

Nature of the Griffiths Phase

A. J. Bray^(a)*Schlumberger-Doll Research, Ridgefield, Connecticut 06877*

(Received 7 May 1987)

Arguments are given that, for random spin systems, the density of states $\rho(\mu)$ of the inverse of the susceptibility matrix vanishes as $\rho(\mu) \sim \exp(-A/\mu)$, for $\mu \rightarrow 0$, throughout the "Griffiths phase." The amplitude A vanishes at the onset of magnetic long-range order, and diverges at the transition between "Griffiths" and "paramagnetic" phases. For an $O(m)$ spin system, with $m \rightarrow \infty$, the spin autocorrelation function $C(t)$ is found to have the "stretched-exponential" form, $\ln C(t) \sim -(At)^{1/2}$, in the Griffiths phase.

PACS numbers: 75.40.-s, 64.60.Fr, 75.10.Hk

A dilute ferromagnet is in the "Griffiths phase"¹ if its temperature T is between the critical temperature $T_c(p)$ for the onset of magnetic long-range order and the critical temperature $T_c(1)$ of the pure (i.e., nondilute) system. The concept of the Griffiths phase is, however, more general. A random magnetic system, specified by a particular bond probability distribution, is in its Griffiths phase if it is above its own particular ordering temperature T_c , but below the highest ordering temperature T_G allowed by the distribution.² (In particular, $T_G = \infty$ for unbounded distributions.)

In this Letter I derive, using heuristic arguments, a seemingly quite general result for the density of states (i.e., eigenvalue spectrum) $\rho(\mu)$ of the inverse χ^{-1} of the susceptibility matrix χ in the Griffiths phase. The matrix χ describes the response of a spin system $\{S_i\}$ to external fields $\{h_i\}$: $\chi_{ij} = \partial \langle S_i \rangle / \partial h_j$. Its eigenvalues must be non-negative for thermodynamic stability. For a *pure* (i.e., translationally invariant) system, the eigenvectors of χ^{-1} are plane waves: The vanishing of the smallest eigenvalue signals the onset of long-range order at T_c , while the corresponding eigenvector specifies (with some exceptions³) the nature of the order. By contrast, for a *random* system it has been argued² that, in the thermodynamic limit, $\rho(\mu)$ has weight at arbitrarily small μ for all T in the range $T_c < T < T_G$. However, the corresponding eigenvectors must be *localized*^{2,4}: As $T \rightarrow T_c$, the "mobility edge" separating localized and extended states finally reaches $\mu = 0$.^{2,4}

The above result for $\rho(\mu)$ is a manifestation of the "Griffiths singularities"⁵ exhibited by the free energy for $T_c < T < T_G$. Physically, these are a consequence of the occurrence, with finite probability per site, of arbitrarily large regions which locally are characteristic of a system with a higher transition temperature $T'_c > T$. (For a dilute ferromagnet, these could be regions of fully occupied sites/bonds.) By estimating, at varying levels of sophistication, both the probability for such a region to occur, and its contribution to $\rho(\mu)$, and summing over regions of all sizes, we find the generic form $\rho(\mu) \sim \exp(-A/\mu)$, for $\mu \rightarrow 0$. The amplitude A is a function of the parameters (e.g., p , T) describing the system. In general, I ex-

pect (and obtain explicitly below) that $A \rightarrow 0$ for $T \rightarrow T_c$, since² $\rho(\mu)$ should be an algebraic function of μ for $T = T_c$, while $A \rightarrow \infty$ for $T \rightarrow T_G$, since $\rho(\mu)$ should have a gap [$\rho(\mu) = 0$ for $\mu < \mu_c(T)$] for $T > T_G$.

Griffiths singularities have important consequences for the *dynamics* of the system.^{1,6} In particular, relaxation is *nonexponential* for $T_c < T < T_G$. Heuristic arguments to this effect, based on the (free) energetics of domain walls, have been given for dilute Ising ferromagnets⁶ and spin-glasses.¹ Below we note that for vector spin systems, the dynamics (with no conservation laws, i.e., "model A" of Hohenberg and Halperin⁷) can be solved exactly, in the limit of infinite-dimensional spins, in terms of the statics. This gives, for the spin autocorrelation function $C(t)$, the result $\ln C(t) \sim -(At)^{1/2}$ for long times, i.e., a "stretched-exponential" form.

I now present the arguments which lead to these results. For simplicity, I restrict explicit computations to dilute ferromagnets, although generalizations to other systems are straightforward in principle. Let p be the site-occupation probability and d the spatial dimension. The probability $\text{Pr}(L)$ that a given site belongs to a "compact" cluster of occupied sites, with linear dimension L , is of order $\exp[-c(p)L^d]$, with $c(p) = \ln(1/p)$, where we have neglected a prefactor which depends algebraically on L . [We will systematically neglect such prefactors in the following: they lead only to algebraic prefactors in $\rho(\mu)$.] According to finite-size scaling,⁸ the susceptibility per site of such a region, for L large and T close to $T_G = T_c(1)$, has the form

$$\chi(L) = L^d (T_G - T)^{2\beta f} L^{1/\nu} (T_G - T),$$

i.e., $\chi(L) \sim L^d m^2$ for $L \gg \xi$, where m , ξ are the magnetization per site and correlation length, respectively, of the *pure* system at the given temperature. The physical origin of the weight near the origin in $\rho(\mu)$ is the existence of eigenvectors localized on (or near) such clusters. A trial eigenvector of the susceptibility matrix χ , having elements $L^{-d/2}$ for sites in the cluster, and zero for other sites, yields the eigenvalue estimate $\mu^{-1} = L^{-d} \sum_{i,j} \chi_{ij} = \chi(L) \sim L^d m^2$ for $L \gg \xi$ (recall that $\{\mu\}$ are the eigenvalues of χ^{-1}). Thus the contribution of a compact clus-

ter of linear dimension L to $\rho(\mu)$ is an eigenvalue $\mu_{\min} \sim L^{-d} m^{-2}$. Of course, there are other states localized on the cluster (an isolated cluster of L^d sites contributes L^d eigenvalues in total) but it can be shown that these do not contribute to the *leading* behavior of $\rho(\mu)$ for $\mu \rightarrow 0$ (modifying prefactors only), in much the same way as higher states bound in deep wells do not affect the leading behavior of the "Lifshitz tail" in the density of states for a particle in a random potential.⁹ Note the variational nature of these arguments: Exact localized states associated with compact clusters presumably have eigenvectors which extend somewhat beyond the edge of the clusters. Also, it may pay (see below) to consider large regions which are characteristic of a system with a higher mean occupation probability p' (and hence a higher, but not maximal, bulk T_c) rather than regions of completely occupied sites. For both of the above reasons, the derived density of states will be a *lower bound* on the true density of states.

Putting together the probability per site $\text{Pr}(L)$ for a large compact cluster to occur, with the smallest eigenvalue $\mu_{\min}(L)$ which such a cluster produces, yields for the (normalized) density of states

$$\begin{aligned} \rho(\mu) &\sim \sum_L \text{Pr}(L) \delta(\mu - \mu_{\min}(L)) \\ &\sim \sum_L \exp(-cL^d) \delta(\mu - L^{-d} m^{-2}) \\ &\sim \exp(-c/m^2 \mu), \quad \mu \rightarrow 0, \end{aligned} \quad (1)$$

$$\ln \text{Pr}(L, p') = -L^d \{p' \ln(p'/p) + (1-p') \ln[(1-p')/(1-p)]\} = -L^d f_p(p'). \quad (2)$$

Equation (2) gives correctly the extensive part of $\ln \text{Pr}$, which is adequate for present purposes. As expected, $\ln \text{Pr}(L, p')$ is maximal for $p' = p$, when the extensive part vanishes. For $p' = 1$, one recovers $f_p(1) = \ln(1/p) = c(p)$. To estimate the smallest eigenvalue associated with the cluster, we use the same trial eigenvector as before to obtain once more $\mu_{\min} \sim L^{-d} m^{-2}$, where now $m = m(p')$ is the magnetization per site of a bulk system with site-occupation probability p' . The best variational estimate for $\rho(\mu)$ is obtained by maximizing with respect to p' :

$$\begin{aligned} \rho(\mu) &\simeq \max_{p'} \sum_L \text{Pr}(L, p') \delta(\mu - \mu_{\min}(p', L)) \simeq \max_{p'} \sum_L \exp[-L^d f_p(p')] \delta(\mu - L^{-d} [m(p')]^{-2}) \\ &= \max_{p'} \exp\{-(1/\mu) f_p(p')/[m(p')]^2\} = \exp(-A/\mu), \end{aligned} \quad (3)$$

where

$$A = \min_{p'} \{f_p(p')/[m(p')]^2\}. \quad (4)$$

Hence the form $\rho(\mu) \sim \exp(-A/\mu)$ is recovered, but with an improved estimate for the amplitude A .

As a special case, consider a point (p, T) that is close to the boundary with the ferromagnetic phase, i.e., T is close to $T_c(p)$. Then we anticipate that the optimal value of p' will correspond to a point (p', T) which is close to (p, T) (but in the ferromagnetic phase). For this case, $f_p(p')$ and $m(p')$ can be expanded as $f_p(p') \propto (p' - p)^2$, $m(p') \propto (p' - p_c)^\beta$, where $p_c \equiv p_c(T)$ is the occupation probability corresponding to the phase boundary at temperature T [i.e., $p_c(T)$ is the inverse function of $T_c(p)$]. Thus, according to Eq. (4), we have

again neglecting algebraic prefactors.

Equation (1) has the general form $\rho(\mu) \sim \exp(-A/\mu)$, with $A = c(p)/m^2$. In particular, A diverges as expected [as $[T_c(1) - T]^{-2\beta}$], for $T \rightarrow T_G = T_c(1)$. The same type of argument can be used to estimate the form of $\rho(\mu)$ for $T = T_c(1)$. For this case, finite-size scaling yields $\chi(L) \sim L^{2-\eta}$, giving $\rho(\mu) \sim \exp(-c'/\mu^{d/(2-\eta)})$ for $\mu \rightarrow 0$. For T close to $T_c(1)$, crossover from this form to Eq. (1) occurs for $\mu \approx \mu_{cr} \propto [T_c(1) - T]^\gamma$ [where $\gamma = (2 - \eta)\nu$ is the usual susceptibility exponent], Eq. (1) holding for $\mu \ll \mu_{cr}$.

It should be emphasized again that, given the variational nature of the arguments leading to Eq. (1), this result gives only a lower bound on $\rho(\mu)$, although we expect the μ dependence ($\ln \rho \propto -1/\mu$) to be correct. In particular, while Eq. (1) gives the expected divergence of the amplitude A for $T \rightarrow T_c(1)$, it fails to predict that A vanishes for $T \rightarrow T_c(p)$. To rectify this shortcoming, we consider now a variational treatment based on large compact regions containing a fraction $p' > p$ of occupied sites. In order for such a region to produce an arbitrarily small eigenvalue for $L \rightarrow \infty$ it is necessary to choose a p' corresponding to a point in the ordered phase, i.e., such that $T_c(p') > T$. In fact we will choose p' variationally, to maximize $\rho(\mu)$. The probability (per site) $\text{Pr}(L, p')$ that a site belongs to a compact region of size L containing $p'L^d$ occupied sites is given by (dropping, as usual, power-law prefactors)

to minimize $g(p') = (p' - p)^2 / (p' - p_c)^{2\beta}$ with respect to p' . The minimum occurs for $p' - p_c = [\beta / (1 - \beta)] \times (p_c - p)$, justifying the assumption that $p' - p$ is small for p near p_c , while the value of g at the minimum is $g_{\min} \propto (p_c - p)^{2/(1-\beta)}$.¹⁰ In conclusion, the use of p' as an additional variational parameter preserves the result $\rho(\mu) \sim \exp(-A/\mu)$ obtained by using $p' = 1$, but in addition shows that A vanishes at the onset of ferromagnetic order [as $[p_c(T) - p]^{2/(1-\beta)}$ or, equivalently, as $[T - T_c(p)]^{2/(1-\beta)}$], as required on physical grounds.

Turning now to dynamics, we consider the spin autocorrelation function $C(t) = N^{-1} \sum_i \langle S_{ia}(t) S_{ia}(0) \rangle$, where the two subscripts are site ($i = 1, 2, \dots, N$) and spin ($a = 1, 2, \dots, m$) indices, and the angle brackets indicate a thermal average. Expanding $S_{ia}(t)$ in terms of the

eigenvectors v_μ of χ^{-1} , $S_{ia}(t) = \sum_\mu S_\mu v_\mu(i)$, gives $C(t) = N^{-1} \sum_\mu \langle S_\mu(t) S_\mu(0) \rangle$. For dynamics with no conservation laws (model A of Hohenberg and Halperin⁷), it can be shown that,¹¹ in the limit $m \rightarrow \infty$, each mode μ relaxes independently at a rate proportional to μ : $\langle S_\mu(t) S_\mu(0) \rangle \propto (1/\mu) \exp(-\mu t)$, where we have absorbed the kinetic coefficient into the time scale. Thus

$$\begin{aligned} C(t) &\propto N^{-1} \sum_\mu (1/\mu) \exp(-\mu t) \\ &\rightarrow \int_0^\infty d\mu [\rho(\mu)/\mu] \exp(-\mu t) \\ &\simeq \int_0^\infty (d\mu/\mu) \exp(-A/\mu - \mu t) \\ &\sim \exp\{-2(At)^{1/2}\}, \text{ as } t \rightarrow \infty, \end{aligned} \quad (5)$$

$$F = \frac{1}{2} [r + V(\mathbf{x})] \sum_a \phi_a^2(\mathbf{x}) + \frac{1}{2} \sum_a [\nabla \phi_a(\mathbf{x})]^2 + \frac{u}{4m} \sum_{a,b} \phi_a^2(\mathbf{x}) \phi_b^2(\mathbf{x}),$$

where $a, b = 1, 2, \dots, m$ and $V(\mathbf{x})$ is a "white-noise" random temperature fluctuation due to the disorder:

$$[V(\mathbf{x}) V(\mathbf{x}')]_{\text{av}} = 2v \delta(\mathbf{x} - \mathbf{x}'). \quad (6)$$

In the limit $m \rightarrow \infty$ the standard decoupling (or "Hartree") approximation⁴ becomes exact.² The matrix χ^{-1} becomes the operator

$$H = -\nabla^2 + r + V(\mathbf{x}) + U(\mathbf{x}), \quad (7)$$

where

$$U(\mathbf{x}) = u \langle \phi_a^2(\mathbf{x}) \rangle_T = u \sum_\mu \frac{1}{\mu} |\langle \mathbf{x} | \mu \rangle|^2, \quad (8)$$

$\langle \dots \rangle_T$ indicates a thermal average, and $\{\mu\}$, $\{|\mu\rangle\}$ are the eigenvalues and eigenfunctions of H . Equations (6)–(8) specify a nonlinear Anderson problem for which we require the density of states $\rho(\mu)$ at small "energy" μ .

Consider first the case $u = 0$. The density of states at low energy is given by the "Lifshitz argument"^{9,12}. Low-energy states are associated with unusually large negative fluctuations in the potential $V(\mathbf{x})$. The probability distribution for V is $P\{V\} \propto \exp\{-\int d\mathbf{x} V^2(\mathbf{x})/4v\}$. A fluctuation of depth $\simeq V_0$ extending over a region of size $\simeq l$ occurs with probability (per unit volume) $\simeq \exp\{-V_0^2 l^d/4v\}$, giving rise to a bound state with energy $\mu \simeq r - V_0 + 1/l^2$ and a density of states $\rho(\mu) \simeq \exp\{-(r - \mu + 1/l^2)^2 l^d/4v\}$. Choosing l variationally

where the final result follows on evaluation of the μ integral by steepest descents for large t . An algebraic prefactor of the form t^{-x} , obtained from carrying the steepest descent calculation to higher order, has been omitted from Eq. (5), since the value of x would be changed by the algebraic prefactors in $\rho(\mu)$ which we have omitted. Equation (5) has the "stretched-exponential" (or "Kohlrausch") form. For $T = T_G$, $\ln \rho(\mu) \sim -1/\mu^{d/(2-\eta)}$ implies $\ln C(t) \sim -t^{d/(d+2-\eta)}$. For T just below T_G , crossover to Eq. (1) occurs for $t \gg t_{cr} \simeq (T_G - T)^{-2\Delta}$, with $\Delta = \beta + \gamma = (v/2)(d+2-\eta)$.

The large- m limit may also provide the key to a more systematic treatment of the statics, since in this limit the determination of the spectrum of χ^{-1} reduces to a self-consistent Anderson problem. Specifically we consider a "Ginzburg-Landau-Wilson" (or " ϕ^4 ") theory, in the continuum limit, with free-energy density

to maximize ρ yields $l \simeq 1/(r - \mu)^{1/2}$ and $\ln \rho(\mu) \sim -(1/v)(r - \mu)^{(4-d)/2}$ for $(r - \mu) \rightarrow \infty$, provided that $d < 4$. More systematic studies⁹ confirm this form and, in addition, obtain the power-law prefactors as well as the correct amplitude in the exponent.

For the nonlinear problem ($u > 0$) there are no negative eigenvalues. Instead the "Lifshitz tail" is moved to the vicinity of $\mu = 0$. A variant of the Lifshitz argument for this case proceeds as follows. Very small eigenvalues are again associated with large, rare, negative fluctuations in V [these correspond to regions with a locally high T_c (i.e., r_c)]. Suppose $V(\mathbf{x}) \simeq -V_0$ over a region of size $\simeq l$ centered around the point \mathbf{x}_0 , leading to a low-lying state localized in this region. To keep the corresponding eigenvalue positive, there must be a compensating large, positive contribution from $U(\mathbf{x})$: The term in (8) associated with the smallest eigenvalue μ_0 with eigenfunction $|\mu_0\rangle$ localized near \mathbf{x}_0 must provide a large contribution to the sum for \mathbf{x} near \mathbf{x}_0 . We will assume that in this region the remaining terms sum to (roughly) a constant which we will absorb into r . Since $|\mu_0\rangle$ has spatial extent $\simeq l$, $|\langle \mathbf{x} | \mu_0 \rangle| \simeq l^{-d/2}$ for $|\mathbf{x} - \mathbf{x}_0| < l$. Hence $\mu_0 \simeq r - V_0 + 1/l^2 + u/\mu_0 l^d$, giving a density of states $\rho(\mu) \simeq \exp[-(r - \mu + 1/l^2 + u/\mu l^d)^2 l^d/4v]$. Again, this has to be extremized with respect to the "localization length" l . The result has the form (we consider $\mu < r$ only, otherwise we are not sampling the tail of the distribution)

$$\ln \rho(\mu) \simeq -(1/v)(r - \mu)^{(4-d)/2} f((u/\mu)(r - \mu)^{(d-2)/2}) \quad (9)$$

$$\sim \begin{cases} -(1/v)(r - \mu)^{(4-d)/2}, & (u/\mu)(r - \mu)^{(d-2)/2} \ll 1, \\ -(u/v)(r - \mu)/\mu, & (u/\mu)(r - \mu)^{(d-2)/2} \gg 1, \end{cases} \quad (10)$$

for $2 < d < 4$.¹³ Equation (9) is the result of the linear theory discussed above. Equation (10) shows that for $\mu \rightarrow 0$,

one recovers the generic form $\rho(\mu) \sim \exp(-A/\mu)$, with $A \propto ur/v$. Again, A vanishes at the onset ($r=0$) of long-range order, as expected. In contrast to the dilute lattice models discussed above, however, the Griffiths phase extends to infinite temperature (i.e., to $r=\infty$) in this model, because the local temperature fluctuations are unbounded.

Note that the localization length $l(\mu)$ (i.e., the value of l which dominates the computation of ρ for given μ) first decreases with decreasing μ , $l \simeq (r-\mu)^{-1/2}$, when (9) applies, then increases, $l \simeq \{u/\mu(r-\mu)\}^{1/d}$, when (10) applies. Thus there is a *maximum localization length* for a given temperature. Physically, this length should be the correlation length,¹⁴ and this is indeed found¹¹ in the present model, where we obtain $l_{\min} \simeq 1/\sqrt{r}$ in the "classical regime," $\mu/r^{(4-d)/2} \ll 1$, and $l_{\min} \simeq (u/r)^{1/(d-2)} \propto r^{-\nu}$ in the "critical regime," $u/r^{(4-d)/2} \gg 1$.

In summary, a number of arguments show that the density of states of the inverse susceptibility matrix has the generic form $\rho(\mu) \sim \exp(-A/\mu)$ in the Griffiths phase. For $O(m)$ models with $m \rightarrow \infty$, the autocorrelation functions have a "stretched-exponential" form. In my view, the $m=\infty$ model provides a potentially useful starting point for a field-theoretic derivation¹⁵ of $\rho(\mu)$ in the Griffiths phase, following the approach used to obtain the Lifshitz tail in the linear theory.¹⁶

It is a pleasure to thank B. I. Halperin for a very helpful discussion.

^(a)Permanent address: Department of Theoretical Physics,

The University of Manchester, Manchester M139PL, United Kingdom.

¹M. Randeria, J. P. Sethna, and R. G. Palmer, Phys. Rev. Lett. **54**, 1321 (1985).

²A. J. Bray and M. A. Moore, J. Phys. C **15**, L765 (1982).

³E.g., spin-glasses.

⁴J. A. Hertz, L. Fleishman, and P. W. Anderson, Phys. Rev. Lett. **43**, 942 (1979).

⁵R. B. Griffiths, Phys. Rev. Lett. **23**, 17 (1969).

⁶D. Dhar, in *Stochastic Processes: Formalism and Applications*, edited by G. S. Agarwal and S. Dattagupta (Springer, Berlin, 1983).

⁷P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. **49**, 435 (1977).

⁸See, e.g., M. N. Barber, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic, New York, 1983), Vol. 8, and references therein.

⁹B. I. Halperin and M. Lax, Phys. Rev. **148**, 722 (1966); J. Zittartz and J. S. Langer, Phys. Rev. **148**, 741 (1966).

¹⁰Note that β is here the order-parameter exponent of the dilute system.

¹¹A. J. Bray, to be published.

¹²I. M. Lifshitz, Zh. Eksp. Teor. Fiz. **44**, 1723 (1963) [Sov. Phys. JETP **17**, 1159 (1963)], and Adv. Phys. **13**, 483 (1964), and Nuovo Cimento **3**, 716 (1956), and Usp. Fiz. Nauk **83**, 617 (1964) [Sov. Phys. Usp. **7**, 549 (1965)].

¹³The $m=\infty$ ferromagnet has no phase transition for $d \leq 2$.

¹⁴I.e., the "range" of χ_{ij} , or of $\chi(\mathbf{x}, \mathbf{x}') = \langle \mathbf{x} | H^{-1} | \mathbf{x}' \rangle$.

¹⁵The related problem of the density of zeros of the partition function in the Griffiths phase has been addressed using field theory by J. L. Cardy and A. J. McKane, Nucl. Phys. **B257**, 383 (1985).

¹⁶J. L. Cardy, J. Phys. C **11**, L321 (1978).