$$\langle \mathcal{O}(y)\rangle = \frac{1}{Z}\int [D\phi]\, e^{-S[\phi]}\, \mathcal{O}(y),$$

with Z the partition function,

$$Z = \int [D\phi]\, e^{-S[\phi]}\, \mathcal{O}(y).$$

Let us do the change of variable $\phi \to \hat{\phi}$. Suppose that under such transformation the operator $\mathcal{O}(y)$ transforms as $\mathcal{O}(y) \to \mathcal{O}(y) + \epsilon(y)(\delta\mathcal{O})(y)$. We then get (after renaming $\hat{\phi}$ into ϕ):

$$\int [D\phi] e^{-S[\phi]} \Big[\int d^D x\, \epsilon(x)\, (\partial_\mu J^\mu)(x)\Big]\, \mathcal{O}(y) = \int [D\phi] e^{-S[\phi]}\, \epsilon(y)(\delta\mathcal{O})(y),$$

for any $\epsilon(x)$. Hence

$$\langle (\partial_\mu J^\mu)(x)\, \mathcal{O}(y)\rangle = \delta(x-y)\, \langle (\delta\mathcal{O})(y)\rangle.$$

Generalization to products of operators is clear.

We can also use the Stoke's theorem to write differently the same equation[3]. Let us pick a ball $\mathcal{B}$ and integrate the current normal to the ball (i.e. evaluate the flux of this current through $\mathcal{B}$). Let us call this flux $\int_{\mathcal{B}} n \cdot J$. Then:

$$\langle \big(\int_{\mathcal{B}} n\cdot J\big) \cdot \prod_{k=1}^{N} \mathcal{O}_k(y_k)\rangle = \sum_{y_k \text{ inside } \mathcal{B}} \langle \mathcal{O}_1(y_1)\cdots(\delta\mathcal{O}_k)(y_k)\cdots\mathcal{O}_N(y_N)\rangle. \tag{42}$$

[3]This way of writing the Ward identities will be useful in conformal field theories, see Chapter 8.

These relations are called "Ward identities". They express the fact that $(\partial_\mu J^\mu)(x) = 0$ away from field insertions but that there is a contact term coding for the field transformation when the current is near to the operator:

$$(\partial_\mu J^\mu)(x)\,\mathcal{O}(y) \simeq \delta(x-y)(\delta\mathcal{O})(y), \quad \text{locally} \tag{43}$$

where $(\delta\mathcal{O})(y)$ is the infinitesimal transformation of the field. Here, locally means inside any correlation functions with other fields inserted at points away from x and y.

Ward identities are analogous to Gauss law in electrodynamic –but within statistical field theory– because, if we view the variation $(\delta\mathcal{O})$ of a field $\mathcal{O}$ as coding for its charge, we infer from the Ward identities that the flux of the current through a ball $\mathcal{B}$ is equal to the sum of all charges inside this ball. They give constraints and information on correlation functions.

• *Charge currents.*

Let us give a simple example: $u(1)$ symmetry for a complex field ϕ with action

$$S[\phi] = \int d^D x \,\big[\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi^*) + V(\phi\phi^*)\big].$$

The Lagrangian is clearly invariant under the $u(1)$ symmetry $\phi \to e^{i\alpha}\phi$ with α real. The infinitesimal transformation is $\phi \to \phi + i\alpha\phi$ and $\phi^* \to \phi^* - i\alpha\phi^*$. It is clear that the measure $[D\phi]$ is invariant under this transformation, so that the Jacobian is one. The Noether current is (we have absorbed a factor i)

$$J^\mu = i(\phi\,\partial^\mu\phi^* - \phi^*\partial^\mu\phi^*)$$

This current is conserved $\partial_\mu J^\mu = 0$ away from operator insertions (and inside correlation functions).

Let us look at what happens if there are field insertions. We start with the expectation values $\langle\phi(y_1)\cdots\phi(y_p)\cdot\phi^*(z_1)\cdots\phi^*(z_q)\rangle$. When doing the change of variables, we have to implement the field transformation. Hence we get

$$\begin{aligned}&\langle(\partial_\mu J^\mu)(x)\,\phi(y_1)\cdots\phi(y_p)\cdot\phi^*(z_1)\cdots\phi^*(z_q)\rangle\\ &= i\big(\sum_j \delta(x-y_j) - \sum_k \delta(x-z_k)\big)\langle\phi(y_1)\cdots\phi(y_p)\cdot\phi^*(z_1)\cdots\phi^*(z_q)\rangle.\end{aligned}$$

That is, locally within any correlation functions (with other field insertions away from the point x and y) we have

$$\begin{aligned}(\partial_\mu J^\mu)(x)\,\phi(y) &\simeq i\delta(x-y)\phi(y),\\ (\partial_\mu J^\mu)(x)\,\phi^*(y) &\simeq -i\delta(x-y)\phi^*(y), \quad \text{locally.}\end{aligned}$$

The operator ϕ has charge $+1$, its complex conjugated ϕ^* charge -1. Note that if we integrate the previous Ward identity over a ball of infinitely large radius, the l.h.s. vanishes (assuming that J^μ decrease fast enough at infinite (which is a statement about the absence of symmetry breaking – invariance of the vacuum), while the r.h.s. is proportional to $(p-q)$. Hence the above expectation is non-zero only if $p = q$, i.e. only if the total charge is balanced (a result that we could have obtained directly from the invariance of the measure). This is charge conservation.

• *The stress-tensor.*

Most of the field theories we are studying here are translation and rotation invariant. As a consequence there are Ward identities associated to those symmetries. They are related –by construction or by definition– to properties of the stress-tensor which codes for the response of the action —or more generally for the statistical measure— under a small space diffeomorphism. If we change variable $x \to x + \epsilon\,\xi(x)$ with $\epsilon \ll 1$ and $\xi^\mu(x)$ some vector field then, by definition, the action varies as

$$\delta S[\Phi] = \epsilon \int d^D x (\partial_\mu \xi^\nu(x))\, T^\mu_\nu(x),$$

with the stress-tensor. See the exercise Section for an illustration.

• *Anomalies.*

We here make a few comments on cases for which the Jacobian associated to the change of variable $\phi \to \hat{\phi}$ is non-trivial. In such case, the current is not anymore conserved. This is call a (quantum) anomaly [4].

Recall that the transformations we look at are local in the field variables. They are of the form $\phi(x) \to \hat{\phi}(x) = \phi(x) + \epsilon(x)(\delta\phi)(x) + \cdots$. Thus it is natural to expect that the default for the Jacobian to be trivial is going to be local in the small parametr $\epsilon(x)$. That is: we expect to first order in $\epsilon(x)$ that

$$\mathrm{Det}\big[\frac{D\phi}{D\hat{\phi}}\big] = \mathrm{Det}\big[\mathbb{I} - \epsilon\frac{D\delta\phi}{D\hat{\phi}} + \cdots\big] = 1 + \int d^D x\, \epsilon(x)\, F(x) + \cdots,$$

for some field functional $F(x)$. Such ansatz is justified by using a formal generalization to infinite determinant of the expansion $\mathrm{Det}[1+\epsilon M] = 1+\epsilon\mathrm{Tr}(M)+\cdots$ valid for finite dimensional matrices. To compute $F(x)$ is of course much harder, it requires regularizing the infinite dimensional Jacobian, and it is very much model dependent.

Going back to the previous derivation of the Ward identities but taking this non-trivial Jacobian into account, it is easy to deduce (following exactly the same steps) that the following relation,

$$\partial_\mu J^\mu(x) = F(x),$$

is valid inside any correlation functions (with other fields inserted away from point x). That is: due to the non-triviality of the Jacobian, the current conservation law is transformed into the new equation of motion above. This is call an anomaly: something which was expected to be zero by classical consideration turns not to vanish in the statistical theory because of fluctuations.

6.3 Generating functions

We consider Euclidean field theory with action (typically) of the form $S[\phi] = \int d^D x \left(\frac{1}{2}(\partial\phi)^2 + V(\phi)\right)$ and its generalizations (with more field components, non-trivial background, etc...). In this Section we introduce standard generating function for field correlation functions. They are

[4] A baby example illustrating this fact has been described in the Chapter 2: there, we have seen that the irregularity of the Brownian path requires regularizing the stochastic integrals and this induces possibly non trivial Jacobians, depending on the regularization scheme.

analogous to generating functions of random variables and/or to thermodynamical functions of statistical physics.

• *Generating functions.*

QFT or SFT deals with random objects and their statistics. The random variables are the fields and the field configurations. As for any random variables one may introduce their generating functions (which almost characterize their distributions). Here it amounts to introduce a source so that the action now becomes (by definition)

$$S[\phi; j] := S[\phi] - \int d^D x\, j(x)\phi(x).$$

One defines the partition function in presence of the source

$$Z[j] := \int [D\phi]\, e^{-S[\phi;j]}. \tag{44}$$

A convenient normalization is $Z[j = 0] = 1$ which fixes the normalization of the measure (but this may hides some properties, say the dependence on the background metric,...). Alternatively one may not normalize the generating function but consider the ration $Z[j]/Z[0]$. It may be worth comparing this definition with that of the generating functions in the case of the free field.

This is a generating function for correlation functions. The latter can be obtained by differentiation:

$$\langle \phi(x_1) \cdots \phi(x_N) \rangle = \frac{1}{Z[0]} \frac{\delta^N Z[j]}{\delta j(x_1) \cdots \delta j(x_N)}\Big|_{j=0}.$$

As for random variables, one may (preferably) introduce the generating function of the cumulants which in diagrammatic perturbation theory are going to correspond to connected correlation functions. It is usually denoted as $W[j]$ and defined by

$$W[j] = \log Z[j], \quad Z[j] = e^{W[j]}, \tag{45}$$

normalized by $W[j = 0] = 0$ (that is: we use the normalization $Z[j = 0] = 1$ as otherwise we would have to consider the ratio $Z[j]/Z[0]$). It is the generating function of (so-called) connected correlation functions:

$$\langle \phi(x_1) \cdots \phi(x_N) \rangle^{\mathrm{c}} = \frac{\delta^N W[j]}{\delta j(x_1) \cdots \delta j(x_N)}\Big|_{j=0}.$$

We could alternatively write $W[j]$ as a series expansions:

$$W[j] = \sum_{N \geq 0} \frac{1}{N!} \int d^D x_1 \cdots d^D x_N \, \langle \phi(x_1) \cdots \phi(x_N) \rangle^{\mathrm{c}} \, j(x_1) \cdots j(x_N).$$

Of course there is an analogy with thermodynamical potentials: $Z[j]$ is the partition function, and $W[j]$ is minus the free energy (up to a coefficient proportional to the temperature). Hence (as usual in thermodynamics) we expect $W[0]$ to be proportional to the space volume, and $W[j] - W[0]$ to be finite if the source are localized in space.

Of course one can write all these formula in Fourrier space. See below.

Before closing this subsection, let us recall the formula for the generating function in the case of Gaussian free field

$$Z[j] = \exp\big[\frac{1}{2} \int d^D x d^D y\, j(x)\, G(x, y)\, j(y)\big],$$

with $G(x,y)$ the Green function. Hence $W[j]$ is quadratic for a free Gaussian theory.

• *The effective action.*

One may introduce other functions via Legendre transform as done with thermodynamic functions. In field theory, a particularly important one is the so-called effective action (or vertex function) defined as the Legendre transform of $W[j]$:

$$\Gamma[\varphi] := \int d^D x\, j(x)\varphi(x) - W[j], \quad \varphi(x) = \frac{\delta W[j]}{\delta j(x)}. \tag{46}$$

The field $\varphi(x)$ is sometimes called the background field. One may of course expand the effective action in power of φ:

$$\Gamma[\varphi] = \sum_N \frac{1}{N!} \int d^D x_1 \cdots d^D x_N\, \varphi(x_1)\cdots\varphi(x_N)\, \Gamma^{(N)}(x_1, \cdots, x_N),$$

or equivalently

$$\Gamma^{(N)}(x_1, \cdots, x_N) = \frac{\delta^N \Gamma[\varphi]}{\delta\varphi(x_1)\cdots\delta\varphi(x_N)}\Big|_{\varphi=0}.$$

The functions $\Gamma^{(N)}$ are called the N-point vertex functions. In perturbation theory they will be related to one-particle irreducible diagrams (see below).

The effective action contains all the information on the theory, in a synthetic way if not the most possible compact way.

• *The effective potential and vacuum expectation values.*

As usual with Legendre transform, the inverse formula for reconstructing the function $W[j]$ from its Legendre transform $\Gamma[\varphi]$ is:

$$W[j] = \int d^D x\, j(x)\varphi(x) - \Gamma[\varphi], \quad j(x) = \frac{\delta\Gamma[\varphi]}{\delta\varphi(x)}.$$

The relation $j(x) = \frac{\delta\Gamma[\varphi]}{\delta\varphi(x)}$, or reciprocally $\varphi(x) = \frac{\delta W[j]}{\delta j(x)}$ allow to express the source j as a function of the background field φ, or reciprocally the background field as a function of the source j.

Let $\varphi_{\rm vac}(x) := \langle\phi(x)\rangle|_{j=0}$ be the field expectation value (in absence of external source), also called the vacuum expectation value. Because $\langle\phi(x)\rangle|_{j=0} = \frac{\delta W[j]}{\delta j(x)}|_{j=0}$ by definition of the generating function $W[j]$, we have :

$$\varphi_{\rm vac}(x) = \varphi(x)|_{j=0} = \langle\phi(x)\rangle|_{j=0}.$$

From the relation $j(x) = \frac{\delta\Gamma[\varphi]}{\delta\varphi(x)}$ taken at $j=0$ we learn that the field expectation at zero external source is a minimum of the effective action:

$$\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)}\Big|_{\varphi_{\rm vac}} = 0. \tag{47}$$

Hence the name of effective action: the field expectation value is the minimum of the effective action.

One also defines the effective potential $V_{\rm eff}(v)$ as the effective action evaluated on constant field configuration (in particular in case of translation invariance). For a constant field configuration the effective action is expected to be proportional to the space volume (by analogy with thermodynamic function which are extensive and thus proportional to the volume of the system). The definition of the effective potential is

$$\Gamma[\varphi]|_{\varphi(x)=v} = \text{Vol. } V_{\rm eff}(v), \tag{48}$$

with Vol. the volume of the system. For homogeneous system, the vacuum expectation value is thus the minimum of the effective potential. This is important –and very useful– in analyzing physical phenomena, in particular in case of symmetry breaking (because then the existence of non trivial minimum of the effective potential will be the sign of symmetry breaking).

In case of non-homogeneous background, the vacuum expectation $\langle\phi(x)\rangle$ is not uniform, it may be x-dependent. Eq.(47) then yields a series of equations satisfied by $\varphi_{\rm vac}(x)$, which may be thought of as the equations of motion for $\varphi_{\rm vac}(x)$. In general, these equations may be non-local in space because the effective action can be non-local. However, if we are only looking at slowly varying vacuum configurations, so that $\varphi_{\rm vac}(x)$ is a smooth, slowly varying, function we may (sometimes) do a gradient expansion of the expansion and approximate it by truncating this expansion to lowest order.

• *Generating functions in Fourier space.*

Of course we can write all these generating function in Fourier space. For instance for the effective action, we decompose the background field on its Fourier components via $\hat\varphi(k) = \int d^Dx\, e^{-ik\cdot x}\,\varphi(x)$, and reciprocally $\varphi(x) = \int \frac{d^Dk}{(2\pi)^D}\, e^{ik\cdot x}\,\hat\varphi(k)$. Then the effective action becomes

$$\Gamma[\varphi] = \sum_N \frac{1}{N!}\int \frac{d^Dk_1}{(2\pi)^D}\cdots\frac{d^Dk_N}{(2\pi)^D}\,\hat\varphi(-k_1)\cdots\hat\varphi(-k_N)\,\tilde\Gamma^{(N)}(k_1,\cdots,k_N),$$

with N-point functions

$$\Gamma^{(N)}(x_1,\cdots,x_N) = \int\prod_{j=1}^N\frac{d^Dk_j}{(2\pi)^D}\,\prod_{j=1}^N e^{ik_j\cdot x_j}\,\tilde\Gamma^{(N)}(k_1,\cdots,k_N),$$

If translation invariance holds (which is often the case) then

$$\tilde\Gamma^{(N)}(k_1,\cdots,k_N) = (2\pi)^D\,\delta(k_1+\cdots+k_N)\,\hat\Gamma^{(N)}(k_1,\cdots,k_N).$$

by momentum conservation (imposing the conservation law $k_1+\cdots+k_N=0$ is a consequence of translation invariance).

Similar formula hold for the other generating functions, say that of connected correlation functions.

• *Two-point correlation and vertex functions.*

There are of course relations between the vertex functions and the connected correlation functions. We shall see below that all correlation functions can (perturbatively) reconstructed from the vertex functions, so that the latter contain all the information on the theory. Let us start from the two point functions. We have the connected 2-point function $G_c^{(2)}(x,y)$ and the

2-point vertex function $\Gamma^{(2)}(x,y)$. By translation invariance, we can write them in Fourier space as (we have extracted the delta-function ensuring momentum conservation).

$$G_c^{(2)}(x,y) = \int \frac{d^Dk}{(2\pi)^D}\, e^{ik\cdot(x-y)}\, \hat{G}_c^{(2)}(k), \quad \Gamma_c^{(2)}(x,y) = \int \frac{d^Dk}{(2\pi)^D}\, e^{ik\cdot(x-y)}\, \hat{\Gamma}^{(2)}(k).$$

We wrote $G_c^{(2)}(k,p) = (2\pi)^D\, \delta(k+p)\, \hat{G}_c^{(2)}(k)$, and similarly for Γ, in order to extracte the delta-function ensuring momentum conservation.

We are now going to prove that these two functions are kernels of inverse operators. In Fourier space this translates into

$$\hat{\Gamma}^{(2)}(k)\, \hat{G}_c^{(2)}(k) = 1. \tag{49}$$

Thus knowing $\hat{\Gamma}^{(2)}(k)$ is enough to reconstruct the connected two-point function. Indeed recall that $\varphi(x_1) = \frac{\delta W[j]}{\delta j(x_1)}$. Differentiating this relation with respect to $\varphi(x_2)$ and then taking the limit $j=0$ yields:

$$\begin{aligned}
\delta(x_1-x_2) &= \frac{\delta}{\delta\varphi(x_2)} \frac{\delta W[j]}{\delta j(x_1)}\Big|_{j=0} = \int d^Dy\, \frac{\delta j(y)}{\delta\varphi(x_2)} \frac{\delta^2 W[j]}{\delta j(x_1)\delta j(y)}\Big|_{j=0} \\
&= \int d^Dy\, \langle\phi(x_1)\phi(y)\rangle^c \frac{\delta j(y)}{\delta\varphi(x_2)}\Big|_{j=0} \\
&= \int d^Dy\, \langle\phi(x_1)\phi(y)\rangle^c\, \Gamma^{(2)}(y,x_2) = \int d^Dy\, G_c^{(2)}(x_1,y)\, \Gamma^{(2)}(y,x_2)
\end{aligned}$$

In the first line we used the chain rule to compute the derivative. In the second and third line we used the fact that $W[j]$ and $\Gamma[\varphi]$ are the generating function of the connected and vertex functions, respectively. In Fourier space, this gives the above relation $\hat{\Gamma}^{(2)}(k)\, \hat{G}_c^{(2)}(k) = 1$.

6.4 Perturbation theory and Feynman rules

There is not so many ways to evaluate a path integral: if it is not gaussian, the only available generic methods are either via saddle point approximations, via symmetry arguments, or via perturbation theory. One has to remember that, although generic and sometimes useful, perturbation theory give, at best, series which are only asymptotic series (not convergent series) and they miss all non perturbative effects (say exponentially small but important effect).

We shall examplify the perturbation theory in the case of the ϕ^4 theory for which the interaction potential is $V(\phi) \propto \int d^Dx\, \phi^4(x)$.

• *Perturbative expansion.*

To compute perturbatively we split the action in its quadratic part plus the rest:

$$S[\phi] = S_0[\phi] + S_I[\phi].$$

In the case of ϕ^4 theory, the interacting part is

$$S_I[\phi] = \frac{g}{4!} \int d^Dy\, \phi^4(y).$$

Let $\langle \cdots \rangle_0$ denote the expectation with the measure defined by the unperturbed action $S_0[\phi]$. We may be interested in computing the partition function and the N-point functions. The partition function reads

$$Z = \int [D\phi]\, e^{-S_0[\phi]-S_I[\phi]} = Z_0 \, \langle e^{-\frac{g}{4!}\int d^D y\, \phi^4(y)} \rangle_0,$$

with $Z_0 = 1$ by convention. The partition function Z is also called the vacuum expectation. The correlation functions are

$$\langle \phi(x_1) \cdots \phi(x_N) \rangle = \frac{\langle \phi(x_1) \cdots \phi(x_N)\, e^{-\frac{g}{4!}\int d^D y\, \phi^4(y)} \rangle_0}{\langle e^{-\frac{g}{4!}\int d^D y\, \phi^4(y)} \rangle_0}$$

Notice that one has to divide by the partition function in the formula for the N-point functions in order to preserve the normalization (the expectation value of 1 is 1).

It is convenient to introduce the "un-normalized" expectation values:

$$\mathcal{G}^{(N)}(x_1, \cdots, x_N) := \langle \phi(x_1) \cdots \phi(x_N)\, e^{-\frac{g}{4!}\int dy\, \phi^4(y)} \rangle_0,$$

so that

$$\langle \phi(x_1) \cdots \phi(x_N) \rangle = \mathcal{G}^{(N)}(x_1, \cdots, x_N)/Z.$$

Expanding in power of g we get the perturbative series:

$$\begin{aligned} \mathcal{G}^{(N)}(x_1, \cdots, x_N) &= \sum_{k \geq 0} (-g)^k \, \mathcal{G}_k^{(N)}(x_1, \cdots, x_N), \\ \mathcal{G}_k^{(N)}(x_1, \cdots, x_N) &= \frac{1}{k!}(\frac{1}{4!})^k \int d^D y_1 \cdots d^D y_k \, \langle \phi(x_1) \cdots \phi(x_N)\, \phi^4(y_1) \cdots \phi^4(y_k) \rangle_0 \end{aligned}$$

Of course we can write similar formula for the perturbative series in Fourier space.

They are evaluated using Wick's theorem with propagator $G_0(x-y)$. It is important to remember that the theory comes equipped with its UV cut-off (a the lattice spacing or $\Lambda = 1/a$ the momentum cut-off). The unperturbed propagator is

$$G_0(x-y) = \int_{|k|<\Lambda} \frac{d^D k}{(2\pi)^D} \, \frac{e^{ik\cdot(x-y)}}{k^2+m^2}.$$

It is convenient to denote its Fourier transform by $\Delta(k)$:

$$\Delta(k) = \frac{1}{k^2+m^2}. \tag{50}$$

It is called the propagator.

Notice that doing this expansion amounts to exchange (path) integration and series expansion, a process which is (often) ill-defined. In general we are going to get, at best, series which are only asymptotic series (not convergent series) missing non perturbative effects.

• *Feynman rules.*

The perturbation expansion is going to be coded into diagrams representing the different terms or contractions involved in Wick's theorem.

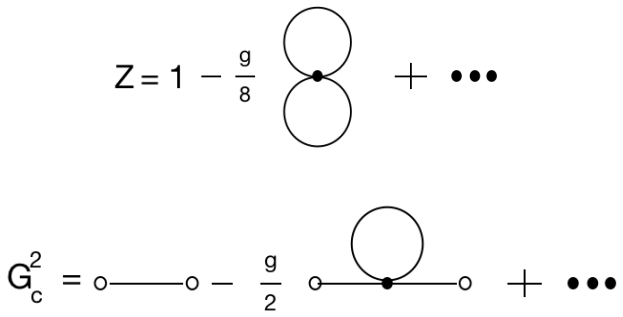

Figure 1: *First order expansion of the partition function and the connected two-point function.*

Let us look at the first few terms. The first order in the vacuum expectation is

$$Z = 1 - \frac{g}{4!}\int d^D y\, \langle \phi^4(y)\rangle_0 + \cdots .$$

To evaluate this integrated expectation value, it is better and simpler to imagine splitting the four coincident points in four neighbor points at position $y_1, \cdots y_4$. Then we need to evaluate $\langle \phi(y_1)\phi(y_2)\phi(y_3)\phi(y_4)\rangle_0$ using Wick's theorem. We get three terms,

$$\langle \phi(y_1)\phi(y_2)\rangle_0 \langle \phi(y_3)\phi(y_4)\rangle_0 + \langle \phi(y_1)\phi(y_3)\rangle_0 \langle \phi(y_2)\phi(y_4)\rangle_0 + \langle \phi(y_1)\phi(y_3)\rangle_0 \langle \phi(y_2)\phi(y_4)\rangle_0.$$

It is better to draw the diagrams with the four points slightly split (because then the enumeration of those diagrams is simpler). We now remember that the points are actually coinciding, so that we get three times the same diagrams corresponding to $\langle \phi(x)\phi(x)\rangle_0^2$. Hence (by translation invariance)

$$Z = 1 - \frac{g}{8}\int d^D y\, G_0(0)^2 + \cdots = 1 - \frac{g}{8}\,\mathrm{Vol.}\, G_0(0)^2 + \cdots .$$

Recall that

$$G_0(0) = \int_{|k|<\Lambda} \frac{d^D p}{(2\pi)^D}\, \frac{1}{p^2 + m^2}.$$

Notice that $G_0(0) \simeq \frac{S_{D-1}}{(2\pi)^D}\, \Lambda^{D-2} + \cdots$ at large Λ, with S_D the volume of the unit sphere in dimension D: $S_D = 2\pi^{D/2}/\Gamma(D/2)$. Hence it diverges for $D > 2$.

Let us now look at the two point function. At first order we have

$$\mathcal{G}^{(2)}(x_1, x_2) = G_0(x_1 - x_2) - \frac{g}{4!}\int d^D y\, \langle \phi(x_1)\phi(x_2)\phi^4(y)\rangle_0 + \cdots .$$

Again, each term is evaluate using Wick's theorem which is done by pairing the points. To correctly count the number of pairings it is again simpler to slightly split the point y into $y_1, \cdots, y_4$. They are two type diagrams: connected or disconnected (the disconnected one are cancelled when dividing by the vacuum expectation value). The number of ways to get the disconnected diagrams is 3 (the same as above for Z), the number of ways to obtained the

connected diagram is $(4 \cdot 3)$ (use the splitting method to avoid mistake when counting the number of ways to obtain a given diagram). Hence

$$\mathcal{G}^{(2)}(x_1, x_2) = G_0(x_1 - x_2) - \frac{g}{8} G_0(x_1 - x_2) G_0(0)^2 - \frac{g}{2} \int d^D y \, G_0(x_1 - y) G_0(y - x_2) G_0(0) + \cdots$$

The factor $\frac{1}{8}$ comes from $\frac{1}{4!} \times 3$ and $\frac{1}{2}$ comes from $\frac{1}{4!} \times (4 \cdot 3)$. We are only interested in the connected diagrams as the other ones can be deduced form those. See Figure 6.4 for the graphical representation of this formula.

We can understand what are the Feynman rules in position space for connected diagrams. We write them for the ϕ^4 theory (but the generalization is clear):
(i) draw all topologically distinct connected diagrams with N external lines and each internal vertex attached to 4 lines;
(ii) to each line associate a factor $G_0(x - x')$ (where x and x' can be either an internal or external vertex);
(iii) to each internal vertex associate a factor $(-g)$;
(iv) integrate over internal vertices with measure $\int \prod_k d^D y_k$;
(v) multiply by the symmetry factor 1/(integer) (this symmetry factor is more easily computed by splitting virtually the internal points: it combines the number of equivalent pairings with the factor $k!(1/4!)^k$ coming form the expansion).

The elements, line and vertices, of the Feynamn graphs are:

$$x\underline{\qquad\quad}y \;= G_0(x - y) \quad ; \quad {}^{y_1}_{y_2}\!\times^{y_3}_{y_4} \;= (-g)\,.$$

The convention is to sum over all internal labels $y_k$with the measure $\prod_k d^D y_k$.

A way to draw all diagrams may consist in drawing all external points as small circles with on leg emerging from each of them, all internal vertices with dots at the center of small four legs crossing (or a different number of legs if the interaction is different from ϕ^4), and connect these seeds of legs in all possible ways. Given a diagram obtained that way, the symmetry factor is $k!(1/4!)^k$ times the number of ways of obtaining it by connection the emerging legs. It may a good idea to practice a bit.

• *Feynman rules in momentum space.*

We can write the perturbative expansion in momentum space. Let us look only at the connected diagrams. Consider for instance the connected two-point functions.

$$\mathcal{G}_c^{(2)}(x_1, x_2) = \int \frac{d^D k}{(2\pi)^D} \, e^{ik \cdot (x_1 - x_2)} \, \mathcal{G}_c^{(2)}(k),$$

Recall its perturbative expansion given above. If Fourier space, this reads

$$\begin{aligned}(k^2 + m^2)\, \mathcal{G}_c^{(2)}(k) &= 1 - \frac{g}{2} \Delta(k)\, G_0(0) + \cdots \\ &= 1 - \frac{g}{2} (\frac{1}{k^2 + m^2}) \int_{|k| < \Lambda} \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 + m^2} + \cdots\end{aligned}$$

with $\Delta(k) = 1/(k^2 + m^2)$. The important point is that integration on the internal positions imposes momentum conservation at the vertex.

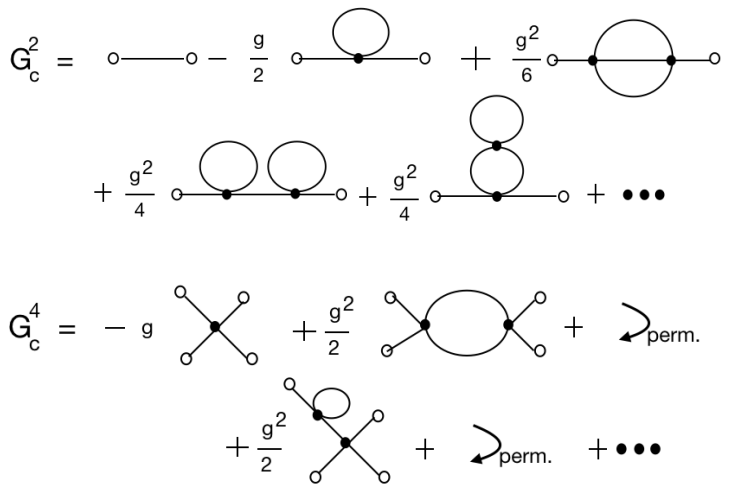

Figure 2: *Second order expansion of the* 2 *and* 4 *point connected functions.*

We then get the Feynman rules in the momentum space for the connected diagrams (again we write them for the ϕ^4 theory):
(i) draw all topologically distinct connected diagrams with N external lines and each internal vertex attached to 4 lines;
(ii) assign momenta flowing along each line so that the external lines have momenta k_j and momentum is conserved at each internal vertex;
(iii) to each internal vertex associate a factor $(-g)$;
(iv) integrate over remaining loop momenta with measure $\int \prod_k d^D p_k/(2\pi)^D$;
(v) multiply by the symmetry factor 1/(integer)

We can look at more diagrams, say at some of the order two diagrams for the four-point functions. See Figure 6.4.

Another way to read the Feynman diagram in Fourier space is to write directly the action is Fourier space. Recall that we define the Fourier components of the field via $\phi(x) = \int \frac{d^D k}{(2\pi)^D}\, e^{ik\cdot x}\, \hat{\phi}(k)$. We can write the free and the interaction parts of the action as

$$\begin{aligned} S_0[\Phi] &= \int \frac{d^D k}{(2\pi)^D}\, \hat{\phi}(-k)(k^2+m^2)\hat{\phi}(k), \\ S_I[\Phi] &= \frac{g}{4!}\int \frac{d^D k_1}{(2\pi)^D}\cdots\frac{d^D k_4}{(2\pi)^D}\,(2\pi)^D\delta(k_1+k_2+k_3+k_4)\,\hat{\phi}(k_1)\hat{\phi}(k_2)\hat{\phi}(k_3)\hat{\phi}(k_4). \end{aligned}$$

From which we read the graphical representation:

$$^{k}\!\rightarrow\!\!\!\!-\!\!\!\!\leftarrow^{p} = (2\pi)^D\delta(k+p)\,\Delta(k) \quad ; \quad \begin{matrix} k_1 & k_3 \\ k_2 & k_4 \end{matrix} = (-g)\,(2\pi)^D\delta(k_1+k_2+k_3+k_4).$$

The convention is to sum over al the internal labels p_k with the measure $\prod_k d^D p_k/(2\pi)^D$.

A point which could be tricky sometimes (it actually is often tricky...) is the symmetry factor. One can give a formal definition, by look at the order of the symmetry group of the

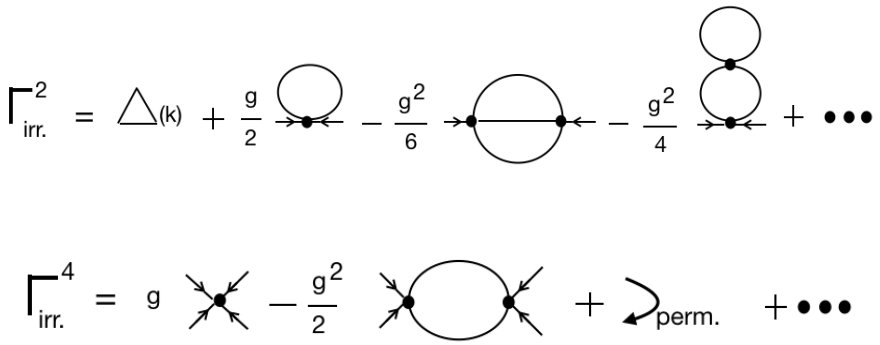

Figure 3: *Second order expansion of the* 2 *and* 4 *point vertex functions.*

diagram (based on the fact that indexing the point in the splitting method amounts to index the lines arriving at a vertex). But this is probably not useful at this level as one can always go back to the Wick's theorem expansion to get it (via the splitting method).

6.5 Diagrammatics

• *Connected and one-particle irreducible (1-PI) graphs.*

It is clear that the generating function of connected graph is the logarithm of the generating function of all graphs (connected and disconnected). This simply follows usual combinatorics encoded into the exponential function. Let us recall the order two expansion (at order g^2 included) of the 2 and 4 point connected functions. See Figure 6.4.

We see that diagrams of higher order includes sub-diagrams of lower order. This leads to the following definition of irreducible graphs, also called one-particle irreducible diagrams and denote 1-PI. By definition, the irreducible graphs are the connected graphs which do not become disconnected if one of their lines is cut. See Figure 6.5.

All these diagrams have external legs, which are simple multiplicative factors in the momentum representation. These factors are the propagators $\Delta(k)^{-1}$ with k the momentum of the corresponding leg. Thus we define the truncated irreducible diagrams as the irreducible diagrams but with the external legs removed (i.e. truncated). We represent these truncated diagrams by erasing the external propagators but indicating the incoming momenta by arrows.

Finally, we define the irreducible N-point functions. Their differ for $N = 2$ and $N \geq 2$:
— For $N \neq 2$: the irreducible N-point functions is "minus" the sum of the N-point irreducible diagrams;
— For $N \neq 2$: we add to this sum the zeroth order contribution so that the irreducible 2-point functions is the propagator minus the sum of all 2-point irreducible diagrams.

These generating functions are also called vertex functions.

• *Two-point functions*

It can be proved that the effective action $\Gamma[\varphi]$ is the generating function of the irreducible N-point functions. See below and the exercise Section. This justifies that we denote them with the same letter.

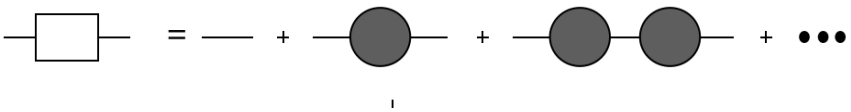

Figure 4: *Series expansion of the* 2 *point connected functions.*

The last statement can be checked on the two point function. According to the above definition, let us decompose the two-point vertex function as

$$\hat{\Gamma}^{(2)}(k) = \hat{\Gamma}^{(2)}_0(k) - \Sigma(k), \tag{51}$$

with $\hat{\Gamma}^{(2)}_0(k) = \Delta^{-1}(k)$ the propagator, so that $\Sigma(k)$ is the sum of irreducible graphs containing at least one loop which remains once the external legs have been removed.

We now explain the relation between $\Sigma(k)$ and the connected two-point function $\hat{G}^{(2)}_c(k)$. Let us look at their diagram expansions. Each time we cut a link which disconnect a diagram of the two point functions, we produce two diagrams with two external line each, that is we produce two diagrams which contribute to the diagram expansion of these functions. Hence, cutting all the lines up to the point we reach the irreducible components produces product of diagrams of $\Sigma(k)$. As a consequence, the connected two-point function can be decomposed as a chain of Σ joined by intermediate propagators. See Figure 6.5.

In equation, this translates into

$$\begin{aligned} \hat{G}^{(2)}_c(k) &= \Delta(k) + \Delta(k)\Sigma(k)\Delta(k) + \Delta(k)\Sigma(k)\Delta(k)\Sigma(k)\Delta(k) + \cdots \\ &= \frac{1}{\Delta^{-1}(k) - \Sigma(k)} = \hat{\Gamma}^{(2)}(k)^{-1}. \end{aligned}$$

Thanks to the relation $\hat{\Gamma}^{(2)}(k)\,\hat{G}^{(2)}_c(k) = 1$, this shows that the diagrammatic definition of the 2-point irreducible function coincide with the vertex function defined via the effective action. We will see below that this statement hold for all N-point functions.

• *Effective action and 1-PI diagrams*

First let us observe that any connected diagram can be decomposed uniquely as a tree whose vertices are its 1-PI components (some of the irreducible components may reduced to single vertex). This simplest proof is graphical. As a consequence, all connected correlation functions can written as tree expansion with propagator the two-point connection function $\Gamma^{(2)}$ and as vertices the irreducible 1-PI vertex functions $\Gamma^{(N)}$. See Figure 6.5.

One can prove that the effective action is the generating function for 1-PI diagrams. See the exercise Section. All correlation functions, connected or not, can but recursively reconstructed from the 1-PI diagrams. Thus those diagrams contain all information on the theory, and all information on the theory is contained in the effective action or equivalently in the irreducible functions.

See the exercise Section for a proof of the equivalence between the generating function of 1-PI diagrams and the effective action.

We stop here for the (slightly boring) diagrammatic gymnastic.

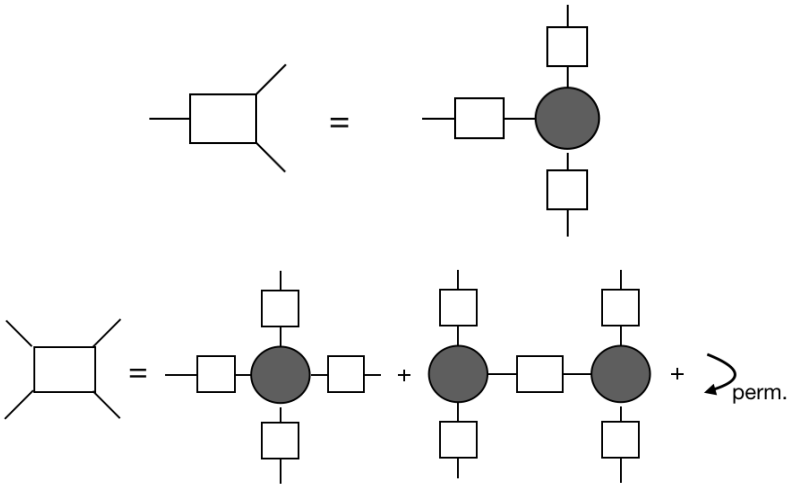

Figure 5: *The* 3 *and* 4 *point connected functions in terms of the vertex functions.*

• *The Euler characteristic of a graph and two useful formula.*

The give here two tools which are often useful in analysing perturbative graphs: one relates the topology of the graph to its order of the perturbative expansion, the other is useful representation of elements of Feynman loop integrants.

— Let G be a graph, and let V its number of vertices, E its number of edges and F its number of faces. Then the Euler characteristic is defined as $\chi(G) = V - E + F$. One has $\chi(G) = 2 - g$ where g is the so-called genus $g \leq 0$). This number codes the topology of the surface on which the graph can be drawn: g is the (minimal) number of handle of that surface, $g = 0$ corresponds to the sphere, $g = 1$ to the torus, etc. One has to pay attention that in this formula one has to count the external vertices —so that V the number of internal vertices (equal to the order in the perturbative expansion) plus the number of external points— as well as the external face of the graph —so that F is the number of loops plus one. For instance, for a tree the number of faces is 1. It is a good exercise to check this on examples.

— Two useful formulas to compute Feynman integral:

$$\begin{aligned} \frac{1}{p^2+m^2} &= \int_0^\infty du\, e^{-u(p^2+m^2)}, \\ \frac{1}{a_1 \cdots a_n} &= \frac{1}{(n-1)!} \int_{[x_j>0]} \prod_{j=1}^n dx_j \, \frac{\delta(x_1 + \cdots x_n - 1)}{[x_1 a_1 + \cdots x_n a_n]^n}. \end{aligned}$$

This leads to the so-called parametric representation of the Feynman graphs.

6.6 One-loop effective action and potential

Here, we compute the effective action at one-loop (i.e. first order in perturbation theory) for the ϕ^4 theory with action $S[\phi] = \int d^D x \big(\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4\big)$. This amounts to a Gaussian integral around the solution of the classical equations of motion. We are going to prove that, to

leading order in the loop counting parameter $\hbar$ (see below), the effective action is:

$$\Gamma[\varphi] = S[\varphi] + \frac{\hbar}{2}\,\mathrm{Tr}\log\big[S''[\varphi]\big] + O(\hbar^2), \tag{52}$$

where $S''[\varphi]$ is the Hessian of the action (i.e. the matrix operator of double derivatives) evaluated at the field configuration φ. Thus, the correction to the effective action at one-loop is given by the logarithm of the Hessian of the classical action (in a way very similar to the WKB approximation).

• *The one-loop effective action*

The strategy, which follows closely the definition, is to first compute the generating function $W[j]$ as function of the external source and then do the Legendre transformation. The partition function with source is

$$Z[j] = \int [D\phi]\, e^{-\frac{1}{\hbar}S[\phi;j]}, \quad S[\phi;j] := S[\phi] - (j,\phi).$$

We have introduce the notation $(j,\phi) := \int d^D x\, j(x)\phi(x)$, and the coefficient $\hbar$ which serves as loop counting parameter. It justifies the saddle point approximation (one-loop) as $\hbar \to 0$. The classical equation of motion (for ϕ_c) is,

$$j(x) = \frac{\delta S[\phi]}{\delta\phi(x)}\Big|_{\phi_c}.$$

In the case of ϕ^4 this reads

$$-\nabla^2\phi_c + m^2\phi_c + \frac{g}{3!}\phi_c = j.$$

We expand around this classical solution (saddle point): $\phi = \phi_c + \hbar\hat{\phi}$. We only keep the term quadratic in $\hat{\phi}$. Then (here the dots refer to higher order in $\hbar$)

$$S[\phi_c + \hat{\phi}; j] = S[\phi_c; j] + \frac{\hbar^2}{2}\int d^D x d^D y\, \hat{\phi}(x)\, S''[\phi_c](x,y)\, \hat{\phi}(y) + \cdots$$

The quadratic term S'' is the Hessian of S. It defines a kernel acting on functions. At this order (one-loop) the integration over $\hat{\phi}$ is Gaussian. ,

$$Z[j] = Z[0] \times \big[\mathrm{Det}[S''[\phi_c]]\big]^{-1/2}\, e^{-\frac{1}{\hbar}S[\phi_c;j]}\,.$$

Since $W[j] = \hbar\,\log(Z[j]/Z[0])$ (here we put back the factor $\hbar$ in the definition), we thus have

$$W[j] = -S[\phi_c] + (j,\phi_c) - \frac{\hbar}{2}\mathrm{Tr}\log\big[S''[\phi_c]\big] + O(\hbar^2).$$

To compute the effective action, we have to implement the Legendre transform. The background field φ is determined by

$$\varphi(x) := \frac{\delta W[j]}{\delta j(x)} = \frac{\delta}{\delta j(x)}\Big((j,\phi_c) - S[\phi_c]\Big) - \frac{\hbar}{2}\frac{\delta}{\delta j(x)}\mathrm{Tr}\big[S''[\phi_c]\big] + O(\hbar^2).$$

Recall that here ϕ_c is viewed as function of j. Thanks to the equations of motion, $j(x) = \frac{\delta S[\phi]}{\delta\phi(x)}\big|_{\phi_c}$, it is easy to verify that

$$\frac{\delta}{\delta j(x)}\big((j,\phi_c) - S[\phi_c]\big) = \phi_c.$$

The second term is of order $\hbar$. Hence, at this one-loop order, the background field is the classical field:

$$\varphi(x) := \frac{\delta W[j]}{\delta j(x)} = \phi_c(x) + O(\hbar).$$

Then we compute the effective action by Legendre transform: $\Gamma[\varphi] := \langle j,\varphi\rangle - W[j]$. We use the fact that $S[\varphi] = S[\phi_c] + (\varphi - \phi_c, S'[\phi_c]) + O(\hbar)$ and $j = S'[\phi_c]$, to get the announced result: $\Gamma[\varphi] = \Gamma_{1-\text{loop}}[\varphi] + O(\hbar^2)$ with

$$\Gamma_{1-\text{loop}}[\varphi] = S[\varphi] + \frac{\hbar}{2}\text{Tr}\log\big[S''[\varphi]\big].$$

One can compute diagrammatically this action for ϕ^4. The Hessian $S''[\varphi]$ is an operator acting on function via

$$f(x) \to \big(S''[\varphi]\cdot f\big)(x) := -\nabla^2 f(x) + m^2 f(x) + \frac{g}{2}\phi^2(x)\, f(x).$$

We aim at computing $\text{Tr}\log\big[S''[\varphi]\big]$. We factorize the free part so that

$$\Gamma_{1-\text{loop}}[\varphi] = \frac{1}{2}\text{Tr}\log\big[-\nabla^2 + m^2\big] + \frac{1}{2}\text{Tr}\log\big[1 + \frac{g}{2}(-\nabla^2+m^2)^{-1}\varphi^2\big]$$

We can then expand (perturbatively) the logarithm in power series of g. Then (up to the trivial pre-factor, trivial in absence of background)

$$\Gamma_{1-\text{loop}}[\varphi] = \sum_{k=1}^{\infty} \frac{g^k}{2^{k+1}}\,\frac{(-)^{k+1}}{k}\,\Gamma^{(k)}_{1-\text{loop}}[\varphi],$$

with $\Gamma^{(k)}_{1-\text{loop}}[\varphi] = \text{Tr}\log\big[(-\nabla^2+m^2)\varphi^2\big]^k$, or else

$$\Gamma^{(k)}_{1-\text{loop}}[\varphi] = \int d^Dx_1\cdots d^Dx_k\,\varphi^2(x_1)G_0(x_1-x_2)\varphi^2(x_2)\cdots G_0(x_{k-1}-x_k)\varphi^2(x_k)G_0(x_k-x_1).$$

with G_0 the Green function (the inverse) of $(-\nabla^2+m^2)$. These are closed polygonal diagrams with k vertices. They also are all the irreducible one-loop diagrams with k vertex and two truncated external lines at each vertex. These diagrams can be computed for a constant background field which is enough to determine the effective potential.

Notice that for a general potential $V(\phi)$ we would just have to replace $m^2 + \frac{g}{2}\varphi^2$ by $V''(\varphi)$ in the previous formula.

• *One-loop effective action for the Φ^4 theory*

The effective action can be computed for a constant background field φ. This determine the effective potential. This Section can be viewed as part of the main text or as an exercise (in that case the reader is advise not to read the answer before trying to compute independently the one-loop effective potential).

Recall that the effective action is $\Gamma_{1-\text{loop}}[\varphi] = S[\varphi] + \frac{\hbar}{2}\text{Tr}\log\big[S''[\varphi]\big]$ with $S''[\varphi] = -\nabla^2 + V''(\varphi)$. We will specify the potential to the ϕ^4 theory in a little while. For constant back-ground field, the effective action is proportional to the volume of the space and the proportionality coefficient is by definition the effective potential:

$$\Gamma_{1-\text{loop}}[\varphi] = \text{Vol.}\ V^{\text{eff}}_{1-\text{loop}}(\varphi).$$

For constant back-ground field the trace of the differential operator can be computed by going to Fourier space. Thus the one-loop correction to the effective potential is

$$V^{\text{eff}}_{1-\text{loop}} = V(\varphi) + \frac{\hbar}{2}\int \frac{d^Dk}{(2\pi)^D}\ \log\big[k^2 + V''(\varphi)\big].$$

This is clearly UV-divergent. We have to introduce a cut-off Λ and write:

$$V^{\text{eff}}_{1-\text{loop}} = V(\varphi) + \frac{\hbar}{2}\int_{|k|<\Lambda} \frac{d^Dk}{(2\pi)^D}\ \log\Big[\big(k^2 + V''(\varphi)\big)/\Lambda^2\Big]. \tag{53}$$

We have introduced the cut-off scale Λ^2 in the logarithm to make it dimensionless.

Let now fix the dimension $D = 4$. Recall that $S_D/(2\pi)^D = 2/(4\pi)^2$ in $D = 4$. We can then explicitly do the integral over the momentum to obtain [5]

$$V^{\text{eff}}_{1-\text{loop}}(\varphi) = V(\varphi) + \frac{\hbar}{32\pi^2}\Big[\text{const.}_\Lambda + \Lambda^2\, V''(\varphi) + \frac{1}{2}[V''(\varphi)]^2\log\big[\frac{V''(\varphi)}{\Lambda^2}\big] - \frac{1}{4}[V''(\varphi)]^2 + O(\Lambda^{-2})\Big].$$

Here const._Λ is a Λ-dependent diverging constant.

Let us now go back to the ϕ^4 theory to write explicitly $V(\varphi) = \frac{m_0^2}{2}\varphi^2 + \frac{g_0}{4!}\varphi^4$ and $V''(\varphi) = m_0^2 + \frac{g_0}{2}\phi^2$. (We introduce the notation m_0 and g_0 for consistency with the following discussion on renormalization). Let us introduce an arbitrary scale μ^2 in order to extract the logarithmic divergence by writing $\log\big[\frac{V''(\varphi)}{\Lambda^2}\big] = \log\big[\frac{V''(\varphi)}{\mu^2}\big] - \log\big[\frac{\Lambda^2}{\mu^2}\big]$. Gathering all the terms, the one-loop effective potential can be written as

$$V^{\text{eff}}_{1-\text{loop}}(\varphi) = \frac{1}{2!}A_\Lambda\,\varphi^2 + \frac{1}{4!}\,B_\Lambda\,\varphi^4 + \frac{\hbar}{(8\pi)^2}V''(\varphi)\log[\frac{V''(\varphi)}{\mu^2}], \tag{54}$$

with

$$\begin{aligned} A_\Lambda &= m_0^2 + \frac{\hbar g_0}{2}\Big(\frac{\Lambda^2}{(4\pi)^2} - \frac{m_0^2}{(4\pi)^2}\log(\frac{\Lambda^2}{\mu^2})\Big) + O((\hbar g_0)^2), \\ B_\Lambda &= g_0 - \hbar g_0^2\frac{3}{2(4\pi)^2}\log(\frac{\Lambda^2}{\mu^2}) + O(g_0(\hbar g_0)^2) \end{aligned}$$

All the diverging terms are at most of degree 4 in φ. They can thus be absorbed in a renormalization of the mass m and the coupling constant g, see following Chapter. It is an interesting exercise to analyse this potential, its minima, as a function of the parameters m_0, g_0 and Λ.

[5] After a change of variable to $x = k^2$, the integral to reduce to $\int x dx\ \log(x+a)$. This is evaluated by integration by part and using $\int \frac{x^2dx}{x+a} = \frac{x^2}{2} + a^2\log(x+a) - ax$.

6.7 The $O(N)$ vector models

We here present basic properties of the $O(N)$ vector model at large N. It provides an instance of models which can be solved via a saddle point approximation —a method which is a valuable tool in statistical field theory. This approximative method, which can be applied when there exist an appropriate small parameter —in a way similar to the small noise problem in stochastic diffusion equations— becomes exact when $N \to \infty$.

The action of the $O(N)$ vector model is

$$S[\vec{\Phi}] = \int d^D x \, \Big[\frac{1}{2} (\nabla \vec{\Phi})^2 + \frac{1}{2} m_0^2 \, \vec{\Phi}^2 + \frac{g}{4!} \, (\vec{\Phi}^2)^2 \Big],$$

where $\vec{\Phi} = (\Phi^1, \cdots, \Phi^N)$ is a N-component scalar field.

There are various ways to solve this model. We can for instance use diagrammatic expansion techniques and derive exact equations for the propagator which are valid in the large N limit. See the Exercise Section for detail computations concerning this diagrammatic approach. Another way consists in using a saddle point approximation and this is the method we shall pursue.

• *The Hubbard-Stratonovich transform.*

This is a technique, based on Gaussian integrals, which allows to disentangle the interaction at the prize of introducing an extra field. It is similar to the trick we used when representing the lattice Ising partition function in terms of a scalar field. It amounts to represent the contribution of the interaction terms $\int d^D x \, (\vec{\Phi}^2)^2$ by a Gaussian integral. So, let $\sigma(x)$ be a scalar field. Then, by a standard Gaussian integral we have:

$$e^{-\frac{g}{4!} \int d^D x \, (\vec{\Phi}^2)^2} = \int [D\sigma] \, e^{-\int d^D x \, \left[\frac{N}{2} \sigma^2 + i \sqrt{\frac{g}{12}} \, \vec{\Phi}^2 \sigma \right]}.$$

As a consequence, instead of dealing with a theory field with a single field $\vec{\Phi}$ with a quadratic interaction, we can deal with a theory involving two fields with action

$$S[\vec{\Phi}; \sigma] = \int d^D x \, \Big[\frac{N}{2} \sigma^2 + \frac{1}{2} (\nabla \vec{\Phi})^2 + \frac{1}{2} m_0^2 \, \vec{\Phi}^2 + i \sqrt{\frac{g}{12}} \, \vec{\Phi}^2 \sigma \Big].$$

The advantage we gain is that now the action is quadratic in $\vec{\Phi}$, although there is an interaction term between $\vec{\Phi}$ and σ.

• *Saddle point approximation.*

Since the action $S[\vec{\Phi}; \sigma]$ is quadratic in $\vec{\Phi}$, we can easily integrate over this field. The part of the action involving $\vec{\Phi}$ is $\frac{1}{2} \int d^D x \, \vec{\Phi} \big[-\Delta_x + m_0^2 + i \sqrt{\frac{g}{3}} \, \sigma(x) \big] \vec{\Phi}$, with $\Delta_x = \nabla_x^2$ the Laplacian. By integrating over $\vec{\Phi}$ we thus get the following contribution to the Boltzmann weights:

$$[D\sigma] \, e^{-\frac{N}{2} \int d^D x \, \sigma(x)^2} \times \Big| \mathrm{Det} \Big[-\Delta_x + m_0^2 + i \sqrt{\frac{g}{3}} \, \sigma(x) \Big] \Big|^{-N/2},$$

or equivalently, we get the following effective action for σ:

$$S_{\text{eff}}[\sigma] = \frac{N}{2} \Big(\int d^D x \, \sigma(x)^2 + \log \mathrm{Det} \Big[-\Delta_x + m_0^2 + i \sqrt{\frac{g}{3}} \, \sigma(x) \Big] \Big).$$

The noticeable point is that this action is proportional to N so that the σ-path integral can be evaluated via a saddle point approximation in the large N limit.

It is clear that the saddle point is for a uniform configuration $\sigma(x) = \sigma_0^c$. For such configuration the effective reads (by using the formula $\log \mathrm{Det}\, M = \mathrm{Tr}(\mathrm{Log}\, M)$ and representing the action of the Laplacian in Fourier space):

$$S_{\mathrm{eff}}[\sigma] = \frac{N}{2}\,\mathrm{Vol.}\left(\sigma_0^2 + \int \frac{d^D k}{(2\pi)^D} \log\big[k^2 + m_0^2 + i\sqrt{\frac{g}{3}}\,\sigma_0\big]\right).$$

The saddle is thus for σ_0^c with

$$\sigma_0^c + i\sqrt{\frac{g}{3}} \int \frac{d^D k}{(2\pi)^D}\, \frac{1}{k^2 + m_R^2} = 0,$$

with $m_R^2 = m_0^2 + i\sqrt{\frac{g}{3}}\,\sigma_0^c$. This equation is analysed in the Exercise Section, and we recommend to do the corresponding exercise.

Let us just comment, that inserting the saddle point value σ_0^c in the action $S[\vec{\Phi}, \sigma]$ yields an effective action for the field $\vec{\Phi}$,

$$S_{\mathrm{eff}}[\vec{\Phi}] = \int d^D x\, \big[\frac{1}{2}(\nabla\vec{\Phi})^2 + \frac{1}{2} m_R^2\, \vec{\Phi}^2\big],$$

with again $m_R^2 = m_0^2 + i\sqrt{\frac{g}{3}}\,\sigma_0^c$, so that m_R is the renormalized mass. The critical theory is for $m_R = 0$ (because the physical correlation length is $\xi = 1/m_R$). The above saddle point equation fixes the value of the bare critical mass m_0^c as a function of the cut-off and the interaction strength g. By looking at the behaviour of the correlation length as m_0 approach its critical value m_0^c one get information on the correlation length critical exponent ν, which here takes the value $\nu^\infty = 1$.

A refined analysis of this model, including the diagrammatic expansion, the analysis of the critical theory, as well as a study of fluctuations around the saddle point approximation and $1/N$ corrections, is given in a dedicated exercise (that we recommend!). See the Exercise Section.

6.8 Exercises

• *Exercise: Ward identities for the stress-tensor.*

The aim of this exercise is to derive the Ward identities associated to translation symmetry. This will allows us to make contact with the stress tensor.

We consider a scalar field ϕ in D-dimensional Euclidean flat space with action

$$S[\phi] = \int d^D x \, \big[\frac{1}{2}(\nabla\phi)^2 + V(\phi)\big].$$

Translations act on the field as $\phi(x) \to \phi(x-a)$ for any vector a. The infinitesimal transformation is $\phi(x) \to \phi(x) - \epsilon a^\mu (\partial_\mu \phi)(x)$.
(i) let us consider an infinitesimal transformation $\phi(x) \to \phi(x) - \epsilon^\mu(x)(\partial_\mu\phi)(x)$ with the space dependent vector fields $\epsilon(x)$.
Prove that the variation of the action is (assuming that the boundary terms do not contribute)

$$\delta S[\phi] = -\int d^D x \, (\partial^\mu \epsilon^\sigma)(x)\, T_{\mu\sigma}(x) = \int d^D x \, \epsilon^\sigma(x)\, (\partial^\mu T_{\mu\sigma})(x),$$

with $T_{\mu\sigma}(x)$ the so-called stress-tensor ($g_{\mu\sigma}$ is the Euclidean flat metric):

$$T_{\mu\sigma}(x) = \partial_\mu\phi\, \partial_\sigma\phi - g_{\mu\sigma}\big[\frac{1}{2}(\nabla\phi)^2 + V(\phi)\big].$$

(ii) Prove that the stress tensor is conserved, that is: $\partial_\mu T_{\mu\nu}(x) = 0$ inside any correlation functions away from operator insertions.
(iii) Prove the following Ward identities (here we use the notation $\partial_j^\nu = \partial/\partial_{y_j^\nu}$):

$$\langle (\partial_\mu T_\mu^\nu)(x)\, \phi(y_1)\cdots\phi(y_p)\rangle = \sum_j \delta(x-y_j)\, \partial_j^\nu \, \langle \phi(y_1)\cdots\phi(y_p)\rangle,$$

in presence of scalar field insertion of the form $\phi(y_1)\cdots\phi(y_p)$.
(iv) Do the same construction but for rotation symmetry.

• *Exercise: Ward identities for dilatation.*

Consider a free massless Gaussian field in dimension two with action $S[\phi] = \frac{1}{2}\int d^2x\, (\nabla\phi)^2$.
(i) Show that action is dilatation invariant.
Prove that the associated Noether current for dilatation is $D_\mu = T_{\mu\sigma}x^\sigma$.
Show that its conservation imposes that the stress-tensor is traceless: $\partial^\mu D_\mu = T_\mu^\mu = 0$.
(ii) Derive the Ward identities associated to dilatation.

• *Exercise: Two-point correlation and vertex functions.*

Prove that the two-point connected correlation function and the two-point vertex function are inverse one from the other, that is:

$$\hat\Gamma^{(2)}(k)\, \hat G_c^{(2)}(k) = 1,$$

as mentioned in the text.

• *Exercise: The effective potential and magnetization distribution functions*

The aim of this exercise to probability distribution function of the total magnetization is governed by the effective potential — and this gives a simple interpretation of the effective potential.

Let $M_\phi := \int d^D x\, \phi(x)$ be the total magnetization. It is suppose to be typically extensive so let m_ϕ be the spatial mean magnetization, $m_\phi = \text{Vol.}^{-1} M_\phi$.
(i) Find the expression of the generating function of the total magnetization, $\mathbb{E}[e^{zM_\phi}]$, in terms of the generating function $W[\cdot]$ of connected correlation functions.
Recall that if the source $J(x)$ is uniform, i.e. $J(x) = j$ independent of x, then $W[J]$ is extensive in the volume: $W[J(x) = j] = \text{Vol.}\, w(j)$.
(ii) Let $P(m)dm$ be the probability density for the random variable m_ϕ. Show that at large volume, we have

$$P(m) \simeq e^{-\text{Vol.}\, V_{\text{eff}}(m)},$$

with $V_{\text{eff}}(m)$ the effective potential, defined as the Legendre transformed of $w(j)$.

This has important consequence, in particular the most probable mean magnetization is at the minimum of the effective potential, and phase transition occurs when this minimum changes value.

• *Exercise: Effective action and one-particle irreducible diagrams.*

The aim of this exercise is to prove the equality between the effective action and the generating function of 1PI diagrams. To simplify matter, we consider a 'field' made of N ($N \gg 1$) components ϕ^j, $j = 1, \cdots, N$. We view ϕ^j as random variables.

Let us define a 'partition function' $Z_\epsilon[J]$ by

$$Z_\epsilon[J] = \int [D\phi]\, e^{-\epsilon^{-1}\left[\Gamma[\phi] - (J,\phi)\right]}, \quad \text{with } [D\phi] = [\prod_j \frac{d\phi^j}{\sqrt{2\pi\epsilon}}]$$

with J a source $(J, \phi) = J_j \phi^j$, and $\Gamma[\phi]$ an action which we define via its (formal) series expansion (summation over repeated indices is implicit):

$$\Gamma[\phi] = \frac{1}{2}\Gamma^{(2)}_{jk}\phi^j\phi^k - \sum_{n\geq 3}\frac{1}{n!}\Gamma^{(n)}_{j_1\cdots j_n}\phi^{j_1}\cdots\phi^{j_n}.$$

We shall compute this partition function in two different ways: via a saddle point approximation or via a perturbation expansion.

(i) Justify that this integral can be evaluating the integral via a saddle-point when $\epsilon \to 0$.
Prove that

$$\log Z_\epsilon = \frac{1}{\epsilon}\, W[J]\,\big(1 + O(\epsilon)\big),$$

where $W[J]$ is the Legendre transform of the action Γ: $W[J] = (J, \phi_*) - \Gamma(\phi_*)$ with ϕ_* determined via $\frac{\partial\Gamma}{\partial\phi^j}(\phi_*) = J_j$.
Hint : Do the computation formally which amounts to assume that the integral converges and that there is only one saddle point.

Let us now compute $Z_\epsilon[J]$ in perturbation theory. Let us decompose the action as the sum of its Gaussian part plus the rest that we view as the interaction part: $\Gamma[\phi] = \frac{1}{2}\Gamma^{(2)}_{jk}\phi^j\phi^k - \hat{\Gamma}[\phi]$.

(ii) Write

$$Z_\epsilon[J] = \int [D\phi]\, e^{-\frac{1}{2\epsilon}\Gamma^{(2)}_{jk}\phi^j\phi^k}\, e^{\epsilon^{-1}\,\hat{\Gamma}[\phi]}\, e^{\epsilon^{-1}\,(J,\phi)}.$$

We view J/ϵ as source, and we aim at computing the connected correlation function using Feynman diagrams perturbative expansion.
Show that the propagator is $\epsilon\, G^{jk}$ with $G = (\Gamma^{(2)})^{-1}$ and the vertices are $\epsilon^{-1}\,\Gamma^{(n)}_{j_1\cdots j_n}$ with $n \geq 3$.
(iii) Compute the two-, three- and four-point connected correlations $G_{(n)}$, $n = 1, 2, 3$, at the level tree, defined by

$$G^{j_1\cdots j_n}_{(n)} = \frac{\partial^n}{\partial J_{j_1}\cdots\partial J_{j_n}} \log Z_\epsilon[J]\Big|_{\text{tree}}.$$

Show that they are of order ϵ^{-1}. Draw their diagrammatic representations (in terms of propagators and vertices) and compare those with the representations of the connected correlation functions in terms of 1PI diagrams.
(iv) Prove that, when $\epsilon \to 0$, the leading contribution comes only from the planar tree diagrams and that all these diagrams scale like $1/\epsilon$. That is:

$$\log Z_\epsilon[J] = \frac{1}{\epsilon}\Big(\text{planar tree diagrams} + O(\epsilon)\Big).$$

Hint : Recall that, for a connected graph drawn on a surface of genus g (i.e. with g handles, $g > 0$), one has $V - E + L + 1 = 2 - g$ with V its number of vertices, E its number of edges and L its numbers of loops (this is called the Euler characteristics). Then, argue that each Feynman graph contributing to the N point connected functions is weighted by (symbolically) $(\epsilon\, G)^E\, (-\epsilon^{-1}\,\Gamma^{(n)})^{V_{\text{int}}}\, (\epsilon^{-1}\, J)^N$ with $V_{\text{int}} + N$ total number of vertices.

(v) By inverting the Legendre transform, deduce the claim that the effective action is the generating function of 1-PI diagrams.

• *Exercise: Computation of the one-loop effective potential.*

Prove the formula for the one-loop effective potential of the ϕ^4-theory given in the text. Namely

$$V^{\text{eff}}_{1-\text{loop}}(\varphi) = \frac{1}{2!}A_\Lambda\,\varphi^2 + \frac{1}{4!}\,B_\Lambda\,\varphi^4 + \frac{\hbar}{(8\pi)^2}V''(\varphi)\log[\frac{V''(\varphi)}{\mu^2}],$$

with

$$\begin{aligned} A_\Lambda &= m_0^2 + \frac{\hbar g_0}{2}\Big(\frac{\Lambda^2}{(4\pi)^2} - \frac{m_0^2}{(4\pi)^2}\log(\frac{\Lambda^2}{\mu^2})\Big) + O((\hbar g_0)^2), \\ B_\Lambda &= g_0 - \hbar g_0^2\frac{3}{2(4\pi)^2}\log(\frac{\Lambda^2}{\mu^2}) + O(g_0(\hbar g_0)^2) \end{aligned}$$

with μ^2 an arbitrary scale that we introduced by dimensional analysis.
[...Analyse this potential and conclude...]

• *Exercise: Computation of one-loop Feynman diagrams.*

[...To be completed...]

• *Exercise: The $O(N)$ vector model in $D = 3$.*

[... See the exercise booklet...]

7 Conformal field theory: basics

This chapter deals with field theory invariant under conformal transformations, called conformal field theory. These are important because they are fixed point of the renormalization group. They describe statistical field theory with infinite correlation length. They have numerous applications. They will also provide simple examples of statistical field theory which will illustrate basics (but fundamental and generic) properties of statistical field theories.

7.1 The group of conformal transformations

Conformal transformation act locally as dilatation and rotation, but with dilatation coefficients and rotation angles which may vary from point to point. We first look at the group of conformal transformations and its structure, which is different in dimension two and higher.

• *Conformal transformations in dimension D*

A conformal transformation on metric manifold, —that is on a manifold equipped with a metric allowing to measure locally distances and angles— is a transformation which locally modify the metric by a dilatation factor possibly depending on the position the manifold.

To simplify matter, let us consider the Euclidean space in dimension D, equipped with the flat Euclidean metric $ds^2 = dx_1^2 + \cdots + dx_D^2$. The metric tensor $g_{\mu\nu}$, is defined by $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ where the summation over the repeated indices is implicit, $g_{\mu\nu} = \delta_{\mu\nu}$ with $\delta_{\mu\nu}$ Kronecker symbol.

Under a diffeomorphism $x \to y = f(x)$, the metric tensor transforms as $g_{\mu\nu} \to \hat{g}_{\mu\nu} := \frac{\partial x^\sigma}{\partial y^\mu} \frac{\partial x^\rho}{\partial y^\nu} g_{\sigma\rho}$. A conformal transformation is thus a transformation $x \to y = f(x)$, locally defined on $\mathbb{R}^d$, such that the two metrics $\hat{g}$ and g are locally proportional with a proportionality coefficient which may vary from point to point, namely

$$\hat{g}_{\mu\nu}(x) = \left(\frac{\partial x^\sigma}{\partial y^\mu}\right)\left(\frac{\partial x^\rho}{\partial y^\nu}\right) g_{\sigma\rho}(x) = e^{2\varphi(x)}\, g_{\mu\nu}(x),$$

where φ is the local dilatation factor, called the conformal factor. Clearly, such transformation preserves angles, locally.

Our aim is now to identify the set of conformal transformations in $\mathbb{R}^D$, equipped with the Euclidean metric. Clearly, conformal transformations form a group, because they can be composed and the composition of two conformal transformations is again a conformal transformation. We look at infinitesimal transformations $x^\mu \to x^\mu + \epsilon\, \xi^\mu(x) + \cdots$ generated by a vector field $\xi^\mu(x)$. To first order in ϵ, the condition for conformal invariance reads

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 2(\delta\varphi)\, \delta_{\mu\nu}, \tag{55}$$

with $e^{2\epsilon(\delta\varphi)}$ the infinitesimal dilatation coefficient and $\xi_\mu := \xi^\nu \delta_{\nu\mu}$. The dilatation factor is not independent from the generating vector field ξ^μ. Indeed, contracting the indices yields $D(\delta\varphi) = (\partial_\mu \xi^\mu)$.

Identifying all possible infinitesimal conformal transformations amounts to find the general solution of this condition. This can be done via simple manipulation (e.g taking derivatives, contracting indices, etc). See the exercise Section for a more detailed description. The outcome is that all infinitesimal conformal transformations, $x^\mu \to x^\mu + \epsilon\, \xi^\mu(x) + \cdots$, in dimension $D > 2$, are generated by vector field ξ^μ of the form:

$$\xi_\nu(x) = a_\nu + k x_\nu + \theta_{\nu\sigma} x^\sigma + [(b \cdot x) x_\nu - \frac{1}{2}(x \cdot x) b_\nu], \tag{56}$$

where a_ν, k, b_μ and $\theta_{\nu\sigma} = -\theta_{\sigma\nu}$ are constant and parametrize the infinitesimal transformations. The infinitesimal dilatation factor is $(\delta\varphi) = \big(k + (b \cdot x)\big)$.

Each in term in the above equation for ξ^μ possesses a simple interpretation:
— the first correspond to translation: $x_\nu \to x_\nu + a_\nu$,
— the second to dilatation: $x_\nu \to x_\nu + k x_\nu$,
— the third to rotation: $x_\nu \to x_\nu + \theta_{\nu\sigma} x^\sigma$,
— the last to so-called special conformal transformation: $x_\nu \to x_\nu + [(b \cdot x)x_\nu - \frac{1}{2}(x \cdot x)b_\nu]$.

The dimension of the group of conformal transformation is thus $D + 1 + D + \frac{D(D-1)}{2} = \frac{1}{2}(D+1)(D+2)$. One can show –the exercise Section– that this group is isomorphic to $so(D+1, 1)$.

• *Conformal transformations in $D = 2$*

In dimension $D = 2$, the special conformal transformations are the so-called homographic transformations, also called the Mobius transformations. In complex coordinates $z = x + iy$ they read

$$z \to f(z) = \frac{az + b}{cz + d}, \quad a,\, b,\, c,\, d \in \mathbb{C}.$$

These are the only holomorphic bijections of the complex plane with a point at infinity added (it is important to add the point at infinity; with this point added the complex plane is isomorphic to the 2D sphere). By composition, they form a group isomorphic to the group of linear transformation of $\mathbb{C}^2$ with unit determinant whose elements are:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C})/\{\pm 1\} = \mathrm{PSL}(2, \mathbb{C}).$$

Looking back to the previous analysis of the conformal transformations, we see that in dimension $D = 2$ the dilatation factor as to be a harmonic function: $\Delta(\partial \cdot \xi) = 0$. Hence locally $(\partial \cdot \xi) = \partial_z v(z) + \partial_{\bar{z}} \bar{v}(\bar{z})$ with $v(z)$ holomorphic. This corresponds to infinitesimal transformations

$$z \to z + \epsilon v(z)$$

with $v(z)$ locally holomorphic. These transformations are in general defined only locally, not globally on the sphere, because there is no holomorphic vector on the sphere except $v_n(z) = z^{n+1}$ with $n = 0,\ \pm 1$.

As a consequence, conformal transformations in 2D are locally holomorphic transformations:

$$z \to w = f(z)$$

Let $z_1 = z_0 + \delta z_1$ and $z_2 = z_0 + \delta z_2$ two neighbour points of z_0. The two small vectors δz_1 and δz_2 joining z_0 to z_1 or z_2 are transformed into two vectors δw_1 and δw_2 joining $w_0 = f(z_0)$ to $w_1 = f(z_1)$ or $w_2 = f(z_2)$. To first order, we have:

$$\delta w_1 = f'(z_0)\, \delta z_1, \quad \delta w_2 = f'(z_0)\, \delta z_2.$$

The angle between the vectors δw_1 and δw_2 is therefore identical to that between the vectors δz_1 and δz_2. Each of these vectors has been rotated, by an angle equal to the argument of $f'(z_0)$, and has been dilated proportionally to the modulus of $f'(z_0)$. Conformal transformations act locally as rotations and dilatations but with dilatation factors or rotation angles which may vary from point to point.

Let us give a few examples of (simple) conformal transformations:
— The transformation

$$z \to w = \frac{z-i}{z+i}$$

is holomorphic, without singularity, on the upper half-plane $\mathbb{H} = \{z \in \mathbb{C},\ \mathrm{Im} z > 0\}$, and the image of the upper half-plane is the unit disk centred at the origin, $\mathbb{D} = \{w \in \mathbb{C},\ |w| < 1\}$.

More generally, Riemann's theorem asserts that any planar domain with the topology the unit disk, i.e. any connexe, simply connected, open set of the complex plane, different from $\mathbb{C}$, is in conformal bijection with the unit disk, and hence also in bijection with the upper half-plane.
— The transformation

$$z \to w = \beta \log z, \quad w \to z = e^{w/\beta}$$

maps the complex z-plane (with the point at the origin and at infinity removed) to the the w-cylinder with radius β. The circle of radius e^τ centred at the origin in the complex plane if mapped to the circle loop of the cylinder at azimutal altitude τ/β.

• *A classical application of conformal transformations.*

Les transformations conformes trouvent de nombreuses applications dans tous les problèmes de physique classique régis par l'équation de Poisson-Laplace en deux dimensions,

$$\Delta_z \phi(z,\bar{z}) = 0 \quad \text{avec} \quad \Delta_z = \partial_x^2 + \partial_y^2 = 4\,\partial_z \partial_{\bar{z}}.$$

Toute solution de cette équation est appelée une fonction harmonique. Comme le laplacien se factorise en $\Delta_z = 4\,\partial_z \partial_{\bar{z}}$, les fonctions harmoniques se décomposent en la somme d'une fonction holomorphe et d'une fonction anti-holomorphe,

$$\phi(z,\bar{z}) = \varphi(z) + \bar{\varphi}(\bar{z}).$$

De cette décomposition découle l'invariance conforme de l'équation de Poisson-Laplace. En effet si $\phi(z,\bar{z})$ est solution, alors $\phi(f(z), \overline{f(z)})$ est aussi solution. Alternativement, si $w = f(z)$, le laplacien dans les coordonnées w s'écrit simplement en terme du laplacien dans les coordonnées z: $\Delta_w = 4\partial_w \partial_{\bar{w}} = |f'(z)|^2\,\Delta_z$.

Cette équation s'applique évidemment aux problèmes d'électrostatique où ϕ est le potentiel électrostatique dans le vide. Ces derniers sont généralement formulés sous la forme de problèmes de Dirichlet pour lesquels on cherche la valeur du potentiel électrostatique dans un domaine planaire $\mathbb{D}$, qui est l'extérieur d'un conducteur parfait, sachant la valeur $\phi|_{\mathbb{D}}$ de ce potentiel au bord du domaine, c'est à dire sur le conducteur.

Elle s'applique également en mécanique des fluides à la description des mouvements bidimensionnels de fluides incompressibles et irrotationnels. En effet, si la vorticité $\omega = \nabla \wedge \mathbf{u}$ est nulle, le champ des vitesses $\mathbf{u}$ dérive d'un gradient $\mathbf{u} = -\nabla\phi$ et la condition d'incompressibilité $\nabla \cdot \mathbf{u} = 0$ impose alors $\Delta\phi = 0$. Les conditions aux limites, appelées conditions de Neumann, dépendent de la forme du domaine dans lequel le fluide se déplace. Elles imposent que le gradient de ϕ soit tangent à la surface du domaine de sorte que l'écoulement soit tangent à cette surface.

7.2 Conformal invariance in field theory

The aim of this section is to understand the basic simple echoes of conformal invariance in statistical field theory. One of the take-home message is that conformal invariance fixes the structures and the values of the one, two and three point field correlation functions.

Before starting let us first recall that to any symmetry correspond conserved currents. For conformal symmetries, there are three conserved currents respectively associated to translation, dilatation and special conformal transformation. They are of the form $J^\mu_\xi = T_{\mu\nu}\xi^\nu$ with ξ^ν any of the vector fields generating conformal transformations, i.e. $\xi_\nu(x) = a_\nu + kx_\nu + \theta_{\nu\sigma}x^\sigma + [(b\cdot x)x_\nu - \frac{1}{2}(x\cdot x)b_\nu]$ and $T^{\mu\nu}$ the stress-tensor. (Recall that the stress-tensor is the generator of diffeomorphisms). The conservation laws are:

$$\partial^\mu(T_{\mu\nu}\xi^\nu) = 0.$$

This equivalent to the conservation for $T_{\mu\nu}$ and for its traceless-ness:

$$\partial^\mu T_{\mu\nu} = 0, \quad T^\mu_\mu = 0.$$

Thus a conformal field theory is characterized by a trace-less, conserved, stress-tensor. Of course, these relations are valid away from field insertions (cf. the discussion about symmetries and Ward identities).

• *Global conformal invariance and the 2-point functions*

We first describe simple consequences of conformal invariance on the two and three point functions. We first use a step-by-step constructive approach, that we shall formalize/generalize later.

Let us first look at 2 point functions. Pick two field ϕ_1 and ϕ_2 located at x_1 and x_2 and consider their correlation functions (in flat space, with the Euclidean matric)

$$G^{(2)}(x_1, x_2) := \langle \Phi_1(x_1)\Phi_2(x_2)\rangle.$$

By definition a conformal field theory is such that the correlation functions of local fields are invariant under the global conformal transformation.

To simplify matter, let assume that the fields are scalar fields (no spin). We shall generalize later, especially in dimension 2. We shall prove that the two point functions of 'scaling fields' have to be of the following form:

$$\langle \Phi_1(x_1)\Phi_2(x_2)\rangle = \frac{\text{const.}}{|x_1 - x_2|^{2h_1}}\ \delta_{h_1;h_2}.$$

the number h_1 and h_2 are the so-called scaling dimension of the fields Φ_1 and Φ_2 respectively. What re scaling fields and how scaling dimensions are are defined is explained in a few lines. In particular, the 2-point functions in a conformal field theory are non vanishing only if the two fields have identical scaling dimension. We can (usually) normalize the fields such that the above constant is 1.

Global translation is easy to implement: it says that $\langle \Phi_1(x_1)\Phi_2(x_2)\rangle = \langle \Phi_1(x_1+a)\Phi_2(x_2+a)\rangle$ for any shift a. To write the condition of rotation invariance, we have to specify how the fields transform under rotation (say, whether they are scalar, vectors, etc). To do that, we have to associate a representation of the rotation group $SO(D)$ to each fields (or more precisely to each multiplet of fields), so that the fields under rotation as $\Phi(x) \to (\rho(R)\cdot\Phi)(R^{-1}x)$, with $R \to \rho(R)$ the representation of rotation group associated to the field Φ. Rotation invariance is then the statement that

$$\langle \Phi_1(x_1)\,\Phi_2(x_2)\rangle = \langle \Phi_1^R(R^{-1}x_1)\,\Phi_2^R(R^{-1}x_2)\rangle, \quad \Phi_j^R = \rho_j(R)\cdot\Phi_j,$$

for any rotation $R \in SO(D)$. A scalar field transform trivially under rotation so that $\Phi = \Phi^R$, by definition. Then, translation and rotation invariance impose that the two point functions of for scalar fields depends only in the distance between the two points,

$$G^{(2)}(x_1, x_2) = F_2(r), \quad r^2 = |x_1 - x_2|^2.$$

Now to use symmetry under dilatation, we have to specify how fields transform under dilatation. Again this depends on the nature of the field –that is: how it transform under the conformal group. For scalar field, this is specified by a number, called the field scaling dimension. Namely, by definition a field of scaling dimension h transform under dilatation $x \to y = \lambda\, x$ as

$$\Phi(x) \to \hat{\Phi}(y) = \lambda^{-h}\, \Phi(x = y/\lambda).$$

Dilatation invariance is then the statement that

$$\langle \Phi_1(y_1)\Phi_2(y_2)\rangle = \lambda^{-h_1-h_2}\, \langle \Phi_1(y_1/\lambda)\Phi_2(y_2/\lambda)\rangle.$$

By taking derivative with respect to λ and integrating the resulting differential equation, it is easy to verify that this implies that

$$G^{(2)}(x_1, x_2) = \text{const.}\, r^{-(h_1+h_2)}.$$

Notice that the above transform law tells that the scalar field of scaling dimension h transform as $\Phi(x) \to \hat{\Phi}(x) = \Phi(y) + \epsilon(\delta\Phi)(y)$ with $(\delta\Phi)(y) = [h + y^\nu \partial_\nu]\Phi(y)$ for an infinitesimal dilatation $x \to y = x + \epsilon x$.

Next we should impose invariance under global special conformal transformation. These are infinitesimal diffeomorphism transformation $x^\mu \to y^\mu = x^\mu + \epsilon\, \xi^\mu(x)$, for which distances (defined via the metric) are scaled by a factor $\delta\varphi = D(\partial.\xi)$. So, we impose that the field transforms as above but with a position dependent scaling factor, that

$$\Phi(x) \to \hat{\Phi}(y) = \left[e^{\varphi(x(y))}\right]^{-h} \Phi(x(y)), \quad x^\mu(y) = x^\mu - \epsilon\, \xi^\mu(y) + \cdots .$$

with $\varphi = \epsilon\delta\varphi$. For infinitesimal transformation this corresponds to:

$$\Phi(x) \to \hat{\Phi}(y) = \Phi(y) - \epsilon\, [h(\delta\varphi)(y) + \xi^\mu(y)\partial_\mu]\Phi(y).$$

Of course this reproduces the above transformation for translation, rotation or dilatation. For special conformal transformation, $\xi_\nu(x) = (b \cdot x)x_\nu - \frac{1}{2}(x \cdot x)b_\nu$ and $\delta\phi = (b \cdot x)$ this gives the field varaition

$$\delta\Phi(x) = \epsilon \left[h(b \cdot x) + [(b \cdot x)x^\nu - \frac{1}{2}(x \cdot x)b^\nu]\partial_\nu\right]\Phi(x).$$

Such fields are called 'scalar conformal fields' (or pseudo-primary fields, or scaling fields) and h is their scaling dimensions. Not all fields are scaling fields, for instance if Φ is a scaling fields its derivatives are not.

The invariance of the two point then under conformal then reads

$$\langle \Phi_1(y_1)\, \Phi_2(y_2)\rangle = e^{-h_1\varphi(x(y_1))}\, e^{-h_2\varphi(x(y_2))}\, \langle \Phi(x(y_1))\, \Phi(x(y_2))\rangle,$$

This yields an extra condition on $G^{(2)}$. Since we already know that $G^{(2)}(x_1, x_2) = \text{const.}\, r^{-(h_1+h_2)}$, it gives a constraint on the scaling dimensions h_1 and h_2 which is that $G^{(2)}$ vanishes unless $h_1 = h_2$. Thus we get

$$G^{(2)}(x_1, x_2) = \text{const.}\, r^{-(h_1+h_2)}\, \delta_{h_1;h_2}.$$

See the exercise Section for a detailed proof of all the technical steps. Notice that global dilatation invariance only is not enough we need the special conformal transformation to prove that two scaling dimensions have to be identical.

• *Global conformal invariance and 3-point functions*

Similar computations can be done for 3-point functions:

$$G^{(3)}(x_1, x_2, x_3) = \langle \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)\rangle.$$

We look for the correlation of scalar conformal fields Φ_j, $j = 1, 2, 3$, of respective scaling dimensions h_j. These are fields which, by definition, transform as follows under infintesimal conformal transformations $x \to y = x + \epsilon\xi(x)$:

$$\Phi(x) \to \hat{\Phi}(y) = \Phi(y) - \varepsilon\left[h\,(\delta\varphi)(y) + \xi^\mu(y)\partial_\mu\right]\Phi(y),$$

with h the scaling dimension fo the field Φ and $(\delta\varphi)(y) = D^{-1}(\partial_\mu\xi^\mu)(y)$ the infinitesimal conformal factor. Invariance of the 3-point functions then demands that

$$\sum_{j=1,2,3}\left[h_j\,(\delta\varphi)(x_j) + \xi_\mu(x_j)\partial_j^\mu\right]G_3(x_1, x_2, x_3) = 0.$$

It is a simple matter of computation to verify that this determines the three point functions as (See the exercise Section)

$$G_3(x_1, x_2, x_3) = \frac{c_{123}}{|x_1 - x_2|^{h_1+h_2-h_3}|x_2 - x_3|^{h_2+h_3-h_1}|x_1 - x_3|^{h_1+h_3-h_2}},$$

with c_{123} a constant (which depends on the structure of the theory). No condition is imposed on the scaling dimension h_1, h_2 and h_3. Whether the three point function is non vanishing and what is the value of the above constant depend on the specific theory one is considering. Note that once the fields Φ_j have been normalized by their two point functions the above constant is specified (there is no freedom to redefine it).

Notice that these results for the 2- and 3-point functions are valid in any dimensions.

• *Invariance*

In previous Chapters, we discussed symmetries and associated Ward identities, but we restricted ourselves to cases in which the symmetry transformations were preserving the geometrical data involved in the specification of the statistical field theory. We here go back to this point but for symmetry transformation involving the geometrical data.

We aim at comparing statistical field theory over two spaces $\mathfrak{M}^0$ and $\mathfrak{M}$, respectively equipped with a metric g^0 and g. As data, we have series of field correlation functinos of the form

$$\langle \Phi_1(y_1)\cdots\rangle_{\mathfrak{M},g}, \quad \langle \Phi_1(x_1)\cdots\rangle_{\mathfrak{M}^0,g^0}.$$

with $y_1, \cdots$ points in $\mathfrak{M}$ and $x_1, \cdots$ points in $\mathfrak{M}^0$. These series of objects cannot be compared because they are not defined over the same set. So let us give ourselves a mapping from $\mathfrak{M}^0$ to $\mathfrak{M}$, say $x \to y = f(x)$, or reciprocally the inverse mapping $y \to x$ from $\mathfrak{M}$ to $\mathfrak{M}^0$. As discussed above, via this mapping we get a metric $\hat{g}^0$ on $\mathfrak{M}$ by transporting that from $\mathfrak{M}^0$:

$$d\hat{s}^2 = \hat{g}^0_{\mu\nu}dy^\mu dy^\nu = g^0_{\sigma\rho}(\frac{\partial x^\sigma}{\partial y^\mu})(\frac{\partial x^\rho}{\partial y^\nu})dy^\mu dy^\nu.$$

We can then use it to push forward all correlation functions $\langle \Phi_1(x_1) \cdots \rangle_{\mathfrak{M}^0,g^0}$ into correlation functions over $\mathfrak{M}$, by viewing the points x in $\mathfrak{M}^0$ as function of the point y in $\mathfrak{M}$. Along such transformation the field Φ are transformed into $\Phi \to \hat{\Phi}$ —we will make explicit this transformation below in the meantime you may think about how vectors, forms, etc, transform on diffeomorphism. Thus from the correlation functions in $\mathfrak{M}^0$ we get correlation functions

$$\langle \hat{\Phi}_1(x(y_1)) \cdots \rangle_{\mathfrak{M}^0,g^0}, \quad y_j \in \mathfrak{M},$$

which are correlation functions over $\mathfrak{M}$. In these correlation functions, distances are measured with the metric $\hat{g}^0$ defined above, by construction. Hence, by pushing forward the correlation functions from $\mathfrak{M}^0$ to $\mathfrak{M}$, we got to series of correlations functions over $\mathfrak{M}$ differing by the metric used to measure distances (one is the original one g, the other is the induced one $\hat{g}^0$).

Conformal invariance is the statement that these correlation functions are identical if the two metrics g and $\hat{g}^0$ are conformally equivalent, that is $\hat{g}^0(y) = e^{\varphi(y)} g(y)$:

$$\langle \Phi_1(y_1) \cdots \rangle_{\mathfrak{M},g} = \langle \hat{\Phi}_1(x(y_1)) \cdots \rangle_{\mathfrak{M}^0,g^0}, \quad \text{if } \hat{g}^0 = e^{\varphi} g.$$

We have to specify how the fields transform. As discussed above, for scalar conformal fields of scaling dimension h, $\hat{\Phi}_1(x(y)) = \left[e^{\varphi(x(y))}\right]^{-h} \Phi(x(y))$ so that conformal invariance reads

$$\langle \Phi_1(y_1) \cdots \rangle_{\mathfrak{M},g} = \langle \left[e^{\varphi(x(y_1))}\right]^{-h} \Phi_1(x(y_1)) \cdots \rangle_{\mathfrak{M}^0,g^0}, \quad \text{if } \hat{g}^0 = e^{\varphi} g.$$

This means that, if conformal invariance holds, we can transport correlation functions from $\mathfrak{M}^0$ equipped with the metric g^0 to $\mathfrak{M}$ equipped with the metric g, provided that the metric $\hat{g}^0$ and g are conformally equivalent.

The important special case we consider above is when $\mathfrak{M}$ and $\mathfrak{M}^0$ are both the D-dimensional Euclidean space and g and g^0 are both the Euclidean flat metric.

7.3 Fields and operator product expansions

We now introduce an important concept in field theory (which turns out to be especially powerful in two dimensional conformal field theory). Let us look at two operators at nearby positions x and y. Say $x = R + r$ and $y = R - r$ with $r \ll R$. When viewed from a point far away from x and y, this product operator looks as defined at medium point r or equivalent at one of the two points, say y. Thus it is 'natural' to expect that this product can be expanded in all the other local operators. That it is, we expect —or assume— that we can write the expansion:

$$\Phi_j(x)\, \Phi_k(y) = \sum_l C_{jk}^l(x,y)\, \Phi_l(y),$$

where the sum is over all set of local operators. Such expansion is called the *operator product expansion* (OPE).

The existence of the operator product expansion is one of the basics assumption of statistical field theory (which may either view as a tautological statement or as axiom, impossible to prove but part of the definition of what a statistical field theory could be...).

The group of conformal transformations acts on the space of local fields of a conformal field theory (by definition and/or constitutive properties of a conformal field theory). We can hence

group the fields into conformal multiplets, forming representation and of the group of conformal transformations, and we can organize the space of fields as sum of those representations (assuming complete reducibility).

By a choice of basis, we order the basis field Φ_l according to their scaling dimension (under global dilatation). Scaling invariance then fixes the way the function $C^l_{jk}(x,y)$ scales: $C^l_{jk}(x,y) \propto |x-y|^{h_l-h_j-h_k}$ (if all the fields involved are spinless). Hence

$$\Phi_j(x)\,\Phi_k(y) = \sum_l \frac{c^l_{jk}}{|x-y|^{h_j+h_k-h_l}}\,\Phi_l(y) + \cdots,$$

The numbers c^l_{jk} are called the OPE coefficients. Of course they coincide with the coefficient of the three point functions. Here the dots refers to contribution to spin full fields (say derivative of spinless fields) that we do not write explicitly because their writing is a bit cumbersome (but without any conceptual difficulties).

This OPE can be checked on the three point functions. Let $G^{(3)}(x_1,x_2,x_3)$ be the expectations of three scalar conformal fields, Φ_1, Φ_2 and Φ_3 of respective scaling dimension h_1, h_2 and h_3, at three different positions. Its explicit expression was just derived above. Let us then fuse $x_1 \to x_2$ in this formula and observe the result. We get

$$\begin{aligned} G^{(3)}(x_1,x_2,x_3) &\simeq_{x_1\to x_2} \frac{1}{|x_1-x_2|^{h_1+h_2-h_3}}\,\frac{c_{123}}{|x_2-x_3|^{h_3}} + \cdots \\ &\simeq_{x_1\to x_2} \frac{1}{|x_1-x_2|^{h_1+h_2-h_3}}\,c_{123}\langle\Phi_3(x_2)\Phi_3(x_3)\rangle + \cdots \end{aligned}$$

Hence, the fusion of $\Phi_1(x_1)$ and $\Phi_2(x_2)$ effectively generate the field Φ_3 if the OPE coefficient c_{123} does not vanishes. This coefficient is then identified with the three point normalization coefficient. Recall that conformal fields with different scaling dimension have vanishing two point function so that only the contribution from Φ_3 in the OPE survives when computing the limit of three point function when $x_1 \to x_2$.

The OPE can be used to write series expansion for correlations. As explained above, the two- and three- point functions of a CFT are fixed by conformal symmetry. Let us thus look at a four-point function

$$\langle\Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)\Phi_4(x_4)\rangle.$$

We can image evaluating this correlation function by series expansion by fusing using the OPE the fields Φ_1 and Φ_2 in one hand and Φ_3 and Φ_4 on the other hand. We then are reduced to compute two-point functions of the fused fields. The result is

$$\sum_{k,l} C^k_{12}(x_1,x_2)\,\delta_{k;l}|x_2-x_4|^{-h_k+h_l}\,C^l_{34}(x_3,x_4).$$

We can alternatively choose to fuse the fields Φ_2 and Φ_3 in one hand and Φ_1 and Φ_4 on the other hand. The result is then

$$\sum_{k,l} C^k_{23}(x_2,x_3)\,\delta_{k;l}|x_3-x_4|^{-h_k+h_l}\,C^l_{14}(x_1,x_4).$$

This is two expressions have to equal. This yields a relation fulfilled by the OPE coefficients. In many respect, it is similar to the condition for associativity of a product algebra. This relation is called crossing symmetry.

We will encounter examples of OPE in the following sections.

7.4 Massless gaussian free field in 2D

We aim at describing the statistical field theory of a massless free bosonic field in two dimension. This is the simplest example of 2D conformal field theory. We will explicit the transformation properties of the basic fields of the theory, the structure of both the Ward identities and the operator product expansion.

We are going to use complex coordinate $z = x + iy$, $\bar{z} = x - iy$ and $\partial_z = \frac{1}{2}(\partial_x + i\partial_y)$, $\partial_{\bar{z}} = \frac{1}{2}(\partial_x - i\partial_y)$.

• *Action and Green function*

We aim at describing the statistical field theory of a free scalar bosonic field ϕ defined by the action S

$$S[\phi] = \frac{g}{4\pi} \int d^2x \, (\partial_\mu \phi)(\partial^\mu \phi) = \frac{g}{\pi} \int dz d\bar{z} \, (\partial_z \phi)(\partial_{\bar{z}} \phi),$$

Here we made a slight abuse of notation: d^2x is the Lebesgue measure on $\mathbb{R}^2$ (d^2x is the infinitesimal flat area dA) which we denote in complex coordinate as $dzd\bar{z}$. The (classical) equation of motion is $\Delta\phi = 0$ which is conformally invariant in 2D. It is in particular simple to verify that the action is invariant under dilatation, if we assume that ϕ is a scalar field, invariant under dilatation (recall that a scalar field a scaling dimension $(D-2)/2$ in dimension D).

The parameter g is dimensionless. It depends on the chosen normalization of the field ϕ. To simplify the notation we shall choose $g = XXXXX$.....

The action S is actually invariant under conformal transformation $z \to w = w(z)$, if we assume that the field transforms as

$$\phi(z, \bar{z}) \to \hat{\phi}(w, \bar{w}) = \phi(z(w), \bar{z}(\bar{w})).$$

Indeed, we have (we put $g = XXXX$)

$$\begin{aligned} S[\hat{\phi}] &= \frac{g}{\pi} \int dw d\bar{w} \, (\partial_w \hat{\phi})(\partial_{\bar{w}} \hat{\phi}) = \frac{g}{\pi} \int dz d\bar{z} \, |\partial_z w(z)|^2 (\partial_w \hat{\phi})(\partial_{\bar{w}} \hat{\phi}) \\ &= \frac{g}{\pi} \int dz d\bar{z} \, (\partial_z \phi)(\partial_{\bar{z}} \phi) = S[\phi]. \end{aligned}$$

where we use, on one hand, the Jacobian to change variable from w to z is $|\partial w/\partial z|^2$ (recall that d^2x is the infinitesimal area), and one the other hand, the chain rule to write $\partial_z \phi = (\partial w/\partial z)\partial_w \hat{\phi}$ and similarly for the complex conjugate.

Recall the 2D Green function (of the Laplacian):

$$G(z, w) = -\log |z - w|^2, \quad \text{up to a constant.}$$

It satisfies $(-\Delta)G = 2\pi \, \delta^{(2)}(z - w)$. The Green function is only determined up to a constant because the Laplacian possesses a constant zero mode. The Laplacian is not invertible and we cannot define/compute the gaussian integral. We thus have to deal with this constant zero mode. A way to it is to define the theory on a finite domain and then let the size of this domain increases infinitely. The simplest is to consider the theory on a disc $\mathbb{D}_R$ of radius R and impose Dirichlet condition at the boundary, that is $\phi = 0$ at the boundary. By integration by part, we can alternatively write the action as (with Δ the Laplacian)

$$S[\phi] = -\frac{g}{4\pi} \int d^2x \, \phi(x) \, (\Delta\phi)(x),$$

because there is no contribution from the boundary terms thanks to the Dirichlet boundary condition. With the Dirichlet boundary condition, the Laplacian is invertible (there is then no zero mode because an harmonic fonction which vanish at the boundary is zero). The Green function is then

$$G_R(z,w) = -2\log\left(\frac{R|z-w|}{|zw-R^2|}\right).$$

Note it has the same short distance expansion $G_R(z,w) \simeq -\log|z-w|^2 + \mathrm{reg}(z,w)$, and it satisfies $(-\Delta)G_R = 2\pi\,\delta^{(2)}(z-w)$ with the boundary condition $G_R(z,R) = 0$. In the large R limit we have

$$G_R(z,w) \simeq_{R\to\infty} -\log(|z-w|^2/R^2).$$

Since the theory is Gaussian, the expectation values of product of fields (or vertex operators) are computable using the two point function,

$$\langle\phi(z,\bar z)\phi(w,\bar w)\rangle_{\mathbb{D}_R} = G_R(z,w),$$

using Wick's theorem. As discussed in previous Chapter, the generating function can (of course) be computed exactly

$$\langle e^{-\int d^2x\, J(x)\phi(x)}\rangle_{\mathbb{D}_R} = e^{-\frac{1}{2}\int d^2x d^2y\, J(x)G_R(x,y)J(y)}.$$

The factor R is going to be irrelevant as long as we consider correlation functions invariant under translation of the field ϕ. That is the large volume limit $R\to\infty$ (IR limit) exists for all correlation functions invariant under translation of the field (and this invariance is going to be related to some $u(1)$ symmetry). We shall denote these correlation functions as $\langle\cdots\rangle_{\mathbb{C}}$.

• *$u(1)$ current.*

The theory possesses two conserved currents $J_\mu := \partial_\mu\phi$ and $\hat J_\mu := \epsilon_{\mu\nu}\partial_\nu\phi$ (one is topological, the other is the Noether current associated to a $u(1)$ symmetry): $\partial_\mu J_\mu = 0$ and $\partial_\mu\hat J_\mu = 0$. In complex coordinate, this translates into two currents, J and $\bar J$, defined by:

$$\begin{aligned} J = i\partial_z\phi \quad &, \quad \partial_{\bar z}J = 0, \\ \bar J = -i\partial_{\bar z}\phi \quad &, \quad \partial_z J = 0 \end{aligned}$$

Thus, J holomorphic: it only depends on the z-coordinate, $J(z)$, while $\bar J$ is anti-holomorphic, it only depends on the $\bar z$ coordinate, $\bar J(\bar z)$. This conservation law are equivalent to the equation of motion $\partial_z\partial_{\bar z}\phi = 0$.

As usual in the statistical field theory, these conservation laws are going to be valid when inserted in correlation functions provided all other fields are located away from the current position. For instance

$$\partial_{\bar z}\langle J(z)\, J(w_1)\cdots V_{\alpha_1}(v_1,\bar v_1)\cdots\rangle_{\mathbb{D}_R} = 0,$$

with $V_{\alpha_1}(v_1,\bar v_1)$ some operator (constructed using the field ϕ). This equation can be checked explicitly by computing using Wick's theorem these correlation functions (one the operator $V_{\alpha_1}(v_1,\bar v_1)$ has been specified) and using the fact the Green function is a harmonic function. This equation is similar to those we proved when looking at symmetries and associated Ward identities.

We can compute explicitly all the current correlation functions. Since the currents are invariant under a translation o fϕ, these correlation functions have a large volume limit $R \to \infty$. To simplify matter, we only consider those. The current two point function reads

$$\langle J(z_1)J(z_2)\rangle_{\mathbb{C}} = -\partial_{z_1}\partial_{z_2}G(z_1,z_2) = \frac{1}{(z_1-z_2)^2},$$

as a result of the Wick's contraction between the two currents. This is (of course) compatible with the fact that the chiral/holomorphic current has dimension one (recall that $J = i\partial\phi$ where ϕ has scaling dimension zero and ∂ scaling dimension one). All other multipoint correlation functions can be computed using the Wick's theorem.

We can as well use the Wick's theorem to find the operator product expansion (OPE) between two currents. Indeed, inside any correlation functions

$$J(z_1)J(z_2) = \frac{1}{(z_1-z_2)^2} + :J(z_1)J(z_2):,$$

where, again, the double dots $:\cdots:$ means normal ordering (forbidding self-contraction of the field inside the double dots). This is proved by imaging how the Wick's theorem would apply inside a correlation function (via Wick's contractions). Since $:J(z_1)J(z_2):= -:\partial_{z_1}\phi(z_1)\partial_{z_2}\phi(z_2):$, we can expand this formula in Taylor series and write the current-current OPE

$$J(z_1)J(z_2) = \frac{1}{(z_1-z_2)^2} + J^2(z_2) - \frac{1}{2}(z_1-z_2)\partial_{z_2}J^2(z_2) + \cdots,$$

with $J^2(z) = -:(\partial_z\phi)^2(z):$ by definition of the operator we name J^2. This computation illustrate how operator product expansions arise in statistical field theory (and it proves its existence in this peculiar example). We may notice that it involves composite operators, such as $(\partial\phi)^2$, made of regularized version of product of operators.

Because ϕ is a scalar, the current $J = i\partial_z\phi$ transform like a holomorphic one form: under a conformal transformation $z \to w = w(z)$, it transforms as

$$J(z) \to \hat{J}(w) = [z'(w)]\, J(z(w)).$$

because $\partial_w\hat{\phi}(w) = (\partial z/\partial w)\,\partial\phi(z(w))$, as a direct consequence of the chain rule for derivative. This may also be written in a more geometrical way as $dw\,\hat{J}(w) = dz\,J(z)$, a formula which encodes the fact the current a holomorphic one form.

• <u>*The stress-tensor and its OPE*</u>

The classical stress tensor, deduced form the action, is $T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2$. By construction it is conserved $\partial_\mu T_{\mu\nu} = 0$ (because of the equation of motion) and traceless $g^{\mu\nu}T_{\mu\nu} = 0$ (because $g^{\mu\nu}g_{\mu\nu} = 2$ in dimension two). The traceless-ness of the the stress-tensor ensures the conservation law of the current $D_\mu = T_{\mu\nu}x^\nu$ associated to dilatation: $\partial_\mu D_\mu = 0$. Actually the conservation law and the traceless-ness of the stress-tensor ensures the conservation law of all the Noether currents $D^\xi_\mu := T_{\mu\nu}\xi^\nu$ associated to conformal transformations $x_\mu \to x_\mu + \epsilon\xi_\mu$, because the vector fields ξ_μ satisfy $\partial_\mu\xi_\nu + \partial_\nu\xi_\mu = (\delta\phi)\delta_{\mu\nu}$.

In complex coordinates z and $\bar{z}$, we have:

$$T_{zz} = -\frac{1}{2}(\partial_z\phi)^2,\ T_{\bar{z}\bar{z}} = -\frac{1}{2}(\partial_{\bar{z}}\phi)^2,\ T_{z\bar{z}} = 0.$$

The conservation law $\partial_\mu T_{\mu\nu} = 0$ then becomes

$$\partial_{\bar{z}} T_{zz} = 0,\ \partial_z T_{\bar{z}\bar{z}} = 0,$$

which are of course consequences of the equation of motion $\partial_z \partial_{\bar{z}} \phi = 0$.

Since product of fields at same points these have to be regularized/renormalized in the statistical field theory. This can be done by normal ordering, in a way similar as what we did with the vertex operators. That is, in the statistical/quantum field theory we defined the stress tensor for a massless bosonic field by

$$T_{zz} = -\frac{1}{2} : (\partial_z \phi)^2 :,\ T_{\bar{z}\bar{z}} = -\frac{1}{2} : (\partial_{\bar{z}} \phi)^2 : .$$

Since normal ordering amounts to cancel all the self contraction in the Wick theorem, this is equivalent to regularized the product $(\partial_z \phi)(\partial_z \phi)$ by slightly splitting the two field apart at nearby position z and $w \to z$ and subtracting the Wick contraction between these two field. Hence

$$T(z) := T_{zz}(z) = \frac{1}{2} \lim_{z' \to z} \Big[-\partial_z \phi(z) \partial_z \phi(z') - \frac{1}{(z - z')^2} \Big],$$
$$\bar{T}(\bar{z}) := T_{\bar{z}\bar{z}}(\bar{z}) = \frac{1}{2} \lim_{z' \to z} \Big[-\partial_{\bar{z}} \phi(\bar{z}) \partial_{\bar{z}} \phi(\bar{z}') - \frac{1}{(\bar{z} - \bar{z}')^2} \Big].$$

It is worth comparing this definition with the structure of the OPE of the currents. In particular we see that the stress tensor is the first regular term in the OPE of two currents. Correlation function of product of the stress tensor can be computed. For instance there two point function is

$$\langle T(z_1) T(z_2) \rangle = \frac{c/2}{(z_1 - z_2)^4}, \quad \text{with } c = 1.$$

This is of course compatible with the fact that T is a chiral (holomorphic) field of dimension 2 (because it is the product of two derivatives of the scalar field ϕ).

We leave as an exercice to compute the OPE between two stress tensor ($c = 1$):

$$T(z_1) T(z_2) = \frac{c/2}{(z_1 - z_2)^4} + \Big[\frac{2}{(z_1 - z_2)^2} + \frac{1}{(z_1 - z_2)} \partial_z \Big] T(z_2] + \cdots .$$

This is an important formula for the general theory of conformal field theory. See below (or in a more advance course).

The stress tensor has dimension $h = 2$. Because ϕ is a scalar and $T = -\frac{1}{2}(\partial\phi)^2$ it naively resemble a holomorphic 2-form. However, the regularization/subtraction we use to define it modifies the way it transforms: under a conformal transformation $z \to w = w(z)$, the holomorphic component of the stress-tensor transforms as ($c = 1$):

$$T(z) \to \hat{T}(w) = [z'(w)]^2 \, T(z(w)) + \frac{c}{12} \, S(z; w),$$

where $S(z; w)$ is the so-called Schwarzian derivative

$$S(z; w) = \Big[\frac{z'''(w)}{z'(w)} \Big] - \frac{3}{2} \Big[\frac{z''(w)}{z'(w)} \Big]^2.$$

Let us do the computation. Recall that the definition of T which is the w-coordinate reads

$$\hat{T}(w) = \frac{1}{2} \lim_{w' \to w} \Big[\hat{J}(w)\hat{J}(w') - \frac{1}{(w-w')^2} \Big].$$

The important point is that we use the same subtraction as before but in the w-coordinate, i.e. $-1/(w-w')^2$. Now, recall that ϕ is scalar so that $\hat{\phi}(w) = \phi(z(w))$ and hence $\hat{J}(w) = z'(w)\, J(z(w))$. Thus

$$\hat{J}(w)\hat{J}(w') = [z(w)z'(w')]\, J(z(w))J(z(w')).$$

Recall now the OPE of the currents found above only involves J^2 at leading orders and thus can be written as $J(z_1)J(z_2) = (z_1 - z_2)^{-2} + 2T(z_2) + \cdots$. Hence

$$\frac{1}{2}\hat{J}(w)\hat{J}(w') = [z'(w)]^2\, T(z(w)) + \frac{[z'(w)z'(w')]}{2(z(w)-z(w'))^2} + \mathrm{reg}\cdots$$

Equivalently,

$$\hat{T}(w) = [z'(w)]^2\, T(z(w)) + \frac{1}{2} \lim_{w' \to w} \Big[\frac{[z'(w)z'(w')]}{(z(w)-z(w'))^2} - \frac{1}{(w-w')^2} \Big].$$

The extra term has a finite limit equals to the Schwarzian derivative (as it is easy to verify by Taylor expansion $z(w+\delta w) - z(w) = (\delta w)z'(w) + \frac{1}{2}(\delta w)^2 z''(w) + \frac{1}{6}(\delta w)^3 z'''(w) + \cdots$).

This is an example of an anomaly, or extra contribution, arising as echoes of renormalization, which is due to the fact the regularization breaks the symmetry. Here, the definition of the stress-tensor needs the regularization consisting in subtracting $1/(z-z')^2$. But the quantities subtracted is not invariant under the transformation symmetry. That is: the subtraction in the image space $1/(w-w')^2$ is not equal to the image of the subtraction $1/(z-z')^2$. There is this extra contribution because the conformal transformation has distorted the quantities we subtracted. This reflects —or originates from— the fact that the fields are not ultra-local on the lattice but codes configuration on a small neighbourhood around their position.

• *Vertex operators and their correlation functions*

Vertex operators are operators of the form $V_\alpha(z,\bar{z}) = e^{i\alpha\phi(z,\bar{z})}$. As these are defined from product of field localized at identical position they need to be regularized (this is an UV regulairzation, not to be confused with the previous, armless, IR regularization).

By Gaussian integration (or by definition of the generating function of Gaussian variables) we have:

$$\langle \prod_j e^{i\alpha_j \phi(z_j,\bar{z}_j)} \rangle = \exp\big(-\frac{1}{2} \sum_{j,k} \alpha_j \alpha_k G(z_j, z_k) \big).$$

To be able to take the IR limt $R \to \infty$ we have to impose that $\sum_j \alpha_j = 0$ (which is charge neutral condition).

We have a divergence fro the diagonal term in the sum, that if of he form $\frac{1}{2}\sum_j \alpha_j^2 G(z_j, z_j)$. This is infinite. So we have to regularized it by defining the theory on a lattice (and then taking the limit in which mesh of the lattice shrinks to zero). On the (say a square) lattice the Green function at coincident point is finite and equal to $\log(1/a^2)$. This yields a diverging factor $\prod_j a^{\alpha_j^2}$ which can be absord in the definition of the operator/observable.

Thus we define the (re-normalized) vertex operator as the UV limit

$$V_\alpha(z,\bar z) =: e^{i\alpha\phi(z,\bar z)} := \lim_{a\to 0} a^{-\alpha^2}\, e^{i\alpha\phi(z,\bar z)}|_{\text{lattice}}.$$

The double dots : ... : are usually called 'normal order', this terminology refers to the operator formalism. It amounts to suppress all self contractions in the Wick's theorem.

Naively, since ϕ is a scalar field, $e^{i\alpha\phi}$ should also be a scalar field. But this naive guess forget about the renormalization procedure which will modify the transformation laws of the vertex operators under conformal transformation. One has to take into account the renormalization procedure which amounts to multiply by $a^{-\alpha^2}$. Recall the definition of the renormalized operator above. This definition is formulated in the z-plane. In the image w-plane the definition reads,

$$\hat V_\alpha(w,\bar w) = \lim_{\hat a\to 0} \hat a^{\alpha^2}\, e^{i\alpha\phi(z(w),\bar z(w))}|_{\text{lattice}},$$

where $\hat a$ in the UV lattice cut-off in the image w-plane. However, this short distance cut-off is not the same the initial cut-off a in the z-plane because such transformations dilate distances. We have for an operator located at point z we ratio of the distances is $a/\hat a = |z'(w)|$. As a consequence,

$$\hat V_\alpha(w,\bar w) = |z'(w)|^{\alpha^2}\, V_\alpha(z(w),\bar z(\bar w)),$$

The 'anomalous' factor $|z'(w)|^{\alpha^2}$ are again an echo of the renormalization procedure does not transform covariantly under the conformal transformation. From there we read the scaling dimension of the vertex operator V_α is α^2.

This can be alternatively (more rigorously) by using a regularization directly defined in the continuum (not calling the lattice formulation of the model) which amounts to smear the field over a small neighbourhood. See the exercice Section.

The correlation functions of the vertex operators are (recall that $\sum_j \alpha_j = 0$):

$$\begin{aligned}\langle \prod_j V_{\alpha_j}(z_j,\bar z_j)\rangle &= \exp\big(-\frac{1}{2}\sum_{j\neq k}\alpha_j\alpha_k G(z_j,z_k)\big)\\ &= \prod_{j<k}|z_j-z_k|^{2\alpha_j\alpha_k}\end{aligned}$$

In particular we have the formula for the two point functions:

$$\langle V_\alpha(z,\bar z)V_{-\alpha}(w,\bar w)\rangle = |z-w|^{-2\alpha^2}.$$

Again we read that the scaling dimension of V_α is $h_\alpha = \alpha^2$. Remark that this (so called 'anomalous') scaling dimension (which would naively be zero because ϕ is dimensionless) is an echo of the renormalization procedure used in the definition of the operator.

From these formulas we can read the operator product expansion (OPE) of vertex operators. Indeed, let us single out two vertex operators $V_{\alpha_1}(z_1,\bar z_1)$ and $V_{\alpha_2}(z_2,\bar z_2)$ in the above N-point correlation functions. Since normal ordering amounts to suppress all self contractions in the Wick's theorem, we learn that insertion of the product $V_{\alpha_1}(z_1,\bar z_1)\,V_{\alpha_2}(z_2,\bar z_2)$ is equivalent to the insertion of the normal order product $|z_1-z_2|^{2\alpha_1\alpha_2}\ :e^{i\alpha_1\phi(z_1,\bar z_1)+i\alpha_2\phi(z_2,\bar z_2)}:$. Hence, by Taylor expansion we have the operator product expansion:

$$\begin{aligned}V_{\alpha_1}(z,\bar z)\,V_{\alpha_2}(w,\bar w) &= |z-w|^{2\alpha_1\alpha_2}\ :e^{i\alpha_1\phi(z,\bar z)+i\alpha_2\phi(w,\bar w)}:\\ &= |z-w|^{2\alpha_1\alpha_2}\big(V_{\alpha_1+\alpha_2}(w,\bar w)+\cdots\big)\end{aligned}$$

We localized one of the field at the origin (without loss of generalities by translation invariance). This OPE holds true in any correlation function.

In particular for $\alpha_1 = -\alpha_2 = \alpha$:

$$\begin{aligned} V_\alpha(z,\bar z)\, V_{-\alpha}(0,0) &= |z|^{-2\alpha^2}\ :e^{i\alpha(\phi(z,\bar z)-\phi(0,0))}: \\ &= |z|^{-2\alpha^2}\big(1+i\alpha(z\partial_z\phi(0,0)+\bar z\partial_{\bar z}\phi(0,0))+\cdots\big) \end{aligned}$$

We leave as an exercice to compute the higher order term (but remember that $\partial\bar\partial\phi = 0$ in any correlation function).

For later use, let us compute the OPE between the vertex operators and the $u(1)$-current or the stress-tensor. For instance, we have

$$J(z)\ V_\alpha(w) \simeq \frac{\alpha}{z-w}\, V_\alpha(w) + \cdots ,$$

a formula which can be derived from the Wick's theorem or from the z-derivative at $\varepsilon = 0$ of the OPE between the two vertex operators $V_\epsilon(z,\bar z)$ and $V_\alpha(0)$. Similarly, one has

$$T(z)\, V_\alpha(w) \simeq \frac{\alpha^2/2}{(z-w)^2} V_\alpha(w) + \frac{1}{z-w}\,\partial V_\alpha(w) + \cdots ,$$

which again follow from Wick's theorem and $T(z) = \frac{1}{2} :J^2(z):$.

7.5 2D conformal field theory

Here we extract the basics principles and structures of the 2D massless boson field theory which remain valid for generic/arbitrary conformal field theory. These structures emerge from the Ward identities associated to the conformal transformations.

In $2D$ the conservation laws, $\partial^\mu T_{\mu\nu} = 0$ and $T^\mu_\mu = 0$, arising form conformal invariance read in complex in complexe coordinates

$$T_{z\bar z} = 0,\ \partial_{\bar z} T_{zz} = 0,\ \partial_z T_{\bar z\bar z} = 0,$$

These relations are valid away from field insertions.

• *Ward identities and primary field OPEs*

Recall the Ward identities (or its generalized form with product of operators) for translation, rotation and dilatation symmetries. Recall also that (scalar) conformal field transformations read $\delta_\xi\Phi(y) = \big[h\,(\partial\cdot\xi)(y) + \xi^\mu(y)\partial_\mu\big]\,\Phi(y)$ or equivalently as $\Phi_h(z) \to |z'(w)|^{2h}\,\Phi_h(w)$. Write them in complex conformal coordinates. This leads to the integral form of the (chiral) ward identities:

$$\oint dz\epsilon(z)\langle T(z)\Phi(y)\rangle = \langle\delta_\epsilon\Phi(y)\rangle.$$

This is equivalent to the following OPE between primary fields and the stress-tensor:

$$T(z)\Phi_h(w) = \frac{h}{(z-w)^2}\Phi_h(w) + \frac{1}{z-w}\partial_w\Phi_h(w) + \cdots$$

Give a few consequences of these relations, e.g. for multipoint correlation functions with a single stress-tensor.

• *The stress-tensor, its OPE and the Virasoro algebra*

Explain the stress-tensor OPE:

$$T(z_1)T(z_2) = \frac{c/2}{(z_1-z_2)^4} + \big[\frac{2}{(z_1-z_2)^2} + \frac{1}{(z_1-z_2)}\partial_{z_2}\big]T(z_2] + \cdots .$$

Here c is the so-called central charge (and $c \neq 1$ in general, $c = 1$ only for free massless boson).

Explain the relation between this OPE and the transformation rules of the stress-tensor under conformal transformations:

$$T(z) \to \hat{T}(w) = [z'(w)]^2\, T(z(w)) + \frac{c}{12}\, S(z;w),$$

or its infinitesimal version.

Elements of consequences: determine all correlation functions of the stress tensor (on the sphere/plane) using its analytic and pole structure. For instance:

$$\begin{aligned} \langle T(z_1)T(z_2)\rangle &= \frac{c/2}{(z_1-z_2)^4}, \\ \langle T(z_1)T(z_2)T(z_3)\rangle &= .... \end{aligned}$$

Idem for multipoint correlation functions T and Φ insertions.

• *The Casimir effect (again)*

7.6 Operator formalism in 2D

• *Bosonic free field*

• *Virasoro representations and field spaces*

• *Null vectors and differential equations*

[.... It is probably better to reserve this last section to a more advanced course.....]

7.7 Exercises

• *Exercise 7.7.1: Conformal mappings in 2D.*

(i) Verify that the map $z \to w = \frac{z-i}{z+i}$ is holomorphic map from the upper half plane $\mathbb{H} = \{z \in \mathbb{C},\ \mathrm{Im} z > 0\}$ to the unit disc $\mathbb{D} = \{w \in \mathbb{C},\ |w| < 1\}$ centred at the origin 0.
(ii) Similarly verify that the map $w \to z = e^{w/\beta}$ is holomorphic map from the cylinder with radius β to the complex z-plane with the origin and the point at infinity removed.

• *Exercise 7.7.2: The group of conformal transformations.*

The aim of this exercise is to fill the missing steps in determining all infinitesimal conformal transformations in the flat Euclidean space $\mathbb{R}^D$. Recall that an infinitesimal transformation $x^\mu \to x^\mu + \varepsilon\, \xi^\mu(x) + \cdots$ is conformal if $\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 2(\delta\varphi)\, \delta_{\mu\nu}$ with $D(\delta\varphi) = (\partial_\mu \xi^\mu)$.

(i) Take derivatives of the previous equation to deduce that $D\, \Delta \xi_\nu = (2-D)\partial_\nu(\partial \cdot \xi)$, with Δ the Euclidean Laplacian.
(ii) Take further derivatives, either w.r.t ∂_ν or w.r.t. ∂_μ, to get two new equations: $(D-1)\Delta(\partial \cdot \xi) = 0$, and $2(2-D)\, \partial_\mu \partial_\nu (\partial \cdot \xi) = D\, \Delta(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu)$.
(iii) Deduce that $(2-D)\partial_\mu \partial_\nu (\partial \cdot \xi) = 0$, and hence that, in dimension $D > 2$, the conformal factor $\delta\varphi(x)$ is linear in x.

Let us write $\delta\varphi(x) = k + b_\nu x^\nu$ with k and b_ν integration constantes. We thus have

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 2(k + b_\sigma x^\sigma)\, \delta_{\mu\nu}.$$

A way to determine ξ consists in getting information on the difference $\partial_\mu \xi_\nu - \partial_\nu \xi_\mu$.
(iv) By taking derivates of the previous equation w.r.t ∂_σ and permuting the indices, deduce that $\partial_\nu(\partial_\sigma \xi_\mu - \partial_\mu \xi_\sigma) = 2(b_\sigma \delta_{\mu\nu} - b_\mu \delta_{\nu\sigma})$, and hence, by integration, that

$$\partial_\sigma \xi_\mu - \partial_\mu \xi_\sigma = 2(b_\sigma x_\mu - b_\mu x_\sigma) + 2\theta_{\mu\sigma},$$

where $\theta_{\sigma\mu} = -\theta_{\mu\sigma}$ are new integration constants.
(v) Integrate the last equations to prove that

$$\xi_\nu(x) = a_\nu + k x_\nu + \theta_{\nu\sigma} x^\sigma + [(b \cdot x) x_\nu - \frac{1}{2}(x \cdot x) b_\nu],$$

where a_ν are new, but last, integration constants.
(vi) Find the explicit formula for all finite –not infinitesimal– conformal transformations in dimension D.
(vii) Verify that the Lie algebra of the group of conformal transformation in dimension D is isomorphic to $so(D+1,1)$.

• *Exercise 7.7.3: The two- and three-point conformal correlation functions.*

The aim of this exercise is to fill the missing steps in determining the two and three point function of conformal fields in conformal field theory. Let $G^{(2)}(x_1, x_2) = \langle \Phi_1(x_1) \Phi_2(x_2) \rangle$ be the two point function of to scalar conformal fields of scaling dimension h_1 and h_2 respectively.
(i) Prove that translation and rotation invariance implies that $G^{(2)}$ is a function of the distance $r = |x_1 - x_2|$ only.
(ii) Prove that dilatation invariance of the 2-point function demands that

$$[h_1 + x_1 \cdot \partial_1 + h_2 + x_2 \cdot \partial_2] G_2(x_1, x_2) = 0.$$

Deduce that $G_2(x_1, x_2) = \text{const.}\, r^{-(h_1+h_2)}$.
(iii) Prove that invariance under special conformal transformations (also called inversions) implies that

$$\sum_{j=1,2} \big[h_j(b\cdot x_j) + [(b\cdot x_j)x_j^\nu - \frac{1}{2}(x_j\cdot x_j)b^\nu]\partial_{x_j^\nu}\big] G_2(x_1,x_2) = 0.$$

Deduce that $G^{(2)}(x_1, x_2)$ vanishes unless $h_1 = h_2$.

Let us now look at the three point functions of scalar conformal fields. Let $G^{(3)}(x_1, x_2, x_3) = \langle \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)\rangle$, be their correlation functions.
(iv) Prove that invariance under infinitesimal conformal transformations demands that

$$\sum_{j=1,2,3} \big[h_j\, D^{-1}(\partial.\xi)(x_j) + \xi^\mu(x_j)\partial_{x_j^\mu}\big] G^{(3)}(x_1,x_2,x_3) = 0,$$

for any conformal vector $\xi^\mu(x)$. See previous exercise.
(v) Integrate this set of differential equations to determine the explicit expression of $G^{(3)}(x_1, x_2, x_3)$ up to constant.

• *Exercise 7.7.4:* Diff $\mathbb{S}^1$ *and its central extension.*

The aim of this exercise is to study the Lie algebra Diff $\mathbb{S}^1$ of vector fields in the circle and its central extension the Virasoro algebra. Let $z = e^{i\theta}$ coordinate on the unit circle. A diffeomorphism is on application $\theta \to f(\theta)$ from $\mathbb{S}^1$ onto $\mathbb{S}^1$. Using the coordinate z, we can write it as $z \to f(z)$ so that it is, at least locally, identified with a holomorphic map (again locally holomorphic). They act on functions $\phi(z)$ by composition: $\phi(z) \to (f\cdot\phi)(z) = \phi(f^{-1}(z))$. For an infinitesimal transformation, $f(z) = z + \epsilon\, v(z) + \cdots$ avec $\epsilon \ll 1$, the transformed function is

$$(f\cdot\phi)(z) = \phi(z) + \epsilon\, \delta_v\phi(z) + \cdots, \quad \text{with } \delta_v\,\phi(z) = -v(z)\,\partial_z\phi(z).$$

(i) Take $v(z) = z^{n+1}$, with n integer. Verify that $\delta_v\phi(z) = \ell_n\phi(z)$ with $\ell_n \equiv -z^{n+1}\partial_z$. Show t

$$\big[\ell_n, \ell_m\big] = (n-m)\,\ell_{n+m}.$$

This Lie algebra is called the Witt algebra.
(ii) Let us consider the (central) extension of the Witt algebra, generated by the ℓ_n and the central element c, with the following commutation relations

$$\big[\ell_n, \ell_m\big] = (n-m)\ell_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m;0}, \quad [c, \ell_n] = 0.$$

Verify that this set of relation satisfy the Jacobi identity. This algebra is called the Virasoro algebra.
(iii) Prove that this is the unique central extension of the Witt algebra.

• *Exercise 7.7.5: The stress-tensor OPE in 2D CFT*

Let ϕ be a massless Gaussian free field in 2D with two point function $\langle\phi(z,\bar z)\phi(w,\bar w)\rangle = -\log(|z-w|^2/R^2)$. Recall that the (chiral component of the) stress-tensor of a massless 2D Gaussian field is $T(z) = -\frac{1}{2} : (\partial_z\phi)^2(z) :$. Prove, using Wick's theorem, that it satisfies the OPE

$$T(z_1)T(z_2) = \frac{c/2}{(z_1-z_2)^4} + \big[\frac{2}{(z_1-z_2)^2} + \frac{1}{(z_1-z_2)}\partial_z\big]T(z_2] + \text{reg.}$$

• *Exercise 7.7.6: Transformation of the stress-tensor in 2D CFT.*

Recall that the transformation rules for the stress-tensor in 2D CFT, under a conformal transformation $z \to w = w(z)$. :

$$T(z) \to \hat{T}(w) = [z'(w)]^2\, T(z(w)) + \frac{c}{12}\, S(z;w),$$

with $z'(w)$ the derivative of z w.r.t. w and $S(z;w)$ the Schwarzian derivative: $S(z;w) = [\frac{z'''(w)}{z'(w)}] - \frac{3}{2}[\frac{z''(w)}{z'(w)}]^2$.
(i) Let us consider two conformal transformations $z \to w = w(z)$ and $w \to \xi = \xi(w)$ and their composition $z \to \xi = \xi(z)$. Prove that consistency of the stress-tensor transformation rules demands that:

$$S(z;\xi) = S(w;\xi) + [\xi'(w)]^2\, S(z,w).$$

Verify this relation from the definition of $S(z;w)$.
(ii) Use this formula to compute the stress-tensor expectation for a CFT defined over a infinite cylinder of radius R. Show that

$$\langle T(z)\rangle_{\text{cylinder}} = -c\,\frac{\pi}{12\,R^2}.$$

• *Exercise 7.7.7: Regularization of vertex operators*

In the text, we use the connection with lattice model to argue for the anomalous transformation of vertex operators in gaussian conformal field theory. The aim of this exercise is to derive (more rigorously) this transformation within field theory (without making connection with lattice models).

Let $\phi(z,\bar{z})$ a Gaussian free field normalized by $\langle \phi(z,\bar{z})\phi(w,\bar{w})\rangle = -\log(|z-w|^2/R^2)$ with R the IR cut-off tending to infinity. In order to regularized the field we introduce a smeared version ϕ_ϵ of ϕ defined by integrating it around a small circle, of radius ϵ, centred at z:

$$\phi_\epsilon(z,\bar{z}) = \int_0^{2\pi} \frac{d\theta}{2\pi}\, \phi(z_\epsilon(\theta), \bar{z}_\epsilon(\theta)),$$

with $z_\epsilon(\theta)$ be point on this circle, $0 < \theta < 2\pi$. The small radius ϵ play the role of UV cutoff.

(i) Prove that (notice that we consider the smeared at the same central position z but with two different cutoff ϵ and ϵ')

$$\langle \phi_\epsilon(z,\bar{z})\phi_{\epsilon'}(z,\bar{z})\rangle = \min(\log(R/\epsilon)^2, \log(R/\epsilon')^2).$$

In particular $\langle \phi_\epsilon(z,\bar{z})^2\rangle = \log(R/\epsilon)^2$.
(ii) Verify that $\langle e^{i\alpha\phi_\epsilon(z,\bar{z})}\rangle = (\epsilon/R)^{\alpha^2}$, for α real. Let us define the vertex operator by

$$V_\alpha(z,\bar{z}) = \lim_{\epsilon\to 0} \epsilon^{-\alpha^2}\, e^{i\alpha\phi_\epsilon(z,\bar{z})}.$$

Argue that this limit exists within any expectation values.
(iii) Let us now consider a conformal transformation $z \to w = w(z)$ or inversely $w \to z = z(w)$. Show that a small circle of radius $\hat{\epsilon}$, centred at point w, in the w-plane is deformed into a small

close curve in the z-plane which approximate a circle of radius $\epsilon = |z'(w)|\,\hat{\epsilon}$, centred at $z(w)$. Deduce that under such conformal transformation the vertex operator transforms as follows:

$$\hat{V}_\alpha(w,\bar{w}) = |z'(w)|^{\alpha^2}\, V_\alpha(z,\bar{z}).$$

That is: the anomalous scaling transformation of the vertex operator arises from the fact that the regularization scheme/geometry is not preserved by the conformal transformations.

- *Exercise 7.7.8: Free field in a finite domain*

  ...Dirichlet Green function in a finite domain and free field in this domain...

8 The renormalisation group

The aim of this important chapter is to present the basics ideas of the renormalization group (RG), the techniques associated to its implementation, and the output of its analysis. Following K. Wilson, RG transformations will be introduced by doing real space renormalization via block spin transformations. This amounts to find iteratively the effective hamiltonians coding for the interactions of coarse grained blocks of spins of increasing sizes. It may alternatively be viewed as arising from a partitioning of the configuration space though a conditioning procedure (the conditioning amounts to fix the effective spins labelling the block spin configurations). The way the interacting hamiltonians evolve under RG transformations is embedded into so-called beta functions which generate the RG flows on the space of coupling constants. RG field transformations are coded into so-called matrices of anomalous dimensions. Critical theories are in correspondence with fixed points of the renormalization group. Their existence explains the origin of scaling exponents and scaling functions and the universality of critical phenomena. RG transformations can be analyzed perturbatively around fixed points. At leading orders closed to a fixed point RG data are given by simple data of the corresponding conformal field theory (scaling field dimensions, OPE coefficients). The analysis of the RG transformations yields a scheme to define renormalized field theories via scaling limits of lattice regularized theories.

8.1 Block spins and RG transformations

• *Block-spin transformations in the 1D Ising model.*

We start with 1D Ising hamiltonian $H[s] = -J\sum_i s_i s_{i+1}$ with $s_i = \pm$ (we absorb the inverse temperature β in the energy scale J so that $J \propto 1/T$). The partition sum is $Z = \sum_{[s]} e^{-H[s]}$. There are many ways to solve this (trivial) problem, say by using the transfer matrix formalism. But here we will do it using a complicated method to illustrate the idea of the RG.

Imagine grouping the spins by blocks of size 3, i.e. $(\cdots][s_1s_2s_3][s_4s_5s_6][\cdots)$. Each blocks may be in $2^3 = 8$ configurations. We can group these eight configurations in two disjoint sets to which we assign an effective spin s'. We can for instance choose the majority rule so that $s' = +$ if the three internal spins of the block are $[++-]$ or a permutation thereof, and $s' = -$ if the internal spins are $[--+]$ up to permutation. It actually will be simpler if choose to assign to each block the spins of the middle site, so that the effective spin for the block $[s_1s_2s_3]$ is $s' = s_2$, or alternatively $s' = \pm$ for the configurations $[s_1 \pm s_3]$. We then imagine computing the partition function in two steps: first summing over the internal spins of each blocks conditioned on their effective spins and second on the block effective spins.

Consider two adjacent blocks, say $(\cdots][s_1s_2s_3][s_4s_5s_6][\cdots)$, and denote by $s'_1 := s_2$ and $s'_2 := s_5$ the two block spins. Doing the partial sum induces effective interaction between the block spins. The Boltzmann weights are of the form:

$$\cdots e^{Js_1s'_1} \times e^{Js'_1s_3}\, e^{Js_3s_4}\, e^{Js_4s'_2} \times e^{Js'_2s_6} \cdots .$$

We sum over s_3 and s_4 at s'_1 and s'_2 fixed (the other spins $s_1,\ s_6, \cdots$ do not play a role). Using $e^{Jss'} = \cosh J(1 + xss')$ with $x = \tanh J$, we may write this as the product of three terms $(\cosh J)^3(1+xs'_1s_3)(1+xs_3s_4)(1+xs_4s'_2)$. The sum is done by expanding this product. It yields

$$2^2(\cosh J)^3\,(1 + x^3\, s'_1s'_2).$$

Up to a multiplicative constant (independent of the spins) this expression for the interaction between the blocks spins (the local Boltzmann weights) is of the same form as that for the spin of the original model but with a new interaction constant J'

$$x' = x^3, \quad \text{i.e. } \tanh J' = (\tanh J)^3.$$

The (new) hamiltonian for the block spin is thus identical the original 1D Ising hamiltonian up to an irrelevant constant,

$$H'([s']) = N\mathfrak{e}(J) - J' \sum_i s'_i s'_{i+1},$$

with $\log \mathfrak{e}(J) = 2^{2/3}(\cosh J)/(\cosh J')^{1/3}$. The original partition function can thus be rewritten as

$$Z(J) = \sum_{[s]} e^{-H[s]} = \sum_{[s']} e^{-H'[s']} = e^{-N\mathfrak{e}(J)}\, Z(J').$$

We thus have effectively reduced the number of degrees of freedom from 2^N to $2^{N/3}$.

By iteration the effective coupling transforms as $x_n \to x_{n+1} = x_n^3$ at each step. There is only two fixed points: $x = 1$ which corresponds to zero temperature and $x = 0$ which corresponds to infinite temperature. Since $x < 1$, unless $T = 0$, the effective couplings x_n converge to zero, and the effective temperature increases towards infinite temperature. Hence, the long distance degrees of freedom are effectively described by an infinite temperature: they are in the disordered paramagnetic phase (a statement that we already knew: no phase transition in 1D).

Since the system is in the disordered phase, its correlation length is finite. The 'physical' correlation length has of course the dimension of a length, but we can measure it in units of the lattice spacing a. This dimensionless correlation length only depends on J, or equivalently on x'. Since the block spin transformations preserve the long distance physics, the dimension-full correlation length is preserved by these transformations. Since the lattice size has be dilated by a factor 3, i.e. $a \to \lambda a$ ($\lambda = 3$), the dimensionless correlation length transform as

$$\xi(x') = \frac{1}{3}\xi(x).$$

Since $x' = x^3$, this implies that

$$\xi(x) = \frac{\text{const.}}{\log x} = \frac{\text{const.}}{\log(\tanh J)}.$$

Of course it is always finite (because the system is in the disordered phase) but it diverges exponentially close to zero temperature: $\xi \simeq e^{\text{const.}/T}$ near $T \to 0$.

• *Framework for a general theory: RG transformations.*

Consider a lattice model with spin variables s_i on each lattice site (of mesh size a) with hamiltonian H (again the inverse temperature is included in H). As we will soon see, we need to think about the hamiltonian as having all possible interactions included (compatible with the symmetry). That is: we have to think as H being the "more general hamiltonian" $H([s]|\{g\})$ with coupling constants $\{g\}$ (a possibly infinite number) of the form (if it preserves the symmetry)

$$H([s]|\{g\}) = \sum_{ij} g^{(2)}_{ij} s_i s_j + \sum_{ijkl} g^{(4)}_{ijkl}\, s_i s_j s_k s_l + \cdots .$$

Let us do a block spin transformation. Each block is supposed to be of dimensionless size λ (this is its size measured in lattice units), i.e. its 'physical' size is λa. At each block we affect an effective spin s^λ, say by the majority rule or via the middle spin. Each effective spin corresponds to a set of configuration of the original spins (in each block). That is, each block spin configuration $[s^\lambda]$ indexes a partition of the set of original spin configurations. So we can decompose the sum of the original spin configurations as

$$\sum_{[s]}(\cdots) = \sum_{[s^\lambda]} \sum_{[s]\downarrow[s^\lambda]} (\cdots), \tag{57}$$

where $[s] \downarrow [s^\lambda]$ means that the configuration $[s]$ belongs to the set of configurations indexed by $[s^\lambda]$, i.e. the configuration $[s]$ yields the block spin configuration $[s^\lambda]$. What we have done here is simply to partition (or to condition) the configuration space, the partition being indexed by the $[s^\lambda]$.

In particular for the partition function we have

$$Z[\{g\}] = \sum_{[s]} e^{-H([s]|\{g\})} = \sum_{[s^\lambda]} \sum_{[s]\downarrow[s^\lambda]} e^{-H([s]|\{g\})} = \sum_{[s^\lambda]} Z'([s^\lambda]|\{g\}),$$

with $Z'([s^\lambda]|\{g\})$ the conditioned partition function obtained by conditioning

$$Z'([s^\lambda]|\{g\}) := \sum_{[s]\downarrow[s^\lambda]} e^{-H([s]|\{g\})}.$$

We may define an effective hamiltonian H' for the effective block spins via $H'([s^\lambda]|\{g\}) := -\log Z'([s^\lambda]|\{g\})$.

The RG hypothesis (which is here a bit tautological as we consider the most general hamiltonians) is that this hamiltonian H' is of the same nature as the orginal one (up to an additive constant) but for new coupling constant $\{g^\lambda\}$:

$$H'([s^\lambda]|\{g\}) = N\mathfrak{e}^\lambda(\{g\}) + H([s^\lambda]|\{g^\lambda\}), \tag{58}$$

or equivalently

$$Z'([s^\lambda]|\{g\}) = e^{-N\mathfrak{e}^\lambda(\{g\})}\, e^{-H([s^\lambda]|\{g^\lambda\})},$$

In other words, Z' is the new Boltzmann weight for the effective block spins (Boltzmann weights and partition functions are always defined up to a multiplicative factor, so the function $\mathfrak{e}$ does not matter much).

The RG transformation is the map from $\{g\}$ to $\{g^\lambda\}$:

$$\{g\} \to \{g^\lambda\} = \mathcal{R}^\lambda(\{g\}). \tag{59}$$

Of course, we can iterate these transformations. We can first group spins into blocks of λ and then group the new effective spins into blocks of size λ'. This will of course be equivalent to group the spins in blocks of size $\lambda \cdot \lambda'$ (dilatation factors multiply). Hence

$$\mathcal{R}^{\lambda'\cdot\lambda} = \mathcal{R}^{\lambda'} \cdot \mathcal{R}^\lambda,$$

and the RG transformations form a group (a pseudo-group has they cannot be inverted: $\lambda > 1$ always).

Since they form a (pseudo)-group, the RG transformations are generated by the infinitesimal transformations. Here, we are implicitly making the assumption that we can view the scaling factor λ as a continuous variable (initially, on the lattice this scale factor was discrete but may assume that when describing the long distance physics we may take it to be continuous). So let g_α coordinates for the set of coupling constants and define the vector field

$$\beta_\alpha(\{g\}) := \lambda \partial_\lambda g_\alpha^\lambda|_{\lambda=1}.$$

These are called "beta-functions" (they are actually vector fields). They generate the RG map $\{g\} \to \{g^\lambda\}$ in the sense that $\{g^\lambda\}$ is solution of the flow equation

$$\lambda \partial_\lambda g_\alpha^\lambda = \beta_\alpha(\{g^\lambda\}), \tag{60}$$

with initial condition $\{g^{\lambda=1}\} = \{g\}$. The solutions g^λ are called the "running coupling constants".

As in the case of the 1D Ising model, the physical correlation length remains unchanged under RG transformations (because this is just a reorganization of the statistical sum). However the lattice mesh is rescaled from a to λa under the transformation. Hence, the dimensionless correlation length satisfies

$$\xi(\{g\}) = \lambda\, \xi(\{g^\lambda\}).$$

In practice we can never compute $\mathcal{R}^\lambda$ acting on an infinite set of variables. So we will have to do approximations (constructive field theory and/or exact renormalization group aims at controlling exactly these RG transformations) by truncating the set of coupling constant (keeping only the relevant ones, or those which are expected to be the most relevant ones).

Of course we cannot iterate the RG group ad-finitum, we have to stop once the size of the block is comparable to the correlation length, that is $a \ll \lambda\, a \ll a\xi$. This is self consistent because spin inside a block of size much smaller than $a\xi$ are correlated (they are a distance much smaller that the correlation length) and hence behave almost collectively. This is the physical rational behind the renormalization group idea.

The understanding of many physical phenomena, ranging from fundamental interactions to classical or quantum extended systems, requires extracting the relevant large-scale degrees of freedom, which can manifest themselves in different guises, say collectives modes, shapes, structures or variables. Extracting these relevant variables is of course one of the main aim of the renormalization group, but making the RG work often requires having recognized the appropriate setup (in the opposite case, one cannot make the RG transformations contracting towards a relevant effective model, and as an illustration, the absence of such appropriate setup is for instance why the RG has not been yet successfully applied to turbulence).

8.2 Momentum RG transformations in field theory

We can alternatively formulate the RG transformations in momentum space. This idea is then to integrate recursively over shells of high momenta and to rescale space appropriately.

To make it more precise let us assume that we look at a scalar field ϕ with action

$$S[\phi] = S_0[\phi] + S_1[\phi].$$

where S_0 is a free quadratic action and S_1 is coding for the interaction. To simplify matter, we choose S_0 to be the action of a massless scalar field in dimension $D > 2$ with a (smooth) momentum cut-off $\Lambda = 1/a$.

The field theory defined by S_0 is therefore Gaussian with two-point function

$$G(x-y) = \langle \phi(x)\phi(y)\rangle|_{S_0} = \int \frac{d^D k}{(2\pi)^D} \frac{e^{ik\cdot(x-y)}}{k^2}\, \varphi_\Lambda(|k|),$$

where $\varphi_\Lambda(|k|)$ is the smooth cut-off function: $\varphi_\Lambda(k) \simeq 1$ for $|k| \leq \Lambda$ and $\varphi_\Lambda(k) \simeq 0$ for $|k| \geq \Lambda$, that is: φ_Λ varies only abruptly from 1 to 0 when $|k|$ crosses the cut-off value Λ. Let us denote by $d\mu_G[\phi]$ this Gaussian measure: (in)-formally $d\mu_G[\phi] = [D\phi]\, e^{-S_0[\phi]}$. The partition function of the interacting theory is $Z = \int d\mu_G[\phi]\, e^{-S_1[\phi]}$.

To implement the RG idea, we want to pick a rescaling factor $\lambda > 1$ and to integrate over all modes with momenta in the shell between $\lambda^{-1}\Lambda$ and Λ. The part of the Gaussian covariance with support in this shell is Γ_λ, with

$$\Gamma_\lambda(x) = \int \frac{d^D k}{(2\pi)^D} \frac{e^{ik\cdot(x-y)}}{k^2} \big[\varphi_\Lambda(|k|) - \varphi_\Lambda(\lambda|k|)\big],$$

because the difference $\varphi_\Lambda(|k|) - \varphi_\Lambda(\lambda|k|)$ is significantly non zero only for $|k| \in [\lambda^{-1}\Lambda, \Lambda]$. Note then that $G = G_\lambda + \Gamma_\lambda$ with $G_\lambda(x) = \lambda^{-(D-2)}\, G(x/\lambda)$, by construction. This decomposition of the covariance tells us that we can decompose the field ϕ as a sum $\phi = \hat\phi_\lambda + \varphi_\lambda$ where $\hat\phi_\lambda$ and φ_λ have covariances G_λ and Γ_λ respectively (because the sum of two independent Gaussian variables is a Gaussian variable with covariance equals to the sum of the two covariances).

Since G_λ is defined from G be rescaling, we can construct the field $\hat\phi_\lambda$ by taking an independent copy of ϕ, which we denote $\hat\phi$, and set $\hat\phi_\lambda(x) = \lambda^{-(D-2)/2}\, \hat\phi(x/\lambda)$. We thus get the decomposition of the Gaussian measure as

$$d\mu_G[\phi] = d\mu_G[\hat\phi]\, d\mu_{\Gamma_\lambda}[\varphi_\lambda], \quad \phi = \hat\phi_\lambda + \varphi_\lambda,$$

with $d\mu_G[\hat\phi]$ the initial Gaussian measure but for the independent variable $\hat\phi$ and $d\mu_{\Gamma_\lambda}[\varphi_\lambda]$ the Gaussian measure with covariance Γ_λ.

With all this preparation we can now integrate over the modes with momenta in the shell $[\lambda^{-1}\Lambda, \Lambda]$ by integrating over the field φ_λ. This indeed gives a precise formulation of integrating over the high momenta and rescaling space appropriately. For the Boltzmann weights, or equivalently for the action, this yields the definition of the flow $S_1 \to S_\lambda$ by

$$e^{-S_\lambda[\hat\phi]} = \int d\mu_{\Gamma_\lambda}[\varphi_\lambda]\, e^{-S_1[\hat\phi_\lambda + \varphi_\lambda]}.$$

As above with block spin transformations, this map is such that it preserves the partition function:

$$Z = \int d\mu_G[\phi]\, e^{-S_1[\phi]} = \int d\mu_G[\phi]\, e^{-S_\lambda[\phi]},$$

by construction. This defines the RG flow of the (interacting) part of the action: $\mathcal{R}^\lambda : S_1 \to S_\lambda$. As above it forms a semi-group under composition: $\mathcal{R}^\lambda \circ \mathcal{R}^{\lambda'} = \mathcal{R}^{\lambda\lambda'}$. The RG flow is here defined on the space of actions and not on the space of coupling constants, but these two formulations are equivalent because coupling constants are simply all possible parameters specifying an action.

Anticipating a little on what we are going to describe in lattice models, we can also describe how operators/observables evolve under RG transformations. Let $F_1[\phi]$ be some functional of the field which we identify with an observable. We can again define its RG transform by

decomposing the field as $\phi = \hat{\phi}_\lambda + \varphi_\lambda$ and by integrating over the modes φ_λ. This yields the definition of the transformed observable F_λ as

$$F_\lambda[\hat{\phi}] = e^{+S_\lambda[\hat{\phi}]} \times \int d\mu_{\Gamma_\lambda}[\varphi_\lambda]\, e^{-S_1[\hat{\phi}_\lambda+\varphi_\lambda]}\, F_1[\hat{\phi}_\lambda + \varphi_\lambda].$$

Note the presence of the prefactor $e^{+S_\lambda[\hat{\phi}]}$ which ensures that the identify function is mapped to the identity. This transformation is such that expectations are preserved, namely

$$\langle F_1[\phi]\rangle|_{S_0+S_1} = \langle F_\lambda[\hat{\phi}]\rangle|_{S_0+S_\lambda}.$$

These rules are analogous to those for RG transformation of operators in lattice models, see eq.(66).

8.3 RG fixed points and universality

We now look at what are the consequences of the existence of fixed points of the RG transformations. To simplify matter we assume that the RG flow is defined on a set of $\mathcal{N}$ coupling constant ("rigorously" this number is, or could be, infinite). We also assume that the RG transformations have enough analytical properties (smoothness, regularities,...) to justify the formal developments we are going to make.

• *RG fixed points.*

A fixed point (in the coupling constant manifold) is a point $\{g_*\}$ such that

$$\mathcal{R}^\lambda(\{g_*\}) = \{g_*\}.$$

Fixed points are zero of the beta-functions (because the beta-functions are the vector field generating the RG transformations):

$$\beta_\alpha(\{g_*\}) = 0. \tag{61}$$

At a fixed point the correlation length is either infinite or zero (because it satisfies $\xi(\{g_*\}) = \lambda\xi(\{g_*\})$ and by iteration $\xi(\{g_*\}) = \lambda^n\, \xi(\{g_*\})$ for any $\lambda > 1$). Fixed points with zero correlation length are called trivial fixed point. Critical fixed points are those with infinite correlation length.

• *RG eigenvalues at a fixed point.*

We aim at describing the behaviour of the RG flow closed to a fixed point (this is actually a standard exercice in dynamical systems). Let us linearize the flow in the vicinity of the point. Let $\{g^\alpha\}$ be some local coordinates in the space of all couplings). We can then linearize the RG flow (to first order) and write near a fixed point (with coordiantes $\{g^\alpha_*\}$)

$$\mathcal{R}^\lambda(\{g^\alpha_* + \varepsilon\, \delta g^\alpha + \cdots\}) = \{\varepsilon \sum_\sigma \mathcal{B}^\lambda_{\alpha\sigma}\, \delta g^\sigma + \cdots\},$$

with $g^\alpha = g^\alpha_* + \varepsilon\, \delta g^\alpha + \cdots$. The matrix $\mathcal{B}^\lambda_{\alpha\sigma}$ is simply the matrix of derivatives of $\mathcal{R}^\lambda$ at the fixed point. The nature of the flow locally around the fixed point is determined by the eigenvalues of $\mathcal{B}^\lambda_{\alpha\sigma}$, because we can then change variables locally to diagonalize the flow. Indeed, let ψ^i be

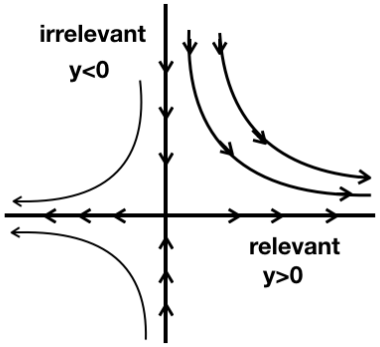

Figure 6: *Schematic representation of coupling constant RG flow.*

the left eigenvectors, with component ψ^i_α. These eigenvalues are necessarily of the form λ^{y_i} for some y_i because of the group law of the RG transformation: $\sum_\alpha \psi^i_\alpha \mathcal{B}^\lambda_{\alpha\sigma} = \lambda^{y_i} \psi^i_\sigma$. Let us define locally new coordinates u^i by $u^i = \sum_\alpha \psi^i_\alpha \delta g^\alpha$. The new coordinates transform diagonally (at in a small neighbourhood of the fixed point)

$$\mathcal{R}^\lambda(\{u^i\}) = \{\lambda^{y_i} u^i\}.$$

The number y_i are called the RG eigenvalues and the u^i scaling variables (or scaling fields).

There are three cases:
— $y_i > 0$: the u^i are said to be "relevant". These directions are pushed away from the fixed point from by the RG flow;
— $y_i = 0$: the u^i are said to be "marginal". The nature of the flows depends on the higher order terms: it can be marginally relevant or marginally irrelevant.
— $y_i < 0$: the u^i are said to be "irrelevant". These directions are attracted to the fixed point by the RG flow;

Suppose that they are n relevant directions (n finite!). These n directions flow away from the fixed points, the $\mathcal{N} - n$ remaining ones are attracted by the fixed point. Suppose that n is relatively small (or at least finite!... and this is always the case we shall consider —it corresponds to renormalizable field theories). Then, there is, locally around the fixed point, a hypersurface of (high) dimension $\mathcal{N} - n$ whose points flow towards the fixed point under RG transformations. This hypersurface is called the "critical surface". For all points on this surface the long distance physics is critical and described by the fixed point because all those points flow towards the fixed points. In the remaining directions the flow moves the point away from the fixed points. Thus n (relevant) parameters have to be adjusted for the system to be critical (to be on the critical surface). For instance, for the Ising model, two parameters have to be adjusted, the temperature T and the magnetic field h, so there are two relevant directions whose coordinates can be named u_t and u_h.

The very large dimensionality of the critical hypersurface explains (or expresses) the universality of the critical phenomena: all points on this hypersurface possess the same critical behavior, i.e. all hamiltonians on the hypersurface, irrespective of the interactions they encode, have the same critical behavior. See Figure 6.

• *Operator scaling dimensions.*

We now relate the RG eigenvalues to the scaling dimensions of the operators at the conformal point. The continuum theory corresponding to a fixed point has no typical scale and it is scale invariant. It is thus described by a conformal field theory (with an action that we denote S_*). That is: conformal field theories are fixed point of the renormalization group. Exploring the neighbour of the critical point amounts to perturb the critical action S_* with by some operators $\mathcal{O}_\alpha$.

$$S = S_* + \sum_\alpha (\delta g)^\alpha \sum_{\mathbf{n}} \mathcal{O}_\alpha(\mathbf{n}),$$

with $\mathbf{n}$ labeling the integer points on the lattice. However, we can change basis of operators. Recall that a conformal field theory comes equipped with its set of operators $\Phi_i(x)$ of scaling dimension Δ_i. They transform homogeneously under dilatation $\Phi_i \to \lambda^{-\Delta_i}\Phi_i$. Using this basis, we can alternatively write

$$S = S_* + \sum_i u^i \sum_{\mathbf{n}} \Phi_i(\mathbf{n}).$$

the behaviour near a critical point is thus governed by scaling operators at the critical point. In particular the critical exponents directly are related to the scaling dimension of the operators at the critical point. This statement is made more precise by looking at the RG transform close to a fixed point.

A comment about the normalization of the field on the lattice and on the continuum is needed. We initially start with dimensionless fields defined on the integer lattice point, say $\Phi_i^{\text{latt}}(\mathbf{n})$ where $\mathbf{n}$ refers to the integer point of lattice of mesh size a (and we temporally added the label 'latt' to specify that these fields are defined within the lattice model). At the fixed point, their two point functions are power law with unit normalization (because they are dimensionless), i.e.

$$\langle \Phi_i^{\text{latt}}(\mathbf{n}) \Phi_i^{\text{latt}}(\mathbf{m}) \rangle_* = \frac{1}{|\mathbf{n}-\mathbf{m}|^{2\Delta_i}}.$$

In the continuous limit ($a \to 0$), the scaling field are defined as (recall how we define the scaling limit of free random paths or how we define the renormalized vertex operators in conformal field theory)

$$\Phi_i^{\text{cont.}}(x) = \lim_{\substack{a\to 0, \mathbf{n}\to\infty \\ x = a\mathbf{n} \text{ fixed}}} a^{-\Delta_i}\, \Phi^{\text{latt}}(\mathbf{n}) = \lim_{a\to 0} a^{-\Delta_i}\, \Phi^{\text{latt}}(x/a).$$

They are now dimension-full (with the dimension of $[\text{length}]^{-\Delta_i}$) with finite two-point correlation functions (defined in the fixed point conformal field theory) normalized to

$$\langle \Phi_i^{\text{cont.}}(x) \Phi_i^{\text{cont.}}(y) \rangle_* = \frac{1}{|x-y|^{2\Delta_i}}.$$

Away from the critical point the action can thus be written as

$$S = S_* + \sum_i u^i \sum_{\mathbf{n}} \Phi_i^{\text{latt}}(\mathbf{n}) = S_* + \sum_i u^i\, a^{\Delta_i - D} \int d^D x\, \Phi_i^{\text{cont.}}(x).$$

The two above expressions are equivalent (in the limit $a \to 0$) because the discrete sum over lattice points approximates the integral: $a^D \sum_{\mathbf{n}} \Phi_i^{\text{latt}}(\mathbf{n}) = a^D \sum_{\mathbf{n}} a^{\Delta_i} \Phi_i^{\text{cont.}}(\mathbf{n}) = \int d^D x\, \Phi_i^{\text{cont.}}(x)$.

The prefactor $a^{\Delta_i - D}$ makes the action dimensionless (because u_i is dimensionless while Φ_i as dimension Δ_i).

Under dilatation (RG transformation) the scaling variable transforms as $u_i \to \lambda^{y_i} u_i$, hence consistency of the coupling tells us that

$$y_i = D - \Delta_i. \tag{62}$$

Critical exponents are thus encoded in the scaling dimensions of the operators at the conformal points. Operators are gathered in three classes:
— Relevant operators have dimension $\Delta_i < D$;
— Marginal operators have dimensions $\Delta_i = D$;
— Irrelevant operators have dimension $\Delta_i > D$.

8.4 Scaling functions and critical exponents

We now explained why RG transformations and the existence of fixed points imply that the existence of scaling functions and critical exponents. These exponents are going to be related to the RG eigenvalues as we are going to explain.

• *Universal scaling functions.*

Let us go back to the basic definition of the RG transformations coding the way the partition function decomposes. Let $f(\{g\})$ be the free energy per site, so that the partition function for N site is $Z[\{g\}] = e^{-Nf(\{g\})}$. We now prove that RG transformations imply that

$$f(\{g\}) = \mathfrak{e}^\lambda(\{g\}) + \lambda^{-D} f(\{g^\lambda\}),$$

with $\mathfrak{e}^\lambda(\{g\})$ a function, which is expected to be smooth, representing a local contribution to the free energy. The factor λ^{-D} comes from the fact that a RG transformation by a scale λ reduces the number of sites from N to $N\lambda^{-D}$.

Recall that

$$Z[\{g\}] = \sum_{[s^\lambda]} Z'([s^\lambda]|\{g\}),$$

with $Z'([s^\lambda]|\{g\}) = e^{-N\mathfrak{e}^\lambda(\{g\})}\, e^{-H([s^\lambda]|\{g^\lambda\})}$, so that

$$Z[\{g\}] = e^{-N\mathfrak{e}^\lambda(\{g\})} \sum_{[s^\lambda]} e^{-H([s^\lambda]|\{g^\lambda\})}.$$

Doing the sum of the block spin $[s^\lambda]$ but remembering that the number of blocks is $N\lambda^{-D}$ proves the result.

The function $\mathfrak{e}^\lambda(\{g\})$ is expected to be non singular because it comes from summing over the spins inside the blocks (so it comes from finite sum of Boltzmann weights and hence cannot developed singularities). As a consequence the singular part in the free energy transforms homogeneously under RG transformations:

$$f_{\text{sing}}(\{g\}) = \lambda^{-D} f_{\text{sing}}(\{g^\lambda\}).$$

After a few iterations of the RG transformations, we may expect to be close to a fixed point. There we can use the scaling variables. The relation then becomes

$$f_{\text{sing}}(\{u^i\}) = \lambda^{-D} f_{\text{sing}}(\{\lambda^{y_i} u^i\}). \tag{63}$$

Similarly for the correlation length

$$\xi(\{u^i\}) = \lambda\, \xi(\{\lambda^{y_i}\, u^i\}). \tag{64}$$

When iterating the RG transformations (λ increases) only the relevant and marginal variables with positive eigen-values remain (See the section to corrections to scaling to look for the effect of the irrelevant variables).

There are many ways to express these above relations. Let us for instance single out one of the variable, say u_t (recall that the set of $\{u^i\}$ refers to a collection of scaling variables u_T, u_h associated to external parameters, the temperature, the magnetic field, etc...). Then let us pick $\lambda = |u_T|^{-1/y_T}$ so that $\lambda_T^y u_T = \pm 1$ (or any other reference point $u_T^{(0)}$ still in the perturbative domain). Then we get

$$\xi(u_T, \{u^j\}_{j\neq T}) = |u_T|^{-1/y_T}\, \xi(\pm 1, \{\frac{u^j}{|u_T|^{y_i/y_t}}\}_{j\neq T}) =: |u_T|^{-1/y_T}\, \hat{\xi}_\pm(\{\frac{u^j}{|u_T|^{y_i/y_T}}\}_{j\neq T}),$$

and

$$f_{\rm sing}(u_T, \{u^j\}_{j\neq T}) = |u_T|^{D/y_T}\, f_{\rm sing}(\pm 1, \{\frac{u^j}{|u_T|^{y_i/y_T}}\}_{j\neq T}) =: |u_T|^{D/y_T}\, \hat{f}_\pm(\{\frac{u^j}{|u_T|^{y_i/y_T}}\}_{j\neq T}).$$

Recall that we only kept the relevant variables.

The simplest case if when there is only one relevant scaling variable. Let us still denote it u_T. Then, the correlation length ξ is a scaling function of that variable:

$$\xi(u_T) \propto |u_T|^{-1/y_T}.$$

This is the only scale of the problem (if we are dealing with an infinite volume system) and all physical quantities can be expressed in terms this scale. That is: we can measure everything in terms of the correlation length instead of the scaling variable u_T.

In cases with only two relevant scaling variables, say u_T and u_h as in the Ising model, we have

$$\xi(u_T, u_h) = |u_T|^{-1/y_T}\, \hat{\xi}_\pm(\frac{u_h}{|u_T|^{y_h/y_T}}),$$

and

$$f_{\rm sing}(u_T, u_h) = |u_T|^{D/y_T}\, \hat{f}_\pm(\frac{u_h}{|u_T|^{y_i/y_T}}).$$

The functions $\hat{\xi}_\pm$, $\hat{f}_\pm$ are called "universal scaling functions".

• *Critical exponents.*

The existence of critical exponents follows from these relations.

— At zero magnetic field $u_h = 0$ and the correlation length scales like $\xi \simeq |u_T|^{-1/y_T}$, so that

$$\nu = 1/y_T.$$

— The specific heat at zero magnetic field is $\partial^2 f/\partial u_T^2|_{u_h=0} \simeq |u_T|^{D/y_T - 2}$, so that

$$\alpha = 2 - D/y_T.$$

— The spontaneous magnetization is $\partial f/\partial u_h|_{u_h=0} \simeq |u_T|^{(D-y_h)/y_T}$ so that

$$\beta = (D - y_h)/y_T.$$

— The magnetic susceptibility is $\partial^2 f/\partial u_h^2|_{u_h=0} \simeq |u_T|^{(D-2y_h)/y_T}$ so that

$$\gamma = (2y_h - D)/y_T.$$

— The magnetization at critical temperature is $\partial f/\partial u_h|_{u_T=0}$. To compute requires knowing the behaviour of $\hat{f}_\pm$ at infinity. Demanding that $f_{\rm sing}(u_T, u_h)$ has finite limit as $u_T \to 0$ requires that $\hat{f}_\pm(x) \simeq x^{D/y_h}$ (so that the various powers of $|u_T|$ cancels). Hence $\partial f/\partial u_h|_{u_T=0} \simeq u_h^{D/y_h-1}$, so that

$$\delta = y_h/(D - y_h).$$

Since the renormalization eigenvalues y_i are given by the dimensions of the operators at the fixed point, all critical exponents are determined from these dimensions.

8.5 Corrections to scaling and finite size effects

Corrections to scaling comes from taking irrelevant variables into account. Finite size effects are treated by looking at the RG transformation fixing the size of the system L fixed. Let us concentrate on the finite size effects.

Suppose that we consider a critical system but in a finite size box of linear system L, say a square box $L \times L \times L$. The free energy is going to depend on the coupling constants $\{g\}$ and on the size L. But, since L is dimension-full, the dependence of the free energy on L comes via the ratio L/a with a the lattice microscopic mesh size – or alternatively the free energy depends on the coupling constants $\{g\}$ and on the number of lattice sites $N = (L/a)^D$. Let us now imagine implementing a RG transformation –say a block spin transformation– scaling a to λa at L fixed. As explained above, this reduces the number of lattice sites –and hence of degrees of freedom– from N to $\lambda^{-D}N$, or equivalently $L/a \to \lambda^{-1}\, L/a$. Hence, under a RG transformation, the singular part of the free energy transforms as

$$f_{\rm sing}(\{g\}, L/a) = \lambda^{-D}\, f_{\rm sing}(\{g^\lambda\}, \lambda^{-1} L/a),$$

with g^λ the RG dressed coupling constants. Here we are implicitly making the hypothesis that the coupling constants transform in the same way in infinite and finite volume. Similarly, the correlation length transform as

$$\xi(\{g\}, L/a) = \lambda\, \xi(\{g^\lambda\}, \lambda^{-1} L/a).$$

As in infinite volume, we may expect that we approach a fixed point after a few iterations of the RG transformations. There we can use the scaling variables and write the above relation as

$$\begin{aligned} f_{\rm sing}(\{u^i\}, L/a) &= \lambda^{-D}\, f_{\rm sing}(\{\lambda^{y_i}\, u^i\}, \lambda^{-1}\, L/a), \\ \xi(\{u^i\}, L/a) &= \lambda\, \xi(\{\lambda^{y_i}\, u^i\}, \lambda^{-1}\, L/a). \end{aligned}$$

As in infinite volume, the scaling form can found by singling out on variable, say u_T, and choosing the RG scale $\lambda = |u_T|^{-1/y_T}$.

Let us for instance assume that there is only one relevant scaling variable, again say u_T. Then the scaling relation for the correlation length, in finite volume, reads

$$\xi(u_T, L/a) = |u_T|^{-1/y_T}\, \hat{\xi}_\pm(|u_T|^{1/y_T}\, L/a) = |u_T|^{-1/y_T}\, \hat{\xi}_\pm(L/\xi_\infty),$$

where we introduce the (dimension-full) correlation length in infinite volume $\xi_\infty = a\,|u_T|^{-1/y_T}$. In other words, the correction to infinite volume limit is obtained by comparing the only two relevant macroscopic length: the size of the system and the system correlation length.

Similarly, the behaviour the finite size correction to the critical behaviour of the specific heat can be obtained form the scaling relation $f_{\rm sing}(u_T, L/a) = \lambda^{-D}\, f_{\rm sing}(\lambda^{y_T}\, u_T, \lambda^{-1}\, L/a)$, in case there is only one relevant variable. For the specific heat $\chi \sim \partial^2 f/\partial^2 u_T$ this gives

$$\chi \sim |u_T|^{-\alpha}\, \varphi_\pm(L/\xi_\infty) = |u_T|^{-\alpha}\, \varphi_\pm(|u_T|^{\nu}\, L/a),$$

with again $\xi_\infty = a\,|u_T|^{-\nu}$ the infinite volume correlation length (with $\nu = 1/y_T$ the correlation length exponent). The infinite volume is recovered in the limit $L \to \infty$ so that $\varphi(\infty)$ is supposed to be finite. At finite volume, there is no sharp phase transition (because there is then a finite number of degrees of freedom and the partition is a polynomial in the Boltzmann weights), and the heat capacity is non singular so that $\varphi(x) \sim x^{\alpha y_T}$ for $x \to 0$. As a consequence, the heat capacity is not singular in finite volume, but it becomes a rounded fonction of u_T with a maximum value scaling as $L^{\alpha/\nu}$ and a width scaling as $L^{-1/\nu}$.

8.6 Field transformations

We now look at how operators transform under the RG. This will give us access to the anomalous dimensions.

• *Block-spin transformation on observables.*

Recall that an observable $\mathcal{O}$ is some kind of function (a measurable function), $[s] \to \mathcal{O}([s])$ on the configuration space. In a way similar to what has been done with the partition function, RG transformations for observables $\mathcal{O} \to \mathcal{O}^\lambda$ can be defined using conditioned sums as follows

$$\mathcal{O}^\lambda([s^\lambda]) = \frac{1}{Z'([s^\lambda]|\{g\})} \sum_{[s]\downarrow[s^\lambda]} e^{-H([s]|\{g\})}\, \mathcal{O}([s]). \tag{65}$$

Of course the transformed operators depend on both on the scale transformation factor λ and on the coupling constants. Recall that we set $Z'([s^\lambda]|\{g\}) = e^{-N\mathfrak{e}^\lambda(\{g\})}\, e^{-H([s^\lambda]|\{g^\lambda\})}$ and that $Z(\{g\}) = \sum_{[s^\lambda]} Z'([s^\lambda]|\{g\})$. Hence by construction, the transformed operators are such that

$$\frac{1}{Z(\{g\})} \sum_{[s^\lambda]} e^{-N\mathfrak{e}^\lambda(\{g\})} e^{-H([s^\lambda]|\{g^\lambda\})} \mathcal{O}^\lambda([s^\lambda]) = \frac{1}{Z(\{g\})} \sum_{[s]} e^{-H([s]|\{g\})}\, \mathcal{O}([s]).$$

The l.h.s. is the expectation with respect to the transformed action with coupling constants $\{g^\lambda\}$, the r.h.s. is the original expectation with the original action with coupling constant $\{g\}$. The relation thus tells us that expectations of the transformed operators $\mathcal{O}^\lambda$ with respect to the transformed hamiltonian is the same as the expectation of the initial operator with the initial hamiltonian:

$$\langle \mathcal{O}^\lambda \rangle_{\{g^\lambda\}} = \langle \mathcal{O} \rangle_{\{g\}}. \tag{66}$$

This is a simple but important equation.

• *Local field transformations and mixing matrix.*

Suppose now that the observable is a product of local observables Φ_k, localized at different lattice points (say $\mathbf{n}_k$):

$$\mathcal{O} = \prod_k \Phi_k(\mathbf{n}_k),$$

where each Φ is a local operator sensitive to the local configuration of the spin variables (hence the definition of local operators involved open neighborhoods of their insertion points). It is reasonable to expect that the transformed operators are still going to be quasi-local (because if the rescaled lattice λa is much smaller than the correlation length ξ, the spins inside a block behave almost collectively). It can thus be decomposed on the set of local operators so that we have (or more precisely we expect)

$$\Phi^\lambda(\mathbf{n}) = \Gamma^\lambda(\{g\}) \cdot \Phi(\mathbf{n}/\lambda),$$

where Γ^λ is a matrix – coding for the decomposition of the transformed operator on the basis of the original operators. It is called the mixing matrix. We wrote the above equation in a matrix form: if we prefer to keep indices it reads $\Phi^\lambda_\alpha(\mathbf{n}) = \sum_\sigma [\Gamma^\lambda]^\sigma_\alpha \, \Phi_\sigma(\mathbf{n}/\lambda)$ where α, σ indexes all possible local operators. The rescaling of the position of the operators comes about from the same argument for the rescaling of the correlation length: After a block spin transformation the dimensionless distances between the operators (counted by the number of lattice site to cross to go from operator to the other) as been divided by λ. (Here $\mathbf{n}$ is the integer dimensionless distance, counted in unit of lattice spacing).

The mixing matrices clearly depend both on the rescaling factor λ and on the coupling constant, so that we should have more precisely written $\Gamma^\lambda(\{g\})$. They inherit a composition law —more precisely a cocycle structure— from that of the RG transformations. Applying successively twice a block spin transformations on the operators implies

$$\Gamma^{\lambda'}(g) \cdot \Gamma^\lambda(g^{\lambda'}) = \Gamma^{\lambda\lambda'}(g).$$

Of course $\Gamma^{\lambda=1}(g) = \mathbb{I}$. At a fixed point Γ^λ acts multiplicatively and $\Gamma^\lambda = \lambda^{-\Delta_i}$ on a scaling field of dimension Δ_i.

The same relation applies (locally) if we consider an operator made of products of operators localized at different position, say $\otimes_k \Phi_k(\mathbf{n}_k)$. Under RG transformations, each of this operators transforms as above with an associated Γ^λ_k matrix. For operators and their expectations the fundamental RG relation is thus the equality:

$$\langle \prod_k \Gamma^\lambda_k(g) \cdot \Phi_k(\mathbf{n}_k/\lambda) \rangle_{\{g^\lambda\}} = \langle \prod_k \Phi_k(\mathbf{n}_k) \rangle_{\{g\}}. \tag{67}$$

Again: this tells that the expectations of the transformed operators, at the transformed positions and with the transformed action, is the original expectations. This is a fundamental relation (the lattice analogue of the so-called Callan-Symanzik equation, see below).

The matrix $\Gamma^\lambda(g)$ is called the matrix of anomalous dimensions. If we sit at a fixed point g_* then it si transformed multiplicatively: $\Gamma^{\lambda'}(g_*) \cdot \Gamma^\lambda(g_*) = \Gamma^{\lambda\lambda'}(g_*)$. The scaling operators Φ_i of the fixed (conformal) theory are those diagonalising this matrix so that

$$\Phi^\lambda_i(\mathbf{n}) = \Gamma^\lambda(g_*) \cdot \Phi_i(\mathbf{n}/\lambda) = \lambda^{-\Delta_i} \, \Phi_i(\mathbf{n}/\lambda),$$

with Δ_i the scaling dimension of Φ_i.

• *Relation with the beta-functions.*

The mixing matrices are not independent of the beta functions, because local operators are dual to coupling constants. Let us slightly deformed the initial action/hamiltonian into

$$H(\{g\}) \to H(\{g+\delta g\}) = H(\{g\}) + \sum_\alpha \delta g^\alpha \sum_{\mathbf{n}} \Phi_\alpha(\mathbf{n}).$$

To first order in δg the expectation values are

$$\langle [\cdots] \rangle_{\{g+\delta g\}} = \langle [\cdots] \rangle_{\{g\}} - \sum_\alpha \delta g^\alpha \sum_{\mathbf{n}} \langle [\cdots] \Phi_\alpha(\mathbf{n}) \rangle_{\{g\}}.$$

Let us now do a block spin transformation with scale factor λ. By taking into account factors λ^D coming from the dilatation of the volume, consistency gives

$$\mathcal{R}^\lambda(g+\delta g)^\alpha = \mathcal{R}^\lambda(g)^\alpha + \sum_\sigma \delta g^\sigma \, \lambda^D \, [\Gamma^\lambda(g)]^\alpha_\sigma + \cdots,$$

This means that the mixing matrix $\Gamma^\lambda(g)$ is, up to a factor λ^D the derivative of the renormalization group, a non so-surprising result in view of the duality between local fields and coupling constants. Let us define the so-called "matrix of anomalous dimensions" $\gamma(g)$ (whose interpretation will be further developed below) by

$$\gamma(g) = -\lambda \partial_\lambda \Gamma^\lambda(g)||_{\lambda=1}.$$

Recall the definition of the beta-function, $\beta(g) = \lambda \partial_\lambda g^\lambda|_{\lambda=1}$. Then, we have the relation

$$\partial_\alpha \beta^\sigma(g) = D\delta^\sigma_\alpha - \gamma^\sigma_\alpha(g). \tag{68}$$

At a fixed point, it is reduces to the relation $y_i = D - \Delta_i$ between the RG eigenvalues and the dimensions of the scaling fields.

It is important to note that the fundamental relation $\langle \mathcal{O}^\lambda \rangle_{\{g^\lambda\}} = \langle \mathcal{O} \rangle_{\{g\}}$, which involve both for the coupling constants and the fields renormalization, codes the fact that these renormalization/redefinition ensures that the correlation functions are cut-off independent, at leat asymptotically as $a \to 0$.

8.7 The perturbative renormalization group

The aim of this section is to compute the β-function perturbatively near a fixed point and to analyse the consequences of this formula.

• *One-loop beta-functions.*

A fixed point is described by a conformal field theory. Let S_* be its action. Away from the critical point, the field theory action is a perturbation of S_* by some operators Φ_i:

$$S = S_* + \sum_i g_i \, a^{\Delta_i - D} \int d^D x \, \Phi_i(x).$$

The operators Φ_i are supposed to be relevant operators (as otherwise they don't change the long distance physics) of dimensions Δ_i. The factors $a^{\Delta_i - D}$ have introduced to make the action explicitly dimensionless. The operators Φ_i are assumed to be normalized by their two point function

$$\langle \Phi_i(x)\Phi_j(y)\rangle_* = \frac{1}{|x-y|^{2\Delta_i}}\,\delta_{i;j}.$$

Here and below $\langle\cdots\rangle_*$ refers to the expectation values at the fixed point. A conformal field theory is characterized by its operator product expansion (OPE):

$$\Phi_i(x)\Phi_j(y) = \sum_k \frac{C_{ij}^k}{|x-y|^{\Delta_i+\Delta_j-\Delta_k}}\,\Phi_k(y) + \cdots .$$

The coefficients C_{ij}^k are the structure constant of the OPE, they are determined by the 3-point functions of the scaling operators at the conformal point (see Chapter 8). Existence of OPE is expected in any conformal field theory but not really proved in full generalities (unless we tautologically put the existence of OPEs as part of the axioms defining a conformal field theory).

We are going to prove that to lowest orders, the beta function is given by:

$$\beta^k(g) := a\partial_a g^k = (D-\Delta_k)g^k - \frac{S_D}{2}\sum_{ij} C_{ij}^k\, g^i g^j + \cdots . \tag{69}$$

We see that the one-loop beta-function is fully determined by the OPE structure of the conformal fixed point. We even can absorb the factor S_D in a redefinition of the coupling constant. The rules is that all relevant operators compatible with the symmetries which can be generated under OPE should be included in the beta function.

Let us now imagine computing the partition functions by perturbing the conformal field theory. (We will deal with the correlation function a bit later. That is we first deal with the 'measure' and then with the observables.). It is defined by the action S,

$$Z\times\langle\mathcal{O}(z)\cdots\rangle := \int [D\varphi]\, e^{-S}\, Z\times\langle\mathcal{O}(z)\cdots\rangle = \int [D\varphi]\, e^{-S_* - \sum_i g_i\, a^{\Delta_i - D}\int d^D x\,\Phi_i(x)}\mathcal{O}(z)\cdots .$$

We expand in perturbation theory up to second order (also called "one-loop" order).

$$\begin{aligned} Z\times\langle\mathcal{O}(z)\cdots\rangle \;&=\; \langle\mathcal{O}(z)\cdots\rangle_* - \sum_i g^i\, a^{\Delta_i - D}\int d^D x\,\langle\Phi_i(x)\,\mathcal{O}(z)\cdots\rangle_* \\ &\quad + \frac{1}{2!}\sum_{ij} g^i g^j\, a^{\Delta_i - D} a^{\Delta_j - D}\int d^D x_1 d^D x_2\,\langle\Phi_i(x_1)\Phi_j(x_2)\,\mathcal{O}(z)\cdots\rangle_* + \cdots \end{aligned}$$

We insert operators $\mathcal{O}(z)\cdots$ just to remember that we are dealing with the measure but do not really take care of them yet (we shall do it just in a while when computing the matrix of anomalous dimensions). The integrals are (of course) UV divergent because of the singularity in the OPE (due to large fluctuations in the microscopic model). They are actually also IR divergent, those divergency can be treated by defining the systems is in finite box. But all we are interested in to understand the RG behavior closed to the fixed point are the UV divergences so we will not care about the IR divergences. To cure the UV divergences we imagine that the system is defined on a lattice so that the field insertions are always at a distance a away each

other. That is: the integral have to be understand with a short distance cut-off (and implicitly also an IR cut-off), namely:

$$\int_{|x_1-x_2|>a} d^D x_1 d^D x_2 \, \langle \Phi_i(x_1)\Phi_j(x_2) \cdots \rangle_*$$

The renormalization problem consists now in answering the following question: How to make the coupling constant a-dependent such that the partition function is unchanged when slightly dilating the cut-off $a \to \lambda a$ with $\lambda = 1 + \varepsilon$ close to one? This is equivalent to answering the question: How make the coupling constants a-dependent in order to compensate for these divergences? Thus let us replace $g_i \to g_i(a)$ and demands that the derivative of the partition function vanishes. We get (keeping only the important terms, the others terms either do not contribute or contribute but to higher orders to the beta functions or to higher terms in the perturbative expansion such as $\Phi_i\Phi_j$ but already renormalized by the lower order terms in the beta functions):

$$\begin{aligned} 0 \;=\; & \Big[-\sum_i (a\partial_a g^i)\, a^{\Delta_i - D} \int d^D x\, \Phi_i(x) \; - \sum_i g^i\, (\Delta_i - D)\, a^{\Delta_i - D} \int d^D x\, \Phi_i(x) \\ & -\frac{1}{2} \sum_{ij} g^i g^j\, a^{\Delta_i - D} a^{\Delta_j - D} \int d^D x_1 d^D x_2\, \delta(|x_1 - x_2| - a)\, \Phi_i(x_1)\Phi_j(x_2) \; + \cdots \Big] \end{aligned}$$

The last term comes from slightly moving the short distance cut-off (say, the Dirac δ-function $\delta(|x_1 - x_2| - a)$ comes form the derivative of the Heaviside function $\Theta(|x_1 - x_2| - a)$ defining the cut-off integrals[6]: this term reflects the block spin transformation). As $a \to 0$ we can evaluate the last integrals using the OPE $\Phi_i(x + a)\Phi_j(x) \simeq \frac{C_{ij}^k}{|a|^{\Delta_i + \Delta_j - \Delta_k}}\, \Phi_k(x)$. The Dirac δ-function $\delta(|x_1 - x_2| - a)$ reduces the integral to the angular variables over the D-dimensional sphere with volume $a^D S_D$, with $S_D = 2\pi^{D/2}\Gamma(D/2)$. Hence to lowest order, the term linear in $\int d^D x\, \Phi_k(x)$, are

$$\begin{aligned} 0 \;=\; & \Big[-\sum_i (a\partial_a g^i)\, a^{\Delta_i - D} \int d^D x\, \Phi_i(x) \; - \sum_i g^i\, (\Delta_i - D)\, a^{\Delta_i - D} \int d^D x\, \Phi_i(x) \\ & -\frac{1}{2} \sum_{ijk} C_{ij}^k g^i g^j\, a^{\Delta_k - D}\, S_D \int d^D x\, \Phi_k(x) \; + \cdots \Big] \end{aligned}$$

Demanding this to vanish yields the announced claim. Computing the beta function to higher order is more complicated (in particular the short distance cut off we use is not very much adapted and easy to deal with at higher orders).

• *One-loop anomalous dimensions.*

We now perturbatively compute the matrix of anomalous dimension. The computation is very similar. As we know there is operator mixing. We have to consider families of operators

[6]Be careful with the signs: $\partial_a \Theta(|x - y| - a) = \delta(|x - y| - a)$, or equivalently $\int_{|x-y|>a+da} - \int_{|x-y|>a} = -\int_{a+da>|x-y|>a}$

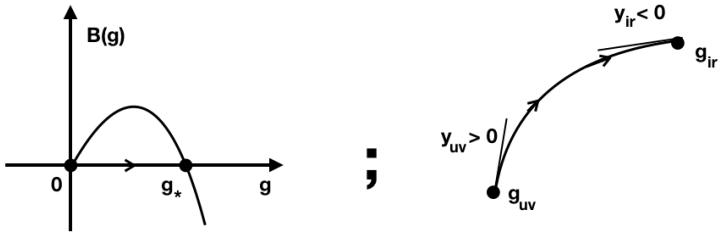

Figure 7: *Graph of a beta function with a nearby zero; Schematic representation of a RG flow between two fixed points, from UV to IR.*

$\mathcal{O}_\alpha$, of dimension Δ_α, whose correlation function are perturbatively defined by the expansion

$$\begin{aligned}\langle \mathcal{O}_\alpha(y)\cdots\rangle &= \langle \mathcal{O}_\alpha(y)\cdots\rangle_* - \sum_\alpha g^i\, a^{\Delta_i - D} \int d^D x\, \langle \Phi_i(x)\, \mathcal{O}_\alpha(y)\cdots\rangle_* \\ &\quad + \frac{1}{2!}\sum_{ij} g^i g^j\, a^{\Delta_i - D} a^{\Delta_j - D} \int d^D x_1 d^D x_2\, \langle \Phi_i(x_1)\Phi_j(x_2)\, \mathcal{O}(y)_\alpha\cdots\rangle_* + \cdots\end{aligned}$$

We know look how to redefined them and the coupling constants,

$$g^i \to g^i(a), \quad \mathcal{O}_\alpha \to a^{-\Delta_\alpha}\, \Gamma^\sigma_\alpha(a)\, \mathcal{O}_\sigma,$$

such as to compensate the a-divergences. We adopt the same computational strategy as before. The cut-off integral $\int d^D x\, \Phi_i(x)\, \mathcal{O}_\alpha(z)$ yields $\Phi_i(y+a)\, \mathcal{O}_\alpha(y)$ which can be evaluated using the OPE structure

$$\Phi_i(y+a)\, \mathcal{O}_\alpha(y) \simeq \frac{C^\sigma_{i\alpha}}{|a|^{\Delta_i + \Delta_\alpha - \Delta_\sigma}}\, \mathcal{O}_\sigma(y).$$

Looking at the terms linear in $\mathcal{O}_\sigma$ in the a-derivative of the expectation of $\langle \mathcal{O}_\alpha(y)\cdots\rangle$ give to lowest order

$$(a\partial_a \Gamma\cdot\Gamma^{-1})^\sigma_\alpha + \Delta_\alpha\, \delta^\sigma_\alpha + S_D \sum_i g^i\, C^\sigma_{i\alpha} = 0.$$

Hence, the matrix of anomalous dimensions at one-loop order is

$$\gamma^\sigma_\alpha = \Delta_\alpha\, \delta^\sigma_\alpha + S_D \sum_i g^i\, C^\sigma_{i\alpha} + \cdots. \tag{70}$$

Again it is completely determined by the OPE structure of the fixed point. Not that we have $\gamma^j_i = D\delta^j_i - \partial_i \beta^j$ as we should. Again higher order in the coupling constant are more difficult to computed (and are not encoded in simple universal structures as the OPEs coefficients).

8.8 The Wilson-Fisher fixed point

Use the perturbative expression of the beta-function to perturbatively learn about new (nearby) fixed point.

• *Almost marginal perturbations.*

Consider perturbing a fixed point by an almost marginal operator Φ of dimension $\Delta = D - \epsilon$ with $\epsilon \ll 1$. The beta function is then of the form $\beta(g) = \epsilon\, g + O(g^2)$. The one-loop terms depend on the OPE structure at the fixed point. Suppose for simplicity the OPE is such that $\Phi \times \Phi = \Phi$ and that no other operator is generated by the RG. The beta function is then

$$\beta(g) = \epsilon\, g - S_D \frac{C}{2} g^2 + \cdots .$$

The flow is as described in the Figure. The remarkable fact is that this beta function vanishes for g of order ϵ so that we can trust the approximate one-loop beta function. The new fixed point is at point g_*,

$$g_* = \frac{1}{S_D C}\, \epsilon + O(\epsilon^2),$$

whose sign depends on the sign of the OPE coefficient C. If g has the appropriate sign, the large distance physics of the system is governed by this new fixed point. The new anomalous dimension of the operator Φ at the new IR fixed point is obtained by linearizing te beta function at the new fixed point (or by using the relation $\gamma = D - \partial\beta$): $\beta(g_* + \delta g) = -\epsilon \delta g + \cdots$. The operator Φ is thus irrelevant at the new fixed point (as it should be because we are approaching this new fixed point by the operator) with new dimension $\delta_* = D + \epsilon$.

• *The Wilson-Fisher fixed point.*

We look at the ϕ^4 theory with action

$$S = \int d^D x \Big[\frac{1}{2!} (\nabla\phi)^2 + \frac{1}{2} g_2\, \phi^2 + \frac{1}{4!} \phi^4 \Big],$$

which we view as a perturbation of the massless gaussian fixed point with action $S_* = \frac{1}{2} \int d^D x (\nabla\phi)^2$. In the Gaussian theory, perturbative expectation can be computed using Wick's theorem. The massless Green function in dimension D is

$$\langle \phi(x) \phi(y) \rangle_* = \frac{1}{|x-y|^{D-2}}.$$

Thus ϕ has dimension $(D-2)/2$ at the Gaussian fixed point. To avoid self Wick's contract we define the operator ϕ^2 and ϕ^4 by normal ordering. This amounts to subtract their vacuum expectation (self contractions), so that

$$: \phi^2 := \phi^2 - \langle \phi^2 \rangle_*, \quad : \phi^4 := \phi^4 - 3 \langle \phi^2 \rangle_*\, \phi^2.$$

At the massless Gaussian fixed point, the dimension of the operator $\phi_n =: \phi^n :$ is $n(D-2)/2$. For $3 < D < 4$, the operators $\phi_2 =: \phi^2 :$ and $\phi_4 =: \phi^4 :$ are the only even relevant operators. So the action is

$$S = S_* + \int d^D x \Big[\frac{1}{2!} g_2\, \phi_2 + \frac{1}{4!} g_4 \phi_4 \Big].$$

We should actually also included $(\nabla\phi)^2$ which is marginal with dimension D but it has no effect at one-loop. It is a simple exercice to compute the OPE between ϕ_2 and ϕ_4 to get:

$$\begin{aligned} \phi_2 \times \phi_2 &= 2\,\mathbb{I} + 4\,\phi_2 + \phi_4, \\ \phi_2 \times \phi_4 &= 12\,\phi_2 + 8\,\phi_4, \\ \phi_4 \times \phi_4 &= 24\,\mathbb{I} + 96\,\phi_2 + 72\,\phi_4. \end{aligned}$$

We did not write terms corresponding to operators ϕ_n with $n > 4$ or to higher derivatives. The coefficients here comes from Wick's theorem and thus from computing the number of different pairings.

Let $D = 4 - \epsilon$ so that ϕ_2 has dimension $2 - \epsilon$ and ϕ_4 has dimension $4 - 2\epsilon$ (it is almost marginal). The beta functions are (we absorb the factor $S_4/2$ in the coupling constant):

$$\begin{aligned} \beta_2 = a\partial_a g_2 &= 2\, g_2 - 4\, g_2^2 - 24\, g_2 g_4 - 96\, g_4^2 + \cdots \\ \beta_4 = a\partial_a g_4 &= \epsilon\, g_4 - 72\, g_4^2 - 16\, g_2 g_4 - g_2^2 + \cdots \end{aligned}$$

For $\epsilon \ll 1$ there is a nearby fixed with $g_4^* = O(\epsilon)$ and $g_2^* = O(\epsilon^2)$:

$$g_4^* = \frac{\epsilon}{72} + O(\epsilon^2), \quad g_2 = O(\epsilon^2).$$

The dimension of the operator ϕ_2 at the new fixed point can be read from the linearization of the beta function at this fixed point: $\beta_2 = (2 - 24\, g_4^*) g_2 + \cdots = 2(1 - \frac{\epsilon}{6}) g_2 + \cdots$. The operator ϕ_2 is coupled to the temperature so that $g_2 \sim \delta T$. We thus have $y_T = 2(1 - \frac{\epsilon}{6})$ and hence the critical exponent

$$\nu = \frac{1}{y_T} = \frac{1}{2} + \frac{\epsilon}{12} + O(\epsilon^2).$$

This is the critical exponent for the Ising model in dimension $4-\epsilon$ which, once (naively) extended to dimension 3 by setting $\epsilon = 1$, is in reasonably good agreement with the exponent obtained by numerical simulations. Other exponents are found by computing the dimension of the operators (say ϕ which is coupled to the magnetic field) at the new fixed points. Those are obtained from the OPE coefficient as explained above. We leave this computation as an exercise.

8.9 Scaling limits and renormalized theories

Constructing, or defining, the renormalized theory amounts to define the theory in the continuum by taking the limit of vanish lattice spacing. As we will see this requires approaching simultaneously the critical surface is such way as to preserve the dimension-full correlation length. The finit continuous theory is called the renormalized theory.

• *Real space correlation length and scaling limit.*

Suppose for a little while that there is only one relevant scaling variable u_T. The (dimensionless) correlation length diverges as $\xi \simeq |u_T|^{-1/y_T}$ when close to the fixed point (i.e. $|u_T| \to 0$). The physical dimension-full correlation length thus behaves as $a|u_T|^{-1/y_T}$ with a the lattice mesh size. If we aim at defining a continuous theory with a physical finite correlation length we thus have to take the limit $a \to 0$, $|u_T| \to 0$, with $a|u_T|^{-1/y_T}$ fixed. Equivalently we may take the continuous limit $a \to 0$ fixing a mass scale (inverse of length scale):

$$m := a^{-1} |u_T|^{1/y_T}.$$

That is: to define the continuous field theory with fixed finite mass scale we have to simultaneously approach the critical hyper-surface, $u_T \to 0$ and $u_h \to 0$, as we take the small lattice size limit. This is called a double scaling limit: $a \to 0$, $u_T \to 0$ with $a^{-1}\, |u_T|^{1/y_T}$ fixed. Doing this we fix a dimension-full correlation length $\hat{\xi}$,

$$\hat{\xi} = m^{-1} = a\, |u_T|^{-1/y_T},$$

or equivalently, we scale u_t appropriately as $a \to 0$,

$$u_T(a) = |am|^{y_T}, \quad \text{as } a \to 0.$$

Recall how we defined the continuous (scaling) limit of free random path in Chapter 2.

If ξ is the dimensionless correlation length, the two point functions behave as $G(\mathbf{n}) \simeq e^{-|\mathbf{n}|/\xi}$ with $\xi \simeq |u|^{-1/y_T}$. Here $|\mathbf{n}|$ is the dimensionless distance, counted in unit of the lattice mesh, so that the 'physical distance' is $|x| = a|\mathbf{n}|$. The double scaling limit ensures that the existence of the continuous two-point functions

$$G(x) = \lim_{\substack{a\to 0, \mathbf{n}\to\infty \\ |x|=a|\mathbf{n}| \text{ fixed}}} G(\mathbf{n}) \simeq_{x \text{ large}} \lim_{\substack{a\to 0, \mathbf{n}\to\infty \\ |x|=a|\mathbf{n}| \text{ fixed}}} e^{-|\mathbf{n}||u_T^{1/y_T}} \simeq e^{-m|x|}.$$

If there are more than one relevant coupling constant, say two u_T and u_h, we have to take simultaneously the scaling limit in order to approach the critical surface as $a \to 0$, that is

$$a \to 0, \quad \text{with } m_T = a^{-1}\,|u_T|^{1/y_T} \text{ and } m_h = a^{-1}\,|u_h|^{1/y_h} \text{ fixed.}$$

The continuous theory will then depend on the two independent mass scales m_T and m_h, or alternatively on one mass scale, say m_T, and on the scale invariant ratio of the scaling variables, say $\frac{|u_T|^{y_h}}{|u_h|^{y_T}} \simeq \frac{|T|^{y_h}}{|h|^{y_T}}$.

We can alternatively write these scaling relations in terms of RG transformations. Recall that the scaling variables scale homogeneously under RG transformation, say $u_T \to u_T^\lambda = \lambda^{y_T}\, u_T$. If we pick $u_T(a) = (am_T)^{y_T}$ as initial condition as required for the double scaling limit, then the running coupling constant at scale $\lambda = \ell_R/a$ is fixed, that is:

$$u_T(a)^{\lambda=\ell_R/a} = (\ell_R m_T)^{y_T}.$$

The new length ℓ_R (which makes $\lambda = \ell_R/a$ dimensionless) can be chosen at will. It is called the renormalization scale. In terms of generic coupling constants the scaling limit thus requires taking the continuous limit $a \to 0$ with coupling constants $g(a)$ with fixed values at scale $\lambda = \ell_R/a$, i.e.

$$a \to 0, \quad \text{with } g(a)^{\lambda=\ell_R/a} = g_R \text{ fixed.}$$

Imposing this condition imposes to the coupling constants $g(a)$ to approach the critical hypersurface.

• *Scaling limits of fields.*

To define the continuous limit of expectation of products of operators we have to take their anomalous dimension into account. Consider first scaling operators Φ_i of scaling dimension Δ_i. At the fixed point, the lattice two point functions are

$$\langle \Phi_i(\mathbf{n})\Phi_i(\mathbf{m})\rangle_*^{\text{latt}} = \frac{1}{|\mathbf{n}-\mathbf{m}|^{2\Delta_i}}.$$

The continuous theory is defined by taking the limit $a \to 0$, $\mathbf{n} \to \infty$ with $x = a\mathbf{n}$ fixed. This demands to define the scaling operator in the continuous theory by

$$\Phi_i^{\text{cont.}}(x) = \lim_{a\to 0} a^{-\Delta_i}\, \Phi_i^{\text{latt}}(\mathbf{n} = \frac{x}{a}).$$

This shows that to take the continuous limit we have to dress the field in a-dependent way according to their anomalous dimensions. Hence, away from the fixed point we have to combine this dressing with the running of the coupling constant $g(a)$ explained above. The best we can hope for, is that there exist matrices (acting on the set of operators) $\Gamma(a)$ such that the following limits:

$$\lim_{\substack{a\to 0, \mathbf{n}_k\to\infty \\ x_k=a\mathbf{n}_k \text{ fixed}}} \langle \prod_k \Gamma_k^{-1}(a)\cdot \mathcal{O}_k(\mathbf{n}_k)\rangle^{\text{latt}}_{g(a)} \quad \text{exist},$$

for any operators $\mathcal{O}_k$. Alternatively, we look for a matrix $\Gamma(a)$ (acting on the set of operators) such the field operators in the continuous theory are defined from the lattice ones by

$$\mathcal{O}^{\text{cont.}}(x) = \lim_{\substack{a\to 0, \\ g(a)^{\lambda=\ell_R/a}=g_R \text{ fixed}}} \Gamma^{-1}(a)\cdot \mathcal{O}^{\text{latt}}(\mathbf{n}=\frac{x}{a}).$$

This relation is simply the (formal) generalization of the relation $\Phi^{\text{cont.}}(x) = a^{-\Delta_i}\,\Phi_i^{\text{latt}}(\mathbf{n}=x/a)$ in case of scaling fields.

We shall show that we can construct this matrix from the mxing matrices of the RG transformation. Namely, choosing,

$$\Gamma(a) = \Gamma^{\lambda=\ell_R/a}(g(a)),$$

ensures that the correlation functions $\langle\prod_k \Gamma_k^{-1}(a)\cdot\mathcal{O}_k(\mathbf{n}_k)\rangle^{\text{latt}}_{g(a)}$ are (formally) independent of the lattice cut-off a, asymptotically as $a\to 0$ at $x_k = a\mathbf{n}_k$ fixed.

If these limits exist, we could define the 'renormalized' continuous theory as (see below)

$$\langle\prod_k \mathcal{O}_k(x_k)\rangle^R_{g_R} = \lim_{\substack{a\to 0 \\ g(a)^{\lambda=\ell_R/a}=g_R \text{ fixed}}} \langle\prod_k \Gamma_k^{-1}(a)\cdot\mathcal{O}_k(\mathbf{n}_k=\frac{x_k}{a})\rangle^{\text{latt}}_{g(a)}.$$

Let us argue that the RG transformation (formally) ensures that such limits exist (if the mixing matrices of the RG transformations can be sufficiently controlled, see previous section). Recall from previous sections that RG transformations on fields yield that (or amounts to assume that the following relation holds):

$$\langle\prod_k \Gamma_k^{\lambda}(g)\cdot\mathcal{O}_k(\mathbf{n}_k/\lambda)\rangle^{\text{latt}}_{\{g^\lambda\}} = \langle\prod_k \mathcal{O}_k(\mathbf{n}_k)\rangle^{\text{latt}}_{\{g\}}$$

for a small enough, and where Γ^λ are the RG mixing matrices. Picking $\mathbf{n}_k = \frac{x_k}{a}$ and $\lambda = \ell_R/a$ yields

$$\langle\prod_k \Gamma_k^{\lambda=1/a}(g)\cdot\mathcal{O}_k(\frac{x_k}{\ell_R})\rangle^{\text{latt}}_{\{g^{\lambda=\ell_R/a}\}} = \langle\prod_k \mathcal{O}_k(\mathbf{n}_k=\frac{x_k}{a})\rangle^{\text{latt}}_{\{g\}}.$$

Choosing now the coupling constant to be $g(a)$ such that $g(a)^{\lambda=\ell_R/a} = g_R$ fixed and multiplying both sides of the equation by $\Gamma_k^{-1}(a) = [\Gamma^{\lambda=\ell_R/a}(g(a))]^{-1}$ we get that

$$\langle\prod_k [\Gamma_k^{\lambda=\ell_R/a}(g(a))]^{-1}\mathcal{O}_k(\mathbf{n}_k=\frac{x_k}{a})\rangle^{\text{latt}}_{\{g(a)\}},$$

is independent of a, at least for a small ($a\to 0$) as claimed.

• *Renormalized correlation functions.*

The renormalized continuous theory is thus (tentatively) defined by

$$\langle \prod_k \mathcal{O}_k(x_k) \rangle^R_{g_R} := \lim_{\substack{a \to 0 \\ g(a)^{\lambda=\ell_R/a}=g_R}} \langle \prod_k [\Gamma_k^{\lambda=\ell_R/a}(g(a))]^{-1} \cdot \mathcal{O}_k(\mathbf{n}_k = \frac{x_k}{a}) \rangle^{\text{latt}}_{g(a)}. \tag{71}$$

It is fully defined from data encoding the RG transformation for critical system. Of course proving rigorously that the limit exist requires more work (and this is clearly much beyond the scope of this simple introduction). By construction, the renormalized expectations are functions of the renormalized parameters g_R but not of the lattice coupling constants $g(a)$.

Renormalization procedures to construct continuous field theory encode all the information about critical systems, and reciprocally RG transformations are the data needed construct continuous field theory. The relation between the renormalized and lattice correlation functions can be inverted, so that

$$\langle \prod_k \mathcal{O}_k(\mathbf{n}_k = \frac{x_k}{a}) \rangle^{\text{latt}}_{g(a)} \sim \langle \prod_k \Gamma_k(a) \cdot \mathcal{O}_k(x_k) \rangle^R_{g_R}, \quad \text{with } g(a)^{\lambda=\ell_R/a} = g_R,$$

for $a \to 0$. Hence, the renormalized correlation functions code for the large distance behavior of the lattice correlation functions, provided the coupling constants approach the critical hypersurface (in the appropriate double scaling limit) as the distance are scaled away.

Let us assume that there is only one relevant variable u_T. Then the scaling relation is $u_T(a) = (am)^{y_T}$ with $\hat{\xi} = m^{-1}$ the dimensionfull correlation length, which is the only renormalized parameter / coupling constant of the theory. If Φ_i are scaling fields, then the RG mixing matrix is $\Gamma^\lambda = \lambda^{-\Delta_i}$ so that $\Gamma_i(a) = \Gamma^{\lambda=1/ma} = (ma)^{\Delta_i}$ and

$$\langle \prod_i \Phi_i(x_i) \rangle^{\text{latt}}_{u_T(a)} \simeq \langle \prod_i (ma)^{\Delta_i} \Phi_i(x_i) \rangle^{\text{cont}}_{\hat{\xi}=1/m}.$$

This is again another way to express that $\Phi_i^{\text{cont}} = a^{-\Delta_i} \Phi_i^{\text{latt}}$. It expresses that the correlation length is the only relevant macroscopic length (if we are dealing the system in infinite volume).

• *RG transformations and the Callan-Symanzik equation.*

The renormalization scale ℓ_R at which we fixed the running constant is arbitrary: a change in the renormalization scale $\ell_R \to \lambda' \ell_R$ can be compensate by a change in the renormalized coupling constant $g_R \to g_R^\lambda$ since

$$g(a)^{\lambda=\lambda'\ell_R/a} = [g(a)^{\lambda=\ell_R/a}]^{\lambda'} = g_R^{\lambda'}.$$

For instance, if the coupling constant is a scaling variable u_t then $u_t(a)^{\lambda=\lambda'\ell_R/a} = (\lambda'\ell_R/a)^{y_t}\, u_t(a) = (\ell_R m_t)^{y_t}$ and changing the scale λ' amounts to change the renormalized mass.

This implies that the renormalized expectations also satisfy a RG invariance. Let us first state this invariant relation and then prove it. Namely, the invariance says that

$$\langle \prod_k \Gamma_k^{\lambda'}(g_R) \cdot \mathcal{O}(\frac{x_k}{\lambda'}) \rangle^R_{g_R^{\lambda'}} \quad \lambda' - \text{independent}. \tag{72}$$

This equation reflects the fact that the renormalization length scale ℓ_R was not present in the initial formulation of the problem. It expresses the covariance of the renormalized expectations

w.r.t. to new arbitrary length scale. Taking the derivative w.r.t to λ' gives the so-called Callan-Symanzik equations

$$\Big[\sum_{k=1}^{N} y_k \cdot \partial_{y_k} + \sum_{k=1}^{N} \gamma^{(k)}(g_R) - \sum_i \beta^i(g_R)\partial_{g_R^i}\Big]\langle \mathcal{O}_1(y_1)\cdots\mathcal{O}_N(y_N)\rangle^R_{\{g_R\}} = 0,$$

with, as before,

$$\beta(g_R) = \lambda\partial_\lambda g_R^\lambda|_{\lambda=1}, \quad \gamma(g) = -\lambda\partial_\lambda \Gamma^\lambda(g_R)|_{\lambda=1}.$$

A (maybe not so usual) way to derive this equation is by using the composition law for the RG mixing matrices. Let start as above with a RG invariance of the lattice expectations but for scale dilatation $\lambda\lambda'$ instead of λ'. We have (as above)

$$\langle\prod_k \Gamma_k^{\lambda\lambda'}(g)\cdot\mathcal{O}_k(\mathbf{n}_k/\lambda\lambda')\rangle^{\rm latt}_{\{g^{\lambda\lambda'}\}} = \langle\prod_k \mathcal{O}_k(\mathbf{n}_k)\rangle^{\rm latt}_{\{g\}}$$

Recall the composition law for the mixing matrices $\Gamma^\lambda(g)\cdot\Gamma^{\lambda'}(g^\lambda) = \Gamma^{\lambda\lambda'}(g)$. In particular for $\lambda = \ell_R/a$ and $g(a)^{\lambda=\ell_R/a} = g_R$, this becomes,

$$\Gamma^{\lambda=\ell_R/a}(g(a))\cdot\Gamma^{\lambda'}(g_R) = \Gamma^{\lambda'/a}(g(a)).$$

Hence RG invariance now reads

$$\langle\prod_k \Gamma^{\lambda=1/a}(g(a))\cdot\Gamma^{\lambda'}(g_R)\cdot\mathcal{O}_k(x_k/\lambda'\ell_R)\rangle^{\rm latt}_{\{g_R^{\lambda'}\}} = \langle\prod_k \mathcal{O}_k(\mathbf{n}_k = \frac{x_k}{a})\rangle^{\rm latt}_{\{g(a)\}},$$

for a-small enough. Multiplying by $[\Gamma^{\lambda=1/a}(g(a))]^{-1}$ as above, and taking (formally) the limit $a\to 0$, we get

$$\langle\prod_k \Gamma_k^{\lambda'}(g_R)\cdot\mathcal{O}_k(\frac{x_k}{\lambda'})\rangle^R_{\{g_R^{\lambda'}\}} = \langle\prod_k \mathcal{O}(x_k)\rangle^R_{\{g_R\}}.$$

as claimed.

• *Solutions of the RG equations.*

Let us assume that there is only one coupling constant g with beta-function $\beta(g)$. Given the beta-function, the RG flow equation is $\lambda\partial_\lambda g(\lambda) = \beta(g(\lambda))$ whose solution (with initial condition $g(\lambda = 1) = g$ is:

$$\lambda = \exp\big[\int_g^{g(\lambda)} \frac{dg'}{\beta(g')}\big].$$

Reciprocally this defines a mass scale $m(g)$ solution of $\beta(g)\partial_g m(g) = m(g)$:

$$m(g) = \exp[\int^g \frac{dg'}{\beta(g')}\big].$$

This is the only scale of the problem. It is an interesting exercise to compute it for a relevant perturbation with $\beta(g) = \varepsilon\, g + \cdots$ or for a marginally relevant/irrelevant perturbation $\beta(g) = cg^2 + \cdots$. See the exercise Section.

By construction the integrated form of the Callan-Symanzik equation is that given above. Let us detailed it in the case of a two-point function of scalar fields and for a scalar matrix of

anomalous dimension with only one coupling constant (i.e. there is no mixing and $\gamma(g)$ is a pure number). Let G be the two-point function:

$$G(|x-y|;g) = \langle \Phi(x)\Phi(y)\rangle^R_g.$$

In this case the Callan-Symanzik equation reads

$$[r\partial_r + 2\gamma(g) - \beta(g)\partial_g]G(r;g) = 0.$$

Its solutions are of the form

$$G(r;g) = Z^2(g)\,\hat{G}(m(g)r),$$

with $Z(g)$ solution of $\beta(g)\partial_g Z(g) = \gamma(g)Z(g)$, explicitly $Z(g) = e^{\int^g \frac{\gamma(g')}{\beta(g')}dg'}$. It is called the wave function renormalization factor. And $m(g)$ clearly defines a mass scale. It is by construction RG covariant in the sense that

$$G(r/\lambda; g(\lambda)) = \hat{Z}^2(\lambda)\,G(r;g), \quad \text{for all } \lambda,$$

with $\hat{Z}(\lambda) = Z(g(\lambda))$ solution of $\lambda\partial_\lambda \log \hat{Z}(\lambda) = \gamma(g(\lambda))$. This equation allows to decipher the IR and/or UV behaviours.

Simple, but important, examples of beta functions and solutions of the Callan-Symanzik equations are detailed in the Exercise Section.

8.10 Perturbatively renormalized ϕ^4 theory

We now look at the perturbative RG analysis of the ϕ^4 theory. For $3 < D < 4$ the two operators ϕ^2, ϕ^4 are relevant and $(\nabla\phi)^2$ is marginal. We have to included them in the action which then reads:

$$S = S_* + \int d^Dx \,\big[\frac{z_0}{2}(\nabla\phi)^2 + \frac{m_0}{2}\phi^2 + \frac{g_0}{4!}\phi^4\big],$$

with S_* the Gaussian massless action $S_* = \frac{1}{2}\int d^Dx\,(\nabla\phi)^2$. Alternatively, we may write

$$S = \int d^Dx \,\big[\frac{1}{2}Z_0\,(\nabla\phi)^2 + \frac{1}{2}m_0\,\phi^2 + \frac{1}{4!}g_0\,\phi^4\big]$$

We have introduced a cut-off that we are going to represent as a momentum cut-off $\Lambda = 1/a$. To renormalized the theory we have to find how to make all coupling constants Λ-dependent such that the limit $\Lambda \to \infty$ exists. That is we have to find the functions $Z_0(\Lambda)$, $m_0(\Lambda)$ and $g_0(\Lambda)$ in such way to cancel all possible divergencies in the limit $\Lambda \to \infty$.

Let $G^{(N)}(x_1, \cdots, x_N)$ be the N-point functions $\langle\phi(x_1)\cdots\phi(x_N)\rangle$. The functions $Z_0(\Lambda)$, $m_0(\Lambda)$ and $g_0(\Lambda)$ are such the N-point correlation functions

$$Z_0^{N/2}(\Lambda)\,G^{(N)}(\{x_j\}; g_0(\Lambda), m_0(\Lambda), \Lambda), \quad \text{finite}$$

have a finite limit when $\Lambda \to \infty$ (here Z_0, g_0 and m_0 are implicit functions the cut-off Λ). To make these finite we have to identify physical parameters and re-express Z_0, g_0 and m_0 in terms of these physical parameters.

• *Perturbative one-loop renormalization in $D = 4$.*

Let us do it explicitly, to first order (i.e. one-loop) in dimension $D = 4$. All correlation functions are computed perturbatively using Wick's theorem. We regularized them by introducing a momentum cut-off Λ. We just quote the results when needed. At one-loop the two-point functions is (recall that $\Gamma^{(2)}(p) = G^{(2)}(p)^{-1}$): *[... Draw the diagrams...]*

$$\Gamma^{(2)}(p) = p^2 + m_0 + \frac{g_0}{2} I_2(m_0; \Lambda) + O(g_0^2),$$

where

$$I_2^\Lambda(m) = \int_{|p|<\Lambda} \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 + m^2} = \frac{\Lambda^2}{(4\pi)^2} - \frac{m^2}{(4\pi)^2} \log(\frac{\Lambda^2}{m^2}) + O(1).$$

This divergence may be cancelled by setting

$$m^2 = m_0^2 + \frac{g_0}{2}\Big(\frac{\Lambda^2}{(4\pi)^2} - \frac{m^2}{(4\pi)^2} \log(\frac{\Lambda^2}{\mu^2}) \Big) + O(g_0^2),$$

where μ is an arbitrary mass scale. Since I_2 does not depend on the external momentum, there is no p-dependent divergencies to cancel and hence there is not need for a renormalization of Z_0, i.e.

$$Z_0 = 1 + O(g_0^2).$$

To compute the four point function is a bit more complicated. According to the perturbative Feynman graphs, its structure is: *[... Draw the diagrams...]*

$$\Gamma^{(4)}(p_j) = g_0 - \frac{g_0^2}{2}\Big[I_4^\Lambda(p_1 + p_2; m_0) + I_4^\Lambda(p_1 + p_3; m_0) + I_4^\Lambda(p_1 + p_4; m_0) \Big] + O(g_0^3).$$

The function $I_4^\Lambda(p; m)$ are given by a Feynman diagram and can be computed to be (See the Exercise Section)

$$I_4^\Lambda(p; m) = \int_{|k|<\Lambda} \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 + m^2)((p + k)^2 + m^2)} = \frac{1}{(4\pi)^2} \log(\frac{\Lambda^2}{m^2}) + O(1).$$

This divergence may be cancelled by setting

$$g = g_0 - \frac{3}{2} g_0^2 \Big(\frac{1}{(4\pi)^2} \log(\frac{\Lambda^2}{\mu^2}) + O(1) \Big) + O(g_0^3).$$

where again we have introduced the arbitrary mass scale. Notice the possible extra $O(1)$ term which is related to the way we choose to defined the physical coupling constant. The statement of renormalizability is now that if m_0, g_0 are expressed in terms of m and g, then all correlation functions are finite in the limit $\Lambda \to \infty$ (at m and g fixed).

The procedure becomes simpler (especially at higher orders) if we adopt a slightly different strategy. We start as above from the action with Z_0, m_0 and g_0 but we write

$$m_0^2 = m_R^2 + \delta m^2, \quad g_0 = g_R + \delta g, \quad Z_0 = 1 + \delta z,$$

and we treat the term δm^2, δg and δz as perturbation, so that the Feynman propagator contains the mass m. The corrections δm^2, δg and δz are perturbatively computed to cancel the divergences. The 2 and 4 point functions are then (at one-loop order)

$$\begin{aligned}
\Gamma^{(2)}(p) &= p^2 + m_R + \Big(\frac{g}{2} I_2^\Lambda(m_R) + \delta m^2 + p^2 \delta z \Big) + O(g_R^2), \\
\Gamma^{(4)}(p_j) &= g_R + \Big(-\frac{g_R^2}{2} \big[I_4^\Lambda(p_1 + p_2; m_R) + I_4^\Lambda(p_1 + p_3; m_R) + I_4^\Lambda(p_1 + p_4; m_R) \big] + \delta g \Big) + O(g_R^3).
\end{aligned}$$

As above, the 2-point function is made finite (renormalized) by setting

$$\begin{aligned}\delta z &= 0 + O(g_R^2),\\ \delta m^2 &= \frac{g_R}{2}\Big(- \frac{\Lambda^2}{(4\pi)^2} + \frac{m_R^2}{(4\pi)^2}\log(\frac{\Lambda^2}{\mu^2})\Big) + O(g_R^2).\end{aligned}$$

The renormalized 2-point functions $\Gamma_R^{(2)}$ is defined by the limit $\Lambda \to \infty$ at m_R^2 and g_R fixed. It is $\Gamma_R^{(2)}(p) = p^2 + m_{\rm phys}^2 + O(g_R^2)$ with the physical mass is (the p-dependance is more complicated at higher order):

$$m_{\rm phys}^2 = m_R^2 + g_R \frac{m_R^2}{2(4\pi)^2}\log(\frac{m_R^2}{\mu^2}) + O(g_R^2).$$

The 4-point function is renormalized by setting as above (we do not include the extra $O(1)$ term because it is included in choosing m_R^2 not egal to $m_{\rm phys}^2$)

$$\delta g = g_R^2 \frac{3}{2(4\pi)^2}\log(\frac{\Lambda^2}{\mu^2}) + O(g_R^3).$$

The renormalized 4-point function is similarly defined by the limit $\Lambda \to \infty$ at m_R^2 and g_R fixed. It is (of course) finite and $\Gamma_R^{(4)}(p_j) = g_R - \frac{g_R^2}{2}\sum_{k=2}^4 I_4^R(p_1 + p_k; m_R) + O(g_R^3)$. It may be checked (by power counting) that this is enough to make all N-point function $\Gamma^{(N)}$ finite at one-loop. The renormalized N-point of course dependent on g_R and m_R but also on the arbitrary scale μ.

• *Renormalization of the ϕ^4 theory.*

The renormalizability of the ϕ^4 theory is that this construction extends to arbitrary higher order. The logic is the same as above. We look for three functions $Z_0(\Lambda)$, $m_0(\Lambda)$ and $g_0(\Lambda)$ such the N-point correlation functions $Z_0^{N/2}\, G(\{x_j\}; g_0, m_0, \Lambda)$ have a finite limit when $\Lambda \to \infty$ (here Z_0, g_0 and m_0 are implicit functions the cut-off Λ). Since there is three functions, fixing them requires imposing three conditions: two conditions involve the two-point functions for Z_0 and m_0 and the third involve the 4-point function for g_0 at some reference momenta. For instance, we can choose

$$\begin{aligned}\Gamma^{(2)}(p)|_{p^2=-m_R} &= 0;\\ \frac{\partial\Gamma^{(2)}(p)}{\partial p^2}|_{p^2=\mu^2} &= 1;\\ \Gamma^{(4)}(p_i)|_{p_i=p_i^{\rm ref}} &= g_R, \quad |p_1^{\rm ref} + p_2^{\rm ref}|^2 = \mu^2.\end{aligned}$$

These conditions are parametrized by the renormalized coupling constant m_R and g_R. The need for references momenta introduce the reference scale μ, called the renormalization scale. There are some freedom in choosing these renormalization conditions. Specifying them specify a (so-called) 'renormalization scheme'. Changing them amounts to reparametrization of the renormalized coupling constants (and the theory is covariant under those reparametrization). The renormalized correlation functions are then defined as

$$G_R^{(N)}(\{x_j\}; m_R, g_R, \mu) = \lim_{\Lambda\to\infty} Z_0^{N/2}\, G^{(N)}(\{x_j\}; g_0, m_0, \Lambda).$$

This is analogue of the equation above defining renormalized correlation functions from the lattice data, with Λ playing the role of the (inverse of the) lattice cut-off a and the renormalization scale μ that of the scale ℓ_R. To prove that this procedure works to all orders is of course much more complicated than the one-loop computation we did!

The renormalization scale is arbitrary (as was ℓ_R in the previous section) and there is an equation coding for the covariance under changes of μ. It follows by writing that the 'bare' correlation functions $G^{(N)}(\{x_j\}; g_0, m_0, \Lambda)$ are independent of μ. Indeed the wave function renormalization Z_0 dependence of the parameter and we may choose to express it as function of m_R, g_R and μ. Then we have

$$\mu\partial_\mu\Big[Z_0^{-N/2}(m_R, g_R, \mu)\; G_R^{(N)}(\{x_j\}; m_R, g_R, \mu)\Big] = 0.$$

This is the Callan-Symanzik equation we discussed in previous section.

• *Back to the effective potential*

The renormalization procedure can also be implemented —or read— on the effective potential. Recall that in a previous Chapter we computed the effective potential for the ϕ^4 theory. We got, to one-loop order, that (see the formula eq.(54) in which we set $\hbar = 1$)

$$V^{\rm eff}_{1-{\rm loop}}(\varphi) = \frac{1}{2!}A_\Lambda\,\varphi^2 + \frac{1}{4!}\,B_\Lambda\,\varphi^4 + \frac{1}{(8\pi)^2}V''(\varphi)\log\Big[\frac{V''(\varphi)}{\mu^2}\Big],$$

with $V(\varphi) = \frac{m_0^2}{2}\varphi^2 + \frac{g_0}{4!}\varphi^4$, the bare potential, and

$$\begin{aligned} A_\Lambda &= m_0^2 + \frac{g_0}{2}\Big(\frac{\Lambda^2}{(4\pi)^2} - \frac{m_0^2}{(4\pi)^2}\log(\frac{\Lambda^2}{\mu^2})\Big) + O(g_0^2), \\ B_\Lambda &= g_0 - g_0^2\,\frac{3}{2(4\pi)^2}\log(\frac{\Lambda^2}{\mu^2}) + O(g_0^3) \end{aligned}$$

All the diverging terms are in the terms of degree 2 or 4 in φ. They can thus be absorbed in a renormalization of the mass and the coupling constant. So, let $m_0^2 = m_R^2 + \delta m^2$ and $g_0 = g_R + \delta g$. As above, the one-loop effective potential is made finite by choosing

$$\begin{aligned} \delta m^2 &= -\frac{g_R}{2}\Big(\frac{\Lambda^2}{(4\pi)^2} - \frac{m_R^2}{(4\pi)^2}\log(\frac{\Lambda^2}{\mu^2})\Big) + O(g_R^2), \\ \delta g &= +g_R^2\,\frac{3}{2(4\pi)^2}\log(\frac{\Lambda^2}{\mu^2}) + O(g_R^3) \end{aligned}$$

These are —of course— the exact counter-terms that we had to choose for renormalizing the 2 and 4 point vertex functions at one loop. The renormalized one-loop effective potential then reads

$$V^{\rm eff;R}_{1-{\rm loop}}(\varphi) = V_R(\varphi) + \frac{1}{(8\pi)^2}V_R''(\varphi)\log[\frac{V_R''(\varphi)}{\mu^2}],$$

with $V_R(\varphi) = \frac{1}{2!}m_R^2\varphi^2 + \frac{1}{4!}g_R\varphi^4$. By looking at the φ^2 term we can read what the physical mass is:

$$m^2_{\rm phys} = m_R^2\Big(1 + \frac{2}{(8\pi)^2}\,g_R\,\log\big(\frac{m_R^2}{\mu^2}\big) + O(g_R^2)\Big).$$

It of course coincides with that found by looking at the 2-point vertex function.

In other words, the same procedure works for renormalizing the effective action —to be more precise we should had looked at the complete effective action, not only at the effective potential. This had to be expected because the effective action is the generating function of the vertex functions.

8.11 Exercises

• *Exercise 8.11.1: Ising on the triangular or hierarchical lattice.*

[... Cf. the exercise booklet...]

• *Exercise 8.11.2: Correction to scaling.*

The aim of this exercise is to understand how the irrelevant variables induce sub-leading corrections to scaling behaviours. To simplify matter, let us suppose that the critical system possesses only one relevant scaling variable, say u_t with RG eigen-value $y_t > 0$, and one irrelevant variable, say $u_{\rm irr}$ with RG eigen-value $y_{\rm irr} < 0$. (Of course generic physical systems have an infinite number of irrelevant variables but considering only one will be enough to understand their roles).
(i) By iterating RG transformations as in the main text, show that the singular part of the free energy can be written as

$$f_{\rm sing} = |u_t|^{D/y_t}\, \varphi_\pm(u^0_{\rm irr}\, |u_t|^{|y_{\rm irr}|/y_t}),$$

where $\varphi_\pm$ are functions possibly different for $u_t > 0$ or $u_t < 0$, and $u^0_{\rm irr}$ is the initial value (before RG transformations) of the irrelevant coupling.
(ii) Argue (without formal proof) that the functions $\varphi_\pm$ may raisonnably be expected to be smooth.
Under this assumption, prove that

$$f_{\rm sing} = |u_t|^{D/y_t}\, \big(A_0 + A_1\, u^0_{\rm irr}\, |u_t|^{|y_{\rm irr}|/y_t} + \cdots\big),$$

where A_0 and A_1 are non-universal constants.

• *Exercise 8.11.3: Change of variables and covariance of RG equations.*

Let us consider a theory with a finite number of relevant coupling constants that we generically denote $\{g^i\}$. Let us write the corresponding beta functions as (no summation in the first term)

$$\beta^i(g) = y_i g^i - \frac{1}{2}\sum_{jk} C^i_{jk} g^j g^k + \cdots .$$

(i) Prove that, if all y_i are non-vanishing, then there exist a change of variables from $\{g^i\}$ to $\{u^i\}$, with $u^i = g^i + O(g^2)$, which diagonalizes the beta functions, up to two loops, i.e. such that $\beta^i(u) = y_i u^i + O(u^3)$.
(ii) Prove that, if all y_i are zero, then the second and third Taylor coefficient are invariant under a change of variables from $\{g^i\}$ to $\{u^i\}$, with $u^i = g^i + O(g^2)$.
That is: For marginal perturbation, the second and third loop beta function coeffcients are independent on the renormalization scheme (alias on the choice of coordinate in the coupling constant space)
(iii) Let expand the beta functions to all orders in the coupling constants:

$$\beta^i(g) = y_i g^i - \sum_{n>0}\sum_{j_1,\cdots,j_n} C^i_{j_1,\cdots,j_n} g^{j_1}\cdots g^{j_n}.$$

Prove that, if there is no integers p_i, p_j such that $p_i y_i - p_j y_j \in \mathbb{Z}$, for $i \neq j$ (in such cases, one says they that there is non resonances), then there exists a change of variables from $\{g^i\}$ to

$\{u^i\}$, with u^i a formal power series in the g^i's, with $u^i = g^i + O(g^2)$, which diagonalizes the beta functions as a formal power series in the u^i's.
That is: There exist scaling variables, at least as formal power series.

• *Exercise 8.11.4: Explicit RG flows.*

The aim of this exercise is to study simple, but important, examples beta functions and solutions of the Callan-Symanzik equation.
(i) Consider a field theory with only one relevant coupling constant g and suppose that its beta function is $\beta(g) = yg$.
Show that the RG flow, solution of $\lambda\partial_\lambda g(\lambda) = \beta(g(\lambda))$ is $g(\lambda) = g_1\,\lambda^y$.
Show that the RG mass scale, solution of $\beta(g)\partial_g m(g) = m(g)$ is $m(g) = m_*\,g^{1/y}$.
Consider the two point function $G(r;g)$ of a scaling field Φ of scaling dimension Δ, i.e. $G(r,g) = \langle\Phi(r)\Phi(0)\rangle_g$. Prove (using the Callan-Symanzik equation) that

$$G(r;g) = r^{-2\Delta}\,F(m(g)r),$$

with $m(g)$ the RG mass scale defined above.
(ii) Consider a field theory with only one marginal coupling constant g and suppose that its beta function is $\beta(g) = cg^2$ ($c > 0$ corresponds to marginally relevant, $c < 0$ to marginally irrelevant).
Prove that the RG flow, solution of $\lambda\partial_\lambda g(\lambda) = \beta(g(\lambda))$ is $g(\lambda) = g_\mu/(1 - cg_\mu\log(\lambda/\mu))$.
Notice that $g_\lambda \to 0^+$, if $c < 0$, while g_λ flows up if $c > 0$, as $\lambda \to \infty$ (with $g_\mu > 0$ initially).
Prove that the RG mass scale, solution of $\beta(g)\partial_g m(g) = m(g)$ is $m(g) = m_*\,e^{-1/cg}$.
Notice that this mass scale is non perturbative in the coupling constant.
Consider the two point function $G(r;g)$ of a scaling field Φ whose matrix of anomalous dimension is $\gamma(g) = \Delta + \gamma_0 g$. Prove (using the Callan-Symanzik equation) that $G(r/\lambda; g(\lambda)) = Z(\lambda)^2 G(r,g)$ with

$$Z(\lambda) = \text{const.}\,\lambda^\Delta\,[g(\lambda)]^{\gamma_0/c}.$$

Deduce from this that, in the case marginally irrelevant perturbation (i.e. $c < 0$) and asymptotically for r large,

$$G(r;g_a) \simeq \text{const.}\,r^{-2\Delta}\,[\log(r/a)]^{\gamma_0/c}.$$

This codes for logarithmic corrections to scaling.

• *Exercise 8.11.5: Anomalous dimensions and beta functions.*

(i) Prove the relation $\gamma^\sigma_\alpha(g) = D\delta^\sigma_\alpha - \partial_\alpha\beta^\sigma(g)$ between the matrix of anomalous dimensions and the beta functions.
(ii) Give two proofs of the formula $\gamma^\sigma_\alpha(g) = \Delta_\alpha\delta^\sigma_\alpha + S_D\sum_i g^i C^\sigma_{i\alpha}$ for the matrix of anomalous dimensions to first order in perturbation theory (Here g^i are the perturbative coupling constant and S_D the volume of the D-dimensional unit sphere): one proof comes from using the previous result, the second proof comes from analysing the perturbative expansion of the correlation functions.

• *Exercise 8.11.6: Renormalisation of ϕ^3 in $D = 6$.*

[... Cf. the exercise booklet...]

• *Exercise: Current-current perturbations and applications.*

[...To be completed...]

- *Exercise: Disordered random bound 2D Ising model.*

  [...To be completed...]

9 Miscellaneous applications

9.1 The XY model

• *The physics of the XY model and the Kosterlitz-Thouless transition*

..... Bla Bla....

The rest of this section is a problem whose aim is to study the XY model using techniques from lattice statistical physics and from statistical field theory. The problem is divided in three parts:
IA/IB- The XY model on a lattice;
II- The role of vortices in the XY field theory;
IIIA/IIIB- The XY field theory and the sine-Gordon model.
The lattice models we shall consider are defined on the two dimensional square lattice Λ with mesh size a, $\Lambda = (a\mathbb{Z})^2$, that we shall view as embedded in the Euclidean plane. statistical field theories we shall consider are defined in the 2D Euclidean space $\mathbb{R}^2$.

• *I- The XY model on a lattice*

The XY model is a statistical spin model with spin variables $\vec{S}_i$, on each site i of the lattice Λ, which are two component unit vectors, $\vec{S}_i^2 = 1$. The energy of a configuration $[\vec{S}]$ is defined as $E[\vec{S}] = -\sum_{[ij]} \vec{S}_i \cdot \vec{S}_j$ where the sum runs over neighboor points on Λ. Parametrising the unit spin vectors $\vec{S}_i$ by an angle Θ_i defined modulo 2π, we write the configuration energy as

$$E[\vec{S}] = -\sum_{[ij]} \cos(\Theta_i - \Theta_j).$$

The partition function is $Z = \int [\prod_i \frac{d\Theta_i}{2\pi}] \exp\left(\beta \sum_{[i,j]} \cos(\Theta_i - \Theta_j)\right)$ with $\beta = 1/k_B T$ the inverse temperature.

• *IA- High temperature expansion*

The aim of this section is to study the high temperature ($\beta \ll 1$) behavior of the XY model. It is based on rewriting the Boltzmann sums in terms of dual flow variables.

IA-1 Explain why we can expand $e^{\beta \cos\Theta}$ in series as $e^{\beta\cos\Theta} = I(\beta)\left(1 + \sum_{n\neq 0} t_n(\beta) e^{in\Theta}\right)$, where $I(\beta)$ and $t_n(\beta)$ are some real β-dependent coefficients. We set $t_0(\beta) = 1$.

IA-2 By inserting this series in the defining expression of the partition function and by introducing integer variables $u_{[ij]}$ on each edge $[ij]$ of the lattice Λ, show that the partition function can be written as $Z = I(\beta)^{N_e} \cdot \hat{Z}$ with N_e the number of edges and

$$\hat{Z} = \sum_{[u],\ [\partial u = 0]} \prod_{[ij]} t_{u_{[ij]}}(\beta),$$

where the partition sum is over all configurations $[u]$ of integer edge variables $u_{[ij]}$ such that, for any vertex $i \in \Lambda$, the sum of these variables arriving at i vanishes, i.e. $\sum_j u_{[ij]} = 0$.
Remark: The variables u are attached to the edge of the lattice and may be thought of as 'flow variables'. The condition that their sum vanishes at any given vertex is a divergence free condition. The divergence at a vertex i of a configuration $[u]$ is defined as $(\partial u)_i := \sum_j u_{[ij]}$.

IA-3 Let i_1 and i_2 be two points of Λ and $\langle \vec{S}_{i_1} \cdot \vec{S}_{i_2} \rangle$ be the two-point spin correlation function. Explain why $\langle \vec{S}_{i_1} \cdot \vec{S}_{i_2} \rangle = \mathrm{Re} \langle e^{-i(\Theta_{i_1} - \Theta_{i_2})} \rangle$.
Show that,

$$\langle e^{-i(\Theta_{i_1} - \Theta_{i_2})} \rangle = \frac{1}{\hat{Z}} \cdot \sum_{\substack{[u] \\ [\partial u = \delta_{\cdot;i_1} - \delta_{\cdot;i_2}]}} \prod_{[ij]} t_{u_{[ij]}}(\beta),$$

where the sum is over all integer flow configurations such that their divergence is equal to $+1$ at point i_1, to -1 at point i_2, and vanishes at any other vertex.

IA-4 Show that $t_n(\beta) = t_{-n}(\beta) \simeq \frac{\beta^n}{2^n n!}$ as $\beta \to 0$.
Argue, using this asymptotic expression for the $t_n(\beta)$'s, that the leading contribution to the spin correlation functions at high temperature comes from flow configurations with $u = 0$ or $u = \pm 1$ on each edge of the lattice.

IA-5 Deduce that, at high temperature, the correlation function $\langle \vec{S}_{i_1} \cdot \vec{S}_{i_2} \rangle$ decreases exponentially with the distance between the two points i_1 and i_2.
Show that the correlation length behaves as $\xi \simeq a / \log(2/\beta)$ at high temperature.

• *IB- Low temperature expansion*

The aim of this section is to study the low temperature ($\beta \gg 1$) behavior of the XY model. It consists in expanding the interaction energy $\cos(\Theta_i - \Theta_j)$ to lowest order in the angle variables so that we write the configuration energy as (up to an irrelevant additive constant)

$$E[\vec{S}] = \text{const.} + \frac{1}{2} \sum_{[i,j]} (\Theta_i - \Theta_j)^2 + \cdots .$$

This approximation neglects the 2π-periodicity of the angle variables.

IB-1 Argue that the higher order terms in this expansion, say the terms proportional to $\sum_{[i,j]} (\Theta_i - \Theta_j)^4$, are expected to be irrelevant and can be neglected.

IB-2 Write the expression of the partition function Z of the model within this approximation. Explain why, in this approximation, the theory may be viewed as a Gaussian theory.

IB-3 Let $G_\beta(x)$ be the two-point function of this Gaussian theory. Show that $G_\beta(x) = \beta^{-1} G(x)$ with

$$G(x) = \int_{-\pi/a}^{+\pi/a} \frac{d^2p}{(2\pi/a)^2} \frac{e^{ip \cdot x}}{4 - 2(\cos ap_1 + \cos ap_2)},$$

with p_1, p_2 the two components of the momentum p and a the lattice mesh.

IB-4 Let i_1 and i_2 be two points on Λ and x_1 and x_2 be their respective Euclidean positions. Let $C_\alpha(x_1, x_2) = \langle e^{i\alpha(\Theta_{i_1} - \Theta_{i_2})} \rangle$ with α integer. Show that

$$C_\alpha(x_1, x_2) = e^{-\frac{\alpha^2}{\beta} \left(G(0) - G(x_1 - x_2) \right)}.$$

IB-5 Explain why $G(x)$ is actually IR divergent[7] and what is the origin of this divergence, but that $G(0) - G(x)$ is finite for all x. Show that

$$G(0) - G(x) = \frac{1}{2\pi} \log(|x|/a) + \text{const.} + O(1/|x|).$$

[7] So that, when defining $G(x)$, we implicitly assumed the existence of an IR cut-off, say $|p| > 2\pi/L$ with L the linear size of the box on which the model is considered.

IB-6 Deduce that the correlation functions C_α decrease algebraically at large distance according to

$$C_\alpha(x_1, x_2) \simeq \text{const.}\, (a/|x_1 - x_2|)^{\alpha^2/2\pi\beta}.$$

Compare with the high temperature expansion.

• *II- The role of vortices in the XY field theory*

The previous computations show that the model is disordered at high temperature but critical at low temperature with temperature dependent exponents. The aim of this section is to explain the role of topological configurations, called vortices, in this transition.

We shall now study the model in continuous space, the Euclidean plane $\mathbb{R}^2$, but with an explicit short distance cut-off a. We shall consider the XY system in a disc of radius L.

In the continuous formulation, the spin configurations are then maps Θ from $\mathbb{R}^2$ to $[0, 2\pi]$ modulo 2π. The above Gaussian energy is mapped into the action

$$S_0[\Theta] = \frac{\kappa}{2} \int d^2x (\nabla\Theta)^2,$$

with a coefficient κ proportional to β.

II-1 Argue that the coefficient κ cannot be absorbed into a rescaling of the field variable Θ?

II-2 A vortex, centred at the origin, is a configuration such that $\Theta_{\rm v}^{\pm}(z) = \pm\text{Arg}(z)$, with z the complex coordinate on $\mathbb{R}^2$, or in polar coordinates[8], $\Theta_{\rm v}^{\pm}(r, \phi) = \pm\phi$.
Show that $\Theta_{\rm v}^{\pm}$ is an extremum of S_0 in the sense that $\nabla^2\Theta_{\rm v}^{\pm} = 0$ away from the origin.
Show that $\oint_{C_0} d\Theta_{\rm v}^{\pm} = \pm 2\pi$ for C_0 a small contour around the origin.

II-3 Let a_0 be a small short distance cut-off and let $\mathbb{D}(a_0)$ be the complex plane with small discs of radius a_0 around the vortex positions cut out. Prove that, evaluated on $\Theta_{\rm v}^{\pm}$, the action S_0 integrated over $\mathbb{D}(a_0)$ (with an IR cut-off L) is

$$S^{(1)}_{\rm vortex} = \frac{\kappa}{2} \int_{\mathbb{D}(a_0)} d^2x\, (\nabla\Theta_{\rm v}^{\pm})^2 = \pi\kappa \log\big[L/a_0\big].$$

Give an interpretation of the divergence as $a_0 \to 0$.

II-4 What is the entropy of single vortex configurations? Show that the contribution of single vortex configurations to the free energy is

$$e^{-F^{(1)}_{\rm vortex}} \simeq \text{const.}\, \Big(\frac{L}{a_0}\Big)^2 e^{-\pi\kappa \log[L/a_0]}$$

Conclude that vortex configurations are irrelevant for $\pi\kappa > 2$ but relevant for $\pi\kappa < 2$.

• *III- The XY field theory and the sine-Gordon model*

The aim of this section is to analyse this phase transition using renormalization group arguments via a mapping to the so-called sine-Gordon field theory.

We shall consider a gas of vortices. The field configuration $\Theta_{\rm v}^{(M)}$ for a collection of M vortices of charges q_a centred at positions x_a is given by the sum of single vortex configuration:

$$\Theta_{\rm v}^{(M)} = \sum_{a=1}^{M} q_a\, \text{Arg}(z - z_a).$$

[8]We recall the expression of the gradient in polar coordinates: $\nabla\Theta = (\partial_r\Theta, \frac{1}{r}\partial_\phi\Theta)$. The Laplacian is $\nabla^2 F = \frac{1}{r}\partial_r(r\partial_r)F + \frac{1}{r^2}\partial_\phi^2 F$.

We shall admit that the action of such configuration is

$$S^{(M)}_{\rm vortex} = -2\pi(\frac{\kappa}{2})\sum_{a\neq b} q_a q_b \,\log\big(\frac{|x_a - x_b|}{a_0}\big) + 2\pi(\frac{\kappa}{2})(\sum_b q_b)^2 \log\big(\frac{L}{a_0}\big) + \sum_a \beta\epsilon_c,$$

where ϵ_c is a 'core' energy (which is not taken into account by the previous continuous description).

• *IIIA- Mapping to the sine-Gordon theory*

This mapping comes about when considering a gas of pairs of vortices of opposite charges $\pm$, so that the vortex system is neutral ($\sum_a q_a = 0$). We denote x_j^+ (resp. x_j^-) the positions of the vortices of charge + (resp. $-$).

The vortex gas is defined by considering all possible vortex pair configurations (with arbitrary number of pairs) and fluctuations around those configurations. We set $\Theta = \Theta_{\rm v}^{(2n)} + \theta_{\rm sw}$ and associate to each such configuration a statistical weights e^{-S} with action given by

$$S = S^{(2n)}_{\rm vortex}[x_j^+, x_j^-] + S_0[\theta_{\rm sw}],$$

with $S_0[\theta_{\rm sw}]$ the Gaussian action $\frac{\kappa}{2}\int d^2x (\nabla\theta_{\rm sw})^2$. We still assume a short-distance cut-off a.

IIIA-1 Write the expression of the action $S^{(2n)}_{\rm vortex}[x_j^+, x_j^-]$ for a collection of n pairs of vortices at positions $x_j^\pm$, $j = 1, \cdots, n$.

IIIA-2 Argue that the partition function of the gas of vortex pairs is given by the product $Z = Z_{\rm sw} \times Z_{\rm vortex}$ with $Z_{\rm sw}$ the partition function for the Gaussian free field $\theta_{\rm sw}$ and

$$Z_{\rm vortex} = \sum_{n\geq 0} \frac{\mu^{2n}}{n!\cdot n!} \times \int (\prod_{j=1}^n d^2x_j^+ \prod_{j=1}^n d^2x_j^-)\, \frac{\prod_{i<j}(|x_i^+ - x_j^+|/a)^{2\pi\kappa}(|x_i^- - x_j^-|/a)^{2\pi\kappa}}{\prod_{i,j}(|x_i^+ - x_j^-|/a)^{2\pi\kappa}},$$

with $\mu = \big(\frac{a_0}{a}\big)^{\pi\kappa} e^{-\beta\epsilon_c}$.

IIIA-3 The aim of the following questions is to express $Z_{\rm vortex}$ as a path integral over an auxiliary bosonic field φ. Let $\tilde{S}_\kappa[\varphi] = \frac{1}{2\kappa}\int d^2x (\nabla\varphi)^2$ be a Gaussian action. Show that, computed with this Gaussian action,

$$\langle e^{i2\pi\varphi(x)} e^{-i2\pi\varphi(y)}\rangle_{\tilde{S}_\kappa} = \frac{1}{|x-y|^{2\pi\kappa}}.$$

Hint: The Green function associated to the action $\tilde{S}_\kappa[\varphi]$ is $G(x,y) = -\frac{\kappa}{2\pi}\log\big(|x-y|/a\big)$.

IIIA-4 What is the scaling dimension (computed with the Gaussian action $\tilde{S}_\kappa[\varphi]$) of the operators $(\nabla\varphi)^2$ and $\cos(2\pi\varphi)$?
Deduce that the perturbation $\cos(2\pi\varphi)$ is relevant for $\pi\kappa < 2$ and irrelevant for $\pi\kappa > 2$.
Is the the perturbation $(\nabla\varphi)^2$ relevant or irrelevant?

IIIA-5 Show that $Z_{\rm vortex}$ can be written as the partition function of Gaussian bosonic field with action $S_{sG}[\varphi]$,

$$Z_{\rm vortex} = \int [D\varphi]\, e^{-S_{sG}[\varphi]},$$

where the action S_{sG} is defined as

$$S_{sG}[\varphi] = \int d^2x \Big[\frac{1}{2\kappa}(\nabla\varphi)^2 - 2\mu\cos(2\pi\varphi)\Big].$$

This is called the sine-Gordon action.
Hint: Compute perturbatively the above partition function as a series in μ while paying attention to combinatorial factors.

• *The renormalization group analysis*

IIIB-1 We now study the renormalization group flow of the action S_{sG} for κ close to the critical value $\kappa_c = 2/\pi$. We let $\kappa^{-1} = \kappa_c^{-1} - \delta\kappa$ and write

$$S_{sG}[\varphi] = \tilde{S}_{\kappa_c}[\varphi] - \int d^2x \Big[\frac{1}{2}(\delta\kappa)(\nabla\varphi)^2 + 2\mu\cos(2\pi\varphi)\Big]$$

Show that, to lowest order, the renormalization group equations for the coupling constants $\delta\kappa$ and μ are of the following form:

$$\begin{aligned} (\dot{\delta\kappa}) = \ell\partial_\ell\,(\delta\kappa) &= b\,\mu^2 + \cdots \\ \dot{\mu} = \ell\partial_\ell\,\mu &= a\,(\delta\kappa)\,\mu + \cdots \end{aligned}$$

with a and b some positive numerical constants.
Hint: It may be useful to first evaluate the OPE of the fields $(\nabla\varphi)^2$ and $\cos(2\pi\varphi)$.

IIIB-2 We redefine the coupling constants and set $X = a\,(\delta\kappa)$ and $Y = \sqrt{ab}\,\mu$ such that the RG equations now reads $\dot{X} = Y^2$ and $\dot{Y} = XY$.
Show that $Y^2 - X^2$ is an invariant of this RG flow.
Draw the RG flow lines in the upper half plane $Y > 0$ near the origin.

IIIB-3 We look at the flow with initial condition $X_I < 0$ and Y_I.
Show that if $Y_I^2 - X_I^2 < 0$ and $X_I < 0$, then the flow converges toward a point on the line $Y = 0$.
Deduce that for such initial condition the long distance theory is critical. Compare with section I-B.

IIIB-4 Show that if $Y_I^2 - X_I^2 > 0$ and $X_I < 0$, the flow drives X and Y to large values.
Let $Y_0^2 = Y_I^2 - X_I^2$ with $Y_0 > 0$. Show that the solution of the RG equations are

$$\log\big(\frac{\ell}{a}\big) = \frac{1}{Y_0}\Big[\arctan\big(\frac{X(\ell)}{Y_0}\big) - \arctan\big(\frac{X_I}{Y_0}\big)\Big].$$

IIIB-5 The initial condition X_I and Y_I are smooth functions of the temperature T of the XY model. The critical temperature T_c is such that $X_I + Y_I = 0$. We take the initial condition to be near the critical line $X_I + Y_I = 0$ with $X_I < 0$. We let $X_I = -Y_I(1+\tau)$ in which $\tau \ll 1$ is interpreted at the distance from the critical temperature: $\tau \propto (T - T_c)$.
For $\tau > 0$, we define the correlation length as the length ξ at which $X(\ell)$ is of order 1.
Why is this a good definition?
Show that

$$\xi/a \simeq \text{const. } e^{\text{const.}/\sqrt{\tau}}.$$

IIIB-6 Comment and discuss.

9.2 Self avoiding walks

..... Bla Bla....

The rest of this section is a problem whose aim is to study the self-avoiding walks using techniques from lattice statistical physics and from statistical field theory. This problem deals

with statistical properties of random curves (say polymers) and their connections with statistical field theory. It contains two parts (which are almost independent):
I- Non-interacting paths and their scaling limit;
II- Self-avoiding walks and the $O(n \to 0)$ model.

All lattice models we shall consider are defined on the D dimensional square lattice Λ with mesh size a: $\Lambda = (a\mathbb{Z})^D$. We let $\mathfrak{e}_j$, $j = 1, \cdots, D$ be a basis of orthornormal vectors in $\mathbb{R}^D$, so that points $\mathbf{x} \in \Lambda$ are $\mathbf{x} = a \sum_{j=1}^{D} n_j \mathfrak{e}_j$ with n_j integers.

A path Γ in Λ is an ordered collection of neighbour points in Λ, i.e. $\Gamma = (\mathbf{x}_0, \cdots, \mathbf{x}_i, \cdots, \mathbf{x}_{|\Gamma|})$ with $\mathbf{x}_i, \mathbf{x}_{i+1}$ neighbour on the lattice. We let $|\Gamma|$ be the number of bonds of the path Γ, so that $a|\Gamma|$ is its length. A loop in Λ is a closed path with $\mathbf{x}_0 = \mathbf{x}_{|\Gamma|}$.

A self avoiding walk $\Gamma_{\rm saw}$ is a path such that all $\mathbf{x}_i$ are distinct[9].

The statistical field theories we shall consider are defined in $\mathbb{R}^D$.

• *I- Non-interacting paths and their scaling limit*

Non-interacting paths are random paths whose statistics is specified by assigning a Boltzmann weight $w_\Gamma := \mu^{|\Gamma|}$ to each path Γ, with μ a real number ($\mu > 0$). Let $\mathbf{x} \in \Lambda$. We shall deal with paths Γ starting at the origin 0 and ending at point $\mathbf{x}$, i.e. paths $\Gamma = (\mathbf{x}_0, \cdots, \mathbf{x}_i, \cdots, \mathbf{x}_{|\Gamma|})$ with $\mathbf{x}_0 = 0$ and $\mathbf{x}_{|\Gamma|} = \mathbf{x}$. Let $Z(\mathbf{x})$ be the partition function conditioned on paths from 0 to $\mathbf{x}$:

$$Z(\mathbf{x}) = \sum_{\Gamma:\, 0 \to \mathbf{x}} \mu^{|\Gamma|}.$$

The probability of a given path Γ from 0 to $\mathbf{x}$ is $\mu^{|\Gamma|}/Z(\mathbf{x})$.

• *IA: The discrete lattice model*

IA-1: Show that

$$Z(\mathbf{x}) = \sum_{N \geq 0} \mu^N \, W_N^{\rm free}(\mathbf{x}) = \delta_{\mathbf{x};0} + \sum_{N > 0} \mu^N \, W_N^{\rm free}(\mathbf{x}),$$

with $W_N^{\rm free}(\mathbf{x})$ the number of paths from 0 to $\mathbf{x}$ with N bonds.

IA-2: Show (without long computation) that $Z(\mathbf{x})$ satisfies the following difference equation:

$$Z(\mathbf{x}) = \delta_{\mathbf{x};0} + \mu \sum_{j=1}^{D} \big(Z(\mathbf{x} + a\mathfrak{e}_j) + Z(\mathbf{x} - a\mathfrak{e}_j) \big).$$

IA-3: Compute the Fourier transform of $Z(\mathbf{x})$ and give an explicit expression of $Z(\mathbf{x})$ as an integral over the Brouillon zone.

IA-4: Let $\Delta^{\rm dis.}$ be the discrete Laplacian and write $\Delta^{\rm dis.} = \Theta - 2D\,\mathbb{I}$ with Θ the lattice adjacency matrix and $\mathbb{I}$ the identity matrix. We view Θ as acting on functions via $(\Theta \cdot f)(\mathbf{x}) = \sum_{j=1}^{D} \big(f(\mathbf{x} + a\mathfrak{e}_j) + f(\mathbf{x} - a\mathfrak{e}_j) \big)$. Show that:

$$Z(\mathbf{x}) = \langle 0 | \, \frac{1}{\mathbb{I} - \mu\,\Theta} \, |\mathbf{x}\rangle,$$

[9] Hence, none of the points of a self-avoiding walk $\Gamma_{\rm saw}$ is visited twice by $\Gamma_{\rm saw}$.

with $|\mathbf{x}\rangle$ the δ-function peaked at $\mathbf{x}$, i.e. $\langle \mathbf{y}|\mathbf{x}\rangle = \delta_{\mathbf{y};\mathbf{x}}$.

IA-5: Prove that there exists a critical μ_c such that $Z(\mathbf{x})$ converges for $|\mu| < \mu_c$ but diverges as $\mu \to \mu_c$. What is the value of μ_c?

• *IB: The scaling limit*

We now study the continuous limit $a \to 0$ of the above discrete model.

IB-1: Show that, for μ and $\mathbf{x}$ fixed, $\lim_{a\to 0} a^{-D}\, Z(\mathbf{x}) = \text{const.}\, \delta(\mathbf{x})$, with an explicit formula for the 'const.' to be determined. Give a geometrical interpretation of this result.

The continuous limit is thus meaningful only if we scale μ appropriately as $a \to 0$. Let $G(\mathbf{x})$ be defined by

$$G(\mathbf{x}) = \frac{1}{2D} \lim_{a\to 0} a^{2-D}\, Z(\mathbf{x}),$$

where the limit is understood with μ an appropriate function of a to be determined below.

IB-2: Show that

$$\big(-\Delta_{\mathbf{x}} + m^2\big) G(\mathbf{x}) = \delta(\mathbf{x}),$$

with $\Delta_{\mathbf{x}}$ the D-dimensional Laplacian, if μ approaches μ_c such that $\mu^{-1} - \mu_c^{-1} = a^2 m^2$ as $a \to 0$ with m fixed.

IB-3: Verify that the scaling limit – i.e. the limit $a \to 0$, $\mu \to \mu_c$ with $a^{-2}(\mu^{-1} - \mu_c^{-1}) = m^2$ fixed – of the explicit expression of $Z(\mathbf{x})$ found above is a solution of this equation.

IB-4: Show that $Z(\mathbf{x})$ can be written in the scaling form $Z(\mathbf{x}) \simeq (ma)^{D-2} F(m\mathbf{x})$, for some function F, when $\mu \to \mu_c$ at $m\mathbf{x}$ fixed.

• *II- Self-avoiding walks and the $O(n \to 0)$ models*

This part develops the connection between self-avoiding walks and a specific $O(n \to 0)$ lattice model on one hand, and between large distance properties of self-avoiding walks and an $O(n \to 0)$ field theory[10] on the other hand.

The $O(n)$ lattice model is defined as follows: a spin variable $\vec{S}_{\mathbf{x}}$, of dimension n, is attached to each lattice site $\mathbf{x} \in \Lambda$ with components $S^a_{\mathbf{x}}$, $a = 1, \cdots, n$. The statistical Boltzmann weight of a spin configuration is $\prod_{\langle \mathbf{x},\mathbf{x}'\rangle}(1+\mu\, \vec{S}_{\mathbf{x}} \cdot \vec{S}_{\mathbf{x}'})$ where the product is over neighbour sites (i.e. product over lattice edges) and μ a real number. The sum over the spin configurations is represented by an integration with measure $d[S]$ on spin configurations so that the partition function Z_n is

$$Z_n = \int d[S] \prod_{\langle \mathbf{x},\mathbf{x}'\rangle} (1 + \mu\, \vec{S}_{\mathbf{x}} \cdot \vec{S}_{\mathbf{x}'}),$$

We choose the measure to be factorized over lattice points, $d[S] = \prod_{\mathbf{x}} dm(\vec{S}_{\mathbf{x}})$, so that spins at different positions are uncorrelated under the measure $d[S]$ and any integral of products of spins $(S^{a_1}_{\mathbf{x}_1} \cdots S^{a_q}_{\mathbf{x}_q})$ factorizes into products of integrals over spins localized at the same position. The measure and spin variables are normalized such that

$$\int d[S]\, (S^a_{\mathbf{x}} S^b_{\mathbf{y}}) = \delta^{ab} \delta_{\mathbf{x};\mathbf{y}} \text{ and } \int d[S] = 1.$$

[10] This connection was first revealed by P.G. de Gennes.

We shall further assume that all higher moments of spins at a given point $\mathbf{x}$ vanish[11] so that $\int d[S]\,(S_{\mathbf{x}}^{a_1}\cdots S_{\mathbf{x}}^{a_p})=0$ as soon as $p\neq 2$.

The $O(n)$ field theory we shall consider is a Φ^4-theory whose action is given below.

• *IIA: The $O(n)$ lattice model and self-avoiding walks*

IIA-1: By expanding the product $\prod_{\langle \mathbf{x},\mathbf{x}'\rangle}(1+\mu\,\vec{S}_{\mathbf{x}}\cdot\vec{S}_{\mathbf{x}'})$ into a sum of clusters of occupied edges (declaring that an edge is occupied if it is weighted by μ), show that Z_n can be written as a sum over non-crossing loop configurations[12] of the form

$$Z_n=\sum_{\mathcal{C}:\,\text{non crossing loops}} n^{\ell(\mathcal{C})}\,\mu^{|\mathcal{C}|},$$

where $|\mathcal{C}|$ is the total number of bonds forming $\mathcal{C}$ and $\ell(\mathcal{C})$ the number of connected components (i.e. the number of loops).

Hint: It may help to draw a picture of the lattice (say in $D=2$) representing each term of this expansion with the edges occupied marked (say with a bold line) and to look at which of those graphs survive after integration over the spin variables.

This formula shows that we can define Z_n for any number n and any fugacity μ by analytic continuation[13]. We adopt this definition from now on.

IIA-2: Show that $Z_{n\to 0}=1$.

We now consider the spin correlation function $\langle S_0^a S_{\mathbf{x}}^b\rangle$ between spins at the origin 0 and a point $\mathbf{x}$ in Λ. It is defined by:

$$\langle S_0^a S_{\mathbf{x}}^b\rangle=\frac{1}{Z_n}\int d[S]\prod_{\langle \mathbf{y},\mathbf{y}'\rangle}(1+\mu\,\vec{S}_{\mathbf{y}}\cdot\vec{S}_{\mathbf{y}'})\,(S_0^a S_{\mathbf{x}}^b).$$

Global $O(n)$-invariance implies that $\langle S_0^a S_{\mathbf{x}}^b\rangle=\delta^{ab}\,G_n(\mathbf{x})$.

IIA-3: Write $G_n(\mathbf{x})$ as a weighted sum of non-crossing loops $\mathcal{C}$ in Λ and a self-avoiding walk Γ_{saw} from 0 to $\mathbf{x}$:

$$G_n(\mathbf{x})=\sum_{\mathcal{C}+(\Gamma_{\text{saw}}:0\to\mathbf{x})} n^{\ell(\mathcal{C})}\,\mu^{|\mathcal{C}|+|\Gamma_{\text{saw}}|}.$$

IIA-4: Show that the analytical continuation of this two point function can be written as a sum over self-avoiding walks Γ_{saw} from 0 to $\mathbf{x}$: $G_{n\to 0}(\mathbf{x})=\sum_{\Gamma_{\text{saw}}:\,0\to\mathbf{x}}\mu^{|\Gamma_{\text{saw}}|}$.

Show that, as a consequence,

$$G_{n\to 0}(\mathbf{x})=\sum_{N\geq 0}\mu^N\,W_N^{\text{saw}}(\mathbf{x})$$

with $W_N^{\text{saw}}(\mathbf{x})$ the number of self-avoiding walks with N bonds from 0 to $\mathbf{x}$.

[11] This is actually an approximation, but it can be shown that higher moments vanish as $n\to 0$, so that we can (hopefully legitimately) neglect them in the limit $n\to 0$ we are interested in.

[12] A loop is said to be non-crossing if all of its edges and vertices are distinct.

[13] Positivity of the Boltzmann weights is then no guaranteed.

• *IIB: An $O(n)$ statistical field theory*

We now consider the $O(n)$-invariant Φ^4 field theory in $\mathbb{R}^D$ with action

$$S = \int d^D x \, \big[\frac{1}{2}(\nabla\vec{\Phi})^2 + \frac{1}{2!}\mu(\vec{\Phi}\cdot\vec{\Phi}) + \frac{1}{4!}g(\vec{\Phi}\cdot\vec{\Phi})^2\big].$$

The field $\vec{\Phi}$ has n components Φ^a, $a = 1, \cdots, n$.

IIB-1: Explain why symmetry and universality arguments allow us to restrict to a $(\vec{\Phi}\cdot\vec{\Phi}) + (\vec{\Phi}\cdot\vec{\Phi})^2$ interaction.

IIB-2: What are the Feynman rules (in momentum space) for this theory?

IIB-3: Compute the beta functions β_μ and β_g at one loop (i.e. at order g^2, μ^2 and $g\mu$) for the mass μ and coupling constant g renormalization and show that :

$$\begin{aligned} \beta_\mu &= 2\mu - 8(n+2)g\mu - 4\mu^2 - 32(n+2)g^2 + \cdots \\ \beta_g &= (4-D)g - 8(n+8)g^2 - \mu^2 - 16g\mu + \cdots \end{aligned}$$

Hint: It may be useful to first evaluate the OPE of the perturbing fields $\phi_2 =: (\vec{\Phi}\cdot\vec{\Phi}) :$ and $\phi_4 =: (\vec{\Phi}\cdot\vec{\Phi})^2 :$.

Assume that the renormalization of $(\nabla\vec{\Phi})^2$ can neglected at this order.

Remark: Only the first coefficients (proportional to μ and to $g\mu$ in β_μ and to g and g^2 in β_g are going to be relevant. If time is lacking, you may restrict in proving that $\beta_\mu = 2\mu - 8(n+2)g\mu + \cdots$ and $\beta_g = (4-D)g - 8(n+8)g^2 + \cdots$, and argue that the other terms are indeed irrelevant.

IIB-4: Show that there is a pertubative IR fixed point in dimension $D = 4-\epsilon$ at $\mu_c = 0 + O(\epsilon^2)$ and $g_c = g_*\epsilon + O(\epsilon^2)$ for some g_*.

What is the value, to first order in ϵ, of the correlation length scaling exponent ν defined by $\xi \simeq |\mu - \mu_c|^{-\nu}$ with ξ the correlation length?

• *IIC: Scaling theory of self-avoiding walks*

Based on universality arguments we may claim that the large distance behaviour of self-avoiding walks is encoded into the $O(n \to 0)$ field theory. We make this hypothesis in the following.

IIC-1: Let $W_N(\mathbf{x})$ be the number of walks with N bonds, free or self-avoiding, from 0 to $\mathbf{x}$: $W_N(\mathbf{x}) = W_N^{\text{free}}(\mathbf{x})$ for free paths, $W_N(\mathbf{x}) = W_N^{\text{saw}}(\mathbf{x})$ for self-avoiding walks. Let $W(\mathbf{x}) = \sum_{N\geq 0} \mu^N W_N(\mathbf{x})$. Show that

$$W_N(\mathbf{x}) = \oint \frac{d\mu}{2i\pi}\, \mu^{-N-1}\, W(\mathbf{x}),$$

where the integration is over a small contour around the origin.

Note: We shall admit the following technical result: If a series $f(\mu) = \sum_{N>0} \mu^N c_N$ is analytic for $|\mu| < \mu_c$ but singular at μ_c with $f(\mu) \simeq \text{const.}\,(\mu_c - \mu)^{-\gamma-1}$ as $\mu \to \mu_c$, then $c_N \simeq \text{const.}\, N^\gamma\, \mu_c^{-N}$ for N large.

IIC-2: Explain why the hypothesis of a scaling theory implies that, for $\mu \to \mu_c$, the function $Z(\mathbf{x})$ behaves as

$$W(\mathbf{x}) \simeq \xi^{-2\Delta} F(|\mathbf{x}|/\xi),$$

with ξ the correlation length and F a smooth function. What is the meaning of Δ?

How does the correlation length ξ depend on μ close to the critical point μ_c?

IIC-3: Let $\langle R^2 \rangle_N$ be the mean square distance defined by

$$\langle R^2 \rangle_N = \frac{\sum_{\mathbf{x}} |\mathbf{x}|^2 W_N(\mathbf{x})}{\sum_{\mathbf{x}} W_N(\mathbf{x})}.$$

Show that if the scaling hypothesis applies then

$$\langle R^2 \rangle_N \simeq \text{const.}\, N^{2\nu}$$

for N large, with ν the correlation length exponent.

Hint: It may be useful to consider both generating functions $\sum_{\mathbf{x}} W(\mathbf{x})$ and $\sum_{\mathbf{x}} |\mathbf{x}|^2 W(\mathbf{x})$.

IIC-4: What is the value of ν for free paths?

What is the value of ν for self-avoiding walks in dimension $D = 4 - \epsilon$ to first order in ϵ, assuming that the $O(n \to 0)$ field theory describes the scaling theory of self-avoiding walks?

Compare this value with the exact value $\nu = 3/4$ in $D = 2$ and the approximate numerical value $\nu \simeq 0.588\cdots$ in $D = 3$.

IIC-5: The fractal dimension D_h of a set embedded in a metric space may be defined through the minimal number $\mathcal{N}_r$ of boxes of radius r needed to cover it by $D_h = \lim_{r\to 0} \log \mathcal{N}_r / \log(1/r)$.

Use the previous scaling theory to find an estimate of the fractal dimension of self-avoiding walks in dimension $D = 4 - \epsilon$.

9.3 Et un...

9.4 Et deux...

9.5 Et trois...

9.6 Zero...