7.2 Perturbative Expansion. Wick's Theorem and Feynman Diagrams

In Section 2.5.1 we have shown how to calculate a path integral for a hamiltonian of the form $\frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 + V_{\text{I}}(q)$ as a series expansion in powers of $V_{\text{I}}(q)$, for any function $V_{\text{I}}(q)$ expandable in powers of q. The result was based on the calculation of a reference gaussian integral (in Chapter 2, the harmonic oscillator). Here, we apply exactly the same strategy to functional integrals.

Furthermore, although most results derived in this chapter will be illustrated with examples corresponding to an action of the form (7.5), the results are more general, applying to any bosonic field theory, and this explains the abstract notation used below.

7.2.1 The gaussian integral

We consider a general gaussian functional integral:

$$\mathcal{Z}_{\text{G}}(J) = \int [\mathrm{d}\phi] \exp\left[-\tfrac{1}{2}\phi K \phi + J \cdot \phi\right], \tag{7.14}$$

where we have used the symbolic notation:

$$\phi K \phi \equiv \int \mathrm{d}^d x\, \mathrm{d}^d y\, \phi(x) K(x,y) \phi(y), \qquad J \cdot \phi \equiv \int \mathrm{d}^d x\, J(x)\phi(x).$$

We assume that the kernel K is symmetric and positive. In expression (7.14), a normalization is implied; we choose $\mathcal{Z}_{\text{G}}(0) = 1$.

We denote the inverse of K by Δ:

$$\int \mathrm{d}^d z\, \Delta(x,z) K(z,y) = \delta^d(x-y). \tag{7.15}$$

To calculate the functional integral (7.14), we simply shift ϕ by ΔJ and find after integration

$$\mathcal{Z}_{\text{G}}(J) = \exp\left(\tfrac{1}{2} J \Delta J\right), \tag{7.16}$$

again with the convention

$$J\Delta J = \int \mathrm{d}^d x\, \mathrm{d}^d y\, J(x) \Delta(x,y) J(y).$$

The identities derived in Section 7.1.1 show that $\mathcal{Z}_{\text{G}}(J)$ is also the generating functional of correlation functions corresponding to a general action quadratic in the field ϕ. The inverse kernel Δ in equation (7.15) is thus the two-point function of the gaussian or *free field theory*:

$$\langle \phi(x_1)\phi(x_2)\rangle_0 = \left.\frac{\delta^2 \mathcal{Z}_{\text{G}}(J)}{\delta J(x_1)\delta J(x_2)}\right|_{J\equiv 0} = \Delta(x_1,x_2), \tag{7.17}$$

where $\langle \bullet \rangle_0$ means gaussian or free field expectation value. In the example of a free massive theory with mass m, which corresponds in expression (7.5) to the choice

$$V(\phi) = \tfrac{1}{2} m^2 \phi^2,$$

the kernel K is a local operator:

$$K(\mathbf{x},\mathbf{y}) = \left(-\nabla^2 + m^2\right) \delta^d(\mathbf{x}-\mathbf{y}), \tag{7.18}$$

where we denote by ∇ the gradient operator in d dimensions. The kernel Δ then takes the form

$$\Delta(\mathbf{x},\mathbf{y}) = \frac{1}{(2\pi)^d} \int \mathrm{d}^d p\, \frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{p^2+m^2}\,. \tag{7.19}$$

7.2.2 Perturbation theory. Wick's theorem

We now consider a more general euclidean action of the form

$$\mathcal{S}(\phi) = \tfrac{1}{2}\phi K \phi + \mathcal{V}_{\mathrm{I}}(\phi). \tag{7.20}$$

Using the property (7.9),

$$\frac{\delta}{\delta J(x)} \mathrm{e}^{J\cdot\phi} = \phi(x)\,\mathrm{e}^{J\cdot\phi},$$

we can express the functional integral

$$\mathcal{Z}(J) = \int [\mathrm{d}\phi] \exp\left[-\mathcal{S}(\phi) + J\cdot\phi\right], \tag{7.21}$$

in terms of $\mathcal{Z}_{\mathrm{G}}(J)$ as

$$\mathcal{Z}(J) = \exp\left[-\mathcal{V}_{\mathrm{I}}\left(\frac{\delta}{\delta J}\right)\right]\mathcal{Z}_{\mathrm{G}}(J) = \exp\left[-\mathcal{V}_{\mathrm{I}}\left(\frac{\delta}{\delta J}\right)\right]\exp\left(\tfrac{1}{2}J\Delta J\right), \tag{7.22}$$

an expression analogous to (2.62). Note that in this chapter we use the convention that a differential operator like $\delta/\delta J$ acts on all factors placed to its right.

The identities (7.22,7.11) can then be combined to calculate all ϕ-field correlation functions as a formal series in powers of the interaction potential $\mathcal{V}_{\mathrm{I}}(\phi)$.

Wick's theorem. Perturbation theory, that is, an expansion in powers of $\mathcal{V}_{\mathrm{I}}$, reduces all calculations to gaussian averages of product of fields. From the expression (7.22), and using the arguments of Section 1.1, we obtain a straightforward generalization of equations (1.9–1.14) or (2.50), which expresses Wick's theorem in field theory:

$$\begin{aligned}\left\langle \prod_1^{2s}\phi(z_i)\right\rangle_0 &= \left[\prod_{i=1}^{2s}\frac{\delta}{\delta J(z_i)}\exp\left(\tfrac{1}{2}J\Delta J\right)\right]\Bigg|_{J\equiv 0} \\ &= \sum_{\substack{\text{all possible pairings}\\ \text{of } \{1,2,\ldots,2s\}}} \Delta(z_{i_1} z_{i_2})\ldots\Delta(z_{i_{2s-1}}, z_{i_{2s}}),\end{aligned} \tag{7.23}$$

where $\langle\bullet\rangle_0$ means gaussian or free field average. Perturbation theory involves a basic ingredient: the two-point function Δ of the gaussian theory (equation (7.17)) which we call the *propagator*.

Graphically, each term in the sum can be represented by a set of contractions corresponding to the particular pairing chosen. For example, for $s=2$ one finds

$$\begin{aligned}\langle\phi(z_1)\phi(z_2)\phi(z_3)\phi(z_4)\rangle_0 &= \overbrace{\phi(z_1)\phi(z_2)}\overbrace{\phi(z_3)\phi(z_4)} + \ 2 \text{ terms}\\ &= \Delta(z_1,z_2)\Delta(z_3,z_4) + \Delta(z_1,z_3)\Delta(z_2,z_4)\\ &\quad + \Delta(z_1,z_4)\Delta(z_2,z_3).\end{aligned}$$

Feynman diagrams. When the interaction terms are local, that is, integrals of polynomials of the field $\phi(x)$ and its derivatives, any perturbative contribution to the n-point correlation function is a gaussian expectation value of the form

$$\left\langle \phi(x_1)\cdots\phi(x_n)\int \mathrm{d}^d y_1\,\phi^{p_1}(y_1)\int \mathrm{d}^d y_2\,\phi^{p_2}(y_2)\cdots\int \mathrm{d}^d y_k\,\phi^{p_k}(y_k)\right\rangle_0 .$$

Therefore, it is a sum of products of propagators integrated over the points corresponding to interaction vertices. It is then possible to give a graphical representation of each product: a propagator is represented by a line joining the two points which appear as arguments; moreover, any point that is common to more than one line corresponds to an argument that has to be integrated over.

Remark. It will be verified in the coming chapters that interacting theories with a propagator of the form (7.19) have large momentum or short distance divergences. Therefore, in what follows we assume either that the field theory has been regularized by replacing continuum space by a lattice as in Section 7.1.2, or the propagator Δ has been replaced by a more complicated function that ensures the convergence of all terms in the perturbative expansion (see Sections 9.5–9.6).

7.2.3 The ϕ^4 example

We now illustrate the previous discussion with the important quartic example,

$$\mathcal{V}_{\mathrm{I}}(\phi) = \frac{g}{4!}\int \mathrm{d}^d x\, \phi^4(x).$$

The two-point function. The two-point function to order g^2 has the expansion

$$\langle \phi(x_1)\phi(x_2)\rangle = (\mathrm{a}) - \frac{1}{2}g\,(\mathrm{b}) + \frac{g^2}{4}\,(\mathrm{c}) + \frac{g^2}{4}\,(\mathrm{d}) + \frac{g^2}{6}\,(\mathrm{e}) + O\left(g^3\right).$$

Note that three additional contributions which factorize into

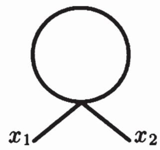
$$\langle \phi(x_1)\phi(x_2)\rangle_0 \left\langle \phi^4(y)\right\rangle_0 , \quad \left\langle \phi(x_1)\phi(x_2)\phi^4(y_1)\right\rangle_0 \left\langle \phi^4(y_2)\right\rangle_0 \text{ and}$$
$$\langle \phi(x_1)\phi(x_2)\rangle_0 \left\langle \phi^4(y_1)\phi^4(y_2)\right\rangle_0 ,$$

cancel in the division by $\mathcal{Z}(J=0)$.

Then (a) is the propagator:

(a): x_1 ——————— x_2

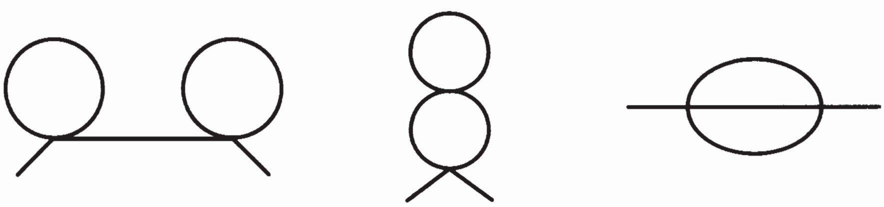
(b), the Feynman diagram which appears at order g, is displayed in figure 7.1:



(b)

Fig. 7.1 Two-point function at order g.

and the diagrams (c),(d),(e) are displayed in figure 7.2:



(c) (d) (e)

Fig. 7.2 Contributions of order g^2 to the two-point function.

Let us explain, for example, in detail the weight 1/6 in front of diagram (e). Expanding the exponential at second order, we have to calculate the gaussian expectation value of

$$\frac{g^2}{2!\,(4!)^2}\int \mathrm{d}^d y_1 \int \mathrm{d}^d y_2 \left\langle \phi(x_1)\phi(x_2)\phi^4(y_1)\phi^4(y_2)\right\rangle_0 ,$$

and we apply Wick's theorem.

First, $\phi(x_1)$ can be associated with any ϕ field of the interaction terms; there are eight choices and one interaction term is distinguished. Then $\phi(x_2)$ must be contracted with a field of the remaining interaction term: four choices. The three remaining fields of the first interaction term can finally be paired with any permutation of the fields of the second one: 3! equivalent possibilities. Multiplying all factors one finds

$$\frac{1}{2}\frac{1}{(4!)^2}\times 8\times 4\times 3! = \frac{1}{6}.$$

Note also that the factor 1/6 multiplying the diagram can be shown to have an interpretation as 1/3!, the combinatorial factor in the denominator reflecting the symmetry under permutation of the three lines joining the two vertices. There exist systematic expressions giving the weight factor of Feynman diagrams in terms of the symmetry group of the graph.

A useful practical remark is the following: the sum of all weight factors at a given order can be calculated from the "zero-dimensional" field theory obtained by suppressing the arguments of the field and all derivatives and integration in the action because the propagator can then be normalized to 1. For example, in the case of the ϕ^4 field considered here the action becomes

$$\mathcal{S}(\phi) = \frac{1}{2}\phi^2 + \frac{g}{4!}\phi^4,$$

and the two-point function is given by

$$Z^{(2)} = \frac{\int \mathrm{d}\phi\,\phi^2 \exp\left[-\mathcal{S}(\phi)\right]}{\int \mathrm{d}\phi\, \exp\left[-\mathcal{S}(\phi)\right]} = 1 - \frac{g}{2} + \frac{2}{3}g^2 + O\left(g^3\right),$$

in which the expressions correspond to ordinary one-variable integrals.

The sum rules are satisfied. For example, at order g^2

$$\frac{2}{3} = \frac{1}{4} + \frac{1}{4} + \frac{1}{6}.$$

The four-point function. The expansion of the four-point function to order g^2 is

$$\begin{aligned}\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle &= [\,(\mathrm{a})_{12}\ (\mathrm{a})_{34}\ + 2\,\text{terms}\,] - \frac{g}{2}[\,(\mathrm{a})_{12}\ (\mathrm{b})_{34}\\ &+5\,\text{terms}\,] - g\,(\mathrm{f}) + g^2\left\{(\mathrm{a})_{12}\left[\frac{1}{4}\left((\mathrm{c})_{34} + (\mathrm{d})_{34}\right) + \frac{1}{6}(\mathrm{e})_{34}\right] + 5\ \text{terms}\right\}\\ &+ \frac{g^2}{4}[\,(\mathrm{b})_{12}\ (\mathrm{b})_{34}\ + 2\,\text{terms}\,] + \frac{g^2}{2}[(\mathrm{g}) + 3\ \text{terms}\,] + \frac{g^2}{2}[(\mathrm{h}) + 2\ \text{terms}\,] + O\left(g^3\right).\end{aligned}$$

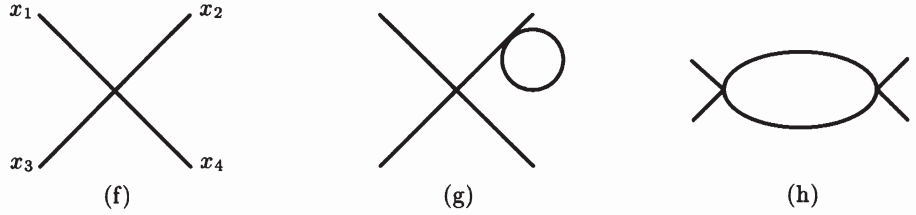


Fig. 7.3 New Feynman diagrams contributing to the four-point function.

The new diagrams (f), (g), (h) are displayed in figure 7.3. The notation for example $(\mathrm{a})_{12}$ means the diagram (a), contributing to the two-point function, with arguments x_1 and x_2. Finally, the additional terms are obtained by exchanging the external arguments to restore the permutation symmetry of the four-point function.

Note that the graphs which involve the two-point functions are disconnected, that is, factorize into a product of functions depending on disjoint subsets of variables. The origin of this phenomenon has already been indicated in Section 1.2.1. The question will be discussed further starting with Section 7.4.

Again, as for the two-point function, we have omitted disconnected diagrams in which one factor has no external arguments. As one can check directly here, and as the general arguments in Section 7.4 will confirm, these diagrams are cancelled by the perturbative expansion of $\mathcal{Z}(J=0)$ in expression (7.11). The diagrams contributing to $\mathcal{Z}(0)$ (the partition function) are called vacuum diagrams.

A final remark: local interaction terms may also involve derivatives of the field $\phi(x)$. Then in expression (7.23) derivatives of the propagator appear. The representation in terms of the Feynman diagrams given above is no longer faithful since it does not indicate where the derivatives are. A more faithful representation can be obtained by splitting points at the vertices and putting arrows on lines.

7.3 Algebraic Properties of Functional Integrals. Field Equations

Functional integrals: perturbative definition. We have seen that path integrals sometimes are ill defined, and a limiting procedure has to be supplied to lift ambiguities. We will discover in Chapter 9 that the problem is even more severe in local quantum field theories for functional integrals. We have exhibited in Section 7.1.2 a possible regularization by a space time lattice. However, this regularization complicates perturbative calculations enormously, and in Chapter 9 more practical regularizations will be proposed. This will lead to a consistency problem: some regularization schemes like dimensional regularization have no interpretation beyond perturbation theory. It is then no longer obvious whether identities proven using standard properties of functional integrals remain valid after regularization. To avoid these difficulties, one can take expression (7.22) as a proper, though perturbative, definition of the functional integral. In particular, one may then avoid the reference to a discretized space and time. However, with this new definition, one has to prove that the usual transformations performed on standard integrals lead to identities that are also true with such a perturbative definition.

The proofs rely on simple but powerful techniques which simultaneously allow a derivation of important identities satisfied by correlation functions like Dyson–Schwinger equations, a form of the quantum field equations when expressed in terms of correlation functions. Finally, by repeating the formal manipulations involved in these derivations,