The Gamma function

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1 Gamma function

The Euler Gamma function is defined as

$$\boxed{\Gamma(z) \equiv \int_0^\infty dt\, e^{-t}\, t^{z-1}}\,. \tag{1}$$

It is easy to show that $\Gamma(z)$ satisfy the recursion relation

$$\boxed{\Gamma(z+1) = z\,\Gamma(z)}\ : \tag{2}$$

indeed, integrating by parts,

$$\begin{aligned}\Gamma(z+1) &= \int_0^\infty dt\, e^{-t}\, t^z \\ &= -e^{-t}\, t^z\Big|_0^\infty + z\int_0^\infty dt\, e^{-t}\, t^{z-1} \\ &= z\,\Gamma(z)\,. \end{aligned} \tag{3}$$

For $z = n$ integer eq. (2) is the recursion relation of the factorial, and thus we have

$$\Gamma(n+1) \propto n!\,; \tag{4}$$

because in addition $\Gamma(1) = 1$ (easly derived from the definition), we have the identification

$$\boxed{\Gamma(n+1) = n!}\,, \tag{5}$$

and in this sense the Gamma function is a complex extension of the factorial.

The sequence of Gamma function computed in all half-integers can be obtained using subsequently the recursion relation (2) and knowing that

$$\boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}} \tag{6}$$

that is easy to compute:

$$\begin{aligned}\Gamma\left(\frac{1}{2}\right) &= \int_0^\infty dt\, e^{-t}\, t^{-\frac{1}{2}} \\ &= 2\int_0^\infty dx\, e^{-x^2} \\ &= 2\frac{\sqrt{\pi}}{2} = \sqrt{\pi}\,. \end{aligned} \tag{7}$$

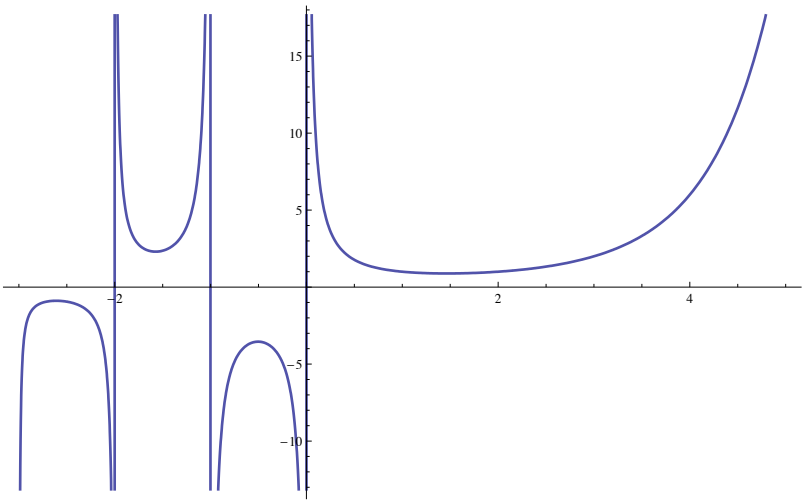

Figure 1: $\mathrm{Re}\,\Gamma(x)$ for real x.

1.1 Analytical structure

First, from the definition (1), we see that

$$\boxed{\Gamma(\bar{z}) = \overline{\Gamma(z)}}\,, \tag{8}$$

that is to say that Gamma is a real function, and in particular $\mathrm{Im}\,\Gamma(x) = 0$ for $x \in \mathbb{R}$.

Inverting eq. (2) we have

$$\Gamma(z) = \frac{1}{z}\,\Gamma(z+1)\,, \tag{9}$$

and when $z = 0$ it diverges (because $\Gamma(1) = 1$ is finite). It is evident that $z = 0$ corresponds to a simple pole of order 1 and residue 1: indeed

$$\operatorname*{Res}_{z=0}\Gamma(z) = \lim_{z\to 0} z\,\Gamma(z) = \lim_{z\to 0}\Gamma(z+1) = 1\,. \tag{10}$$

Formally, we can discover that the Gamma function has simple poles in all negative integers simply iterating the recursion (2):

$$\begin{aligned}\Gamma(-n) &= \frac{1}{-n}\,\Gamma(-n+1)\\ &= \dots\\ &= \frac{(-1)^n}{n!}\Gamma(0) \qquad (n \in \mathbb{N})\,.\end{aligned} \tag{11}$$

This brings to the conclusion that $\Gamma(-n)$ has a simple pole of order 1 and residue $\frac{(-1)^n}{n!}$. More rigorously, this can be shown in the usual way:

$$\begin{aligned}\operatorname*{Res}_{z=0}\Gamma(z-n) &= \lim_{z\to 0} z\,\Gamma(z-n)\\ &= \lim_{z\to 0}\frac{z}{z-n}\,\Gamma(z-n+1)\\ &= \dots\end{aligned}$$

$$= \lim_{z\to 0} \frac{z}{(z-n)(z-n+1)\cdots(z-1)}\,\Gamma(z)$$
$$= \frac{(-1)^n}{n!} \qquad (n \in \mathbb{N})\,. \tag{12}$$

In all the complex plane (except the negative real axis) the Gamma function is well defined. Fon non-integer negative real values the Gamma function can be analytically continued (as we have seen for example for half-integers, positives and negatives).

We can prove these results in a simpler way. Starting from the definition (1) we rewrite

$$\begin{aligned}\Gamma(z) &= \int_1^\infty dt\, e^{-t}\, t^{z-1} + \int_0^1 dt\, e^{-t}\, t^{z-1} \\ &= \int_1^\infty dt\, e^{-t}\, t^{z-1} + \int_0^1 dt\, t^{z-1} \sum_{n=0}^\infty \frac{(-1)^n}{n!}\, t^n \\ &= \int_1^\infty dt\, e^{-t}\, t^{z-1} + \sum_{n=0}^\infty \frac{(-1)^n}{n!}\, \frac{1}{z+n}\end{aligned} \tag{13}$$

where the first integral is an analytic function of z and the second term gives the position of the poles and their residues.

Moreover, the Gamma function has an essential singularity to complex infinity, because $\Gamma\left(\frac{1}{z}\right)$ has a non-defined limit for $z \to 0$. This is to say that the Gamma function is not well defined in the compactified complex plane.

Now we want to show that near a pole in $-n$ one has the expansion

$$\boxed{\Gamma(z-n) = \frac{(-1)^n}{n!}\left[\frac{1}{z} + \psi(n+1) + O(z)\right]} \tag{14}$$

where $\psi(z)$ is the logarithmic derivative of the Gamma function, defined in (41); note that $\psi(n+1)$ has an explicit expression given in eq. (44). First, note that

$$\begin{aligned}\Gamma(z-n) &= \frac{1}{z-n}\,\Gamma(z-n+1) \\ &= \dots \\ &= \frac{1}{(z-n)(z-n+1)\cdots(z-1)\,z\,(z+1)\cdots(z+n)}\,\Gamma(z+n+1) \\ &= \frac{1}{z}\,G(z,n)\end{aligned} \tag{15}$$

where we have defined

$$G(z,n) = \frac{\Gamma(z+n+1)}{(z^2-n^2)\cdots(z^2-1)} = \Gamma(z+n+1)\,\frac{(-1)^n}{(n!)^2}\left[1 + O(z^2)\right]\,. \tag{16}$$

By Taylor expanding $G(z,n)$ around $z=0$ in (15) we get

$$\begin{aligned}\Gamma(z-n) &= \frac{1}{z}\left[G(0,n) + z\,G'(0,n) + O(z^2)\right] \\ &= \frac{(-1)^n}{n!}\left[\frac{1}{z} + \frac{1}{n!}\Gamma'(n+1) + O(z)\right]\,,\end{aligned} \tag{17}$$

that is the claimed result.

1.2 Alternative definitions

There are some alternative definitions of the Gamma function. One, due to Euler, is

$$\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty}\frac{(1+\frac{1}{n})^z}{1+\frac{z}{n}} = \lim_{n\to\infty}\frac{n!\,(n+1)^z}{z(z+1)\cdots(z+n)}\,. \tag{18}$$

It is easy to prove (using the expression with the limit) that with this definition the recursion relation (2) is satisfied, and also that for $z = 1$ gives $\Gamma(1) = 1$.

Another one, due to Weierstrass, is

$$\Gamma(z) = \frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\frac{e^{z/n}}{1+\frac{z}{n}} = e^{-\gamma z}\lim_{n\to\infty}\frac{n!\,\exp\left[z\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)\right]}{z(z+1)\cdots(z+n)}\,, \tag{19}$$

where γ is the Euler-Mascheroni number defined in Section 2.2 The equivalence to the Euler definition can be seen by taking the logarithm of this two forms: the first gives

$$\log\Gamma(z) = -\log z + \sum_{n=0}^{\infty}\left[z\log\left(1+\frac{1}{n}\right) - \log\left(1+\frac{z}{n}\right)\right] \tag{20}$$

and the second gives

$$\log\Gamma(z) = -\log z - \gamma z + \sum_{n=0}^{\infty}\left[\frac{z}{n} - \log\left(1+\frac{z}{n}\right)\right]\,. \tag{21}$$

Comparing this two expressions we note that they are equal because of eq. (48). Note that this two expression are useful in order to numerically approximate $\log\Gamma(z)$, by truncating the series to a certain value of n.

1.3 Euler reflection formula

A useful formula is

$$\boxed{\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}} \tag{22}$$

called the Euler reflection formula. To demonstrate it, consider the alternative definition (18) and note that

$$\begin{aligned}\frac{1}{\Gamma(z)\,\Gamma(1-z)} &= \frac{1}{-z\,\Gamma(z)\,\Gamma(-z)}\\ &= \frac{z(-z)}{-z}\prod_{n=1}^{\infty}\frac{(1+\frac{z}{n})\,(1-\frac{z}{n})}{(1+\frac{1}{n})^{z}\,(1+\frac{1}{n})^{-z}}\\ &= z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)\\ &= \frac{\sin(\pi z)}{\pi}\end{aligned} \tag{23}$$

where we have used

$$\sin(\pi z) = \pi z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)\,. \tag{24}$$

1.4 Stirling approximation

The Stirling approximation for the factorial is

$$\boxed{n! = \sqrt{2\pi n}\, e^{-n}\, n^n} \tag{25}$$

which is valid for large n. To demonstrate it, we perform a saddle point approximation to the integral (1):

$$\begin{aligned}\Gamma(z+1) &= \int_0^\infty dt\, e^{-t}\, t^z \\ &= \int_0^\infty dt\, e^{-t+z\log t}\end{aligned}$$

now we expand the exponent near its maximum and get

$$\begin{aligned}&\simeq \int_0^\infty dt\, \exp\left(-z + z\log z - \frac{1}{2z}t^2\right) \\ &\simeq e^{-z}\, z^z \int_{-\infty}^\infty dt\, e^{-\frac{1}{2z}t^2} \\ &= \sqrt{2\pi z}\, e^{-z}\, z^z\end{aligned} \tag{26}$$

that is the claimed result.

1.5 Derivative recursion relation

Consider the Taylor expansion near $z = 0$ of $\Gamma(z + z_0 + 1)$:

$$\Gamma(z+z_0+1) = \sum_{k=0}^\infty \frac{\Gamma^{(k)}(z_0+1)}{k!}\, z^k\ ; \tag{27}$$

using eq. (2) this is equal to $(z + z_0)\,\Gamma(z + z_0)$, that expanded is

$$\begin{aligned}(z+z_0)\,\Gamma(z+z_0) &= (z+z_0)\sum_{k=0}^\infty \frac{\Gamma^{(k)}(z_0)}{k!}\, z^k \\ &= z_0\,\Gamma(z_0) + \sum_{k=1}^\infty \frac{z^k}{k!}\left[z_0\,\Gamma^{(k)}(z_0) + k\,\Gamma^{(k-1)}(z_0)\right]\ .\end{aligned} \tag{28}$$

Comparing the two expansions, we get the equality ($k > 0$)

$$\boxed{\Gamma^{(k)}(z_0+1) = z_0\,\Gamma^{(k)}(z_0) + k\,\Gamma^{(k-1)}(z_0)}\ . \tag{29}$$

1.6 Inverse Gamma function

We can define an inverse Gamma function

$$\boxed{\Delta(z) \equiv \frac{1}{\Gamma(z)}} \tag{30}$$

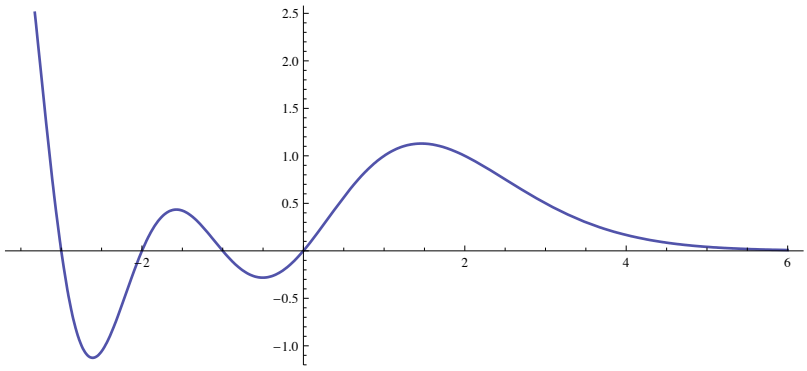

Figure 2: Re $\Delta(x)$ for real x.

which is an entire function (analytical in all the complex plane). The recursion relation (2) becomes

$$\boxed{\Delta(z) = z\,\Delta(z+1)}\,. \tag{31}$$

We are able to produce an integral representation of the Δ function: consider the Laplace transform of the function t^{z-1} with respect to t:

$$\int_0^\infty dt\, e^{-st}\, t^{z-1} = s^{-z}\,\Gamma(z) \tag{32}$$

and then we have that the inverse Laplace tranform of s^{-z} is

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\, e^{ts}\, s^{-z} = \Delta(z)\, t^{z-1} \tag{33}$$

where c has to be choosen to the right of all the singularities of s^{-z}, that is $c > 0$. Now, by choosing $t = 1$, we get

$$\boxed{\Delta(z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\, e^{s}\, s^{-z}}\,, \tag{34}$$

where the integration path can be deformed to be a contour of the negative real axis. Note that for positive integers $z = n + 1$ the branch cut vanishes and only a pole in $s = 0$ remains, and then the path can be closed around $s = 0$ to obtain

$$\frac{1}{n!} = \frac{1}{2\pi i}\oint ds\, e^{s}\, s^{-n-1}\,, \tag{35}$$

that can be also easly demonstrated using the residue theorem.

Now consider the Taylor expansion near $z = 0$ of $\Delta(z + z_0)$:

$$\Delta(z+z_0) = \sum_{k=0}^{\infty} \frac{\Delta^{(k)}(z_0)}{k!}\, z^k\,; \tag{36}$$

using eq. (31) this is equal to $(z + z_0)\,\Delta(z + z_0 + 1)$, that expanded is

$$(z+z_0)\,\Delta(z+z_0+1) = (z+z_0)\sum_{k=0}^{\infty} \frac{\Delta^{(k)}(z_0+1)}{k!}\, z^k$$

$$= z_0\,\Delta(z_0+1) + \sum_{k=1}^{\infty} \frac{z^k}{k!}\left[z_0\,\Delta^{(k)}(z_0+1) + k\,\Delta^{(k-1)}(z_0+1)\right] . \tag{37}$$

Comparing the two expansions, we get the equality ($k \geq 1$)

$$\boxed{\Delta^{(k)}(z_0) = z_0\,\Delta^{(k)}(z_0+1) + k\,\Delta^{(k-1)}(z_0+1)} \tag{38}$$

and, for $z_0 = 0$,

$$\Delta^{(k)}(0) = k\,\Delta^{(k-1)}(1) . \tag{39}$$

1.6.1 Saddle point approximation

The analogous of the Stirling approximation can be done for the $\Delta(z)$ function. The result is:

$$\Delta(z) \simeq \sqrt{\frac{z}{2\pi}}\, e^z\, z^{-z} . \tag{40}$$

2 Gamma related functions

2.1 Log derivative $\psi(z)$

Define the logarithmic derivative of the Gamma function as follows

$$\boxed{\psi(z) \equiv \frac{d}{dz} \log \Gamma(z)} \,. \tag{41}$$

Form the recursion relation (2) it follows that

$$\boxed{\psi(z+1) = \psi(z) + \frac{1}{z}} \,, \tag{42}$$

as we can easly demonstrate:

$$\begin{aligned} \psi(z+1) &= \frac{d}{dz} \log \Gamma(z+1) \\ &= \frac{d}{dz} \log \left[z\, \Gamma(z) \right] \\ &= \frac{1}{z} + \psi(z) \end{aligned} \tag{43}$$

(note that the same result can be obtained from eq. (29) for $k = 1$). For a positive integer n, we can iterate the recursion (42) to obtain

$$\boxed{\psi(n+1) = \psi(1) + 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}} \,, \tag{44}$$

and the value of $\psi(1)$ is

$$\boxed{\psi(1) = -\gamma} \tag{45}$$

where γ is the Euler-Mascheroni number, as demonstrated in Section 2.2.

The functions

$$\boxed{\psi_n(z) \equiv \psi^{(n)}(z) = \frac{d^{n+1}}{dz^{n+1}} \log \Gamma(z)} \tag{46}$$

are also defined. Iterating the derivative in (42) we obtain the recursion relation

$$\boxed{\psi_n(z+1) = \psi_n(z) + (-1)^n\, n!\, \frac{1}{z^{n+1}}} \,. \tag{47}$$

2.2 Euler-Mascheroni number

The Euler-Mascheroni number is defined as

$$\boxed{\gamma \equiv \sum_{n=1}^{\infty} \left[\frac{1}{n} - \log\left(1 + \frac{1}{n}\right) \right]} \,, \tag{48}$$

and its value is

$$\boxed{\gamma = 0.577216\ldots} \,. \tag{49}$$

Note that

$$\begin{aligned}\sum_{n=1}^{\infty}\log\left(1+\frac{1}{n}\right) &= \lim_{N\to\infty}\sum_{n=1}^{N}\log\left(1+\frac{1}{n}\right)\\ &= \lim_{N\to\infty}\sum_{n=1}^{N}\left[\log(1+n)-\log n\right]\\ &= \lim_{N\to\infty}\log(1+N)\\ &= \lim_{N\to\infty}\log N\,, \end{aligned} \tag{50}$$

where we noted that the sum is a telescopic sum and that in the limit $N\to\infty$ $\log(1+N)$ and $\log N$ are equivalent. Thus we can rewrite eq. (48) as

$$\boxed{\gamma = \lim_{N\to\infty}\left[\sum_{n=1}^{N}\frac{1}{n}-\log N\right]}\,. \tag{51}$$

Now we prove that $\boxed{\psi(1)=-\gamma}$. To do this, let us consider eq. (20), and take its derivative:

$$\psi(z) = -\frac{1}{z}+\sum_{n=1}^{\infty}\left[\log\left(1+\frac{1}{n}\right)-\frac{1}{n}\frac{1}{1+\frac{z}{n}}\right]\,. \tag{52}$$

For $z=1$ we get

$$\begin{aligned}\psi(1) &= -1+\sum_{n=1}^{\infty}\left[\log\left(1+\frac{1}{n}\right)-\frac{1}{1+n}\right]\\ &= -\sum_{n=1}^{\infty}\left[\frac{1}{n}-\log\left(1+\frac{1}{n}\right)\right]\,, \end{aligned} \tag{53}$$

in which we recognise the definition (48).

2.3 Generalized Gamma functions

You can also define an incomplete Gamma function (or plica function)

$$\Gamma(z,a) \equiv \int_a^{\infty} dt\, e^{-t}\, t^{z-1} \tag{54}$$

and a truncated Gamma function

$$\gamma(z,a) \equiv \int_0^{a} dt\, e^{-t}\, t^{z-1}\,, \tag{55}$$

that have a branch cut on the negative real axis in the a complex plane. Obviously

$$\Gamma(z) = \Gamma(z,a)+\gamma(z,a)\,. \tag{56}$$

For integer $z = k + 1$, integrating repeatedly by parts we get

$$\Gamma(k+1, a) = k!\, e^{-a} \sum_{n=0}^{k} \frac{a^n}{n!} \tag{57}$$

$$\gamma(k+1, a) = k! \left(1 - e^{-a} \sum_{n=0}^{k} \frac{a^n}{n!} \right) . \tag{58}$$

2.4 Beta function

The Beta function is defined as

$$\boxed{B(a,b) \equiv \int_0^1 dx\; x^{a-1}(1-x)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}} \, . \tag{59}$$

To show the last equality, cosider the product

$$\begin{aligned}
\Gamma(a)\Gamma(b) &= \int_0^\infty dt\, e^{-t}\, t^{a-1} \int_0^\infty du\, e^{-u}\, u^{b-1} \\
&= 4 \int_0^\infty dx\, e^{-x^2}\, x^{2a-1} \int_0^\infty dy\, e^{-y^2}\, y^{2b-1} \\
&= \int_{-\infty}^\infty dx \int_{-\infty}^\infty dy\, e^{-(x^2+y^2)}\, |x|^{2a-1}\, \left|y\right|^{2b-1} \\
&= \int_0^{2\pi} d\theta\, \left|\cos^{2a-1}\theta\, \sin^{2b-1}\theta\right| \int_0^\infty dr\, e^{-r^2}\, r^{2a+2b-1} \\
&= 2\,\Gamma(a+b) \int_0^{\pi/2} d\theta\, \cos^{2a-1}\theta\, \sin^{2b-1}\theta \\
&= \Gamma(a+b) \int_0^1 dx\, x^{a-1}(1-x)^{b-1} \, .
\end{aligned} \tag{60}$$