Relaxation of Disordered Magnets in the Griffiths' Regime

F. Cesi [1], C. Maes [2] and F. Martinelli [3]

[1] Dipartimento di Fisica, Università "La Sapienza", P.le A. Moro 2, 00185 Roma, Italy
e-mail: cesi@vaxrom.roma1.infn.it

[2] Instituut voor Theoretische Fysika, K.U. Leuven, Celestijnenlaan 200D B-3001 Leuven, Belgium and Onderzoeksleider N.F.W.O, Belgium
e-mail: Christian.Maes@fys.kuleuven.ac.be

[3] Dipartimento di Energetica, Università dell' Aquila, Italy
e-mail: martin@mat.uniroma3.it

Abstract. We study the relaxation to equilibrium of discrete spin systems with random many–body (not necessarily ferromagnetic) interactions in the Griffiths' regime. We prove that the speed of convergence to the unique reversible Gibbs measure is almost surely faster than any stretched exponential, at least if the probability distribution of the interaction decays faster than exponential (e.g. Gaussian). Furthermore, if the interaction is uniformly bounded, the *average over the disorder* of the time–autocorrelation function, goes to equilibrium as $\exp[-k(\log t)^{d/(d-1)}]$ (in $d > 1$), in agreement with previous results obtained for the dilute Ising model.

Key Words: Random spin systems, Glauber dynamics, relaxation time, Griffiths singularities, logarithmic Sobolev inequalities

---

Work partially supported by grant CHRX-CT93-0411 of the Commission of European Communities

Mathematics Subject Classification: 82B44, 82C22, 82C44, 60K35

1. Introduction

In the present paper we study the speed of convergence to equilibrium of a single spin–flip stochastic dynamics with a reversible Gibbs measure with *random* interactions in the so called *Griffiths' phase.*

For simplicity (our models are introduced in the next section) consider here a nearest–neighbor Ising model on the d-dimensional lattice with coupling coefficients $J = \{J_{xy}\}$. The $\{J_{xy}\}$, are independent and identically distributed real–valued random variables.

If the $\{J_{xy}\}$ are uniformly bounded, then at all sufficiently high temperatures Dobrushin's uniqueness theory applies and detailed information about the unique Gibbs measure and the relaxation to equilibrium of an associated Glauber dynamics are available using the concept of complete analyticity [DS], [SZ], [MO1] and [MO2]. This regime is usually referred to as the *paramagnetic* phase.

There is then a range of temperatures, below the paramagnetic phase, where, even if the Gibbs state is unique, certain characteristics of the paramegnetic phase like the analyticity of the free energy as a function of the external field disappear. This is the so called *Griffiths' regime* [G] (see also [F] for additional discussion on this and many other related topics). This "anomalous behavior" is caused by the presence of arbitrarily large clusters of bonds associated with "strong" couplings J_{xy}, which can produce a long–range order inside the cluster. Even above the percolation threshold, *i.e.* when one of such clusters is infinite, there may be a Griffiths phase, at temperatures between a certain critical temperature T_c(disordered) and the critical temperature for the "pure system" (*i.e.* the system with "strong" couplings everywhere on $\mathbb{Z}^d$). What happens is that for almost all realizations of the disorder J and for all sitex x there is a finite length scale $l(J, x)$, such that correlations between $\sigma(x)$ and $\sigma(y)$ start decaying exponentially at distances greater than $l(J, x)$.

In [BD] an "elementary" approach was given to the problem of uniqueness of the equilibrium state of disordered systems in the Griffiths regime (see also [FI]). In another paper [D] Dobrushin prepared the mathematical background for the study of (arbitrary order) truncated correlation functions for spin glasses. Bounds on T_c(disordered) have been obtained in [ACCN] and [OPG]. More recent references where, at least for the statics, the situation has been considerably cleared up are [DKP], [GM2], [GM3] and [Be]. In particular, under suitable conditions on the couplings distribution, one proves that the infinite volume Gibbs state is unique with probability one and the static correlation functions decay exponentially fast uniformly in the size of the system and its boundary conditions. Though the system is not completely analytic there is still sufficient *local* analyticity to ensure many high temperature properties.

The effect of the Griffiths' singularities on the dynamical properties are much more serious since, as we will see, the long time behaviour of any associated Glauber dynamics is dominated by the islands of strongly coupled spins produced by large statistical fluctuations in the disorder (see e.g. [B1] and [B2]).

Consider the usual stochastic (or kinetic) Ising model associated to the model discussed above. It is a stochastic spin flip dynamics for which the (almost sure unique) Gibbs state is a reversible measure. Let us denote by $q(J,t)$ the absolute difference between the expectation at time t of e.g. the spin at the origin starting in some initial state, and its equilibrium value.

Up to now only little rigorous information was available about the long time behaviour in the Griffiths' regime of $q(J,t)$ or about its more physically relevant disorder–average $q(t)$. The first rigorous result was obtained in [Z] where the absence of a gap in the spectrum of the Markov generator was proven. This result, in particular, rules out the possibility that $q(J,t)$ decays exponentially fast in t with probability one with a rate independent of J.

In [GM1], for quite general models, an almost sure upper bound of the form

$$q(J,t) \leq c(J)\exp[-\lambda(\log t)^{\nu}], \qquad \nu > 1$$

(faster than any polynomial) was derived. This was further improved for continuous spin systems in [GZ1], [GZ2] to almost sure upper bounds of the form

$$q(J,t) \leq c(J)\exp[-\lambda t^{\delta}], \qquad \delta < 1$$

in $d \leq 2$ dimensions (stretched exponential), and

$$q(J,t) \leq c(J)\exp\big[-\epsilon\exp[\,\theta(\log t)^{1/(d-1)^2}\,]\big]$$

(subexponential) for dimensions $d \geq 3$ and *ferromagnetic* interactions. We refer to the original papers for the precise statement of the results and the models. We only observe that in [GZ1] the probability distribution has only exponential tails and that, in this case, the stretched exponential bound cannot be improved in general (see the remark after Theorem 3.1).

One of the main results of the present paper is the proof that the *almost sure* decay is in fact faster than stretched exponential at least for random interaction whose probability distribution decays faster than exponential. In particular, for bounded interactions, we show that

$$q(J,t) \leq c(J)\,\exp\Big[-t\,\exp\big[-k\,(\log t)^{1-\frac{1}{d}}\,(\log\log t)^{d-1}\big]\Big]$$

Our general assumption, called (H1), (see Section 3 for a precise statement) is that in a cube of side length L, the correlation between $\sigma(x)$ and $\sigma(y)$ start decaying exponentially fast if, $|x-y|$ is greater than, say, $L/2$, with probability (w.r.t the disorder) at least $1-\exp(-cL)$.

We show that this assumption is indeed implied by the assumptions used in [GM2] and [DKP]. In a related paper [CMM] we prove that, at least for the two dimensional diluted Ising model, (H1) holds in a wider region of the Griffiths' phase which extends above the percolation threshold.

We also analyze the average (over the disorder or spatially) $q(t)$ of $q(J,t)$ and prove both upper and lower bounds of the form

$$\exp[-\lambda_1(\log t)^{d/(d-1)}] \le q(t) \le \exp[-\lambda_2(\log t)^{d/(d-1)}(\log\log t)^{-d}] \tag{1.1}$$

for suitable constants λ_1, λ_2. This result agrees with the predictions of [B1], [B2], [DRS] and [RSP] for dilute Ising magnets. Recent computer simulations [J] suggest that behavior in (1.1) is attained only when $q(t)$ is extremely small, while, at intermediate times $q(t)$ is better fitted with a stretched exponential. We hope to come back to this problem in the next future.

Although we consider only discrete spin systems we don't see any serious obstacle to extend our upper bounds to compact continuous spins. The lower bound, on the other side, could be very different if one considers, for instance, the Heisenberg model (see our comment at the beginning of Section 4.3).

The basic questions treated in this paper can of course also be put in the general context of interacting particle systems (with random interactions). But there no reversibility is guaranteed and the methods of this paper or of [GZ1], [GZ2] fail in that case. The only general results so far are contained in [GM1] but they are believed to be far from optimal.

The paper is organized as follows. Section 2 contains the definition of our models.
Section 3 contains the statements of the main results.
Section 4 contains several technical tools that are essential for section 5. In particular we prove that, if the interactions are bounded, then the relaxation time in an arbitrary set V does not grow faster than a exponential of the "surface " $|V|^{\frac{d-1}{d}}$.
Section 5 represents the core of the paper. We prove a rather sharp deterministic upper bound on the logarithmic Sobolev constant for the finite volume Gibbs measure.
Section 6 is devoted to the proof of the main results.
Appendix 1 contains the proof of a key geometric bound.

Acknowledgements. The authors are grateful to the Schrödinger Institute in Wien for the kind hospitality and the opportunity to start this work. A particular thank goes to M. Zahradník for suggesting the main idea in the proof of the geometric proposition contained in the Appendix 1.

2. The model

The lattice. We consider the d dimensional lattice $\mathbb{Z}^d$ with *sites* $x = \{x_1, \dots, x_d\}$ and norm

$$|x| = \max_{i \in \{1,\dots,d\}} |x_i|$$

The associated distance function is denoted by $d(\cdot,\cdot)$. By Q_L we denote the cube of all $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$ such that $x_i \in \{0, \dots, L-1\}$. If $x \in \mathbb{Z}^d$, $Q_L(x)$ stands for $Q_L + x$. We also let B_L be the ball of radius L centered at the origin, *i.e.* $B_L = Q_{2L+1}((-L, \dots, -L))$.

A finite subset Λ of $\mathbb{Z}^d$ is said to be a *multiple* of Q_L if Λ is the union of a finite number of cubes $Q_L(x_i)$ where $x_i \in L\mathbb{Z}^d$.

If Λ is a finite subset of $\mathbb{Z}^d$ we write $\Lambda \subset\subset \mathbb{Z}^d$. The cardinality of Λ is denoted by $|\Lambda|$. $\mathbb{F}$ is the set of all nonempty finite subsets of $\mathbb{Z}^d$. We define the exterior n–*boundary* as $\partial_n^+ \Lambda = \{x \in \Lambda^c : d(x, \Lambda) \le n\}$. Given $r \in \mathbb{Z}_+$ we say that a subset V of $\mathbb{Z}^d$ is r-*connected* if, for all $y, z \in V$ there exist $\{x_1, \dots, x_n\} \subset V$ such that $x_1 = y$, $x_n = z$ and $|x_{i+1} - x_i| \le r$ for $i = 2, \dots, n$.

The configuration space. Our *configuration space* is $\Omega = S^{\mathbb{Z}^d}$, where $S = \{-1, 1\}$, or $\Omega_V = S^V$ for some $V \subset \mathbb{Z}^d$. The single spin space S is endowed with the discrete topology and Ω with the corresponding product topology. Given $\sigma \in \Omega$ and $\Lambda \subset \mathbb{Z}^d$ we denote by σ_Λ the natural projection over Ω_Λ. If U, V are disjoint, $\sigma_U \eta_V$ is the configuration on $U \cup V$ which is equal to σ on U and η on V.

If f is a function on Ω, Λ_f denotes the smallest subset of $\mathbb{Z}^d$ such that $f(\sigma)$ depends only on σ_{Λ_f}. f is called *local* if Λ_f is finite. $\mathcal{F}_\Lambda$ stands for the $\sigma-$algebra generated by the set of projections $\{\pi_x\}$, $x \in \Lambda$, from Ω to $\{-1, 1\}$, where $\pi_x : \sigma \mapsto \sigma(x)$. When $\Lambda = \mathbb{Z}^d$ we set $\mathcal{F} \equiv \mathcal{F}_{\mathbb{Z}^d}$ and $\mathcal{F}$ coincides with the Borel $\sigma-$algebra on Ω with respect to the topology introduced above. By $\|f\|_\infty$ we mean the supremum norm of f. The *gradient* of a function f is defined as

$$(\nabla_x f)(\sigma) = f(\sigma^x) - f(\sigma)$$

where $\sigma^x \in \Omega$ is the configuration obtained from σ, by flipping the spin at the site x. If $\Lambda \in \mathbb{F}$ we let

$$|\nabla_\Lambda f|^2 = \sum_{x \in \Lambda} (\nabla_x f)^2$$

We also define

$$|||f||| = \sum_{x \in \mathbb{Z}^d} \|\nabla_x f\|_\infty$$

The interaction and the Gibbs measures. We consider an abstract probability space $(\Theta, \mathcal{B}, \mathbb{P})$ and a set of real valued random variables $J = \{J_A\}$ with $A \in \mathbb{F}$, with the properties

(1) J_A and J_B are independent if $A \neq B$
(2) J_A and J_{A+x} are identically distributed for all $A \in \mathbb{F}$ and all $x \in \mathbb{Z}^d$
(3) There exists $r > 0$ such that with $\mathbb{P}-$probability 1, $J_A = 0$ if $\operatorname{diam} A > r$. r is called the *range* of the interaction.

The expectation with respect to $\mathbb{P}$ is denoted by $\mathbb{E}(\cdot)$. For $x \in \mathbb{Z}^d$, we let

$$\|J\|_x \equiv \sum_{A \ni x} |J_A|$$

We write $\|J\|_V = \sup\{\|J\|_x : x \in V\}$. Given a *potential* or *interaction* J, and $V \in \mathbb{F}$ we define the Hamiltonian $H_V^J : \Omega \mapsto \mathbb{R}$ by

$$H_V^J(\sigma) = - \sum_{A:\, A \cap V \neq \emptyset} J_A \prod_{x \in A} \sigma(x)$$

For $\sigma, \tau \in \Omega$ we also let $H_V^{J,\tau}(\sigma) = H_V^J(\sigma_V \tau_{V^c})$ and τ is called the *boundary condition*. For each $V \in \mathbb{F}$, $\tau \in \Omega$ the (finite volume) conditional Gibbs measure on $(\Omega, \mathcal{F})$, are given by

$$\mu_V^{J,\tau}(\sigma) = \begin{cases} \left(Z_V^{J,\tau}\right)^{-1} \exp[-H_V^{J,\tau}(\sigma)] & \text{if } \sigma(x) = \tau(x) \text{ for all } x \in V^c \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

where $Z_V^{J,\tau}$ is the proper normalization factor called partition function. We will sometimes drop the superscript J if that does not generate confusion. Given a measurable bounded function f on Ω, $\mu_V f$ denotes the function $\sigma \mapsto \mu_V^\sigma(f)$. Analogously, if $X \in \mathcal{F}$, $\mu_V(X) \equiv \mu_V \mathbb{I}_X$, where $\mathbb{I}_X$ is the characteristic function on X. $\mu(f, g)$ stands for the covariance (with respect to μ) of f and g. The set of measures (2.1) satisfies the DLR compatibility conditions

$$\mu_\Lambda(\mu_V(X)) = \mu_\Lambda(X) \qquad \forall\, X \in \mathcal{F} \qquad \forall\, V \subset \Lambda \subset\subset \mathbb{Z}^d \tag{2.2}$$

A probability measure μ on $(\Omega, \mathcal{F})$ is called a *Gibbs measure* for J if

$$\mu(\mu_V(X)) = \mu(X) \qquad \forall\, X \in \mathcal{F} \qquad \forall\, V \in \mathbb{F} \tag{2.3}$$

Given any two measures μ and ν on $(\Omega, \mathcal{F})$, and given $V \in \mathbb{F}$ such that for, each $X \in \mathcal{F}_V$, $\nu(X) = 0$ implies $\mu(X) = 0$, we define the $\mathcal{F}_V-$measurable function

$$\left.\frac{d\mu}{d\nu}\right|_V : \eta \mapsto \frac{\mu\{\sigma \in \Omega : \sigma_V = \eta_V\}}{\nu\{\sigma \in \Omega : \sigma_V = \eta_V\}} \qquad \eta \in \Omega \tag{2.4}$$

where $0/0$ means 0. We have, of course,

$$\mu(f) = \nu(f \left.\frac{d\mu}{d\nu}\right|_V) \qquad \forall f \in \mathcal{F}_V \tag{2.5}$$

The dynamics. The stochastic dynamics we want to study is determined by the Markov generators L_V^J, $V \subset \mathbb{Z}^d$, defined by

$$(L_V^J f)(\sigma) = \sum_{x\in V} c_J(x,\sigma)(\nabla_x f)(\sigma) \qquad \sigma \in \Omega \tag{2.6}$$

The nonnegative real quantities $c_J(x,\sigma)$ are the *transition rates* for the process.

The general assumptions on the transition rates are

(1) *Finite range interactions.* If $\sigma(y) = \sigma'(y)$ for all y such that $d(x,y) \le r$, then $c_J(x,\sigma) = c_J(x,\sigma')$

(2) *Detailed balance.* For all $\sigma \in \Omega$ and $x \in \mathbb{Z}^d$,

$$\exp\bigl[-H^J_{\{x\}}(\sigma)\bigr] c_J(x,\sigma) = \exp\bigl[-H^J_{\{x\}}(\sigma^x)\bigr] c_J(x,\sigma^x) \tag{2.7}$$

(3) *Positivity and boundedness.* There exist non–negative real numbers c_m, κ_1 c_M and κ_2 such that

$$c_m e^{-\kappa_1 \|J\|_x} \le \inf_{x,\sigma} c_J(x,\sigma) \quad \text{ and } \quad \sup_{x,\sigma} c_J(x,\sigma) \le c_M e^{\kappa_2 \|J\|_x} \tag{2.8}$$

Three cases one may want to keep in mind are

$$c_J(x,\sigma) = \min\bigl\{e^{-(\nabla_x H_{\{x\}})(\sigma)}, 1\bigr\} \tag{2.9}$$

$$c_J(x,\sigma) = \mu^{J,\sigma}_{\{x\}}(\sigma^x) = \Bigl[1 + e^{(\nabla_x H_{\{x\}})(\sigma)}\Bigr]^{-1} \tag{2.10}$$

$$c_J(x,\sigma) = \frac{1}{2}\Bigl[1 + e^{-(\nabla_x H_{\{x\}})(\sigma)}\Bigr] \tag{2.11}$$

Notice that the first two examples, corresponding to the Metropolis and heat–bath dynamics respectively, satisfy (2.8) with $\kappa_2 = 0$. When considering the infinite volume dynamics (see below) we will always assume uniformly bounded transition rates ($\kappa_2 = 0$), even if this assumption can be relaxed (see Theorem 2.3 in [GZ1]).

We denote by $L_V^{J,\tau}$ the operator L_V^J acting on $L^2(\Omega, d\mu_V^{J,\tau})$ (this amounts to choose τ as the boundary condition). Assumptions (1), (2) and (3) guarantee that there exists a unique Markov process whose generator is $L_V^{J,\tau}$, and whose semigroup we denote by $\{T_V^{J,\tau}(t)\}_{t\ge 0}$. $L_V^{J,\tau}$ is a bounded operator on $L^2(\Omega, d\mu_V^{J,\tau})$. The process has a unique invariant measure given by $\mu_V^{J,\tau}$. Moreover $\mu_V^{J,\tau}$ is *reversible* with respect to the process, *i.e.* $L_V^{J,\tau}$ is self–adjoint on $L^2(\Omega, d\mu_V^{J,\tau})$.

A fundamental quantity associated with the dynamics of a reversible system is the gap of the generator, *i.e.*

$$\text{gap}(L_V^{J,\tau}) = \inf \text{spec}\,(-L_V^{J,\tau} \restriction \mathbb{1}^\perp)$$

where $\mathbb{I}^\perp$ is the subspace of $L^2(\Omega, d\mu_V^{J,\tau})$ orthogonal to the constant functions. The gap can also be characterized as

$$\operatorname{gap}(L_V^{J,\tau}) = \inf_{f\in L^2(\Omega,d\mu_V^{J,\tau})} \frac{\mathcal{E}_V^\tau(f,f)}{\operatorname{Var}_V^{J,\tau}(f)} \tag{2.12}$$

where $\mathcal{E}$ is the Dirichlet form associated with the generator L,

$$\mathcal{E}_V^{J,\tau}(f,f) = \frac{1}{2}\sum_{\sigma\in\Omega_V}\sum_{x\in V}\mu_V^{J,\tau}(\sigma)\,c(x,\sigma)\,[(\nabla_x f)(\sigma)]^2 \tag{2.13}$$

and $\operatorname{Var}_V^{J,\tau}$ is the variance relative to the probability measure $\mu_V^{J,\tau}$. When the transition rates are chosen as in (2.11), it is easy to verify that the Dirichlet form takes a particularly simple form

$$\mathcal{E}_V^{J,\tau}(f,f) = \frac{1}{2}\,\mu_V^{J,\tau}(|\nabla_V f|^2) \tag{2.14}$$

We define the *logarithmic Sobolev constant* $c_s(L_V^{J,\tau})$ associated with the generator $L_V^{J,\tau}$ as the infimum over all c such that, for all positive functions f

$$\mu_V^{J,\tau}(f^2\log f) \le c\,\mathcal{E}_V^{J,\tau}(f,f) + \mu_V^{J,\tau}(f^2)\log\sqrt{\mu_V^{J,\tau}(f^2)} \tag{2.15}$$

We define $c_s(L_V^J) = \sup_\tau c_s(L_V^{J,\tau})$ and we denote by $c_s(\mu_V^{J,\tau})$ the logarithmic Sobolev constant corresponding to the rates (2.11) and associated Dirichlet form (2.14). Notice the following simple estimates relating $c_s(L_V^{J,\tau})$ to $c_s(\mu_V^{J,\tau})$:

$$c_m e^{-\kappa_1\|J\|_V} c_s(L_V^{J,\tau}) \le c_s(\mu_V^{J,\tau}) \le c_M e^{\kappa_2\|J\|_V} c_s(L_V^{J,\tau}) \tag{2.16}$$

The infinite volume dynamics. Let μ be a Gibbs measure for J. If the transition rates are bounded, *i.e.* when $\kappa_2 = 0$, then the infinite volume generator L^J obtained by choosing $V = \mathbb{Z}^d$ in (2.6) is well defined on the set of functions f such that $|||f|||$ is finite. The closure of L^J in $L^2(\Omega, d\mu)$ (or in $C(\Omega)$, the metric space of all continuous functions on Ω with the sup–distance) is a Markov generator (see, for instance Theorems 3.9 in Chapter I and 4.1 in Chapter IV of [L]), which defines a Markov semigroup denoted by $T(t)$. L^J is self–adjoint on $L^2(\Omega, d\mu)$.

The block dynamics. We will also consider a more general version of heat–bath dynamics in which more than one spin can flip at once. Let $\mathcal{D} = \{V_1,\dots,V_n\}$ be an arbitrary collection of finite sets $V_i \in \mathbb{F}$ and let $V = \cup_i V_i$. The generator of the Markov process corresponding to $\mathcal{D}$ is defined as

$$L_{\mathcal{D}}^J f = \sum_{i=1}^n (\mu_{V_i}^J f - f)$$

From the DLR condition (2.2) it follows that $L_{\mathcal{D}}^J$ is self–adjoint on $L^2(\Omega, d\mu_V^{J,\tau})$.

3. Main results

In this section we state our hypotheses and our main results on

(i) the growth of the logarithmic Sobolev constant in a cube of side L as a function of L,

(ii) the speed of relaxation to equilibrium for the infinite volume dynamics for a set of potentials J of measure one,

(iii) the speed of relaxation to equilibrium for the *averaged* infinite volume dynamics.

In order to state our hypotheses we need first the following definition. Given $V \subset\subset \mathbb{Z}^d$, $n, \alpha > 0$, we say that the condition $SMT(V, n, \alpha)$ holds if for all local functions f and g on Ω such that $d(\Lambda_f, \Lambda_g) \geq n$ we have

$$\sup_{\tau \in \Omega} |\mu_V^{J,\tau}(f, g)| \leq |\Lambda_f||\Lambda_g| \, \|f\|_\infty \|g\|_\infty \exp[-\alpha \, d(\Lambda_f, \Lambda_g)]$$

Then our hypotheses on the random interactions $J = \{J_A\}_{A \in \mathbb{F}}$ are:

(H1) There exist $L_0 \in \mathbb{Z}_+$, $\alpha > 0$, $\vartheta > 0$ such that for all $L \geq L_0$

$$\mathbb{P}\{ \, SMT(Q_L, L/2, \alpha) \, \} \geq 1 - e^{-\vartheta L}$$

(H2) There exists $\delta > 0$, such that $\mathbb{E}\,(\exp(\, \|J\|_x^{1+\delta} \,)) \equiv G_\delta < \infty$

Some of our result are given in the special case in which the J_A's are bounded, so we let

(H3) There exists $J_0 > 0$ such that with probability 1, we have $|J_A| \leq J_0$ for all $A \in \mathbb{F}$.

Remark. The key hypothesis (H1) is different from the assumption that appears in the basic references on disordered systems in the Griffiths phase (see e.g. [GM3], [DKP], [D], [FI]), [GZ1], [GZ2]). In these references, in fact, one sets a constraint on either the inverse temperature β or the external fields (if present). Here we adopt a more general hypothesis. In Section 3.2 we show that (H1) holds under the assumption of [GM3] or [DKP], while in [CMM] we study the two dimensional diluted Ising model above the percolation threshold.

As far as the second hypothesis is concerned, we observe that it is definitely stronger than the assumption one needs in order to control the equilibrium (see e.g. [DKP] or [GM3]). This fact is, as we hope it will be clear from the proofs of the various results, almost unavoidable when dealing with dynamical problems if one wants to get sufficiently precise results. Relaxation times are in fact much more sensitive than correlation functions to the occurrence of small regions of very large couplings (see also the remark after Theorem 3.1). In any case we have focused more on the bounded case (H3) since it appears to be the most interesting one from the physical point of view.

3.1. General theorems

Theorem 3.1.

(i) Assume (H1) and (H2) and let $A_\delta = (\log G_\delta)^{\frac{1}{1+\delta}} \vee 1$. Then there exist $C(d,r,\alpha,\vartheta)$, and $L_1(d,r,\alpha,\vartheta,L_0,A_\delta)$ such that for all $L \geq L_1$,

$$\mathbb{P}\big\{\, c_s(L^J_{Q_L}) > \exp\big[\, CA_\delta(\log\log L)^{d-\frac{\delta}{1+\delta}}(\log L)^{1-\frac{\delta}{d(1+\delta)}}\,\big]\,\big\} < L^{-1.5} \tag{3.1}$$

If, in addition, (H3) holds (bounded interactions) then there exist C_1 and C_2 depending on d, r, α and ϑ such that for all $L \geq L_1$,

$$\mathbb{P}\big\{\, c_s(L^J_{Q_L}) > C_1\exp\big[\, C_2 J_0\,(\log\log L)^{d-1}(\log L)^{\frac{d-1}{d}}\,\big]\,\big\} < L^{-1.5} \tag{3.2}$$

(ii) Assume (H1) and (H3) and $d \geq 2$. Then for any $\varepsilon \in (0,1]$ there exist positive constants C_3 and L_2 depending on d, r, α, ϑ, J_0 and ε such that for all $L \geq L_2$

$$\mathbb{P}\big\{\, c_s(L^J_{Q_L}) > L^\varepsilon\,\big\} < \exp\big[-C_3\,(\log\log L)^{-d}(\log L)^{\frac{d}{d-1}}\big] \tag{3.3}$$

Remark 1. Using (i) together with the Borel–Cantelli lemma, it follows that, with probability one, $c_s(L^J_{Q_L})$ does not grow faster than the exponential appearing in (3.1). It is quite easy to see that, in this respect, hypothesis (H2) is almost optimal. Let us in fact consider a ferromagnetic model with nearest neighbor couplings $J_{xy} = J_{\{x,y\}}$, with only exponential moments, e.g. such that $\mathbb{P}\{J_{xy} \geq n\} \leq C\exp[-n]$, $n = 0,1,\ldots$. That would correspond to $\delta = 0$ in (H2). Then, with large probability, one can find, in the cube Q_L, a pair of nearest neighbor sites $\{x,y\}$ such that, for some small ε, $J_{xy} = \varepsilon\log L$ and $J_{z,z'} = 0\;\forall z \in \{x,y\}, z' \in \{x,y\}^c$. In this case the Gibbs measure $\mu^\tau_{Q_L(0)}$ will factorize on the product of the Gibbs measure on the pair $\{x,y\}$ with free boundary conditions and of the Gibbs measure $\mu^\tau_{Q_L(0)\setminus\{x,y\}}$. In turn the logarithmic Sobolev constant $c_s(L^{J,\tau}_{Q_L})$ will be at least as large as the logarithmic Sobolev constant for the pair $\{x,y\}$ with free boundary conditions and coupling $J_{xy} = \varepsilon\log L$. A simple computation shows that this latter, for e.g. the heat bath dynamics, is of the order of $\exp[J_{xy}] = L^\varepsilon$, i.e. much larger than our bound in (3.1).

Remark 2. It is easy to check that, in the bounded case (H3), the almost sure bound on the growth of $c_s(L^{J,\tau}_{Q_L})$ that follows from (3.2) is, apart from the $\log\log L$ factor, optimal. To this purpose let us consider the simplest model, namely the diluted Ising model without external field, nearest neighbor interactions J_{xy} taking only two values, 0 and $\bar{J} \gg J_c$ with probability $1-p$ and $p \ll p_c$ respectively. Here J_c denotes the critical inverse temperature for the Ising model while p_c is percolation thereshold for bond percolation in d–dimensions. Since we are below p_c, it is not difficult to check that hypothesis (H1) holds. It is easy to see now that for almost all realizations J there exists $L_0(J)$ such that for all $L \geq L_0(J)$ there exists $x \equiv x(L,J)$, with $|x| \leq L/2$, such that *all* couplings inside the cube $Q_l(x)$, $l = (\varepsilon\log L)^{1/d}$, are equal to $\bar{J}$ and *all* couplings connecting a point inside $Q_l(x)$ with one of its nearest neighbors outside it are zero. Let now Λ be the cube of side $2L$ centered at the origin. By construction $Q_l(x) \subset \Lambda$. Since the couplings across the boundary of $Q_l(x)$ are zero one has

$$c_s(L^J_\Lambda) \geq c_s(L^{\bar J}_{Q_l(x)}) \geq \mathrm{gap}(L^{\bar J}_{Q_l(x)})^{-1}$$

In turn, since $\bar J \gg J_c$, one has (see e.g [M]) that

$$\mathrm{gap}(L^{\bar J}_{Q_l(x)})^{-1} \geq \exp[k(\varepsilon \log L)^{\frac{d}{d-1}}]$$

for a suitable constant k. Actually one can prove a similar lower bound on the logarithmic Sobolev constant even in the more general case discussed in Theorem 3.3 below.

Theorem 3.2. *Assume (H1) and (H2) and uniformly bounded transition rates, i.e.* $\kappa_2 = 0$ *in (2.8). Then*

(a) If $d \geq 1$ *there exists a set* $\bar\Theta \subset \Theta$ *of full measure such that for each* $J \in \bar\Theta$ *there exists a unique infinite volume Gibbs measure* μ^J*. Moreover there exists a constant* k *and, for each* $J \in \bar\Theta$ *and for any local function* f *there exists* $0 < t_0(J, f) < \infty$ *such that for all* $t \geq t_0$

$$\|T^J(t)f - \mu^J(f)\|_\infty \leq \exp\Big[-t\, \exp\big[-k\,(\log t)^{1-\frac{\delta'}{d}}\,(\log\log t)^{d-\delta'}\,\big]\Big] \tag{3.4}$$

where $\delta' \in (0,1)$ *is given by* $\delta' = \delta(1+\delta)^{-1}$*. If in addition (H3) holds (bounded interactions), then for all* $t \geq t_0(J, f)$

$$\|T^J(t)f - \mu^J(f)\|_\infty \leq \exp\Big[-t\, \exp\big[-k\,(\log t)^{1-\frac{1}{d}}\,(\log\log t)^{d-1}\,\big]\Big] \tag{3.5}$$

(b) Assume (H3) and $d \geq 2$*. Then there exists a constant* k *and for any local function* f *there exists* $0 < t_0(J, f) < \infty$ *such that, if* $t \geq t_0(f)$ *then*

$$\mathbb{E}\,\|T^J(t)f - \mu^J(f)\|_\infty \leq \exp\big[-k\,(\log t)^{\frac{d}{d-1}}\,(\log\log t)^{-d}\,\big] \tag{3.6}$$

Remark 1. The "constant" k as well as t_0 may depend on the geometrical parameters d, r and on the various parameters appearing in our hypotheses, like α, ϑ, L_0, δ, J_0.
Remark 2. The almost sure speed of relaxation to equilibrium is faster than any stretched exponential, at least under our assumptions (H2) or (H3). It is possible to show, at least for ferromagnetic systems, that if we assume an exponential tail for the distribution of the couplings ($\delta = 0$), then the almost sure bound cannot be better than a stretched exponential (see also the Remark 1 after Theorem 3.1)

In the general bounded case unfortunately we don't know any almost sure lower bound on speed of relaxation of a local function f that is slower than exponential. We conjecture however, also on the basis of Remark 2 after Theorem 3.1, that for a truly interacting model (thus *not* for example the dilute Ising model below the percolation threshold for which any local function relaxes exponentially fast but with its own exponential rate) our upper bound on the speed of relaxation is, apart from the $\log\log L$ factor, optimal.

The bound (b) on the relaxation for the *averaged* dynamics in the bounded case is, apart from the technical factor $(\log\log t)^d$, optimal as the next Theorem 3.3 shows. We don't give the analogous result in the unbounded case, *i.e.* when only (H2) holds, since the computation of the new exponent of $\log t$ is quite involved and, in our opinion, not particularly interesting from the physical point of view.

An interaction J is said to be *nearest neighbor* (n.n.) if $J_A = 0$ unless $A = \{x, y\}$ and the euclidean distance between x and y equals 1. We also remind the reader that with π_x, for $x \in Z^d$, we denote the projection from Ω over $\{-1, 1\}$, given by $\pi_x : \sigma \mapsto \sigma(x)$.

Theorem 3.3. *Assume uniformly bounded transition rates, i.e.* $\kappa_2 = 0$ *in (2.8), nearest neighbor interactions* J*, and suppose that*

(i) for almost all $J \in \Theta$*, there exists a unique Gibbs measure* μ^J

Then for each $d \geq 2$ *there exist* $J_1(d) > 0$ *such that if*

(ii) $p_1 \equiv \mathbb{P}\{J_{xy} = J_1\} > 0$ *and* $p_2 \equiv \mathbb{P}\{|J_{xy}| \leq 1/4\} > 0$

then we have, for all large enough t*,*

$$\mathbb{E}\,\|T^J(t)\pi_0 - \mu^J(\pi_0)\|_{L^2(\mu^J)} \geq \exp\big[-k\,(\log t)^{\frac{d}{d-1}}\,\big] \tag{3.7}$$

for some k *which depends on* d*,* p_1 *and* p_2*.*

Remark 1. If one assumes (H1) and (H2) then the uniqueness of the Gibbs measure with $\mathbb{P}-$probability one follows.

Remark 2. $\mu^J(\pi_0)$ is clearly equal to 0, by uniqueness of the Gibbs measure and symmetry.

Remark 3. (3.7) is obiously also a lower bound for $\mathbb{E}\,\|T(t)\pi_0 - \mu(\pi_0)\|_\infty$. This lower bound is of the same order, apart from the technical factor $(\log\log t)^d$, as the upper bound given in (b) of Theorem 3.2. Although a similar result was argued in [DRS] for the diluted Ising model, to our knowledge this is the first rigorous lower bound in a truly interacting case.

3.2. Applications

In this Section we discuss the hypotheses in our Theorems from the point of view of standard examples. Clearly the hypotheses (H2) and (H3) refer to the nature of the disorder (the distribution of the interaction potential) while the first hypothesis (H1) needs to be checked in a given disordered equilibrium model. General methods to verify (H1) can be found in [DKP] and [GM3] or the references therein. Here we follow [GM3] to discuss (H1) for the important example of a random-field short-range spin glass with formal Hamiltonian

$$H = -\sum_{<xy>} J_{xy}\sigma(x)\sigma(y) - b\sum_x h_x\sigma(x) - h\sum_x \sigma(x) \tag{3.8}$$

determined by a realization of one-(h_x) and two–body interactions (J_{xy}). To have in mind a specific example satisfying (H2) we could e.g. take the J_{xy} identically distributed independent Gaussian random variables and let the h_x be equal to ± 1 or 0 each with probability 1/3. In the notation of Section 2, $J_A = h_x$ if $A = \{x\}$ and $J_A = J_{xy}$ if the set $A = \{x, y\}$ is a nearest neighbor pair $< xy >$ on the lattice. In (3.8) b and h are just (constant) parameters.

To check (H1) we must consider the finite volume measure μ_V^τ corresponding to (3.8) with $V = Q_L$, and estimate truncated correlation functions. It is an immediate consequence of Corollary 2 in [BM] as applied in the main Theorem of [GM3] that for all local functions f and g

$$|\mu_V^\tau(f,g)| \leq 2|\Lambda_f|\,|\Lambda_g|\,\|f\|_\infty\,\|g\|_\infty \max_{x\in\Lambda_f,\, y\in\Lambda_g} G(x,y) \tag{3.9}$$

where $G(x,y)$ is the two-points connectivity function for independent site percolation on $\mathbb{Z}^d$ with (random) densities $\{p_z, z \in \mathbb{Z}^d\}$ specified below. More precisely, $G(x,y)$ is the probability in the independent site percolation process to find an open path from site x to site y; independently a site z is open with probability p_z and is closed with probability $1-p_z$. The densities are an explicit function of the interaction potential. In the spin-glass example ($b = 0$ in (3.8)) the random densities are given by

$$p_z^{SG} = 1/2\left[\tanh\Big[\sum_{<yz>}|J_{yz}| + h\Big] + \tanh\Big[\sum_{<yz>}|J_{yz}| - h\Big]\right]$$

while in the random-field case ($J_{xy} = J > 0, h = 0$ in (3.8)) we get

$$p_z^{RF} = 1/2\left[\tanh(2dJ + bh_x) + \tanh(2dJ - bh_x)\right]$$

It follows easily that $\{p_z^{SG}, z \in \mathbb{Z}^d\}$ is a one-dependent stationary random field while the $\{p_z^{RF}\}$ are independent and identically distributed. All the above features are quite general and do not depend very much on the model under investigation. As long as the interaction is short range, we will find some independent percolation process with random but almost independent densities which allows a domination like (3.9).

It is not difficult to see that if $\mathbb{E}p_z$ is sufficiently small (typically, below some percolation treshold), then for all sites $x, y \in \mathbb{Z}^d$,

$$\mathbb{E}\, G(x,y) \le e^{-\alpha d(x,y)} \tag{3.10}$$

for some $\alpha > 0$, with $\alpha \to \infty$ as $\mathbb{E}p_z \to 0$.

For example, in order to have $\alpha > 0$ for the random-field case it suffices that $\mathbb{E}p_z^{RF} < p_c(d)$ where $p_c(d)$ is the treshold (or critical) density for Bernoulli site percolation on $\mathbb{Z}^d$; for the general model (3.8) it suffices that $\mathbb{E}p_z < 1/(2d-1)^2$, see [GM3].

The combination of the upper bounds in (3.9) and (3.10) with the Chebyshev inequality yields (H1). In fact the probability that $SMT(Q_L, L/2, \alpha/2)$ does not hold, thanks to (3.9), is bounded by the probability that there exist x an y such that $d(x,y) \ge L/2$ and $2G(x,y) > \exp(-\alpha d(x,y)/2)$. This latter probability is, in turn, not greater than

$$\begin{aligned} L^2 & \sup_{x,y\in Q_L:\, d(x,y)\ge L/2} \mathbb{P}\{\, 2G(x,y) > \exp(-\alpha d(x,y)/2)\,\} \le \\ & \le 2\, L^2\, \mathbb{E}\, G(x,y)\ \exp(\alpha d(x,y)/2) \le e^{-\alpha L/6} \end{aligned}$$

Hence, (H1) is verified.

Another (but very similar) approach to check (H1) can be found in [DKP]. In particular their estimate (2.19) is almost the same as (3.9) above except that they are dominating via a bond percolation process.

Notice that checking (H1) as we have illustrated above requires a rather strong "high temperature" or "strong external field" condition. In certain cases however (e.g. the dilute Ising ferromagnet) one can substantially improve on this. We refer to [CMM] for the details.

4. Preliminaries

In this section we collect several technical results to be used in the next key section. Most of the results presented here, with the notable exception of Theorem 4.12 which seems to us completely new and of independent interest, are rather simple and some of them can actually be found in the literature. We thought however useful, also for future purposes, to put them together in a sort of primitive tool–box for the subject.

4.1. Mixing properties and bounds on relative densities for Gibbs measures

In this first part we give three equilibrium results on finite volume Gibbs measures. The first one is what was called in [MO1] "effectiveness" of property SMT. The second and the third one provide two simple bounds on the relative density between the projection over certain sets of two different Gibbs measures, once one assumes exponential decay of correlations.

Proposition 4.1. *Given $\alpha > 0$ there exist positive numbers $\bar{l}(d,r,\alpha)$, $\gamma_1(d,r)$, $\gamma_2(d,r)$, $\widehat{m}(d,r,\alpha)$ such that, the following holds for all $l > \bar{l}$: let $V \in \mathbb{F}$ such that V can be written as union of (possibly overlapping) cubes of side l and assume that*

(i) $SMT(Q_l(x), l/2, \alpha)$ *holds if* $Q_l(x) \subset V$

(ii) $\gamma_2 \, \|J\|_V \leq \alpha l$

then $SMT(V, \gamma_1 l, \widehat{m}(d,r,\alpha))$ holds.

Proof. It follows from the Lemma A2.1 of [MO1] and Proposition 3.1, Eq's (3.9), (3.11) of [O] that any cube C_i for which (i) and (ii) hold also satisfies condition $\mathcal{C}_{\bar{l}}$ of [OP] for some $\bar{l}$ large enough depending only on α, d, r, provided that the constants γ_1, γ_2 are chosen respectively large and small enough depending only on the dimension d and the range r. Then the result follows from Propositions 2.5.1, 2.5.2, 2.5.3, 2.5.4 of [OP]. We also refer the reader to Appendix A.1 of [MO1] for a simple proof in the attractive case. □

Remark. In the sequel for any given $\alpha > 0$ we will denote by $\widehat{m}(\alpha)$ the constant $\widehat{m}(d,r,\alpha)$ given in the above proposition.

Proposition 4.2. *Let $V \subset \Lambda \subset\subset \mathbb{Z}^d$, and let $x \in \Lambda^c$ such that $d(x,V) > r$. If $U \equiv \Lambda \backslash V$, we have*

$$\sup_{\tau\in\Omega} \Big\| 1 - \frac{d\mu_\Lambda^{J,\tau^x}}{d\mu_\Lambda^{J,\tau}}\Big|_V \Big\|_\infty \leq e^{14\|J\|_x} \sum_{y\in V} e^{2\|J\|_y} \sup_{\tau\in\Omega} \big| \mu_U^\tau\big(e^{-\nabla_x H_U} \, , \, e^{-\nabla_y H_U} \big)\big|$$

Proof. Let $\underline{1} \in \Omega$ be the configuration with all spins equal to $+1$, and let

$$W_{\Lambda,V}(\sigma) = \log Z_U^{\sigma} - \log Z_U^{\sigma_{V^c} \underline{1}_V}$$

It is easy to show that

$$\Big\| 1 - \frac{d\mu_\Lambda^{J,\tau^x}}{d\mu_\Lambda^{J,\tau}}\Big|_V \Big\|_\infty \le e^{2\|\nabla_x W_{\Lambda,V}\|_\infty} \qquad \text{for all } \tau \in \Omega$$

which, using the trivial bound $\|\nabla_x W_{\Lambda,V}\|_\infty \le 4\|J\|_x$, gives

$$\Big\| 1 - \frac{d\mu_\Lambda^{J,\tau^x}}{d\mu_\Lambda^{J,\tau}}\Big|_V \Big\|_\infty \le e^{8\|J\|_x} \|\nabla_x W_{\Lambda,V}\|_\infty \qquad \text{for all } \tau \in \Omega$$

By proceeding as in Lemma 3.1 of [MO2] one can show that

$$\|\nabla_x W_{\Lambda,V}\|_\infty \le e^{6\|J\|_x} \sum_{y \in V} e^{2\|J\|_y} \sup_{\tau\in\Omega} \big| \mu_U^\tau\big(e^{-\nabla_x H_U} \, , \, e^{-\nabla_y H_U} \big)\big|$$

which completes the proof. $\square$

Proposition 4.3. *For each $m > 0$ there exists $C(d,r,m)$ such that the following holds. Let $A \subset\subset \mathbb{Z}^d$, $A_0 \subset A$ and $B_0 \subset \partial_r^+ A$. Let $\bar{A} = A \cup \partial_r^+ A$ and assume that*

- *(i) $md_0 \equiv md(A_0, B_0) \ge \max\{ C \, , \, 100\|J\|_{\bar{A}} \, , \, 10 \, (\log|B_0| + 1) \}$*
- *(ii) $SMT(A \backslash A_0, \, d_0 - 2r, \, m)$ holds*

then for each pair of configurations $\sigma, \tau \in \Omega$ which agree on $\partial_r^+ A \backslash B_0$, we have

$$\Big\| 1 - \frac{d\mu_A^\tau}{d\mu_A^\sigma}\Big|_{A_0} \Big\|_\infty \le e^{-(m/4)d_0} \tag{4.1}$$

Proof. For each $\eta \in \Omega_{A_0}$, consider the event $F_\eta = \{\sigma \in \Omega : \sigma_{A_0} = \eta\}$. Choose a pair of configurations σ, τ which agree on $\partial_r^+ A \backslash B_0$. Then there exists a sequence of interpolating configurations $\gamma_i \in \Omega$ for $i = 1, \dots, n$ such that $n \le |B_0|$, γ_{i+1} differs from γ_i at exactly one site, $\gamma_1 = \sigma$ and γ_n agrees with τ on $\partial_r^+ A$. Thus, for each $\eta \in \Omega_{A_0}$, we can write

$$\left| 1 - \frac{\mu_A^\tau(F_\eta)}{\mu_A^\sigma(F_\eta)} \right| = \left| 1 - \prod_{i=2}^n \frac{\mu_A^{\gamma_i}(F_\eta)}{\mu_A^{\gamma_{i-1}}(F_\eta)} \right| \tag{4.2}$$

If we define

$$a = \sup_{\zeta\in\Omega, \, x \in B_0, \, \eta\in\Omega_{A_0}} \left| 1 - \frac{\mu_A^\zeta(F_\eta)}{\mu_A^{\zeta^x}(F_\eta)} \right|$$

then it is easy to check that, if $a \leq \frac{1}{10}$ and $a|B_0| \leq 1$ then the RHS of (4.2) cannot exceed $ea|B_0|$, so if we show that, for instance, $a \leq e^{-(m/2)d_0}$, the proposition follows.

Let then, for $z \in \mathbb{Z}^d$, $g_z = \exp(-\nabla_z H_{A\setminus A_0})$. By Proposition 4.2, and the SMT property given in the hypotheses, we find

$$a \leq \sup_{x\in B_0} \, e^{14\|J\|_{B_0}} \sum_{y\in A_0} e^{2\|J\|_y} |\Lambda_{g_x}||\Lambda_{g_y}| \left(\|g_x\|_\infty \|g_y\|_\infty\right) e^{-md(\Lambda_{g_x}, \Lambda_{g_y})} \leq$$

$$\leq \, (2r+1)^{2d} \, e^{20\|J\|_{\bar{A}}} \sup_{x\in B_0} \sum_{y\in A_0} e^{-m\,|x-y|-2r}$$

In the second inequality we have used the fact that Λ_{g_x} is contained in a ball of center x and radius r, and the fact that $\|g_x\|_\infty \leq \exp(2\|J\|_x)$. Finally, using the hypothesis on d_0, we easily get

$$a \leq e^{-(m/2)d_0} \quad \square$$

4.2. Some results on the spectral gap of the block dynamics

Here we provide three lower bounds on the spectral gap of the block dynamics with just two blocks.

Proposition 4.4. *Let $V \subset\subset \mathbb{Z}^d$, and let A, B be two (possibly intersecting) subsets of V such that $V = A \cup B$. Let $\mathcal{D} = \{A, B\}$. Assume that*

$$\sup_{\tau\in\Omega} \left\| 1 - \frac{d\mu_A^\tau}{d\mu_V^\tau}\Big|_{\partial_r^+ B} \right\|_\infty \leq \varepsilon < 1 \tag{4.3}$$

Then the gap for the block dynamics on $\mathcal{D}$ satisfies

$$\inf_{\tau\in\Omega} \operatorname{gap}(L_{\mathcal{D}}^\tau) \geq 1 - \sqrt{\varepsilon}$$

Proof. The action of the semigroup $T_{\mathcal{D}}(t)$ associated to the block dynamics is given by

$$T_{\mathcal{D}}(t)f = \sum_{n=0}^{\infty} \frac{t^n}{n!} (L_{\mathcal{D}})^n f$$

Using the explicit expression for $L_{\mathcal{D}}$ and some elementary combinatorics, it is not difficult to show that

$$T_{\mathcal{D}}(t)f = \sum_{n=0}^{\infty} \frac{(2t)^n}{n!} e^{-2t} \frac{1}{2^n} \sum_{X\in\{A,B\}^n} \mu_{X_1} \cdots \mu_{X_n}(f) \tag{4.4}$$

Since $(\mu_A)^2 = \mu_A$ (and similarly for B) the last summation (over X) in (4.4) can be written as

$$\sum_{k=0}^{n-1} \binom{n-1}{k} (\widehat{A}_{k+1} + \widehat{B}_{k+1}) f \tag{4.5}$$

where

$$\widehat{A}_k = (\mu_A \circ \mu_B)^{\lfloor k/2 \rfloor} \circ \mu_A^{k-2\lfloor k/2 \rfloor} \qquad \widehat{B}_k = (\mu_B \circ \mu_A)^{\lfloor k/2 \rfloor} \circ \mu_B^{k-2\lfloor k/2 \rfloor}$$

If now g is an arbitrary bounded measurable function on Ω, such that $\mu_V(g) = 0$, we get

$$\|\mu_A \mu_B \mu_A g\|_\infty \le \|\mu_V \mu_B \mu_A g\|_\infty + \|\mu_V \mu_B \mu_A g - \mu_A \mu_B \mu_A g\|_\infty \tag{4.6}$$

By the DLR property (2.2) the first term on the RHS of (4.6) is equal to $\mu_V(g) = 0$. Furthermore, since the interaction has range r, the function $h \equiv \mu_B \mu_A g$ is $\mathcal{F}_{V^c \cup \partial_r^+ B}$ measurable. This fact together with hypothesis (4.3) and the trivial observation that μ_A and μ_V agree on $\mathcal{F}_{V^c}$ implies

$$\|\mu_A \mu_B \mu_A g\|_\infty \le \varepsilon \|\mu_B \mu_A g\|_\infty \le \varepsilon \|\mu_A g\|_\infty \tag{4.7}$$

Iterating this inequality we get, for each bounded measurable f with $\mu_V(f) = 0$,

$$\|\widehat{A}_k f\|_\infty \le (\sqrt{\varepsilon})^{k-3} \|f\|_\infty \qquad \|\widehat{B}_k f\|_\infty \le (\sqrt{\varepsilon})^{k-3} \|f\|_\infty \tag{4.8}$$

Thus, we get that the sup norm of (4.5) is not greater than

$$\|f\|_\infty \frac{2}{\varepsilon^{3/2}} (1 + \sqrt{\varepsilon})^{n-1}$$

which, inserted back into (4.4) yields

$$\|T_{\mathcal{D}}(t) f\|_\infty \le \|f\|_\infty 4\varepsilon^{-3/2} e^{-2t} \sum_{n=0}^{\infty} \frac{t^n}{n!} (1 + \sqrt{\varepsilon})^n = \|f\|_\infty 4\varepsilon^{-3/2} e^{-(1-\sqrt{\varepsilon})t} \quad \square$$

Proposition 4.5. *Let V, A and B be as in Proposition 4.4. Let also $A_0 = A \cap \partial_s^+ B$, with $s \ge r$, $B_0 = B \cap \partial_r^+ A$ and $\bar{A} = A \cup \partial_r^+ A$. For each $m > 0$ there exists $C(d, r, m)$ such that if*

(i) $md_0 \equiv md(A_0, B_0) \ge \max\{\, C \,,\, 100\|J\|_{\bar{A}} \,,\, 10\,(\log|B_0| + 1)\,\}$
(ii) $SMT(A \backslash A_0,\, d_0 - 2r,\, m)$ *holds*

then

$$\inf_{\tau \in \Omega} \mathrm{gap}(L^\tau_{\{A,B\}}) \ge \frac{1}{2}$$

Proof. Thanks to Proposition 4.4 it is sufficient to show that

$$\sup_{\tau\in\Omega}\Big\|1-\frac{d\mu_A^\tau}{d\mu_V^\tau}\Big|_{A_0}\Big\|_\infty\le\frac{1}{4} \tag{4.9}$$

By the DLR property (2.3) we have

$$\text{LHS of (4.9)}\ \le \sup_{\tau,\sigma\in\Omega\,:\,\tau_{V^c}=\sigma_{V^c}}\Big\|1-\frac{d\mu_A^\tau}{d\mu_A^\sigma}\Big|_{A_0}\Big\|_\infty \tag{4.10}$$

At this point we can use Proposition 4.3 and obtain the result. $\square$

Proposition 4.6. *Let V, A, B be as in Proposition 4.4. Let*

$$N=|\partial_r^+A\cap B|\wedge|\partial_r^+B\cap A|$$

Then there exists $k=k(d,r)$ such that

$$\inf_{\tau\in\Omega}\text{gap}(L^{J,\tau}_{\{A,B\}})\ge\exp[\,-k\|J\|_V N\,]$$

Proof. We can assume $N=|\partial_r^+B\cap A|$. Consider a new interaction J^0 such that A and $V\backslash A$ are decoupled, *i.e.*

$$J^0_X=\begin{cases}J_X & \text{if } X\cap\partial_r^+A\cap B=\emptyset\\ 0 & \text{otherwise}\end{cases}$$

We have clearly $\|J-J^0\|_x=0$ unless x is in a neighborhood of radius r of $\partial_r^+B\cap A$, hence

$$\sum_{x\in V}\|J-J^0\|_x\le k_1N\|J\|_V$$

for some k_1 which depends on d and r. This implies that for all functions f on Ω,

$$\mathcal{E}^{J,\tau}_{\{A,B\}}(f,f)\ge\exp(-4k_1\|J\|_VN)\,\mathcal{E}^{J^0,\tau}_{\{A,B\}}(f,f) \tag{4.11}$$

$$\text{Var}_V^{J,\tau}(f)\le\exp(4k_1\|J\|_VN)\ \text{Var}_V^{J^0,\tau}(f) \tag{4.12}$$

From (4.11), (4.12) and the variational characterization of the gap (2.12), it follows that

$$\text{gap}(L^{J,\tau}_{\{A,B\}})\ge\exp[-8k_1\|J\|_VN]\ \text{gap}(L^{J^0,\tau}_{\{A,B\}}) \tag{4.13}$$

In order to estimate the gap for the block–dynamics with couplings J^0, we just notice that the hypotheses of Proposition 4.4 are satisfied with $\varepsilon=0$, and thus $\text{gap}(L^{J^0,\tau}_{\{A,B\}})\ge1$. $\square$

4.3. Some general results on the spectral gap and the log–Sobolev constant

This is actually the most important part of this section since it contains two key results. The first one, Proposition 4.9, has been essentially proved in [MO2] and, at least in the case of bounded interaction, it roughly says the following. If in a given cube Q_L of side L truncated correlations decay exponentially fast on all length scales larger than l_1, with $l_1 \ll L$, then the logarithmic Sobolev constant in that cube is not larger than the largest among the logarithmic Sobolev constants of all cubes of side l_1 inside Q_L. In order to appreciate this result one should consider that, if hypotheses (H1) and (H3) hold, then with probability one, truncated correlations in a cube of side L centered at the origin decay exponentially fast on all length scales larger than $l_1 \approx \log L$. Thus in this case we would have a logarithmic contraction of the starting length scale, namely from L to $\log L$. This result, together with a very rough estimate of the logarithmic Sobolev constant for a cube (see Proposition 4.10 and Theorem 4.12 below), allows us to conclude immediately that, with probability one, $c_s(L^{\tau}_{B_L})$ cannot grow faster than $\exp[C(\log L)^{d-1}]$. Notice that that in two dimensions this bound is just a power law in the side L.

The second important result is a very general lower bound on the spectral gap of Glauber dynamics (or upper bound on the logarithmic Sobolev constant) in an *arbitrary* set $V \subset\subset \mathbb{Z}^d$. It says that the spectral gap is always larger than a negative exponential of $|V|^{\frac{d-1}{d}}$. Notice that if V is cube then $|V|^{\frac{d-1}{d}}$ is simply its surface. In this case the bound is certainly optimal, at least in our general setting, since it is known that for several models of lattice *discrete spins* in the phase coexistence region, the activation energy between different stable phases is proportional to the *surface* of the region in consideration (see [M] and [CGMS] for more precise statements for the Ising model). Apparently the situation for *continuous* spins systems can be very different. For Heisenberg models, in fact, it is believed on the basis of spin–wave theory (see [B1], [B2]) that, at least for cubic regions, the spectral gap does not go to zero faster than the inverse of the volume. It is a challenging problem to actually prove it!

Definition 4.7. *A cube $C = Q_l(x)$ is said to be α–regular if, letting $n = \lfloor l/(2\gamma_1)\rfloor$*

(i) *$SMT(Q_n(y), n/2, \alpha)$ holds for all $y \in Q_l(x)$*

(ii) *$(\gamma_2 \vee 100)\, \|J\|_{\bar C} \le \widehat{m}(\alpha) l$*

where $\bar C = C \cup \partial_r^+ C$ and the constants γ_1, γ_2 and $\widehat{m}(\alpha)$ are those appearing in Proposition 4.1.

We immediately observe the following

Proposition 4.8. *Assume (H1) and (H2). Then there exist $L_0' \in \mathbb{Z}_+$, $\vartheta' > 0$ (depending on α, ϑ, γ_1, γ_2 and G_0) such that for all $L \ge L_0'$,*

$$\mathbb{P}\{\, Q_L \text{ is } \alpha\text{–regular}\,\} \ge 1 - e^{-\vartheta' L}$$

Furthermore, if V is a union of α−regular cubes of side length l, then $SMT(V, l/2, \widehat{m}(\alpha))$ holds.

Proof. The probability that Q_L is not α−regular is bounded by (we use the exponential Chebyshev inequality)

$$\mathbb{P}\{\text{ (i) does not hold }\} + \mathbb{P}\{\text{ (ii) does not hold }\} \leq$$
$$\leq L^d\, e^{-\vartheta/(3\gamma_1)} + (L+2r)^d\, G_0\, e^{-\widehat{m}L/(\gamma_2 \vee 100)} \leq e^{-\vartheta' L}$$

if L is greater than some L_0'. The second statement follows from Proposition 4.1. $\square$

Proposition 4.9. *Let $l_1 \in \mathbb{Z}_+$ and let $\Lambda \subset\subset \mathbb{Z}^d$ be a multiple of Q_{l_1}, i.e. $\Lambda = \cup_{i=1}^n B_i$ where $B_i = Q_{l_1}(x_i)$ for some $x_i \in l_1\mathbb{Z}^d$. Let, for any $I \subset \{1, \ldots, n\}$, $\Lambda_I = \cup_{i\in I} B_i$. Let also $\mathcal{A}$ be the set of all $I \subset \{1, \ldots, n\}$ such that $\operatorname{diam}(\Lambda_I) \leq 3l_1$. Assume that each B_i is α−regular for $\alpha > 0$. Then there exist two positive constants $\bar{l}_1$ and k depending on α, d, r such that if $m \equiv \widehat{m}(\alpha)$, and*

(i) $l_1 \geq \bar{l}_1$

(ii) $\inf_{I\in\mathcal{A}} \inf_{\tau\in\Omega} \operatorname{gap}(L^\tau_{\Lambda_I}) \geq \exp[-ml_1/2]$

then

$$\sup_{\tau\in\Omega} c_s(\mu^\tau_\Lambda) \leq k \sup_{I\in\mathcal{A}} \sup_{\tau\in\Omega} c_s(\mu^\tau_{\Lambda_I})$$

Proof. Since each B_i is α−regular, using Proposition 4.1, we get that for any $I \subset \{1, \ldots, n\}$, $SMT(\Lambda_I, l_1/2, m)$ holds. This fact, together with hypothesis (ii), allows us to apply Theorem 2.1 of [MO2] and to conclude that there exists $k(d, r, \alpha)$ such that

$$\sup_{\tau\in\Omega} c_s(\mu^\tau_\Lambda) \leq \frac{k}{4} \sup_{i\in\{1,\ldots,n\}} \sup_{\substack{I\subset\{1\ldots n\}\\ I\ni i}} \sup_{\tau\in\Omega} \gamma_i(\mu^\tau_{\Lambda_I}) \tag{4.14}$$

provided that l_1 was taken large enough. Here, for any $I \subset \{1, \ldots, n\}$ containing i, $\gamma_i(\mu^\tau_{\Lambda_I})$ is the smallest constant γ such that the logarithmic Sobolev inequality

$$\mu^\tau_{\Lambda_I}(f^2 \log f) \leq \gamma \mu^\tau_{\Lambda_I}(|\nabla_{\Lambda_I} f|^2) + \mu^\tau_{\Lambda_I}(f^2) \log \sqrt{\mu^\tau_{\Lambda_I}(f^2)}$$

holds for all positive functions f that depend only on the spins in B_i.

It is clear from the above definition that $\gamma_i(\mu^\tau_{\Lambda_I}) \leq c_s(\mu^\tau_{\Lambda_I})$. Assume that the supremum in the RHS of (4.14) is attained over a set $I \notin \mathcal{A}$ (otherwise the proof of the proposition would be finished).

Given i, I such that $i \in I \notin \mathcal{A}$, let I_0 be the largest subset of I such that $d(B_i, B_j) < l_1$ for all $j \in I_0$. By construction $i \in I_0 \in \mathcal{A}$. We claim that

$$\gamma_i(\mu^\tau_{\Lambda_I}) \leq 4 \sup_{\tau\in\Omega} c_s(\mu^\tau_{\Lambda_{I_0}}) \tag{4.15}$$

provided that l_1 is large enough depending on α, d, r. Such a bound clearly completes the proof.

In order to prove (4.15) it is enough to estimate the relative density between the projection over $\mathcal{F}_{B_i}$ of the two Gibbs measures $\mu^\tau_{\Lambda_{I_0}}$ and $\mu^\tau_{\Lambda_I}$ uniformly in the boundary condition τ. More precisely, let

$$g^\tau_{max} \equiv \| \frac{d\mu^\tau_{\Lambda_I}}{d\mu^\tau_{\Lambda_{I_0}}}\Big|_{B_i} \|_\infty \, ; \qquad g^\tau_{min} \equiv \min_{\sigma \in \Omega_{B_i}} \frac{d\mu^\tau_{\Lambda_I}}{d\mu^\tau_{\Lambda_{I_0}}}\Big|_{B_i}(\sigma)$$

Then, using Exercise 6.1.27 of [DeSt], (2.2) and the bound $\gamma_i(\mu^\tau_{\Lambda_{I_0}}) \le c_s(\mu^\tau_{\Lambda_{I_0}})$, we get

$$\gamma_i(\mu^\tau_{\Lambda_I}) \le \sup_{\tau\in\Omega} \frac{g^\tau_{max}}{g^\tau_{min}} \gamma_i(\mu^\tau_{\Lambda_{I_0}}) \le \sup_{\tau\in\Omega} \frac{g^\tau_{max}}{g^\tau_{min}} c_s(\mu^\tau_{\Lambda_{I_0}}) \tag{4.16}$$

We then use the DLR equations and write

$$g^\tau_{max} \le \sup_{\tau,\tau'\in\Omega\,:\,\tau_{\Lambda^c_I} = \tau'_{\Lambda^c_I}} \Big\| \frac{d\mu^\tau_{\Lambda_{I_0}}}{d\mu^{\tau'}_{\Lambda_{I_0}}}\Big|_{B_i} \Big\|_\infty \tag{4.17}$$

$$g^\tau_{min} \ge \inf_{\tau,\tau'\in\Omega\,:\,\tau_{\Lambda^c_I} = \tau'_{\Lambda^c_I}} \min_{\sigma\in\Omega_{B_i}} \frac{d\mu^\tau_{\Lambda_{I_0}}}{d\mu^{\tau'}_{\Lambda_{I_0}}}\Big|_{B_i}(\sigma) \tag{4.18}$$

Thanks to Proposition 4.3 applied to the sets $A \equiv \Lambda_{I_0}$, $A_0 \equiv B_i$ and $B_0 \equiv \partial^+_r \Lambda_{I_0} \cap \Lambda_I$, we know that the RHS of (4.17) is less than 2 while the RHS of (4.18) is greater than 1/2, provided that l_1 is taken large enough depending only on α, d, r. In this way we have proven (4.15), and, by consequence, the proposition. $\square$

Proposition 4.10. *For each $\Lambda \subset\subset \mathbb{Z}^d$ we have*

$$c_s(L^{J,\tau}_\Lambda) \le \Big[4 + 4 \sum_{x\in\Lambda} \|J\|_x + 2|\Lambda| \log 2 \Big] (\mathrm{gap}(L^{J,\tau}_\Lambda))^{-1}$$

The proposition follows from (2.12), Proposition 4.11 below, and from a trivial estimate on $\inf_\sigma \mu^{J,\tau}_\Lambda(\sigma)$.

Proposition 4.11. *Let Ω be a finite set, let μ be a probability measure on $(\Omega, 2^\Omega)$ and assume*

$$\mu_0 \equiv \inf_{x\in\Omega} \mu(x) > 0$$

Then, for each positive function f on Ω, we have

$$\mu(f^2 \log f) \le (4 + 2\log \mu_0^{-1}) \,\mathrm{Var}(f) + \mu(f^2) \log \sqrt{\mu(f^2)}$$

Proof. We can assume $\mu(f^2) = 1$. If we let $f = \mu(f)(1+g)$, we find $\mu(g) = 0$ and $\mu(g^2) = \mathrm{Var}(f)/\mu(f)^2$. Let A be the set of all $x \in \Omega$ such that $|g(x)| < 1$. We can then write

$$\mu(f^2 \log f) = \mu(f^2 \log f\, \mathbb{1}_A) + \mu(f^2 \log f\, \mathbb{1}_{A^c}) \tag{4.19}$$

Let's denote by X_1 respectively X_2 the first and the second term in the RHS of (4.19). Using the inequalities $\log(1+g) \le g$ and $\log \mu(f) \le \log \mu(f^2) \le 0$, we get

$$\begin{aligned} X_1 &\le \mu(f)^2 \mu[(g + 2g^2 + g^3)\mathbb{1}_A] \\ &\le \mu(f)^2 \big[\, 3\mu(g^2) + \mu(g\mathbb{1}_A)\,\big] = 3\,\mathrm{Var}(f) + \mu(f)^2 \mu(g\mathbb{1}_A) \end{aligned} \tag{4.20}$$

To take care of the last term we remember that $\mu g = 0$, so $\mu(g\mathbb{1}_A) = -\mu(g\mathbb{1}_{A^c})$ which implies, using the Schwarz and then the Chebyshev inequalities

$$|\mu(g\mathbb{1}_A)| \le \mu(|g|\mathbb{1}_{A^c}) \le (\mu(g^2)\mu(\mathbb{1}_{A^c}))^{1/2} \le \mu(g^2)$$

Thus we get $X_1 \le 4\,\mathrm{Var}\, f$. As for X_2, we write

$$X_2 \le (\,\sup_{x\in\Omega} \log f(x)\,)\, \mu(f^2\mathbb{1}_{A^c}) \le \log \|f\|_\infty\, \mu(f^2\mathbb{1}_{A^c}) \tag{4.21}$$

Finally we observe that $\|f\|_\infty$ is bounded by $(\mu(f^2)/\mu_0)^{1/2} = \mu_0^{-(1/2)}$ while

$$\mu(f^2\mathbb{1}_{A^c}) = \mu(f)^2\mu((1+2g+g^2)\mathbb{1}_{A^c}) \le 4\mu(f)^2\mu(g^2) = 4\,\mathrm{Var}(f)$$

This concludes the proof. $\square$

Theorem 4.12. *There exist $k(d, r, \kappa_1)$, such that, for each $\Lambda \subset\subset \mathbb{Z}^d$ and for each $\tau \in \Omega$, we have (c_m was defined in (2.8))*

$$\mathrm{gap}(L_\Lambda^\tau) \ge c_m \exp\big[-k\, \|J\|_\Lambda\, |\Lambda|^{\frac{d-1}{d}}\,\big] \tag{4.22}$$

Proof. For each non–negative integer n, let

$$(K_n) = \text{ the inequality (4.22) holds for all } \Lambda \in \mathbb{F} \text{ such that } |\Lambda| \le (3/2)^n$$

We want to show, that (K_n) holds for all $n \in \mathbb{Z}_+$, by proving that there exists $n_0(d,r) \in \mathbb{Z}_+$ such that (K_{n_0}) holds, and such that, for all $n \ge n_0$, (K_n) implies (K_{n+1}).

Assume then that K_{n-1} holds, and take any Λ such that $(3/2)^{n-1} < |\Lambda| \le (3/2)^n$. Let $v = |\Lambda|$. By Proposition A1.1, it is possible to write Λ as the disjoint union of two subsets X and Y, such that

(a) $v/2 - k_1 v^{\frac{d-1}{d}} \le |X| \le v/2$
(b) $\delta_r(X,Y) \le k_1 v^{\frac{d-1}{d}}$

where k_1 depends only on d and r. There exists then $n_0(d,r)$ such that if $n > n_0$ (and thus $v > (3/2)^{n_0-1}$), then $|Y| \le (2/3)|\Lambda|$. So we can apply the inductive hypothesis to both X and Y. Furthermore a simple calculation (see Proposition A1.1 in [CM]) shows that

$$\inf_{\tau\in\Omega} \mathrm{gap}(L_\Lambda^{J,\tau}) \ge \inf_{\tau\in\Omega} \inf_{W\in\{X,Y\}} \mathrm{gap}(L_W^{J,\tau}) \inf_{\tau\in\Omega} \mathrm{gap}(L_{\{X,Y\}}^{J,\tau}) \tag{4.23}$$

where, as usual, the last term refers to the block dynamics. By Proposition 4.6 we know that

$$\inf_{\tau\in\Omega} \mathrm{gap}(L_{\{X,Y\}}^{J,\tau}) \ge e^{-k_2\|J\|_\Lambda v^{\frac{d-1}{d}}}$$

for some $k_2(d,r)$. Together with the inductive hypothesis on X and Y, this gives

$$\inf_{\tau\in\Omega} \mathrm{gap}(L_\Lambda^{J,\tau}) \ge c_m \exp\bigl[-k\|J\|_\Lambda |Y|^{\frac{d-1}{d}} - k_2\|J\|_\Lambda v^{\frac{d-1}{d}}\bigr] \tag{4.24}$$

Since $|Y| \le (2/3)v$, we have

$$\inf_{\tau\in\Omega} \mathrm{gap}(L_\Lambda^{J,\tau}) \ge c_m \exp\bigl[-k\|J\|_\Lambda v^{\frac{d-1}{d}}\bigr] \qquad \text{if } k \ge \frac{k_2}{1-(2/3)^{\frac{d-1}{d}}}$$

In this way we have shown that (K_n) implies (K_{n+1}) for all $n \ge n_0(d,r)$.
All is left is to prove (K_{n_0}). For this purpose we observe that

$$e^{-2\|J-J'\|_\Lambda|\Lambda|} \le \frac{\mu_\Lambda^{J,\tau}(\sigma)}{\mu_\Lambda^{J',\tau}(\sigma)} \le e^{2\|J-J'\|_\Lambda|\Lambda|} \qquad \text{for all } \tau,\sigma,\Lambda,J,J' \tag{4.25}$$

Choose now any Λ with volume not exceeding $(3/2)^{n_0}$ and let $\widetilde{L}_\Lambda$ be the generator of the heat–bath dynamics with $J=0$, *i.e.*

$$\widetilde{L}_\Lambda = \sum_{x\in\Lambda} \widetilde{L}_{\{x\}} = \sum_{x\in\Lambda} (\mu_{\{x\}}^{J=0} - \mathbb{1})$$

Since all $\widetilde{L}_{\{x\}}$ commute, it follows that $\mathrm{gap}(\widetilde{L}_\Lambda) = \mathrm{gap}(\widetilde{L}_{\{x\}}) = 1$ (the last equality can be checked via an explicit calculation). From (2.8), (2.12) and (4.25), it now follows that

$$\mathrm{gap}(L_\Lambda^{J,\tau}) \ge \mathrm{gap}(\widetilde{L}_\Lambda)\, e^{-6\|J\|_\Lambda|\Lambda|}\, c_m e^{-\kappa_1\|J\|_\Lambda} \ge c_m \exp\bigl[-(6+\kappa_1)\|J\|_\Lambda (3/2)^{n_0}\bigr]$$

which implies (K_{n_0}) (and then (4.22)), if we take $k \ge (6+\kappa_1)(3/2)^{n_0}$. $\square$

5. The deterministic problem

This section is the core of the paper. We give a deterministic upper bound on the logarithmic Sobolev constant $c_s(\mu_\Lambda^{J,\tau})$ in the cube $\Lambda \equiv Q_L$. In order not to obscure the discussion of our ideas with less relevant details due to unbounded interactions, we present the main steps of our strategy in the bounded case.

For this purpose, consider first the so–called two dimensional diluted Ising model with nearest neighbor interactions J_{xy} which are either zero or $\bar{J}$, and call "regular" any site $x \in \Lambda$ such that $J_{xy} = 0$ for all neighboring sites y. Let us also consider the set W of all non–regular sites and its connected components (in the obvious sense) $\{W_i\}_{i=1}^n$ inside Λ. Fix a volume scale v and assume that $\sup_i |W_i| \le v$. Then we claim that in this case

$$c_s(\mu_\Lambda^{J,\tau}) \le C_1 L^d \exp[C_2 v^{\frac{d-1}{d}}] \tag{5.1}$$

for suitable constants C_1 and C_2 independent of L and v. The proof follows immediately from Proposition 4.10 if we can prove the key inequality

$$\text{gap}(L_\Lambda^{J,\tau}) \ge C_1' \exp[-C_2' v^{\frac{d-1}{d}}] \tag{5.2}$$

for another pair of constants C_1', C_2'. The above inequality follows from Theorem 4.12 once we observe that, and this is the key feature of the diluted model, the connected components of W are non–interacting since they are separated from each other by a "safety belt" of completely decoupled sites. Therefore the spectral gap of $L_\Lambda^{J,\tau}$ is not smaller than the smallest among the spectral gaps of $L_{W_i}^{J,\tau}$. Using now Theorem 4.12 and the assumption $\sup_i |W_i| \le v$ we get the required bound (5.1).

It is very important to observe that, thanks to some of the results of section 4.2, the above conclusion remains true, modulo some irrelevant constant factors, even if the value $J_{xy} = 0$ is replaced by a very small number J_{min}, provided that $|J_{min}||W| \ll 1$.

This remark suggests how to transpose to a truly interacting model the previous ideas. In a certain sense our original model behaves after a suitable "coarse-graining" quite closely to this diluted model. Let us in fact make a coarse–grained description of the model on a new scale $l_0 \ll L$, by replacing sites with disjoint cubes C_i of side l_0 and declare "regular" those cubes C_i in which truncated correlations decay exponentially fast with rate $\alpha > 0$. In this way, if B is a collection of "non–regular" cubes C_i surrounded by a safety–belt of regular cubes, then the effective interaction of B with any other region outside the safety–belt will be not larger than $|B| \exp(-\alpha l_0)$. Thus, if l_0 is chosen so large that the effective interaction among the connected components of the set W_{l_0} of non–regular cubes C_i is much smaller than one, e.g. if $|W_{l_0}| \exp(-\alpha l_0) \ll 1$, then our system, on scale l_0, will behave like a diluted Ising model. In particular we will be able to apply the results of Section 4.2 and, as a consequence, we will get the bound (5.2) on the spectral gap, with v

equal to the volume of the largest connected component of the set W_{l_0}. We refer the reader to Proposition 5.2 below for a precise formulation of this result in the more general case of unbounded J. Once we have (5.2) then we also get (5.1) simply by applying Proposition 4.10.

Although the above reasoning looks quite appealing from a physical point of view, it is still unsatisfactory for the following reason. In a typical configuration of J, the volume of the set W_{l_0} is roughly $p(l_0)L^d$, where $p(l_0)$ is the probability that a cube C_i is not regular. Using our basic assumption (H1), $p(l_0) \approx \exp(-\vartheta l_0)$ so that the minimal scale l_0 satisfying $|W_{l_0}| \exp(-\alpha l_0) \ll 1$ becomes of order $\log L$. This unfortunately is a too large scale: since v is at least l_0^d, the corresponding bounds (5.2) or (5.1) on the spectral gap or on the logarithmic Sobolev constant, become at least of the order of a power of L.

In order to overcome this difficulty, we appeal to Proposition 4.9. More precisely we introduce an intermediate length scale $l_1 \ll L$ and we assume that the J in Λ are such that the hypotheses of Proposition 4.9 apply for l_1. If this is the case, then Proposition 4.9 basically allows us to replace the initial cube $\Lambda = Q_L$ with a smaller cube $Q_{l_1}(x)$, for a suitable $x \in \Lambda$. Once we have reduced the initial scale L to the new scale l_1, we make the coarse–grained analysis on scale $l_0 \ll l_1$ on the new cube $Q_{l_1}(x)$ and proceed as explained before.

The advantage of the above two–scale analysis is twofold. First of all the shortest scale l_0 is now at most of the order of $\log l_1$ instead of $\log L$. Secondly the prefactor L^d in (5.1) is replaced by l_1^d. If one considers that in a typical configuration the intermediate scale l_1 can be taken already of the order of $\log L$ (see the comments before Proposition 4.9), we see that the smallest scale becomes $l_0 \approx \log\log L$ with an enormous gain in precision. We conclude this short heuristic discussion by observing that it is precisely the coarse–grained analysis on scale $\log\log L$ that is responsible for the various $\log\log L$ factor in Theorem 3.1.

We are now ready for a precise formulation of our results.

Definition 5.1. *Let $l \in \mathbb{Z}_+$, $\alpha > 0$ and let Λ be a mutilple of Q_l and write $\Lambda = \cup_{i=1}^n Q_l(x_i)$ for some $n \in \mathbb{Z}_+$ and $x_i \in l\mathbb{Z}^d$. Let K be the set of all $i \in \{1, \dots, n\}$ such that $Q_l(x_i)$ is not $\alpha-$regular. Then we let*

$$W(\Lambda, l, \alpha) = \{\, x \in \Lambda : \, d\big(x, \cup_{i\in K} Q_l(x_i)\big) \le 2l \,\}$$
$$v(\Lambda, l, \alpha) = \textit{the cardinality of the largest } r-\textit{connected component of } W(\Lambda, l, \alpha)$$

Given $\lambda \ge 0$, we also define a cutoff interaction $J^{(\lambda)}$ as

$$J_A^{(\lambda)} = (\operatorname{sgn} J_A)\,(|J_A| \wedge \lambda) \tag{5.3}$$

Proposition 5.2. *Choose the transition rates c_J as in (2.11). Then, for each $\alpha > 0$ there exists $\bar{l}(d, r, \alpha)$ such that the following holds for all $l_0 \geq \bar{l}$: let V be a multiple of Q_{l_0}, $v = v(V, l_0, \alpha)$ (see Definition 5.1), and let $m = \widehat{m}(d, r, \alpha)$. Let also $\lambda, \gamma \geq 0$, and assume that*

(i) $ml_0 \geq 10\,(1 + \log |W(V, l_0, \alpha)|)$

(ii) For each $r-$connected subset X of V with $|X| \leq v$, we have $\sum_{x\in X} \|J - J^{(\lambda)}\|_x \leq \gamma$

Then,

$$\inf_{\tau\in\Omega} \mathrm{gap}(L_V^\tau) \geq |V|^{-\omega} \exp\big[-\big(\,8\gamma + k\lambda v^{\frac{d-1}{d}} + k' m l_0^d\,\big)\big] \tag{5.4}$$

where ω can be taken equal to $d \log 4/\log(3/2)$, $k = k(d, r)$ is the quantity defined in Theorem 4.12 and $k' = 9^{d-1}k$.

Remark. The reader who does not want to bother with the extra complications due to the unboundedness of the interaction may just consider the bounded case and take λ equal to $\sup_x \|J\|_x$ and $\gamma = 0$.

Proof. Write $V = \bigcup_{i=1}^s C_i$, where $C_i = Q_{l_0}(y_i)$ for some $y_i \in l_0\mathbb{Z}^d$. Let $B = W(V, l_0, \alpha)$ and let A be the union of all those (α–regular) cubes C_i such that $d(C_i, C_j) > l_0$ for all C_j which are not $\alpha-$regular. Let also $A_0 = A \cap \partial_{l_0}^+ B$ and $B_0 = B \cap \partial_r^+ A$. By Proposition A1.1 in [CM], we have

$$\inf_{\tau\in\Omega} \mathrm{gap}(L_V^{J,\tau}) \geq \frac{1}{2}\Big[\inf_{\tau\in\Omega}\ \inf_{D\in\{A,B\}} \mathrm{gap}(L_D^{J,\tau})\,\Big] \inf_{\tau\in\Omega} \mathrm{gap}(L_{\{A,B\}}^{J,\tau}) \tag{5.5}$$

The proof of the Proposition can be organized in the following steps

(a) We can use Proposition 4.5 to show that the gap of the block dynamics generator $L_{\{A,B\}}^{J,\tau}$ is at least 1/2. In order to show that 4.5 does indeed apply to our case, we first notice that $d(A_0, B_0) \geq l_0$, which, together with the fact that all cubes in A are $\alpha-$regular and the trivial inequality $|B_0| \leq |W(V, l_0, \alpha)|$, implies the hypothesis (i) of 4.5. Then we observe that $A\backslash A_0$ can be expressed as a union of $\alpha-$regular cubes C_i. So, by Proposition 4.8, the property $SMT(A\backslash A_0, l_0/2, m)$ holds.

(b) Since the set A is a union of $\alpha-$regular cubes, using the ideas in [MO1] one can prove that $\mathrm{gap}(L_A^{J,\tau})$ is bounded from below by a quantity which does not depend on the size of A. In the Appendix 2, we give a simple proof of the much weaker result

$$\mathrm{gap}(L_A^{J,\tau}) \geq 8|A|^{-\omega} \exp(-k' m l_0^d)$$

Such an inequality, even if far from optimal, is sufficient anyway for our purposes.

(c) For what concerns the gap of $L_B^{J,\tau}$, we write B as the disjoint union of its $r-$connected components $\widetilde{B}_1, \ldots, \widetilde{B}_n$. Since $L_{\tilde{B}_i}^{J,\tau}$ commutes with $L_{\tilde{B}_j}^{J,\tau}$ for all $i \neq j$, it follows that

$$\mathrm{gap}(L_B^{J,\tau}) = \inf_{i\in\{1,\ldots,n\}} \mathrm{gap}(L_{\tilde{B}_i}^{J,\tau})$$

(d) Now we get rid of those couplings which are too strong, by introducing, on each $\widetilde{B}_i$, a cutoff interaction $\{J_X^{(\lambda)}\}_{X\in\mathbb{F}}$ (see (5.3)). By (2.12) and hypothesis (ii), we obtain

$$\operatorname{gap}(L_{\tilde{B}_i}^{J,\tau}) \geq e^{-8\gamma}\ \operatorname{gap}(L_{\tilde{B}_i}^{J^{(\lambda)},\tau})$$

From (5.5), (a), (b), (c) and (d), together with Theorem 4.12 (for the dynamics (2.11) we can take $c_m = 1/2$ and $\kappa_1 = 0$ in (2.8)) and the fact that trivially $\|J^{(\lambda)}\|_{\widetilde{B}_i} \leq \lambda$, we get

$$\inf_{\tau\in\Omega} \operatorname{gap}(L_V^{J,\tau}) \geq \frac{1}{4}\ \min\{\frac{1}{2}\ \inf_i e^{-(8\gamma + k\,\lambda\,|\tilde{B}_i|^{\frac{d-1}{d}})},\ 8|A|^{-\omega}\ \exp(-k' m l_0^d)\}$$

In order to obtain (5.4) we now observe that by definition of v, we have $|\widetilde{B}_i| \leq v$, and that the minimum of the two quantities in braces is greater than their product if l_0 is such that both terms are less than 1. $\square$

Theorem 5.3. *If the transition rates are given by (2.11), then for each $\alpha > 0$ there exist $\bar{l}$, C_1 and C_2 depending on d, r and α such that the following holds for all positive integers $l_0 \geq \bar{l}$: let l_1 be a multiple of l_0 and let Λ be a multiple of Q_{l_1} so that we can write*

$$\Lambda = \textstyle\bigcup_{i=1}^n B_i = \bigcup_{i=1}^s C_i \tag{5.6}$$

where $B_i = Q_{l_1}(x_i)$ and $C_i = Q_{l_0}(y_i)$ for some $x_i \in l_1\mathbb{Z}^d$ and $y_i \in l_0\mathbb{Z}^d$ Let $v = v(\Lambda, l_0, \alpha)$ (see Definition 5.1), and let $m = \widehat{m}(d, r, \alpha)$. Let also $\lambda, \gamma \geq 0$, and assume that

(i) For each $i \in \{1, \dots n\}$ the cube B_i is $\alpha-$regular
(ii) $8\gamma + k\lambda v^{\frac{d-1}{d}} \leq m l_1/4$, where $k(d, r)$ is the quantity defined in Theorem 4.12
(iii) $30d\log l_1 \leq m l_0 \leq (l_1)^{1/(2d)}$
(iv) For each $r-$connected $V \subset \Lambda$ with $|V| \leq v$, we have $\sum_{x\in V} \|J - J^{(\lambda)}\|_x \leq \gamma$

Then we have

$$\sup_{\tau\in\Omega} c_s(\mu_\Lambda^{J,\tau}) \leq C_1 \exp\big[\,8\gamma + C_2\,(\lambda v^{\frac{d-1}{d}} + l_0^d)\,\big]$$

Proof. Let $V \subset \Lambda$ be a union of cubes C_i such that $\operatorname{diam}(V) \leq 3l_1$. The hypotheses (iii) and (iv) tell us that Proposition 5.2 can be applied to V. Therefore we have

$$\operatorname{gap}(L_V^{J,\tau}) \geq |V|^{-\omega}\ \exp\big[-(\,8\gamma + k\lambda v^{\frac{d-1}{d}} + k' m l_0^d\,)\,\big] \geq e^{-ml_1/2} \tag{5.7}$$

where, $\omega = d\log 4/\log(3/2)$, and in the second inequality, we have used hypotheses (ii) and (iii). Thanks to (5.7) we can now apply Proposition 4.9, which, combined with Proposition 4.10 and again with (5.7) implies that

$$\sup_{\tau\in\Omega} c_s(L_\Lambda^{J,\tau}) \leq C'\ \big[\,4 + 4(3l_1)^d(\|J\|_\Lambda + 2\log 2)\,\big]\,(3l_1)^{\omega d}\ \exp\big[\,8\gamma + k\lambda v^{\frac{d-1}{d}} + k' m l_0^d\,\big] \leq$$

$$\leq C_1 \exp\big[\,8\gamma + C_2(\lambda v^{\frac{d-1}{d}} + l_0^d)\,\big]$$

for some C', C_1, C_2 depending on d, r and α. $\square$

6. Proof of the main results

6.1. The upper bounds

In this section we finally prove our main results. Before doing that we need a simple probabilistic estimate on independent random variables.

Lemma 6.1. *Let $\{X_i\}_{i=1}^n$ be real independent random variables such that*

$$\mathbb{E}(\exp(X_i^{1+\delta})) \leq G_\delta < \infty$$

for some $\delta > 0$, for all i. Then, for all $\lambda, \gamma > 0$

$$\mathbb{P}\Big\{\sum_{i=1}^{n}(X_i - \lambda)_+ \geq \gamma\Big\} \leq \exp\big[-\lambda^\delta\gamma + nG_\delta e^{-\lambda^{1+\delta}}\big]$$

Proof. By the Chebyshev inequality, and using $\log(1+x) \leq x$, we obtain

$$\begin{aligned}&\mathbb{P}\Big\{\sum_{i=1}^{n}(X_i - \lambda)_+ \geq \gamma\Big\} \leq e^{-\alpha\gamma}\left[\mathbb{E}e^{\alpha(X_i-\lambda)_+}\right]^n \leq \\ &\leq e^{-\alpha\gamma}\left[1 + e^{-\alpha\lambda}\mathbb{E}\left(e^{\alpha X_i}\mathbb{1}\{X_i \geq \lambda\}\right)\right]^n \leq \exp\left[-\alpha\gamma + ne^{-\alpha\lambda}\mathbb{E}\left(e^{\alpha X_i}\mathbb{1}\{X_i \geq \lambda\}\right)\right]\end{aligned}$$

Now take $\alpha = \lambda^\delta$ and notice that

$$\mathbb{E}\left(e^{\lambda^d X_i}\mathbb{1}\{X_i \geq \lambda\}\right) \leq \mathbb{E}\left(e^{(X_i)^{1+\delta}}\mathbb{1}\{X_i \geq \lambda\}\right) \leq \mathbb{E}\,e^{(X_i)^{1+\delta}} \leq G_\delta \quad \square$$

Proposition 6.2. *Assume (H2). Then there exists $k = k(d,r) > 0$ such that if $v > \log L$ and if we let*

$$A_\delta = (\log G_\delta)^{\frac{1}{1+\delta}} \vee 1 \qquad \lambda = v^{\frac{1}{d(1+\delta)}}A_\delta \qquad \gamma = kv^{1-\frac{\delta}{d(1+\delta)}}$$

then, for all $L \in \mathbb{Z}_+$ (see (5.3))

$$\mathbb{P}\Big\{\exists V \subset Q_L : V \text{ is } r\text{−connected}\ ,\ |V| \leq v,\ \sum_{x\in V}\|J - J^{(\lambda)}\|_x \geq \gamma\Big\} \leq L^{-3d} \tag{6.1}$$

Proof. For each $V \in \mathbb{F}$ we have

$$\mathbb{P}\Big\{\sum_{x\in V}\|J - J^{(\lambda)}\|_x \geq \gamma\Big\} \leq \mathbb{P}\Big\{\sum_{A:\,A\cap V\neq\emptyset}(J_A - \lambda)_+ \geq \gamma/k_1\Big\} \tag{6.2}$$

where k_1 can be taken equal to $\sup\{|A| : \operatorname{diam} A \leq r\}$. Using lemma 6.1 and the fact that the number of sets $A \in \mathbb{F}$ with a diameter not greater than r which intersect V can be bounded by $|V| k_2(d,r)$ we obtain

$$\mathbb{P}\Big\{ \sum_{x\in V} \|J - J^{(\lambda)}\|_x \geq \gamma \Big\} \leq \exp\Big[-\lambda^\delta \frac{\gamma}{k_1} + k_2|V| G_\delta e^{-\lambda^{1+\delta}}\Big] \tag{6.3}$$

Furthermore, since the number of $r-$connected $V \subset Q_L$ such that $|V| \leq v$ is not greater than $L^d \exp(k_3 v)$ for some $k_3(d,r)$, if $v \geq \log L$ and λ is chosen as in the hypothesis, we get

$$\text{RHS of (6.1)} \leq \exp\Big[v(d + k_3 + k_2) - \lambda^\delta \frac{\gamma}{k_1}\Big]$$

If now $k \geq k_1(4d + k_2 + k_3)$, we find

$$\text{RHS of (6.1)} \leq e^{-3dv} \leq L^{-3d} \quad \square$$

Proof of Theorem 3.1. We give the proof in the special case of L which is a power of 2, which is enough to prove Theorem 3.2. A proof which works for all L requires a modification of Theorem 5.3 where one considers more general coverings of Λ with cubes and cuboids with slightly different sidelengths. This generalization is straightforward.

Part (i). By combining hypothesis (H2) with the exponential Chebyshev inequality, one gets

$$\mathbb{P}\{\, \|J\|_{Q_L} \geq 3\,(\log L)^{1-\frac{\delta}{d(1+\delta)}} \} \leq L^d G_\delta \exp\big[-3(\log L)^{1+\delta-\frac{\delta}{d}}\big] \ll L^{-2} \tag{6.4}$$

for all L large enough. Therefore, using (2.16), it is enough to prove (3.2) with $c_s(L^{J,\tau}_{Q_L})$ replaced by $c_s(\mu^{J,\tau}_{Q_L})$ and $L^{-1.5}$ replaced by $3L^{-2}$.

For this purpose we are going to use the key deterministic estimate of $c_s(\mu^{J,\tau}_{Q_L})$ given in Theorem 5.3. The idea is to prove that with probability greater than $1 - 3L^{-2}$, it is possible to choose the four parameters in Theorem 5.3, l_0, l_1, λ and γ in such a way that the deterministic upper bound on $c_s(\mu^{J,\tau}_{Q_L})$ given in that proposition is not greater than

$$\exp\big[\, C A_\delta (\log\log L)^{d-\frac{\delta}{(1+\delta)}} (\log L)^{1-\frac{\delta}{d(1+\delta)}} \,\big]$$

More precisely we define l_0 and l_1 as those powers of 2 (they are uniquely defined) such that

$$\frac{60d}{m}\log\log L \leq l_0 < \frac{120d}{m}\log\log L \qquad \frac{3d}{\vartheta'}\log L \leq l_1 < \frac{6d}{\vartheta'}\log L \tag{6.5}$$

where $m \equiv \widehat{m}(\alpha)$ (see Proposition 4.1) and ϑ' is given in Proposition 4.8. We then take

$$v_* = l_0^d \log L \quad \lambda = v_*^{\frac{1}{d(1+\delta)}} A_\delta \quad \text{and} \quad \gamma = k v_*^{1-\frac{\delta}{d(1+\delta)}} \tag{6.6}$$

and $k(d,r)$ is given in Proposition 6.2. Since l_0 divides l_1 and l_1 divides L, we can write Q_L as in (5.6). We now observe that, if L is large enough, the hypotheses (i) – (iv) of Theorem 5.3 are satisfied for all $J \in \tilde{\Theta} \equiv \cap_{i=1}^{3} \Theta_i$, where

$$\Theta_1 = \{J : \text{ each } B_i \text{ is } \alpha\text{-regular}\}$$
$$\Theta_2 = \{J : v(Q_L, l_0, \alpha) \le v_*\}$$
$$\Theta_3 = \{J : \text{ for each } r\text{-connected } V \subset \Lambda \text{ with } |V| \le v_*, \ \textstyle\sum_{x \in V} \|J - J^{(\lambda)}\|_x \le \gamma \ \}$$

and $v(Q_L, l_0, \alpha)$ has been defined in 5.1. Notice that, for all $J \in \Theta_1$ the bound $\|J\|_{Q_L} \le k' \log L$ holds for some constant k', because of the definition of α−regular cubes and of our choice of l_1. By Theorem 5.3, (6.5) and (6.6), for any $J \in \tilde{\Theta}$, we have

$$c_s(\mu_{Q_L}^{J,\tau}) \le \exp\left[\, C A_\delta (\log\log L)^{d - \frac{\delta}{1+\delta}} (\log L)^{1 - \frac{\delta}{d(1+\delta)}} \,\right]$$

for a suitable constant C independent of J. In order to prove the theorem it is therefore sufficient to estimate from above $\mathbb{P}\big(\tilde{\Theta}^c\big)$.

From Proposition 4.8, it follows

$$\mathbb{P}\big(\Theta_1^c\big) \le L^d e^{-\vartheta' l_1} \le L^{-2d} \tag{6.7}$$

for all L large enough. Let $p(l)$ be the probability that a cube Q_l is not α−regular. Then $p(l)$ goes to zero as $l \to \infty$, and a standard estimate for 2−dependent site percolation implies

$$\mathbb{P}\big(\Theta_2^c\big) \le L^d \left(k_1\, p(l_0)\right)^{k_2 v_* l_0^{-d}} \le L^{-3d} \tag{6.8}$$

for L large enough, where k_1 and k_2 are two suitable geometrical constants. Finally, by Proposition 6.2, we have $\mathbb{P}\big(\Theta_3^c\big) \le L^{-3d}$, and, by consequence $\mathbb{P}\big(\tilde{\Theta}^c\big) \le 3L^{-2}$. This completes the proof in the general unbounded case. The bounded case can be treated in the same way, by choosing $\lambda = J_0$ and $\gamma = 0$.

Proof of part (ii). The proof is the same as in part (i), with a different choice of the three basic parameters l_0, l_1 and v_*. More precisely we define l_0 and l_1 as those powers of 2 (they are uniquely defined) such that (let again $m = \widehat{m}(\alpha)$)

$$\tfrac{60d^2}{(d-1)m} \log\log L \le l_0 < \tfrac{120d^2}{(d-1)m} \log\log L \qquad (\log L)^{\frac{d}{d-1}} \le l_1 < 2(\log L)^{\frac{d}{d-1}}$$

Given $\varepsilon \in (0,1)$ we then let

$$\lambda = J_0 \quad \gamma = 0 \qquad \text{and} \quad v_* = \left(\tfrac{\varepsilon \log L}{2 J_0 C_2}\right)^{\frac{d}{d-1}} \tag{6.9}$$

where C_2 appears in Theorem 5.3. Write Q_L as in (5.6) and define the events $\tilde{\Theta}$, Θ_i as in the proof of part (i). Thanks to Theorem 5.3, we get

$$c_s(\mu_{Q_L}^{J,\tau}) \le C_1 L^{\frac{\varepsilon}{2}} \exp(C_2 l_0^d) \le L^{\varepsilon} \qquad \forall J \in \tilde{\Theta} \tag{6.10}$$

for all L sufficiently large. In order to prove (3.3) it is therefore sufficient to bound from above $\mathbb{P}(\tilde{\Theta}^c)$. As before, we find

$$\mathbb{P}(\Theta_1^c) \le L^d e^{-\vartheta' l_1} = L^d \exp\big[-\vartheta'(\log L)^{\frac{d}{d-1}}\big] \tag{6.11}$$

and

$$\mathbb{P}(\Theta_2^c) \le L^d \,(k_1 p(l_0))^{k_2 v_* l_0^{-d}} \le \exp\big[-C_3\,(\log\log L)^{-d}(\log L)^{\frac{d}{d-1}}\big] \tag{6.12}$$

for a suitable constant C_3 and all L large enough. Clearly (6.11) and (6.12) complete the proof of (ii). $\square$

Proof of Theorem 3.2. The proof of the almost sure bounds (part (a)) is a simple consequence of Theorem 3.1. We prove only (3.4) since the case of bounded interactions (3.5) is identical. Let $\bar{\Theta}$ be the set of interactions J such that for each $J \in \bar{\Theta}$ there exists $L_1(J)$ such that for all $L \ge L_1(J)$ (C is given in Proposition 3.1)

(i) $c_s(L^J_{B_L}) < \exp\big[\,CA_\delta\,(\log L)^{1-\frac{\delta'}{d}}\,(\log\log L)^{d-\delta'}\,\big]$
(ii) $SMT(B_L, \gamma_1(2L+1), \alpha)$ holds
(iii) $\|J\|_{\bar{B}_L} \le \log L$, where $\bar{B}_L = B_L \cup \partial_r^+ B_L$

Using Theorem 3.1, (H1), (H2) and the Borel–Cantelli lemma, one can check that $\mathbb{P}(\bar{\Theta}) = 1$. Moreover, thanks to (ii) and (iii), for all $J \in \bar{\Theta}$ there exists a unique infinite volume Gibbs measure that in the sequel will be denoted by μ^J. Let, in fact, f be any local function on Ω, and take L large enough such that $B_L \supset \Lambda_f$. Then, given two arbitrary boundary conditions τ and η, and using a telescopic interpolation between them, we get

$$\begin{aligned}\sup_{\tau,\eta\in\Omega}\big|\,\mu^{J,\tau}_{B_L}(f) - \mu^{J,\eta}_{B_L}(f)\,\big| &\le |\partial_r^+ B_L| \sup_{x\in\partial_r^+ B_L} \|\nabla_x[\mu^J_{B_L}(f)]\|_\infty = \\ &= |\partial_r^+ B_L| \sup_{x\in\partial_r^+ B_L}\sup_{\tau\in\Omega}\left|\frac{\mu^{J,\tau}_{B_L}(h_x, f)}{\mu^{J,\tau}_{B_L}(h_x)}\right|\end{aligned} \tag{6.13}$$

where $h_x \equiv \exp[-\nabla_x H_{B_L}]$.

Notice that, because of (iii) in the definition of $\bar{\Theta}$, we have $\|h_x\|_\infty \le \exp(2\log L) = L^2$. Therefore, if L is larger than $L_1(J)$ and if $d(\Lambda_f, (B_L)^c) > \gamma_1(2L+1) + r$, we can use $SMT(B_L, \gamma_1(2L+1), \alpha)$, and write

$$\sup_{\tau,\eta\in\Omega}\big|\,\mu^{J,\tau}_{B_L}(f) - \mu^{J,\eta}_{B_L}(f)\,\big| \le kL^{d+4}|\Lambda_f|\|f\|_\infty e^{-\alpha d(\Lambda_{h_x},\Lambda_f)} \tag{6.14}$$

for a suitable constant k, and the uniqueness follows.

In order to prove inequalities (3.4) and (3.5) we first need to recall a standard result on the "finite speed of information propagation" for Glauber dynamics with bounded rates.

Lemma 6.3. *Assume the transition rates uniformly bounded, i.e. $\kappa_2 = 0$ in (2.8). Then there exists a constant k_0 depending on d, r and c_M and for any local function f, there is $A(f)$ such that for all $V \subset\subset \mathbb{Z}^d$, $t \geq 0$ with $d(V^c, \Lambda_f) \geq k_0 t$, we have*

$$\sup_{\tau \in \Omega} \|T(t)f - T_V^\tau(t)f\|_\infty \leq A(f)\, e^{-2t}$$

Proof. One can see for instance Lemma 1.7 in [HS], or Lemma 1 in [S] which makes use of the explicit "graphical construction" of the process. □

Let now $L_t = \lfloor k_1 t \rfloor$ for some $k_1 > k_0$ (k_0 is given in Lemma 6.3) and, for simplicity, let $\Lambda_t = B_{L_t}$. Choose an arbitrary boundary condition τ. Then we have

$$\begin{aligned} \|T^J(t)f - \mu^J(f)\|_\infty \leq \|T^{J,\tau}_{\Lambda_t}(t)f - \mu^{J,\tau}_{\Lambda_t}(f)\|_\infty + \\ + \|T^J(t)f - T^{J,\tau}_{\Lambda_t}(t)f\|_\infty + |\mu^{J,\tau}_{\Lambda_t}(f) - \mu^J(f)| \end{aligned} \tag{6.15}$$

Let us examine separately the three terms appearing in the RHS of (6.15). The first one, using (i) and (iii) above, together with hypercontractivity (see the proof of Theorem 4.1 in [GZ1]), can be bounded from above by

$$\begin{aligned} \|T^{J,\tau}_{\Lambda_t}(t)f - \mu^{J,\tau}_{\Lambda_t}(f)\|_\infty &\leq 2\, |||f|||\, \exp\big[-\frac{t}{2} c_s(L^{J,\tau}_{\Lambda_t})^{-1}\big] \leq \\ &\leq 2\, |||f|||\, \exp\Big[-\frac{t}{2} \exp\big[-CA_\delta\, (\log L_t)^{1-\frac{\delta'}{d}}\, (\log\log L_t)^{d-\delta'}\,\big]\Big] \end{aligned} \tag{6.16}$$

for any sufficiently large t.

The second term in (6.15), thanks to Lemma 6.3 is not greater than $A(f)e^{-2t}$. The last term is bounded by the RHS of (6.14) which, if $k_1 > 3\alpha^{-1}$, is bounded by $A'(f)e^{-2t}$ for large t. This concludes the proof (3.4).

Proof of part (b). Define L_t as in part (a) and, for any $\varepsilon \in (0,1)$, let $\Theta(t,\varepsilon)$ be the set of interactions J such that

(i) $c_s(L^J_{\Lambda_t}) \leq L_t^\varepsilon$

(ii) $SMT(\Lambda_t, (\log L_t)^{\frac{d}{d-1}}, \widehat{m}(\alpha))$.

We can write, for any $\tau \in \Omega$,

$$\begin{aligned} \mathbb{E}\|T^J(t)f - \mu^J(f)\|_\infty \leq |||f|||\,\mathbb{P}\big(\Theta(t,\varepsilon)^c\big) + \sup_{J\in\Theta(t,\varepsilon)} \|T^{J,\tau}_{\Lambda_t}(t)f - \mu^{J,\tau}_{\Lambda_t}(f)\|_\infty + \\ + \|T^J(t)f - T^{J,\tau}_{\Lambda_t}(t)f\|_\infty + \sup_{J\in\Theta(t,\varepsilon)} |\mu^{J,\tau}_{\Lambda_t}(f) - \mu^J(f)| \end{aligned} \tag{6.17}$$

We denote by X_1, X_2, X_3 and X_4 the four terms on the RHS of (6.17). For the last two terms we can proceed as in part (a) and we get

$$X_3 + X_4 \le (A(f) + A'(f))\, e^{-2t} \tag{6.18}$$

Furthermore, we have

$$\begin{aligned} \mathbb{P}\big(\Theta(t,\varepsilon)^c\big) \le \mathbb{P}\{\, c_s(L^J_{\Lambda_t}) \ge L^\varepsilon \,\} + \\ + \mathbb{P}\{\, SMT(\Lambda_t, (\log L)^{\frac{d}{d-1}}, \widehat{m}(\alpha)) \text{ does not hold } \} \end{aligned} \tag{6.19}$$

Of the above two terms the first one is estimated via (ii) of Theorem 3.1, which implies

$$\mathbb{P}\{\, c_s(L^J_{\Lambda_t}) \ge L^\varepsilon \,\} \le \exp\big[-C_3\, (\log\log L_t)^{-d} (\log L_t)^{\frac{d}{d-1}}\big] \tag{6.20}$$

provided that t is large enough. The second term in the RHS of (6.19) can be bounded from above, using Proposition 4.1, by the probability that there exists a cube $Q_l(x)$ in Λ_t, with $l = \lceil (\log L)^{\frac{d}{d-1}} \rceil$, which is not $\alpha-$regular. Using Proposition 4.8 such a probability is bounded from above by

$$L_t^d \exp[-\vartheta'\, (\log L_t)^{\frac{d}{d-1}}] \tag{6.21}$$

provided that t is so large that $L_t \ge L_0$. In this way we have obtained

$$X_1 \le |||f|||\, \Big[\exp[\,-C_3\, (\log\log L_t)^{-d} (\log L_t)^{\frac{d}{d-1}}] + L_t^d \exp[-\vartheta (\log L_t)^{\frac{d}{d-1}}] \Big] \tag{6.22}$$

As for X_2, we use hypercontractivity (see again the proof of Theorem 4.1 in [GZ1]) and the fact that now $c_s(L^J_{\Lambda_t}) \le L_t^\varepsilon$, and we get

$$X_2 \le 2\, |||f|||\, \exp\big[\,-k'\, t^{1-\varepsilon}\,\big] \tag{6.23}$$

for any t sufficiently large. From (6.18), (6.22) and (6.23) we get that for large t the dominant term in (6.17) is the first one and, by consequence (3.6) follows. $\square$

6.2. Proof of the lower bound, Theorem 3.3

The main idea behind the proof is not new (see [DRS]) and it can be summarized as follows. If *all* the couplings J_{xy} in the cube B_L are above the critical value for the standard Ising model, then the spin at the origin reaches the equilibrium after a time t which is at least relaxation time of the cube B_L. Since the relaxation time for the stochastic Ising model in a cube B_L, at low temperature and zero external field, grows like the exponential of the *surface* L^{d-1}, it follows that if $L \approx (\log t)^{\frac{1}{d-1}}$, then at time t the spin at the origin has not yet equilibrated. To complete the argument one has to observe that, under our assumptions, the probability that the J_{xy}'s in B_L are all equal and large is not smaller than an exponential of the volume L^d.

Let us now provide the details. Given $J_1 > 0$ and a positive integer L, let $\Lambda = B_L$ and let $\bar\Theta$ be the set of all interactions $J \in \Theta$ such that hypothesis (i) holds and

(a) $J_{xy} = J_1$ for all $\{x,y\}$ such that $\{x,y\} \subset \Lambda$
(b) $|J_{xy}| \le \frac{1}{4}$ for all $\{x,y\}$ which intersect both Λ and Λ^c (the boundary edges)

If we denote with $m_\Lambda = |\Lambda|^{-1}\sum_{x\in\Lambda}\sigma(x)$ the normalized magnetization in Λ, we can write (remember that $\mu^J(\pi_0) = 0$)

$$\mathbb{E}\,\|T^J(t)\pi_0\|_{L^2(\mu^J)} \ge \mathbb{E}\,\|T^J(t)\,m_\Lambda\|_{L^2(\mu^J)} \ge \mathbb{P}(\bar\Theta)\,\inf_{J\in\bar\Theta}\|T^J(t)\,m_\Lambda\|_{L^2(\mu^J)} \tag{6.24}$$

Choose $J \in \bar\Theta$ and let

$$F_\Lambda = \{\sigma\in\Omega:\; m_\Lambda(\sigma) > \tfrac{1}{2}\}$$

Then we have

$$\|T^J(t)\,m_\Lambda\|_{L^2(\mu^J)} \ge \sqrt{\mu(F_\Lambda)}\;\|T^J(t)\,m_\Lambda\|_{L^2(\mu^J(\cdot\,|\,F_\Lambda))}$$

and

$$\|T^J(t)\,m_\Lambda\|_{L^2(\mu^J(\cdot\,|\,F_\Lambda))} \ge \|T^J(t)\,m_\Lambda\|_{L^1(\mu^J(\cdot\,|\,F_\Lambda))} \ge \mu(T^J(t)\,m_\Lambda\,|\,F_\Lambda) \tag{6.25}$$

For $\sigma\in\Omega$, let $\{\eta_t^\sigma\}_{t\ge0}$ be the process associated with $T^J(t)$ with initial condition $\eta_0^\sigma = \sigma$, and let $\{\eta_t^\mu\}_{t\ge0}$ be the stationary process (the one with initial distribution μ^J). Consider the events

$$G^\sigma_{\Lambda,t} \equiv \{\,\exists s\in[0,t]:\; |m_\Lambda(\eta_s^\sigma) - 1/2| \le 1/(100)\,\} \qquad \sigma\in\Omega\cup\{\mu\}$$

For each $\sigma\in F_\Lambda$, if $|\Lambda| > 100$, we have

$$m_\Lambda(\eta_t^\sigma) \ge \frac{1}{2}\mathbb{1}_{(G^\sigma_{\Lambda,t})^c} - \mathbb{1}_{G^\sigma_{\Lambda,t}} = \frac{1}{2} - \frac{3}{2}\mathbb{1}_{G^\sigma_{\Lambda,t}}$$

which implies

$$\mu(T^J(t)\,m_\Lambda\,|\,F_\Lambda) \ge \frac{1}{2} - \frac{3}{2}\int_\Omega \mu(d\sigma\,|\,F_\Lambda)\,\mathrm{Prob}(G^\sigma_{\Lambda,t}) \ge \frac{1}{2} - \frac{3}{2}\mu(F_\Lambda)^{-1}\,\mathrm{Prob}(G^\mu_{\Lambda,t}) \tag{6.26}$$

If $t_1, t_2, \ldots$ are the (random) times at which the stationary process η_t^μ is updated inside Λ and n_t is the number of updates up to time t, we have, for all $j\in\mathbb{Z}_+$,

$$P(G^\mu_{\Lambda,t}) \le j\mu^J\{|m_\Lambda(\sigma) - 1/2| \le 1/(100)\} + \mathrm{Prob}\{n_t > j\} \tag{6.27}$$

which, taking $j = k|\Lambda|t$ with $k = 2c_M$, can be bounded by (remember that we have $\kappa_2 = 0$)

$$k\,|\Lambda|\,t\;\mu^J\{|m_\Lambda(\sigma) - 1/2| \le 1/(100)\} + e^{-k'|\Lambda|t} \tag{6.28}$$

for a suitable positive constant k'.

The idea is now to

(1) replace μ^J with $\mu_\Lambda^{J_1,\emptyset}$, *i.e.* with the Ising Gibbs measure in Λ with coupling J_1 and free boundary conditions. Thanks to the properties (a) and (b) of $\bar\Theta$ and the DLR condition, the price to pay can be estimated as

$$e^{-|\partial^+\Lambda|/2}\mu_\Lambda^{J_1,\emptyset}(X) \le \mu^J(X) \le e^{|\partial^+\Lambda|/2}\mu_\Lambda^{J_1,\emptyset}(X) \qquad \forall X \in \mathcal{F}_\Lambda \tag{6.29}$$

(2) use the following key result of [P] for the large deviations of the magnetization for the $d-$dimensional Ising model in Λ without external field and with free boundary conditions.

Theorem 6.4. *There exists* $J_1 > 0$ *such that*

$$\mu_\Lambda^{J_1,\emptyset}\{m_\Lambda(\sigma) \ge 1/2\} \ge \frac{1}{3}$$
$$\mu_\Lambda^{J_1,\emptyset}\{|m_\Lambda(\sigma) - 1/2| \le 1/(100)\} \le e^{-2|\partial^+\Lambda|}$$

Proof. See [P]

Remark. The results of [P] are stated for the standard Ising model, namely when the couplings J_{xy} are all *equal* and large enough. We expect the same result to hold also when the J_{xy}'s are not all equal, but just large enough.

Choose now J_1 as in Lemma 6.4, and take $L = L_t$ as the smallest integer for which $|\partial^+\Lambda| \ge 2\log t$. In this way we find

$$\mu(T^J(t)\,m_\Lambda\,|\,F_\Lambda) \ge \frac{1}{2} - \frac{9}{2}\,k\,|\Lambda|\,t\,e^{-|\partial^+\Lambda|} - \frac{9}{2}e^{-k'|\Lambda|t+|\partial^+\Lambda|/2} \ge \frac{1}{3} \tag{6.30}$$

for all t large enough. From (6.24) ... (6.30) it follows

$$\mathbb{E}\,\|T^J(t)\pi_0\|_{L^2(\mu^J)} \ge \frac{1}{3}e^{-|\partial^+\Lambda|/2}\,\mathbb{P}(\bar\Theta) \ge \exp\big[-k''\,(\log t)^{\frac{d}{d-1}}\,\big]$$

for a suitable positive constant k''. $\square$

A1. Appendix 1

Given two subsets A, B of $\mathbb{Z}^d$ we let

$$\delta_r(A,B) = (\partial_r^+ A \cap B) \cup (\partial_r^+ B \cap A)$$

Proposition A1.1. *For each $d, r \in \mathbb{Z}_+$, there exists $k(d,r)$ such that for each $V \subset\subset \mathbb{Z}^d$, and for each $v \in [0, |V|]$ there exists $X_v \subset V$ such that, if we let $S \equiv \lceil 2|V|^{\frac{d-1}{d}} \rceil$ and $Y_v \equiv V \backslash X_v$, we have*

(a) $v - kS \le |X_v| \le v$

(b) $\delta_r(X_v, Y_v) \le kS$

(c) $X_0 = \emptyset$, $X_{|V|} = V$ *and* $X_v \subset X_w$ *if* $v < w$.

Proof. Given $V \subset\subset \mathbb{Z}^d$, we define the *$i-$width* of V as the smallest $k \in \mathbb{Z}_+$ such that there exists $n \in \mathbb{Z}$ with the property that, for all $x \in V$, we have $x_i \in \{n, \dots, n+k-1\}$ where x_i is the i^{th} coordinate of x. We start with the following result

Lemma A1.2. *Let V be a finite subset of $\mathbb{Z}^d$, let $i \in \{1, \dots, d\}$ and let $a \in [0, |V|]$. Let L, S be two positive numbers such that $LS > |V|$. Then there exist $k = k(d) > 0$ and two disjoint subsets of V, W_1 and W_2 such that*

(a) $|W_1| \le a$

(b) The $i-$width of W_2 is less than or equal to L

(c) $|\delta_1(W_1, V\backslash W_1)| \le kS$ *and* $|\delta_1(W_2, V\backslash W_2)| \le kS$

(d) If $W_2 = \emptyset$ then $|W_1| \ge a - S$, while, if $W_2 \ne \emptyset$, then $|W_1 + W_2| > a$.

Proof. If $a = |V|$, we take $W_1 = V$ and $W_2 = \emptyset$ and the Lemma follows. Assume now that $a < |V|$. For $j \in \mathbb{Z}$, let

$$V_j^{(i)} = \{x \in V : x_i = j\} \tag{A1.1}$$

and

$$m = \inf\{j \in \mathbb{Z} : |\bigcup_{k \le j} V_k^{(i)}| > a\}$$

Since $a < |V|$, m is always finite. If $|V_m^{(i)}| \le S$ we set

$$W_1 = \bigcup_{j<m} V_j^{(i)} \qquad W_2 = \emptyset \tag{A1.2}$$

Then properties (a), (b) and (d) are trivial, while, in order to prove (c), we observe that

$$\delta_1(W_1, V\backslash W_1) \subset V_m^{(i)} \cup \{\, x \in \mathbb{Z}^d : d(x, V_m^{(i)}) \le 1 \,\}$$

which implies

$$|\delta_1(W_1, V\backslash W_1)| \le |V_m^{(i)}|(1 + 3^d) \le (3^d + 1)S$$

Consider now the case when $|V_m^{(i)}| > S$. Define

$$m_1 = \sup\{j < m : |V_j^{(i)}| \le S\} \qquad m_2 = \inf\{j > m : |V_j^{(i)}| \le S\}$$

Let then

$$W_1 = \bigcup_{j \le m_1} V_j^{(i)} \qquad W_2 = \bigcup_{j=m_1+1}^{m_2-1} V_j^{(i)}$$

The statements (a) – (d) are easily verified. $\square$

We now prove Proposition A1.1 when $r = 1$ and then we will show that this is enough to treat the case of r arbitrary. We let

$$f_a^{(i)}(V) = W_1 \qquad g_a^{(i)}(V) = W_2$$

where W_1 and W_2 are the subsets of V found in the previous lemma for given values of a and i, with L and S chosen as

$$S \equiv \lceil 2|V|^{\frac{d-1}{d}} \rceil \qquad L \equiv \lceil |V|^{\frac{1}{d}} \rceil$$

Then we define

$$A_1 = f_v^{(1)}(V) \qquad D_1 = g_v^{(1)}(V)$$

and recursively

$$D_i = g_{v-|A_{i-1}|}^{(i)}(D_{i-1}) \qquad E_i = f_{v-|A_{i-1}|}^{(i)}(D_{i-1}) \qquad A_i = A_{i-1} \cup E_i \tag{A1.3}$$

We let then $k = \min\{i : D_i = \emptyset\}$ and we claim that Proposition A1.1 holds with $X_v = A_k$. We observe that necessarily $k \le d$. In fact, previous lemma together with the fact that $D_i \subset D_{i-1}$, imply that the $i-$width of D_{d-1} is no greater than L for all $i \in \{1, \ldots, d-1\}$. By consequence, all the slices of D_{d-1} in the direction perpendicular to the d direction (see (A1.1)) have a cardinality not greater than L^{d-1} which is less than S. Thus, by the proof of the lemma (see (A1.2)), it is clear that $D_d = \emptyset$.

Then we check that, for all $i \in \{1, \ldots, k-1\}$ we have $0 \le v - |A_i| \le |D_i|$, so that definitions (A1.3) make sense for $i \le k$. This can be done by induction. From statement (a) of previous lemma, we get

$$|A_i| \le |A_{i-1}| + |E_i| \le |A_{i-1}| + v - |A_{i-1}| = v$$

On the other hand, since $D_i \neq \emptyset$, statement (d) of Lemma A1.2 implies

$$|A_i| + |D_i| = |A_{i-1}| + |E_i| + |D_i| > |A_{i-1}| + v - |A_{i-1}| = v$$

Since $D_k = \emptyset$, Lemma A1.2 implies $|E_k| > v - |A_{k-1}| - S$, thus, by consequence we have $|A_k| > v - S$. Together with the statement $|A_k| \leq v$, this gives part (a) of the Proposition.

In order to prove part (b), we notice that, thanks to Lemma A1.2, we have

$$|\delta_1(E_i, D_{i-1} \backslash E_i)| \leq kS \qquad |\delta_1(D_i, D_{i-1} \backslash D_i)| \leq kS \tag{A1.4}$$

We then claim that

$$\delta_1(A_i, V \backslash A_i) \subset \delta_1(A_{i-1}, V \backslash A_{i-1}) \cup \delta_1(E_i, D_{i-1} \backslash E_i) \cup \delta_1(D_{i-1}, V \backslash D_{i-1}) \tag{A1.5}$$

$$\delta_1(D_i, V \backslash D_i) \subset \delta_1(D_i, D_{i-1} \backslash D_i) \cup \delta_1(D_{i-1}, V \backslash D_{i-1}) \tag{A1.6}$$

Iterating (A1.6) and using (A1.4) we find $|\delta_1(D_i, V \backslash D_i)| \leq kiS$, which, inserted into (A1.5), together with (A1.4), gives $|\delta_1(A_i, V \backslash A_i)| \leq kd^2 S$ which completes the proof of the Proposition. To obtain (A1.5), we write

$$\delta_1(A_i, V \backslash A_i) = \delta_1(A_{i-1}, V \backslash A_i) \cup \delta_1(E_i, V \backslash A_i) \subset \delta_1(A_{i-1}, V \backslash A_{i-1}) \cup \delta_1(E_i, V \backslash A_i)$$

The last term can be written as

$$\begin{aligned} &\delta_1(E_i, V \backslash A_i) = \delta_1(E_i, D_{i-1} \backslash A_i) \cup \delta_1(E_i, V \backslash (A_i \cup D_{i-1})) \subset \\ &\subset \delta_1(E_i, D_{i-1} \backslash E_i) \cup \delta_1(D_{i-1}, V \backslash D_{i-1}) \end{aligned}$$

which proves (A1.5). Furthermore, to get (A1.6), we observe that

$$\delta_1(D_i, V \backslash D_i) = \delta_1(D_i, D_{i-1} \backslash D_i) \cup \delta_1(D_i, V \backslash D_{i-1}) \subset \delta_1(D_i, D_{i-1} \backslash D_i) \cup \delta_1(D_{i-1}, V \backslash D_{i-1})$$

This proves (b). Property (c) follows from the construction.

Finally we want to show that Proposition A1.1 with δ_1 (*i.e.* for $r = 1$) implies that the same result holds for δ_r but with a different constant k. Choose then $v \in [0, |V|]$, let $s = 2r$ and consider the mapping

$$\mathbb{Z}^d \ni x = (x_1, \ldots, x_d) \mapsto \pi(x) \equiv (\lceil x_1/s \rceil, \ldots, \lceil x_d/s \rceil) \in \mathbb{Z}^d$$

Applying the Proposition A1.1 (with $r = 1$) to the set πV, we get that, for each $u \in [0, |\pi V|]$, πV is the disjoint union of two subsets $\pi V = X'_u \cup Y'_u$ such that properties (a), (b) (with $r = 1$) and (c) hold. Let then

$$w = \sup\{\, u \in [\,0, |\pi V|\,] : \ |(\pi^{-1} X'_u) \cap V| \leq v \,\}$$

We claim that, if we define

$$X_v = (\pi^{-1} X'_w) \cap V$$

then (a), (b) and (c) are satisfied.

By definition of v we have $|X_v| \le v$. Now, let $\bar{w} = (w+1) \wedge |\pi V|$ and let $\Delta = X'_{\bar{w}} \backslash X'_w$. Using (a) we get

$$|\Delta| = |X'_{\bar{w}}| - |X'_w| \le \bar{w} - (w - kS) \le (k+1)S \tag{A1.7}$$

Moreover, by considering both cases $\bar{w} = w+1$ and $\bar{w} = |\pi V|$, it easy to verify that $|(\pi^{-1}X'_{\bar{w}}) \cap V| \ge v$. Thus we obtain

$$|X_v| = |(\pi^{-1}X'_{\bar{w}}) \cap V| - |(\pi^{-1}\Delta) \cap V| \ge v - s^d|D| \ge v - (k+1)s^d S$$

which proves (a).

To prove (b) all we need is to observe that

$$\delta_r(X_v, Y_v) \subset \delta_r(\pi^{-1}X'_w, \pi^{-1}Y'_w) \subset \pi^{-1}\delta_1(X'_w, Y'_w)$$

which implies

$$|\delta_r(X_v, Y_v)| \le s^d kS$$

The proof of (c) is straightforward. $\square$

A2. Appendix 2

Proposition A2.1. *Take the transition rates as in (2.11). Let $l \in \mathbb{Z}_+$ and let V be a multiple of Q_l, i.e.*

$$V = \bigcup_{i=1}^{s} Q_l(x_i)$$

with $x_i \in l\mathbb{Z}^d$. Assume that each $Q_l(x_i)$ for $i = 1, \dots, s$ is $\alpha-$regular for some $\alpha > 0$. Then, if l is larger than some $\bar{l}(d, r, \alpha)$, we have

$$\mathrm{gap}(L_V^{J,\tau}) \geq 8|V|^{-\omega} \exp(-k'ml^d)$$

where $\omega = d \log 4/\log(3/2)$, $k' = 9^{d-1}k$ and k is the constant given in Theorem 4.12.

Proof. We can assume that V is $r-$connected, since, otherwise one could just consider the $r-$connected components of V. Since V is a multiple of Q_l, this implies, if $l > r$, that V is actually connected (*i.e.* $1-$connected). Choose $l \in \mathbb{Z}_+$ and let, for $n = 0, 1, 2, \dots$

$$R_n = \big([0, a_{n+1}) \times [0, a_{n+2}) \times \cdots \times [0, a_{n+d}) \big) \cap \mathbb{Z}^d$$

where $a_n = 6lb^n$ and $b = (3/2)^{1/d}$. Let C_n^* be the set of all volumes $V \subset \mathbb{Z}^d$ such that
(1) V is a multiple of Q_l
(2) $V \subset R_n$ modulo translations and permutations of the coordinates.

Let also $\Theta_r(V, l, \alpha)$ be the set of all interactions J such that each $Q_l(x_i) \subset V$ with $x_i \in l\mathbb{Z}^d$ is $\alpha-$regular. Define

$$g_n = \inf_{\tau \in \Omega} \inf_{V \in C_n^*} \inf_{J \in \Theta_r(V,l,\alpha)} \mathrm{gap}(L_V^{J,\tau})$$

Thanks to the $\alpha-$regularity we know that

$$\|J\|_x \leq 100^{-1} ml \equiv J_0 \qquad \forall x \in V \tag{A2.1}$$

where, as usual, we have set $m = \widehat{m}(\alpha)$ (see Proposition 4.1).

We will show that for all $n \geq 1$ we have

$$g_n \geq \frac{1}{4} g_{n-1} \tag{A2.2}$$

which implies $g_n \geq 4^{-n} g_0$. Using Theorem 4.12 to estimate g_0 we get

$$g_n \geq 4^{-n} \frac{1}{2} \exp(-kJ_0 |R_0|^{\frac{d-1}{d}}) \tag{A2.3}$$

Once we have (A2.3) the proposition easily follows from (A2.1) and from the following observations:

(1) $|R_0| \le (6lb^d)^d = (9l)^d$
(2) Since V is connected, if $V \in C_n^* \backslash C_{n-1}^*$ then $|V| \ge a_n = 6l(3/2)^{n/d}$, which implies

$$4^{-n} \le (6l)^\omega \, |V|^{-\omega}$$

So we are left with the proof of (A2.2). For this purpose we want to use (5.5) and Proposition 4.5. So we let

$$p_1 = \sup\{s \in l\mathbb{Z} : s \le a_n\}$$
$$p_2 = \inf\{s \in l\mathbb{Z} : s \ge a_{n+d} - a_n\}$$

and

$$A = \{x = (x_1, \dots, x_d) \in V : x_d < p_1\} \qquad B = \{x = (x_1, \dots, x_d) \in V : x_d \ge p_2\}$$

Both A and B are in C_{n-1}^*, so $\text{gap}(L_A^{J,\tau}) \ge g_{n-1}$ and the same holds for B. Moreover, if we let $A_0 = A \cap \partial_l^+ B$, $B_0 = B \cap \partial_r^+ A$, we get

$$d(A_0, B_0) = p_1 - p_2 + 1 \ge 2a_n - a_{n+d} - 2l \ge l\,(3/2)^{n/d}$$

One can then check that, thanks also to (A2.1) and Proposition 4.1, the hypotheses of Proposition 4.5 are satisfied, so that the gap for the block dynamics on $\{A, B\}$ is at least 1/2. Combining this fact with formula (5.5) we get (A2.2) $\square$

References

[ACCN] M. Aizenman, J. T. Chayes, L. Chayes and C. M. Newman: *The phase boundary in dilute and random Ising and Potts ferromagnets.* J. Phys. A: Math. Gen. **20**, L313 (1987)

[B1] A. J. Bray: *Upper and lower bounds on dynamic correlations in the Griffiths phase.* J. Phys. A: Math. Gen. **22**, L81 (1989)

[B2] A. J. Bray: *Dynamics of dilute magnets above T_c.* Phys. Rev. Lett. **60**, No 8, 720 (1988)

[BD] L. A. Bassalygo and R. L. Dobrushin: *Uniqueness of a Gibbs field with random potential–an elementary approach.* Theory Prob. Appl. **31**, 572 (1986)

[Be] J. van den Berg: *A constructive mixing condition for 2-D Gibbs measures with random interactions.* Preprint 1996

[BM] J. van den Berg and C. Maes: *Disagreement percolation in the study of Markov fields.* Ann. Prob. **22**, 749 (1994)

[CM] F. Cesi and F. Martinelli: *On the Layering Transition of an SOS Surface Interacting with a Wall. I. Equilibrium Results.* J. Stat. Phys., **82**, no 3/4, 823 (1996)

[CGMS] F. Cesi, G. Guadagni, F. Martinelli and R. Schonmann: *On the 2D Stochastic Ising Model in the Phase Coexistence Region Near the Critical Point.* J.. Stat. Phys. to appear

[CMM] F. Cesi, C. Maes and F. Martinelli: *Relaxation to equilibrium for two dimensional disordered Ising models in the Griffiths phase.* in preparation

[D] R. L. Dobrushin: *A formula of full Semiinvariants.* In 'Cellular Automata and Cooperative Systems', N. Boccara, E. Goles, S. Martinez and P. Picco (eds.), Kluwer Acad. Publ., Dordrecht-Boston-London, pp. 135-140 (1993).

[DS] R. L. Dobrushin and S. Shlosman: *Constructive criterion for the uniqueness of Gibbs fields.* Statistical Physics and Dynamical Systems. Fritz, Jaffe and Szász eds. Birkhauser 347 (1985)

[DeSt] J. D. Deuschel and D. W. Stroock: *Large deviations.* Academic Press, Series in Pure and Applied Mathematics, **137**, (1989)

[DKP] H. Von Dreyfus, A. Klein and J. F. Perez: *Taming Griffiths singularities: infinite differentiability of quenched correlations functions.* Commun. Math. Phys. **170**, 21 (1995)

[DRS] D. Dhar, M. Randeria and J. P. Sethna: *Griffiths singularities in the dynamics of disordered Ising models.* Europhys. Lett., **5**, No 6, 485 (1988)

[F] J. Fröhlich: *Mathematical aspects of the physics of disordered systems.* in "Critical Phenomena, Random Systems, Gauge Theories" Eds. K.Osterwalder and R. Stora, Elsevier (1986)

[FI] J. Fröhlich and J. Z .Imbrie: *Improved perturbation expansion for disordered systems: beating Griffiths singularities.* Commun. Math. Phys. **96**, 145 (1984)

[G] R. Griffiths: *Non–analytic behaviour above the critical point in a random Ising ferromagnet.* Phys. Rev. Lett. **23**, 17 (1969)

[GM1] G. Gielis and C. Maes: *Percolation Techniques in Disordered Spin Flip Dynamics: Relaxation to the Unique Invariant Measure.* Commun. Math. Phys. 177, 83 (1996).

[GM2] G. Gielis and C. Maes: *Local analyticity and bounds on the truncated correlation functions in disordered systems.* Markov Proc. Relat. Fields **1**, 459 (1995)

[GM3] G. Gielis and C. Maes: *The Uniqueness regime of Gibbs Fields with Unbounded Disorder.* J. Stat. Phys. **81**, 829 (1995)

[GZ1] A. Guionnet and B. Zegarlinski: *Decay to equilibrium in random spin systems on a lattice.* Preprint 1996

[GZ2] A. Guionnet and B. Zegarlinski: *Decay to equilibrium in random spin systems on a lattice II.* Preprint 1996

[HS] R. A. Holley and D. W. Strook: *Uniform and L^2 convergence in one dimensional stochastic Ising models.* Commun. Math. Phys. **123**, 85 (1989)

[J] S. Jain: *Anomalously slow relaxation in the diluted Ising model below the percolation threshold.* Physica A, **218**, 279 (1995)

[L] T. M. Ligget: *Interacting particles systems.* Springer–Verlag (1985)

[M] F. Martinelli: *On the two dimensional dynamical Ising model in the phase coexistence region.* J. Stat. Phys. **76**, No. 5/6, 1179 (1994)

[MO1] F. Martinelli and E. Olivieri: *Approach to equilibrium of Glauber dynamics in the one phase region I: The attractive case.* Commun. Math. Phys. **161**, 447 (1994)

[MO2] F. Martinelli and E. Olivieri: *Approach to equilibrium of Glauber dynamics in the one phase region II: The general case.* Commun. Math. Phys. **161**, 487 (1994)

[O] E. Olivieri: *On a cluster expansion for lattice spin systems and finite size condition for the convergence.* J. Stat. Phys. **50**, 1179 (1988)

[OP] E. Olivieri and P. Picco: *Cluster expansion for D–dimensional lattice systems and finite volume factorization properties.* J. Stat. Phys. **59**, 221 (1990)

[OPG] E. Olivieri, F. Perez and S. Goulart–Rosa–Jr.: *Some rigorous results on the phase diagram of the dilute Ising model.* Phys. Lett. **94A**, No 6,7, 309 (1983)

[P] A. Pisztora: *Surface order large deviations for Ising, Potts and percolation models.* Probab. Th. Rel. Fields **104**, 427 (1996)

[RSP] M. Randeria, J. P. Sethna and R. G. Palmer: *Low–frequency relaxation in Ising spin–glasses.* Phys. Rev. Lett. **54**, No. 12, 1321 (1985)

[S] R. H. Schonmann : *Slow droplet–driven relaxation of stochastic Ising Models in the vicinity of the phase coexistence region.* Commun. Math. Phys. **170**, 453 (1995)

[SZ] D. W. Stroock and B. Zegarlinski: *The logarithmic Sobolev inequality for discrete spin systems on a lattice.* Commun. Math. Phys. **149**, 175 (1992)

[Z] B. Zegarlinski: *Strong decay to equilibrium in one dimensional random spin systems.* J. Stat. Phys. **77**, 717 (1994)