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Effect of random defects on the critical behaviour of Ising models †

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Abstract. A cumulant expansion is used to calculate the transition temperature of Ising models with random-bond defects. For a concentration, x, of missing interactions in the simple-square Ising model we find $-T_c^{-1}\,dT_c/dx\,|_{x=0} = 1{\cdot}329$ compared with the mean-field value of one. If the interactions are independent random variables with a width $\delta J/J \equiv \epsilon$, the result is $-T_c^{-1}\,dT_c/d\epsilon^2\,|_{\epsilon=0} = 0{\cdot}312$ compared with the mean-field result of zero. An approximation yields the specific heat in the critical régime as $C \sim C_0/(1 + x\gamma^2 C_0)$, where γ is a constant and C_0 is the unperturbed specific heat at a renormalized temperature. Thus, the specific heat divergence is broadened over a temperature interval ΔT, with $\Delta T/T_c \sim x^{(1/\alpha)}$, where α is the critical exponent for the specific heat, and a maximum value of order x^{-1} is attained. Heuristic arguments show that this smoothing effect occurs if $\alpha > 0$. The relation of our work to that of McCoy and Wu is discussed.

1. Introduction

The nature of collective excitations in impure systems has been the subject of many studies (Murray 1966, Svensson *et al* 1972, Kumar and Harris 1972, Walker *et al* 1972) and a number of exact results for the static properties have been announced (Elliott *et al* 1960, McCoy and Wu 1968, Griffiths 1969, McCoy 1969). Unfortunately, none of these exact results answer the simplest question of experimental interest, namely, how does T_c, the temperature below which long-range order appears, depend on the fraction of sites, x, in a ferromagnet which are occupied by magnetic ions. For a pure system one normally uses a high-temperature expansion to locate $T_c(x)$. However, according to Griffiths' (1969) theorem the thermodynamic functions in the diluted system are not analytic for $T < T_c(1)$, regardless of the behaviour of $T_c(x)$. Therefore, simply examining the radius of convergence of the high-temperature expansions in the dilute system (Rushbrooke and Morgan 1961, Rapaport 1972a, Rushbrooke *et al* 1972) is not, in principle, a valid way to determine $T_c(x)$, although it does give reasonable results.

In this paper we consider Ising models with 'bond' (that is, force constant) defects rather than site defects, since calculations for the former are more reliable than for the latter. The model for which we obtain the most comprehensive results is one in which a fraction, x, of interactions are randomly removed from an Ising system with nearest-

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neighbour interactions. The other model we study is a 'glass' in which each interaction is taken to be an independent random variable with a width in energy ϵJ which is much smaller than the average energy J. We calculate the free energy for these two models in powers of the defect concentration x or the defect strength ϵ, respectively, using a cumulant formulation.

Since our results are based on a resummation of perturbation theory they are not rigorously established. However, we believe that our results for the shift in T_c to lowest order in x or ϵ, respectively, are exact. We find that even for the Ising model the limiting behaviour of T_c differs significantly from the predictions of molecular-field theory (Néel 1936, Behringer 1957). To lowest order in x our results agree with those obtained by other methods (Osawa 1973, Rapaport 1972a).

To obtain higher-order corrections to T_c one needs a theory which is valid for temperatures such that $|T - T_c| \ll x$ for the 'bond' model, for instance. Accordingly, we give an approximate theory which is valid for $|T - T_c(x)| \gg x^{1+\eta}$, where η is a small positive constant. Although the use of this theory throughout the critical régime can not be justified *a priori*, it does give reasonable results and hence we propose it as a simple approximation for critical properties in impure Ising systems. According to this theory, the specific heat, which is of order $|T - T_c|^{-\alpha}$ for the pure system (Fisher 1966), is of order $|T - T_c(x)|^{-\alpha}$ in the impure system as long as $|T - T_c(x)|^{-\alpha} \leqslant x^{-1}$. However, for $|T - T_c(x)|^{-\alpha} > x^{-1}$, that is for $|T - T_c(x)| < x^{1/\alpha}$, the specific heat remains bounded, reaching a maximum value of order x^{-1}. This result agrees with arguments based on estimates of the largest value of the correlation length which is self-consistently stable with respect to configurational fluctuations. These arguments are corroborated by using them to estimate the width of the critical régime in the model of McCoy and Wu (1968), who carried out a rigorous analysis of this régime. This picture of the critical régime differs from that of Watson (1970) or Domb (1972) who both see no evidence for such a smoothing of the singularities due to configurational fluctuations.

2. Shift in T_c for the 'bond' model to first order in defect concentration

In this section we will calculate the shift in the transition temperature of a dilute Ising model correct to leading order in the defect concentration. The model we consider is the so-called 'bond' model, which is governed by the hamiltonian,

$$\mathcal{H} = -J\sum_i (1 - p_i)\sigma(r_i)\sigma(s_i) \tag{2.1a}$$

$$\equiv J\sum_i (1 - p_i)h(i) \tag{2.1b}$$

where the ith term in the hamiltonian, $h(i)$, is an interaction between nearest-neighbour spins r_i and s_i. Here the random variables p_i assume the values 1 or 0 depending on whether the interaction $h(i)$ is or is not removed and the σ assume the values 1 and -1.

The cluster expansion (Horwitz and Callen 1961) for the configurationally-averaged free energy F is written in terms of the configurational averages, denoted by $\langle\!\langle\ \rangle\!\rangle$, as

$$F = F_0 + \sum_i \langle\!\langle p_i \rangle\!\rangle F(i) + \sum_{i<j} \langle\!\langle p_i p_j \rangle\!\rangle F(i,j) + \sum_{i<j<k} \langle\!\langle p_i p_j p_k \rangle\!\rangle F(i,j,k) + \cdots. \tag{2.2}$$

Here F_0 is the free energy of the pure ($p_i = 0$) system, and the other variables F are 'additive' cumulants defined by

$$F(i) = -k_B T \ln \langle \rho(i) \rangle \equiv -k_B T \ln \rho \tag{2.3a}$$

$$F(i,j) = -k_B T \ln \langle \rho(i)\rho(j) \rangle - F(i) - F(j) \tag{2.3b}$$

etc, where $\rho(i) = \exp(Jh(i)/k_B T)$ and

$$\langle A \rangle \equiv \frac{\mathrm{Tr}[A \exp(-\mathscr{H}_0/k_B T)]}{\mathrm{Tr}[\exp(-\mathscr{H}_0/k_B T)]} \tag{2.4}$$

is an unperturbed average, where $\mathscr{H}_0$ is the hamiltonian for the pure system. We treat a random alloy with an average concentration of bond defects, x, so that the configurational average of n different p's is x^n.

We express the averages $\langle \Pi_{k=1}^n \rho(i_k) \rangle$ in terms of the 'multiplicative' cumulants, $\langle \Pi_{k=1}^n \rho(i_k) \rangle^c$, as

$$\langle \rho(i_1) \rangle = \rho \tag{2.5a}$$

$$\langle \rho(i_1)\rho(i_2) \rangle = \rho^2 + \langle \rho(i_1)\rho(i_2) \rangle^c \tag{2.5b}$$

etc, and since $h(i)^2 = 1$ we set

$$\rho(i) = c + s\,h(i) \tag{2.6}$$

where $c \equiv \cosh(J/k_B T)$ and $s \equiv \sinh(J/k_B T)$. Using the cluster property of cumulants (namely that they vanish when any two arguments are infinitely separated from one another) in conjunction with equation (2.6) we obtain the result

$$\langle \prod_{k=1}^n \rho(i_k) \rangle^c = s^n \langle \prod_{k=1}^n h(i_k) \rangle^c, \qquad n \geqslant 2. \tag{2.7}$$

Thus we obtain an expansion of the form

$$\ln \langle \prod_{k=1}^n \rho(i_k) \rangle = n \ln \rho + \ln \{1 + \sum_{\{j\}'} \sum_{\{p\}} \prod_{k=2}^{\infty} [(p_k!)^{-1}(k!)^{-p_k} \times \prod_{l=1}^{p_k} \langle \gamma^k \prod_{m=1}^{k} h(j_{l,m}) \rangle^c]\} \tag{2.8}$$

where $\gamma = s/\rho = s/(c + s\langle h \rangle)$ and the notation $\{j\}'$ indicates that the j's are to be summed over values within the set $i_1, i_2, \ldots, i_n$ with the restriction that no two j's coincide.

We now expand the logarithm in equation (2.8) in powers of γ. The subtraction of the lower-order partial free energies in the cumulant expansion of equation (2.3) can be accomplished by retaining only terms which have the cluster property, that is, 'connected' terms. Thus, we evolve the following diagrammatic prescription for the free energy:

1. Draw n dots labelled by the indices $i_1, i_2, \ldots, i_n$.

2. For each cumulant draw a vertex, indicated by a cross and draw lines connecting the vertex to the dots corresponding to the arguments of the cumulant. (Each cumulant has two or more arguments.)

3. The contribution to the free energy from any diagram is given by $2\Sigma g x^n \gamma^m \hat{C}/Nz$ where n is the number of dots, m is the number of lines, $\hat{C}$ is the product of cumulants, N is the total number of sites, z is the number of nearest neighbours, g is a combinatorial factor (discussed further below), and the sum is over all the i's, under the restriction that no two i's coincide.

4. Diagrams which differ only in the labelling of vertices are topologically equivalent.

5. The sum over all topologically inequivalent connected diagrams yields the quantity $\Delta\Omega$, where

$$\Delta\Omega = \Omega - \Omega_0 - x \ln \rho. \tag{2.9}$$

Here and below we introduce dimensionless quantities per bond by $Nz\Omega/2 = -F/k_B T$ for the free energy, $U = -\partial\Omega/\partial(\beta J)$ for the internal energy, and $C = -\partial U/\partial(\beta J)$ for the specific heat, where $\beta^{-1} = k_B T$. Similar definitions are obtained for Ω_0, U_0, and C_0.

We write the combinatorial factor, g, as

$$g(\Gamma) = \frac{G(\Gamma)}{h(\Gamma)} \tag{2.10}$$

where $h(\Gamma)$ is the symmetry factor for the diagram Γ. In general there is no simple way to construct $G(\Gamma)$. However, for tree diagrams (for which removal of any line causes the diagram to be disconnected) we have obtained the result

$$G(\Gamma) = \prod_p (-1)^{n_p - 1}(n_p - 1)! \tag{2.11}$$

where n_p is the number of branches meeting at the dot p.

We wish to calculate the free energy in the critical régime for small x. In lowest order, we expect to find a shifted, and possibly distorted, singularity. Such a result can not be obtained by calculating the free energy to any finite order in x. What we shall do is analogous to resumming perturbation theory to obtain a Dyson equation from which the shift in energy of a pole is obtained. That is, we retain only the most divergent contributions to the free energy from each power of x. Neglected terms will then be of higher order in x than equally divergent terms which are retained.

To find the most divergent contribution of order x^n we note that the correlation length, ξ, becomes infinite as $T \to T_c$. In particular, for the simple-square Ising model Hecht (1967) gives the result

$$\langle h(i)h(j)\rangle^c = \left(\frac{2\kappa}{\pi}\right)^2 [K_1^2(\kappa R_{ij}) - K_0^2(\kappa R_{ij})], \tag{2.12a}$$

$$\approx \left(\frac{2}{\pi}\right)^2 R_{ij}^{-2} \qquad 1 \ll R_{ij} \ll \kappa^{-1}; \tag{2.12b}$$

$$\approx \left(\frac{2}{\pi}\right) e^{-\kappa R_{ij}} R_{ij}^{-2}, \qquad \kappa^{-1} \ll R_{ij}. \tag{2.12c}$$

Here K_0 and K_1 are modified Bessel functions of the second kind, $\kappa = \xi^{-1}$ and is given explicitly as

$$\kappa = \frac{4J|T - T_c|}{kTT_c} \tag{2.13}$$

and $R_{ij} = R_i - R_j$, where distances are measured in units of the lattice constant. More generally, we will make the scaling assumption (M E Fisher, unpublished) that

$$\langle \prod_{k=1}^{n} h(i_k)\rangle^c \approx \varphi(\kappa r_1, \kappa r_2, \ldots \kappa r_n) \prod_{k=1}^{n} r_k^{\nu^{-1}-d} \tag{2.14}$$

where r_i is the position of the ith site relative to the centre of mass of the n points and φ is a function which decays exponentially at larger separation. In equation (2.14) d is the dimensionality of the lattice and ν is the critical index (Fisher 1966) for the correlation length $\kappa \sim |T - T_c|^{\nu}$ for $T \to T_c$. We may use the asymptotic form in equation (2.14) to estimate sums over the i's of cumulants. For instance, we have

$$S_n \equiv \sum_{i_1, i_2, \ldots i_n} \langle \prod_{k=1}^{n} h(i_k)\rangle^c \approx N\frac{z}{2}\int\left(\prod_{k=1}^{n-1} d^d s_k\right)\varphi(\kappa r_1, \kappa r_2, \ldots, \kappa r_r)\left(\prod_{i=1}^{n} r_i^{\nu^{-1}-d}\right) \tag{2.15}$$

where $s_j = r_j - r_n$. Dimensional arguments show that

$$S_n \approx N\frac{z}{2}\kappa^{d-n/\nu} \sim t^{d\nu-n} \tag{2.16}$$

where $t = |T - T_c|/J$. We can evaluate S_n exactly as

$$S_n = N\frac{z}{2}[-\partial/\partial(\beta J)]^{n-1}U_0(\beta J). \tag{2.17}$$

For T near T_c we set $C \sim |T - T_c|^{-\alpha} = t^{-\alpha}$, so that

$$S_n \sim t^{2-n-\alpha}. \tag{2.18}$$

Using the scaling relation (Kadanoff 1966), $d\nu = 2 - \alpha$, we see that equation (2.16) agrees with equation (2.18). For the simple-square Ising model (Onsager 1944) $C \sim \ln|T - T_c|$, so that $\alpha = 0$, and also $\nu = 1$, rigorously confirming equation (2.16) in that case. Arguments similar to those we give here have previously been used by Rapaport (1972b) and Suzuki (1971).

We shall now use these types of estimate to evaluate the dominant contribution of the free energy in the limit of small x and t with x/t finite. We shall refer to this limit as the linear régime. Such an evaluation should enable us to determine the transition temperature correct to first order in x. Consider first the family of diagrams shown in

Figure 1. Family of diagrams with no extra lines.

figure 1 for which the number of lines is equal to the number of dots and denote contribution to $\Delta\Omega$ from this set of diagrams by $\Delta\Omega^{(0)}$. A diagram of this type having n dots gives a contribution to $\Delta\Omega^{(0)}$ of order

$$x^n t^{d\nu-n} = \left(\frac{x}{t}\right)^n t^{d\nu}, \tag{2.19}$$

where we have used the estimate based on equation (2.14). One may use the same technique to estimate the contributions from other diagrams. For example, in figure 2(a) we study the effect of doubly connecting two points i_1 and i_2. The additional two-point correlation function yields a factor of order $(r^{\nu-1-d})^2$ at large separation. One then concludes that the major contribution now comes from the region for which i_1 and i_2 are not far apart. Hence, in applying the estimates of equation (2.14) we note that this diagram has $n - 1$ widely separated spatial arguments. This reasoning indicates that figure 2(a) gives a contribution to $\Delta\Omega$ of order $(x/t)^n t^{d\nu+1}$. The contribution from figure 2(b) can be estimated directly from equation (2.14) to be of order $(x/t)^n t^{2d\nu-1}$. Since we may safely assume that $d\nu > 1$, the conclusion is that the diagrams of figure 2 give contributions to $\Delta\Omega$ which are negligible in comparison with $\Delta\Omega^{(0)}$. More generally, we conclude that in the linear régime diagrams with more than the minimum number of lines can be neglected in comparison with those of figure 1.

To evaluate $\Delta\Omega^{(0)}$ we write

$$\Delta\Omega^{(0)} = \left(\frac{2}{Nz}\right)\sum_{n=2}^{\infty}(x\gamma)^n \sum_{i_1<i_2<\ldots i_n} \langle h(i_1)h(i_2)\ldots h(i_n)\rangle^c. \tag{2.20}$$

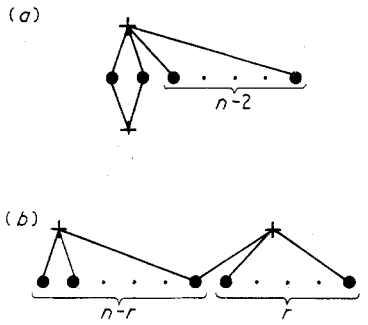

Figure 2. Some diagrams not included in figure 1.

The most singular contribution to $\Delta\Omega^{(0)}$, denoted $\Delta\Omega_0^{(0)}$, is obtained by ignoring the restriction that the i be distinct. Corrections to this approximation involve summations over $n-1$ (or less) distinct i and can be shown to be of relative order t. Thus we write

$$\Delta\Omega_0^{(0)} = \sum_{n=2}^{\infty} (x\gamma)^n (n!)^{-1} \langle h^n \rangle^c \tag{2.21}$$

where we introduce the shorthand notation

$$\langle h^n \rangle^c \equiv \sum_{i_2, i_3, \ldots, i_n} \langle \prod_{k=1}^{n} h(i_k) \rangle^c. \tag{2.22}$$

To evaluate equation (2.21) we use the identity,

$$[-\partial/\partial(\beta J)]^{n-1} \langle A \rangle = \sum_{i_2, i_3, \ldots, i_n} \langle A \prod_{k=2}^{n} h(i_k) \rangle^c \tag{2.23}$$

whereby we may write

$$\langle h^n \rangle^c = [-\partial/\partial(\beta J)]^n \, \Omega_0(\beta J). \tag{2.24}$$

Thus, equation (2.21) yields the result

$$\Delta\Omega_0^{(0)} = \Omega_0(\beta J - x\gamma) - x\gamma U_0(\beta J) - \Omega_0(\beta J) \tag{2.25}$$

and, according to equation (2.9), the free energy is

$$\Omega(\beta J) = \Omega_0(\beta J - x\gamma) + x \ln \rho - x\gamma U_0(\beta J). \tag{2.26}$$

Thus, summing the most divergent contributions in each order of x leads to a renormalization of the temperature scale. The critical behaviour of the diluted system is essentially unmodified, except that the transition now occurs at $k_B T_c(x) = \beta_c(x)^{-1}$, where $\beta_c(x)$ is that value of β for which

$$\beta J - x\gamma = \beta_c J \tag{2.27}$$

where $\beta_c \equiv \beta_c(1)$ is the critical value of β for the pure system. To lowest order in x, equation (2.27) yields

$$T_c(x) = T_c \left[1 - \frac{x\gamma_c k_B T_c}{J} \right], \tag{2.28a}$$

$$\equiv T_c[1 - xs'], \tag{2.28b}$$

where $\gamma_c \equiv \gamma(\beta_c J)$.

For the two-dimensional simple-square Ising model Onsager (1944) gives $-U_0(\beta_c J) = 2^{-1/2}$ and $k_B T_c = 2{\cdot}269\,J$, which yields $s' = 1{\cdot}329$ which is significantly larger than the molecular field result $s' = 1$. For the simple-cubic Ising model we may use the approximate results of Baker (1961) that $k_B T_c = 4{\cdot}510J$ and of Fisher and Burford (1967) that $-U_0(\beta_c J) = 0{\cdot}3284$. These values yield $s' = 1{\cdot}060$. As expected, the result for three dimensions is closer to the molecular field result than for two dimensions. The result, equation (2.28*a*) is identical to that obtained by Rapaport (1972*a*) for the 'annealed' bond model. This agreement is not surprising since, as Rapaport (1972*a*) mentions, differences between the 'quenched' and 'annealed' models are of order x^2 or higher. This result has also been previously obtained by Osawa and Sawada (1973). They deduced the result from a calculation of the thermodynamics to order x assuming that the effect of vacancies could be represented by a shift in the temperature scale. Our evaluation based on equation (2.21) verifies this hypothesis.

Finally, we should point out that the result of equation (2.26) does not imply that the diluted system passes through a sharp transition at $\beta = \beta_c(x)$. It does indicate that deviations from such behaviour can only occur over a range of temperature, ΔT, such that $\Delta T/T_c \ll x$. Thus, a continuous transition over a temperature interval of width, say $\Delta T/T_c \approx x^2$, would be entirely consistent with equation (2.26). Similar comments apply to the other terms in equation (2.26) which are singular for $T = T_c$. These singularities have a smaller amplitude and could be completely modified by higher-order terms. Also, note that it is a result, not an assumption, of the calculation that over temperature intervals of order $\Delta T/T_c \sim x$ the behaviour of the diluted system is obtained from that of the pure system by a shift of the temperature scale. As we shall discuss later, this is not a universal result.

3. Critical behaviour of the bond model

In §2 we treated the diagrams necessary to evaluate the shift in T_c to order x. Now we wish to develop an approximation suitable for investigating the nature of the critical behaviour over temperature intervals which are small compared with x. To do this we must keep terms in the series of order

$$\left(\frac{x}{t}\right)^n t^{d\nu} t^{\rho}, \tag{3.1}$$

where $\eta = \rho/n$ remains finite as $n \to \infty$. It is clear from a generalization of the estimate of equation (2.20) that higher-order diagrams will yield contributions of the form written in equation (3.1). Accordingly, we now investigate diagrams with n points having r extra lines with r/n finite in the limit $n \to \infty$. The sum of these contributions will give correct results for $xt^{\eta-1} \ll 1$, that is for $x \ll t^{1-\eta}$, where η is or order r/n. For any non-zero η, this condition means that we can expect to obtain the correct results for any finite value of x/t in the limit of small x and t.

To study this problem we restrict ourselves to the asymptotic behaviour for large n. Let us estimate the relative size, for large n, of the various processes corresponding to $r > 0$. Consider the case $r = 2$, corresponding to the various diagrams of figure 3. Diagrams of the type in figure 3(*a*) have a combinatorial factor of order

$$g_a \sim (n_1!\, n_2!\, n_3!)^{-1} \tag{3.2a}$$

where $n = n_1 + n_2 + n_3 + 2$. Those of the type shown in figure 3(b) have a combinatorial factor of order

$$g_b \sim (n_4!\, n_5!)^{-1} \tag{3.2b}$$

with $n_4 + n_5 + 2 = n$. To estimate the relative size of these two quantities we set $n_1 = n_2 = n_3 = n/3$ and $n_4 = n_5 = \frac{1}{2}n$, whence we estimate

$$g_a \sim \frac{3^n}{n!}, \tag{3.3a}$$

$$g_b \sim \frac{2^n}{n!}. \tag{3.3b}$$

A similar argument can be given to show that contributions to $\Delta\Omega^{(0)}$ of order x^n involving the coincidence of a pair of indices involves a factor of order $n^2/n!$, which is smaller than either of the above quantities.

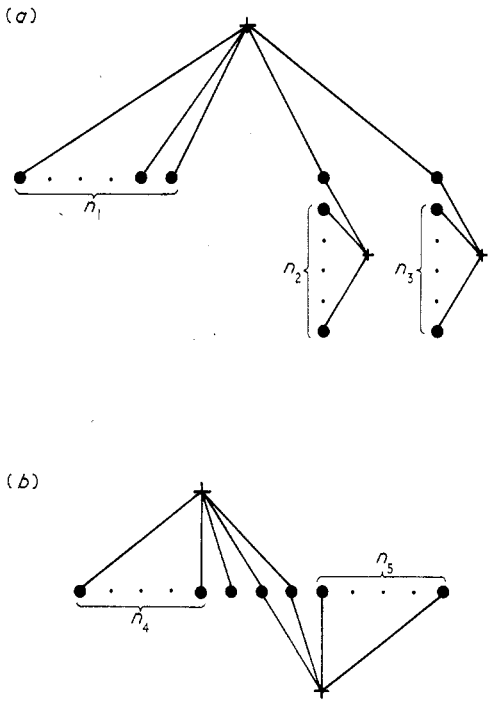

Figure 3. Various ways of adding extra lines to diagrams.

In view of the dominance of the factor g_a, we are led to sum the contributions of diagrams consisting of sections singly connected to one another. An example of such a 'tree' graph is shown in figure 4. Hopefully, summing over these tree graphs will give qualitatively correct results for the critical behaviour. Although our arguments are only convincing for small η, we will use our theory to examine the régime when $t^\alpha \sim x$, which would require extending our justification to the case $\eta = 1$. Although we can not give such a justification, the results we obtain in the régime $t^\alpha \sim x$ are physically reasonable and hence our theory is of interest.

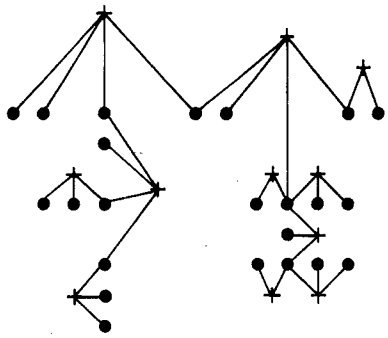

Figure 4. A typical tree graph. In a tree graph there is a unique path between any two given points.

In summing over tree graphs we may use equation (2.11) for $G(\Gamma)$. However, since there is no simple analogue of this relation for $h(\Gamma)$, we proceed indirectly. We will show that

$$\gamma \, \mathrm{d}\Omega_{\mathrm{tree}}/\mathrm{d}\gamma = M \tag{3.4}$$

where Ω_{tree} is the contribution to $\Delta\Omega$ from all tree graphs and M is a diagrammatic function defined below which can be determined more conveniently than Ω_{tree}.

A complete derivation of equation (3.4) including symmetry factors is unwieldy, so we give here a simplified treatment based upon which the interested reader ought to be able to construct a complete proof. The rules for constructing M are similar to those for constructing Ω_{tree} except with regard to the way of counting. Thus, M is calculated from diagrams in which one point, q, is designated as the origin. The combinatorial factor, $\hat{g}(\Gamma)$, for the contribution of Γ to M is given as

$$\hat{g}(\Gamma) = \frac{n_q}{h(\Gamma)} \Big[\prod_p (-1)^{n_p - 1} (n_p - 1)! \Big] \tag{3.5}$$

in the same notation as equation (2.11). Crudely speaking, as q ranges over all points in the diagram Γ, one effectively finds a contribution to M equal to n_l times the contribution of Γ to Ω_{tree}, where n_l is the number of lines (that is, powers of γ) in Γ. This is the content of equation (3.4).

To illustrate these remarks we will calculate M and Ω_{tree} from a few simple tree graphs. Consider first the diagrams of figure 1. The contribution to Ω_{tree}, denoted $\Omega^{(1)}_{\mathrm{tree}}$, from these diagrams was given in equation (2.21). To obtain the corresponding contribution to M, denoted $M^{(1)}$, we note that if there are n lines, there are n choices for the origin. Hence

$$M^{(1)} = \sum_{n=2}^{\infty} n(x\gamma)^n (n!)^{-1} \langle h^n \rangle^c. \tag{3.6}$$

Comparison of this result with equation (2.21) shows that

$$M^{(1)} = \frac{\gamma \mathrm{d}\Omega^{(1)}_{\mathrm{tree}}}{\mathrm{d}\gamma} \tag{3.7}$$

in accord with equation (3.4).

Next we consider the contribution to Ω_{tree} and M from the diagrams of figure 5, denoted $\Omega^{(2)}_{\mathrm{tree}}$ and $M^{(2)}$, respectively. We calculate $M^{(2)}$ as the sum of two terms. The first,

denoted $M_a^{(2)}$, corresponds to placing the origin on the articulation point and the other, denoted $M_b^{(2)}$, to placing the origin on one of the free ends. We write

$$M_a^{(2)} = -x^{-1}[M^{(1)}]^2. \tag{3.8}$$

To understand this equation one should expand each factor $M^{(1)}$ as in equation (3.6). Then there are two types of terms: those involving $(\langle h^n\rangle^c)^2$ and those involving $\langle h^n\rangle^c \langle h^m\rangle^c$ with $m \neq n$. For the former the symmetry factor, $h(\Gamma)$, is clearly 2. The latter terms

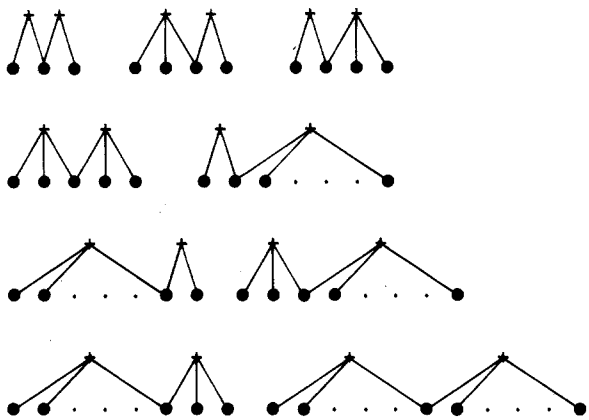

Figure 5. Some simple tree graphs. These are the simplest graphs having one articulation point.

are counted twice in our procedure, so effectively $h(\Gamma)$ is 2 for them as well. Thus the factor $n_q = 2$ indicated in equation (3.5) is compensated by the symmetry factor $h(\Gamma) = 2$. Also we write

$$M_b^{(2)} = -\gamma \sum_{n=1}^{\infty} n(x\gamma)^n (n!)^{-1} \langle h^{n+1}\rangle^c M^{(1)}. \tag{3.9}$$

The factor n occurs because the origin can equally well be placed on any of the n free ends. We take account of the symmetry factor by allowing the origin to be placed only on one of the two cumulants. We may write equation (3.9) for $M_b^{(2)}$ in the form

$$M_b^{(2)} = -x^{-1} M^{(1)} \left(\frac{\gamma \mathrm{d}M^{(1)}}{\mathrm{d}\gamma} - M^{(1)} \right). \tag{3.10}$$

Thus in all we have that

$$M^{(2)} = -x^{-1} M^{(1)} \left(\frac{\gamma \mathrm{d}M^{(1)}}{\mathrm{d}\gamma} \right) \tag{3.11}$$

We evaluate $\Omega_{\text{tree}}^{(2)}$ as

$$\Omega_{\text{tree}}^{(2)} = -\tfrac{1}{2} x\gamma^2 \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \frac{(x\gamma)^{n-1}}{(n-1)!} \frac{(x\gamma)^{m-1}}{(m-1)!} \langle h^n\rangle^c \langle h^m\rangle^c \tag{3.12}$$

where again the symmetry factor is 2. We may write this as

$$\Omega_{\text{tree}}^{(2)} = -[M^{(1)}]^2/(2x). \tag{3.13}$$

Comparing equations (3.11) and (3.13) we verify that for the diagrams of the type shown in figure 5, equation (3.4) is valid.

This procedure may be generalized to an arbitrary tree graph. Let M_1 be the contribution to M from graphs in which the origin is placed on a free end and M_n be the contribution to M from graphs in which the origin is placed on an articulation point at the intersection of n lines. Then the generalization of equation (3.8) is

$$M_n = x\left(\frac{-M_1}{x}\right)^n. \tag{3.14}$$

It may be surprising at first glance that this expression does not seem to take any account of the factor $n!/h(\Gamma)$ associated with the origin. This factor is actually present as the multinomial coefficient involved when M_1 is expanded as a sum of terms and raised to the nth power. The factors $(n_p - 1)!$ associated with the other points are accounted for in a similar way; the fact that the line leading from point p to the origin is distinguished in a tree graph results in having $(n_p - 1)!$ rather than $n_p!$ as occurs for the origin.

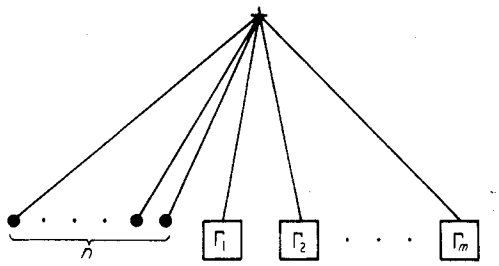

Figure 6. Schematic representation of diagrams contributing to M_1. Here we show a diagram having n free ends and m points to which are attached arbitrarily complicated tree subgraphs. These are denoted $\Gamma_1, \Gamma_2, \ldots \Gamma_m$. The contribution of this family of diagrams to M_1 is given in equation (3.15).

It remains to evaluate M_1 from diagrams of the type shown in figure 6. Here we show a diagram with n free ends any one of which may be selected as the origin and there are m other lines which lead to arbitrarily complicated trees, each of which carries a factor M. Thus we have

$$M_1 = \sum_{\substack{n=1 \\ m+n\geq 2}}^{\infty} \sum_{m=0}^{\infty} n(x\gamma)^n (n!)^{-1} (m!)^{-1} \langle h^{n+m} \rangle^c (-\gamma M)^m. \tag{3.15}$$

The term with $n = 1, m = 0$ does not occur because each cumulant has a minimum of two arguments. The factor $(-1)^m$ in equation (3.15) is needed to reproduce correctly the factor $(-1)^{n_p - 1}$ at each of the m articulation points explicitly appearing in the diagram of figure 6. Thus we find

$$M_1 = x\gamma[U_0(\beta J - x\gamma + \gamma M) - U_0(\beta J)]. \tag{3.16}$$

Combining this result with equation (3.14) we obtain an implicit equation to determine M:

$$M = \sum_{n=1}^{\infty} M_n, \tag{3.17a}$$

$$= \frac{M_1}{[1 + x^{-1} M_1]} = \frac{x M_1}{[x + M_1]} \tag{3.17b}$$

$$= \frac{x\gamma[U_0(\beta J - x\gamma + \gamma M) - U_0(\beta J)]}{1 + \gamma[U_0(\beta J - x\gamma + \gamma M) - U_0(\beta J)]}. \tag{3.17c}$$

Having thus determined M, we may use equations (3.4) and (2.9) to obtain the free energy as

$$\Omega = \Omega_0 + x \ln \rho + \int_0^\gamma d\gamma' M(\gamma', \beta J)/\gamma'. \tag{3.18}$$

We wish to evaluate the specific heat for (a) T near T_c, that is for $|T - T_c| \sim x^{1/\alpha}$ and (b) T near $T_c(x)$, that is for $|T - T_c(x)| \sim x^{1/\alpha}$. We expect to find a finite specific heat in the former régime. In fact, if we evaluate M to lowest order in x using equation (3.17c) we obtain

$$M(\gamma, \beta J) = x\gamma[U_0(\beta J - x\gamma) - U_0(\beta J)]. \tag{3.19}$$

Using equation (3.18) we recover the lowest-order result given in equation (2.26). Using this result we find that

$$U(\beta J) = U_0(\beta J - x\gamma) + \frac{x[s + c\langle h\rangle]}{[c + s\langle h\rangle]}. \tag{3.20}$$

Thus, although the specific heat is singular at $T = T_c$, the amplitude of the singularity is of order x, in the sense that

$$\lim_{x\to 0} C(\beta_c J + x^{1/\alpha}, x) \sim 1. \tag{3.21}$$

A complete analysis based on equation (3.17c) shows that

$$\lim_{x\to 0} C(\beta_c J + x^{1/\alpha}, x) = 0. \tag{3.22}$$

Thus, unlike the linear theory of § 2, equation (3.17c) yields a singularity whose amplitude vanishes more rapidly than x as $x \to 0$. However, since an amplitude factor which decreases as $x \to 0$ is unphysical, we believe that equation (3.22) means that there is no divergence in the specific heat for T near T_c, regardless of the amplitude.

Next we consider the specific heat for T near $T_c(x)$. Divergent behaviour in this régime can only come from the integral in equation (3.18). Thus, for $T \sim T_c(x)$ we set

$$C(\beta J) \sim \int_0^\gamma \frac{d\gamma'}{\gamma'} \frac{\partial^2 M(\gamma', \beta J)}{\partial(\beta J)^2}. \tag{3.23}$$

To analyse this formula we need the results

$$\frac{\partial M(\gamma, \beta J)}{\partial(\beta J)} = \frac{x\gamma}{D(\gamma)}[U_1(\phi(\gamma)) - U_1(\beta J)] \tag{3.24a}$$

$$\frac{\partial M(\gamma, \beta J)}{\partial \gamma} = \frac{x}{D(\gamma)}\{\Delta U(\gamma) + \gamma[M(\gamma, \beta J) - x]U_1(\phi(\gamma))\} \tag{3.24b}$$

which are obtained by differentiating equation (3.17c). Here we define

$$\phi(\gamma) = \beta J - x\gamma + \gamma M(\gamma, \beta J) \tag{3.25}$$

$$D(\gamma) = [1 + \gamma \Delta U(\gamma)]^2 - x\gamma^2 U_1(\phi(\gamma))$$

$$\Delta U(\gamma) = U(\phi(\gamma)) - U(\beta J) \qquad \text{and} \qquad U_n(x) = \left(\frac{d}{dx}\right)^n U(x). \tag{3.26}$$

Differentiating equation (3.24*a*) we find that

$$\frac{\partial^2 M(\gamma, \beta J)}{\partial(\beta J)^2} = A_1 + A_2 + A_3, \tag{3.27}$$

where

$$A_1 = x\gamma U_2(\phi(\gamma))\{[1 + \gamma\Delta U(\gamma)]^2 - x\gamma^2 U_1(\beta J)\}[D(\gamma)]^{-3} \tag{3.28a}$$

$$A_2 = -x\gamma U_2(\beta J)/D(\gamma) \tag{3.28b}$$

$$A_3 = -2x\gamma^2[U_1(\phi(\gamma)) - U_1(\beta J)]^2[1 + \gamma\Delta U(\gamma)]^3[D(\gamma)]^{-3}. \tag{3.28c}$$

We will also use the relation

$$\frac{\partial\phi(\gamma, \beta J)}{\partial\gamma} = -\frac{x}{D(\gamma)} \tag{3.29}$$

which follows when equation (3.24*b*) is substituted in equation (3.25). We now evaluate equation (3.23) using equation (3.27). Since A_2 and A_3 do not diverge for any values of γ', they lead to smooth contributions to the specific heat for T near $T_c(x)$. In analysing the contribution from A_1 it is permissible to set $1 + \gamma'\Delta U(\gamma') = 1$, whence

$$C(\beta J) \approx -\int_0^\gamma d\gamma' \frac{\partial U_1(\phi(\gamma'))}{\partial\gamma'}\left[\frac{1 - x\gamma'^2 U_1(\beta J)}{1 - x\gamma'^2 U_1(\phi(\gamma'))}\right]^2 \tag{3.30}$$

where we have also used equation (3.29). Since the main contribution to this integral comes when $\gamma' \approx \gamma$, we replace the γ' factors which appear explicitly by the factors γ. Then the integral in equation (3.30) can be carried out in closed form with the result

$$C(\beta J) = [C_0(\phi(\gamma)) - C_0(\beta J)]\,[1 - x\gamma^2 U_1(\beta J)]/D(\gamma). \tag{3.31}$$

For $T \approx T_c(x)$ we have $x\gamma^2 U_1(\beta J) \ll 1$ and $1 + \gamma\Delta U \approx 1$, so that

$$C(\beta J) \approx \frac{C_0(\beta J - x\gamma + \gamma M) - C_0(\beta J)}{1 + x\gamma^2 C_0(\beta J - x\gamma + \gamma M)}. \tag{3.32}$$

Thus as $T \to T_c$, that is as $\beta J - x\gamma + \gamma M \to \beta_c J$, the specific heat increases, but only up to a maximum value of order $(x\gamma^2)^{-1}$. This maximum value is of order x^{-1} and, as one would expect, it increases as x decreases. The divergence in the specific heat is smeared out over the temperature range for which $x\gamma^2 C_0(\beta J - x\gamma + \gamma M)$ is of order unity. Clearly, this temperature range is of order $\Delta T/T_c \sim x^{(1/\alpha)}$.

From equation (3.32) one sees that the renormalized critical temperature is given by the solution to

$$\beta J - x\gamma(\beta J) + \gamma(\beta J)M(\gamma(\beta J), \beta J) = \beta_c J. \tag{3.33}$$

We now analyse $T_c(x)$ as determined by this approximation keeping terms of order at most x^2. To this accuracy we may write equation (3.17*c*) as

$$M = x\gamma[U_0(\beta J - x\gamma) - U_0(\beta J)] \tag{3.34}$$

so that equation (3.33) becomes

$$\beta J - x\gamma + x\gamma^2[U_0(\beta J - x\gamma) - U_0(\beta J)] = \beta_c J. \tag{3.35}$$

In order to analyse this equation for $\beta_c(x)$, we assume $U_0(\beta J)$ to be of the form

$$U_0(\beta J) \approx U_0(\beta_c J) + A'(\beta J - \beta_c J) + B'(\beta J - \beta_c J)|\beta J - \beta_c J|^{-\alpha} \tag{3.36}$$

for β near β_c. Inserting this expansion into equation (3.35) shows that $T_c(x)$ has an expansion of the form

$$T_c(x) = T_c[1 - s'x - s''x^2 + s'''x^{2-\alpha} + \ldots]. \tag{3.37}$$

In determining the constants appearing in equation (3.37) we must expand $x\gamma(\beta J)$ in powers of x. We write

$$x\gamma(\beta J) \approx x\gamma(\beta_c J + x\gamma_c) \tag{3.38a}$$

$$\approx x\gamma_c + x^2\gamma_c(c_c + s_c U_c)^{-2} - x\gamma_c^2[A' x\gamma_c + B' x\gamma_c|x\gamma_c|^{-\alpha}], \tag{3.38b}$$

where $U_c = U_0(\beta_c J)$.
Then we find that s' is given by equation (2.28b), $s''' = 0$ and

$$s'' = \left[\frac{\gamma_c}{s_c^2} - \frac{kT_c}{J}\right]\gamma_c^2 \frac{kT_c}{J}. \tag{3.39}$$

It is possible to verify this calculation of $T_c(x)$ for the case of a Cayley tree in which a fraction of sites are randomly removed. For $T > T_c(x)$ one has $\langle\sigma_i\rangle = 0$ and then if the sites k and l are connected one may write

$$\langle\sigma_k\sigma_l\rangle = \frac{\mathrm{Tr}[\sigma_k\sigma_l \prod_{\langle ij\rangle}(c + s\sigma_i\sigma_j)]}{\mathrm{Tr}[\prod_{\langle ij\rangle}(c + s\sigma_i\sigma_j)]} \tag{3.40a}$$

$$= [\tanh(\beta J)]^{d_{kl}}, \tag{3.40b}$$

where d_{kl} is the distance between sites k and l. The configurationally averaged susceptibility at the origin is then

$$\langle\!\langle\chi\rangle\!\rangle = 1 + z(1-x)\tanh(\beta J) + z(1-x)\tanh(\beta J)\sum_{n=1}^{\infty}[(1-x)(z-1)\tanh(\beta J)]^n \tag{3.41}$$

for a tree with z branches at each site. The critical temperature where the susceptibility diverges is given by

$$1 = (1 - x)(z - 1)\tanh(\beta J). \tag{3.42}$$

When $T_c(x)$ is expanded in powers of x the result obtained from equation (3.42) agrees with that of equation (3.37).

We can evaluate the constant s'' in equation (3.37) for the simple-square and simple-cubic Ising models using the values of the constants cited after equation (2.28). We find $s'' = 0{\cdot}436$ and $s'' = 0{\cdot}085$ for the two cases, respectively. In figure 7 we show the resulting curves of T_c against x. In both cases s'' is positive, leading to the downward curvature which one expects on physical grounds. The critical values, x_c, of x for which T_c vanishes are 0·62 and 0·88 for the two cases, respectively. These values are significantly larger than the corresponding values of the critical percolation concentration, 0·5 (Sykes and Essam 1963) and 0·75 (Vyssotsky *et al* 1961, Sykes and Essam 1964). However, such a discrepancy is not surprising in view of the fact that terms of higher order in x must be important in the determination of x_c. Such terms would no doubt lead to a sudden decrease in T_c as x approaches x_c.

We also note that s''', the coefficient of $x^{2-\alpha}$, vanishes. The cancellation leading to this result is probably peculiar to the bond model with nearest-neighbour interactions. The existence, more generally, of such a term could probably be predicted by scaling arguments. Needless to say, detection of such a term via traditional series expansions is problematical.

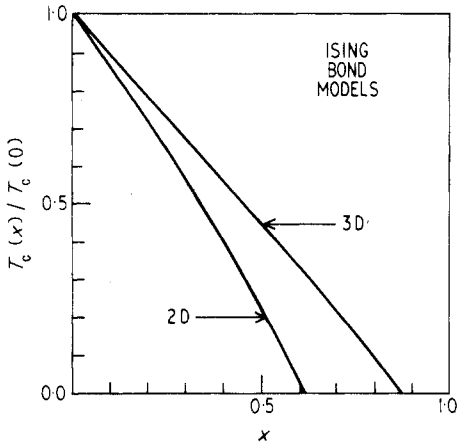

Figure 7. $T_c(x)$ against x according to equation (3.37) for the simple-square (2DI) and the simple-cubic (3DI) Ising models with vacant bonds.

4. Magnetic glass

We now consider a magnetic glass governed by the hamiltonian

$$\mathscr{H} = -\sum_i [J + \delta J(i)]\sigma(r_i)\sigma(s_i) \tag{4.1}$$

where again the sum is over the nearest-neighbouring pairs of spins. The $\delta J(i)$ are independent random variables governed by an even distribution with a width much smaller than J. That is, we impose the conditions

$$\langle\!\langle [\delta J(i)]^{2n+1} \rangle\!\rangle = 0, \tag{4.2a}$$

$$\langle\!\langle [\delta J(i)]^{2n} \rangle\!\rangle \leqslant A\epsilon^{2n} \tag{4.2b}$$

where $\epsilon^2 = \langle\!\langle (\delta J(i)/J)^2 \rangle\!\rangle \ll 1$. We may use the product cumulants to write the free energy as (Horwitz and Callen 1961)

$$F = F_0 + \sum_n \sum_{i_1, i_2, \ldots i_n} (n!)^{-1}(-k_B T)^{1-n} \times \langle\!\langle \prod_{l=1}^{n} \delta J(i_l) \rangle\!\rangle \langle \prod_{m=1}^{n} h(i_m) \rangle^c \tag{4.3}$$

where the indices $i_1, i_2, \ldots i_n$ are summed over all bonds. To treat the case of small ϵ we keep only the most divergent contributions in each order of ϵ^2. To do this we note that as $T \to T_c$ the correlation length diverges, so that, as before, the dominant contribution in equation (4.3) for a given value of n comes from terms with the maximum number of distinct bond indices. Since the δJ are independent random variables obeying equation (4.2*a*), this criterion indicates that the dominant terms are those for which the i consist of $\frac{1}{2}n$ distinct pairs. Thus we write

$$F = F_0 - k_B T \sum_n (2n!)^{-1} \sum_{i_1, i_2, \ldots, i_n} (-\epsilon\beta J)^{2n} \times \langle \prod_{m=1}^{n} h(i_m)^2 \rangle^c \left(\frac{2n!}{2^n n!}\right). \tag{4.4}$$

The factor $(2n!/2^n n!)$ is the number of ways $2n$ indices can be grouped into n pairs. We now use the evaluation of the cumulants for widely separated arguments given in the Appendix to write

$$F = F_0 - \frac{1}{2}\left(\frac{Nz}{2}\right)\epsilon^2 J^2 \beta[1 + U_0(\beta J)^2]$$

$$- k_B T \sum_{n=1}^{\infty} (n!)^{-1} (-\epsilon^2 \beta^2 J^2 U_0(\beta J))^n \sum_{i_1, i_2, \ldots, i_n} \langle \prod_{k=1}^{n} h(i_k) \rangle^c. \tag{4.5}$$

The sums over the i's are done as in equation (2.24) so that

$$\Omega(\beta J) = \Omega_0(\beta J + \epsilon^2 \beta^2 J^2 U_0(\beta J)) + \tfrac{1}{2}(\beta \epsilon J)^2[1 + U_0(\beta J)^2]. \tag{4.6}$$

Thus, as for the bond model, the lowest-order approximation leads to a shift in the temperature scale but no alteration in the type of singularity. The shifted transition occurring at $kT_c(\epsilon^2) \equiv \beta_c(\epsilon^2)$ has the value of β for which

$$\beta J + \epsilon^2 \beta^2 J^2 U_0(\beta J) = \beta_c J. \tag{4.7}$$

To lowest order in ϵ^2 we may write this as

$$T_c(\epsilon^2) = T_c\left(\frac{1 + U_0(\beta_c J)\epsilon^2}{kT_c}\right) \tag{4.8a}$$

$$\equiv T_c(1 - \hat{s}\epsilon^2). \tag{4.8b}$$

We may evaluate the constant $\hat{s}$ using the data cited following equation (2.28). In this way we obtain $\hat{s} = 0{\cdot}312$ for the two-dimensional simple-square Ising model and $\hat{s} = 0{\cdot}073$ for the three-dimensional simple-cubic Ising model. Note that in the mean-field approximation fluctuations are ignored in which case one has $\hat{s} = 0$. It is therefore quite reasonable that the values of $\hat{s}$ is smaller for the three-dimensional model than for the two-dimensional one.

5. Physical estimates of the effects of fluctuations

In this section we give some simple physical arguments to support the more formal calculations of the preceding sections. We first consider the well known argument that a randomly dilute system will behave like an ensemble of systems with a distribution of transition temperatures reflecting the fluctuations in concentration over the system and therefore that the transition in an alloy is rounded. We make this argument more quantitative as follows. Suppose, to the contrary, that the diluted system has a transition temperature $T_c(x)$ and consider a temperature near $T_c(x)$ so that the correlation length, ξ, is of the order $\xi \sim |T - T_c(x)|^{-\nu}$. (Here we assume that for small x the critical index $\nu(x)$ for the dilute system goes smoothly to $\nu(0) \equiv \nu$ as x goes to zero.) Having fixed the correlation length, ξ, let us estimate the width in the aforementioned distribution of T_c. To do this we divide the system into independent volume elements, $\Omega^{(d)}$, having ξ^d spins, where d is the dimensionality. If we made the volume elements any smaller, they could not be considered independent.

The average number of missing interactions in a typical volume element, denoted $\langle\langle n \rangle\rangle \equiv \langle\langle \Sigma p_i \rangle\rangle$, where the sum is over the volume element $\Omega^{(d)}$, is given by

$$\langle\langle n \rangle\rangle = x\xi^d. \tag{5.1}$$

Also, the mean square fluctuation in the number of missing interactions $\langle\langle(\delta n)^2\rangle\rangle \equiv \langle\langle(\Sigma p_i)^2\rangle\rangle - \langle\langle\Sigma p_i\rangle\rangle^2$ is

$$\langle\langle(\delta n)^2\rangle\rangle = \xi^d x(1-x). \tag{5.2}$$

Thus, the probability distribution for the concentration in $\Omega^{(d)}$ has a width in concentration, Δx, of order

$$\Delta x \sim [\xi^d x(1-x)]^{1/2}/\xi^d. \tag{5.3}$$

Since T_c and x are linearly related (that is dT_c/dx is finite), this width in the distribution of concentration implies that the width in the distribution of T_c is of order

$$\frac{\Delta T_c}{T_c} \sim \frac{[\xi^d x(1-x)]^{1/2}}{\xi^d}. \tag{5.4}$$

Now let us see whether the originally assumed value of ξ is self-consistent. In view of the fluctuations in T_c, the correlation length ξ can not be larger than that corresponding to a temperature with $|T - T_c|/T_c = \Delta T_c/T_c$ given by equation (5.4). Thus the condition for self-consistency is

$$\xi \leqslant |\Delta T/T_c|^{-\nu} \tag{5.5a}$$

$$\leqslant \xi^{d\nu/2}[x(1-x)]^{-\nu/2}. \tag{5.5b}$$

Thus, if $\frac{1}{2}d\nu - 1 \geqslant 0$, the original assumption of a correlation length ξ is self-consistent no matter how large ξ is, and the assumption of a sharp transition is self-consistent with respect to concentration fluctuations if $d\nu > 2$. If we use the scaling theory (Kadanoff 1966) relation, $\alpha = 2 - d\nu$, we see that a sharp transition is only possible if $\alpha \leqslant 0$. Thus, the simple intuitive argument that a random alloy does not have a sharp transition is correct if $\alpha > 0$. If $\alpha = 0$, our argument is not precise enough to settle the question as to whether or not the transition is sharp. The critical behaviour of the bond model found in § 3 agrees with this conclusion to the extent that it yielded a finite specific heat maximum. However, the form given in equation (3.32) indicates that the specific heat has a cusp for $\beta J - x\gamma + \gamma M = \beta_c J$. This type of behaviour does not correspond to a sharp transition but it is not obvious that such behaviour necessarily contradicts the arguments given in this section.

For $\alpha > 0$ we can estimate the range of temperatures over which the rounding effect takes place. For small x one can write equation (5.5*b*) as $\xi \leqslant \xi^{d\nu/2}x^{-\nu/2}$ so that ξ_{max}, the maximum self-consistent value of the correlation length, is given by

$$(\xi_{max})^{1-d\nu/2} = x^{-\nu/2}. \tag{5.6}$$

Using the relation $\alpha = 2 - d\nu$, we write this as $\xi^{\alpha/2} = x^{-\nu/2}$. Thus the temperature range ΔT over which the transition is renormalized away by concentration fluctuations is of order

$$\Delta T/T_c \sim x^{1/\alpha}. \tag{5.7}$$

The same type of estimates may be given for the glass model studied in § 4. In this case one evaluates the average interaction strength $\langle\langle J\rangle\rangle$ in the volume element $\Omega^{(d)}$. Since we have set $J(i) = J + \delta J(i)$ and have assumed $\langle\langle\delta J(i)\rangle\rangle = 0$, we obviously have $\langle\langle J\rangle\rangle = J$. For the mean square fluctuation in interaction strength, denoted $\langle\langle(\delta J)^2\rangle\rangle$, we have

$$\langle\langle(\delta J)^2\rangle\rangle = \langle\langle[\sum_i J(i)]^2\rangle\rangle - \langle\langle\sum_i J(i)\rangle\rangle^2 \tag{5.8}$$

where again the sums over i are over all sites in $\Omega^{(d)}$. Thus

$$\langle\langle(\delta J)^2\rangle\rangle = \xi^d J^2 \epsilon^2. \tag{5.9}$$

The condition analogous to equation (5.5*b*) is in this case $\xi \leqslant \xi^{dv/2}\epsilon^{-v}$. If we use $\alpha = 2 - dv$, then the width of the critical régime is

$$\Delta T/T_c \sim \epsilon^{2/\alpha}. \tag{5.10}$$

Similarly, we may use the above argument to estimate the width of the region over which the specific heat singularity is rounded for the model studied by McCoy and Wu (1968). In their two-dimensional model all J's in a given row were set equal to a common random variable. Hence the ξ^2 spins have only ξ independent random interactions. As a result, the mean square fluctuation in the interaction for their model is of order $\xi^{-1/2}\epsilon$ and consequently the maximum self-consistent value of the correlation length, ξ_{max}, is

$$\xi_{max} = (\xi_{max}^{-1/2}\epsilon)^{-v}. \tag{5.11}$$

Since $v = 1$ for the two-dimensional Ising model, we find that

$$\xi_{max} = \epsilon^{-2}. \tag{5.12}$$

Thus, the width in temperature of the critical régime is of order

$$\Delta T/T_c \sim \epsilon^2. \tag{5.13}$$

This estimate agrees with the rigorous result of McCoy and Wu (1968) and therefore corroborates our physical reasoning.

Finally, we consider the trivial 'uniform' model in which all exchange integrals are equal to a common random variable $J + \delta J$, with $\langle\langle \delta J\rangle\rangle = 0$ and $\langle\langle(\delta J)\rangle\rangle^2 = \epsilon^2 \ll 1$. Arguments similar to those given above show that the maximum value of the correlation length is given by $\xi_{max} = \epsilon^{-v}$, so that the width of the critical régime is of order

$$\Delta T/T_c \sim \epsilon. \tag{5.14}$$

This result is obvious, inasmuch as the specific heat of this model is simply given by the convolution of the specific heat of the pure sustem with the distribution function for δJ.

Note that for both the model of McCoy and Wu (1968) and the uniform model there is no shift in the transition temperature for small ϵ. In contrast, for the two models we have studied the transition temperature is shifted by the presence of random defects. The outstanding difference between the 'bond' model and the 'glass' model on the one hand and the uniform-fluctuation model and the model of McCoy and Wu (1968) on the other hand is that the former involve local fluctuations, whereas the latter involve only strongly correlated nonlocal fluctuations. We are tempted to conclude that the former type of model leads to shifts in T_c whereas the latter does not. In both cases there is a width induced in the critical region. Typically, the width is much larger in the latter case than in the former. Since local fluctuations in interaction strengths are more characteristic of real physical systems than are nonlocal fluctuations, we believe that our results are more indicative of what one might expect from real random systems than are those of McCoy and Wu (1968).

Implicit in the above discussion is the assumption that the behaviour of site models and bond models should be comparable for small dilution. In general, this assumption is a difficult one to verify. However, one might argue that since the interactions in the site and bond models have the same symmetry their critical properties should be the same.

The reason the technique we have used for the bond problem does not yield simple results for the site problem is that the analogue of equation (2.3) for the site problem involves

$$Q(i) = \exp\left[-\beta J \sum_{\delta} \sigma_i \sigma_{i+\delta}\right] \tag{5.15}$$

where $i + \delta$ denotes a nearest neighbour of the site i. (Effects due to overlap of defect potentials can be shown to be irrelevant.) In view of equation (5.15) we must generalize equation (2.6) as

$$Q(i) = \prod_{\delta} [\cosh \beta J - \sigma_i \sigma_{i+\delta} \sinh \beta J]. \tag{5.16}$$

In order to evaluate sums of the form of equation (2.20) we would have to analyse the behaviour of correlation functions involving several Q at widely separated lattice sites. Clearly, one could do this approximately but we have not studied this problem in great detail. The interesting question is whether, from the analogue of equation (2.20), one would obtain a Taylor series expansion as we have for the bond problem, or whether one would obtain a form indicating a broadening of the critical regime of order x. Physically, since we believe the site and bond problem are closely related, we believe the former possibility to be the more likely and hence that the results in this paper are qualitatively the same as we would have found for the site problem.

6. Conclusions

In this section we summarize the results of our work.

(i) For Ising models with a small concentration, x, of bond defects it is possible to evaluate the limiting slope of the transition temperature against concentration, $s' \equiv -T_c^{-1}\, \mathrm{d}T_c(x)/\mathrm{d}x\big|_{x=0}$ in terms of properties of the pure system.

(ii) For the two-dimensional simple-square Ising model (2DI) we obtain $s' = 1{\cdot}329$ and for the three-dimensional simple-cubic Ising model (3DI) we obtain $s' = 1{\cdot}060$, compared with the mean-field result $s' = 1$. Our result for s' is the same as that found by Rapaport (1972a) for the annealed system and verifies the result of Osawa and Sawada (1973).

(iii) For a 'glass' model in which each interaction is an independent random variable with a width in energy, ϵJ, much smaller than the average energy, J, we have determined the limiting slope $\hat{s} \equiv -T_c^{-1}\, \mathrm{d}T_c(\epsilon)/\mathrm{d}\epsilon^2\big|_{\epsilon=0}$. We find $\hat{s} = 0{\cdot}312$ for the (2DI) and $\hat{s} = 0{\cdot}073$ for the (3DI), compared with the mean field result $\hat{s} = 0$.

(iv) We have evaluated contributions to $T_c(x)$ of higher order in x for the bond defect model. We find

$$T_c(x) = T_c[1 - s'x - s''x^2 + s'''x^{2-\alpha}], \tag{6.1}$$

where $s''' = 0$ and the constants s' and s'' are given in equations (2.28) and (3.39), respectively. For both the 2DI and the 3DI equation (6.1) gives a physically reasonable result for $T_c(x)$ as can be seen in figure 7.

(v) Estimates of the effects of configurational fluctuations on the correlation length show that for $\alpha > 0$ the critical divergences are smoothed out over a range of temperatures, ΔT, of order $\Delta T/T_c \sim x^{(1/\alpha)}$. This same type of result is found quantitatively by an approximate theory derived in § 3 to treat the critical régime. Both the quantitative theory and the simple estimates show that the specific heat of the random system attains a maximum value of order x^{-1} at the shifted transition temperature.

(vi) We argue that the width of the critical régime found by McCoy and Wu (1968) depends strongly on the fact that in their model the interactions are not independent random variables. We argue that our results for models where the interactions are independent random variables are more characteristic of physically realizable systems than are theirs. In our models there are shifts in T_c and small widths of the critical régime, whereas in the model of McCoy and Wu (1968) there is no shift in T_c and the critical régime is relatively large.

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Appendix. Cumulants of $h(i)^2$

Here we derive an asymptotic evaluation of R_n

$$R_n \equiv \langle \prod_{m=1}^{n} h(i_m)^2 \rangle^c \tag{A.1}$$

valid for widely separated arguments. For $n = 1$ we find

$$R_1 = 1 - U_0^2. \tag{A.2}$$

For $n = 2$ we write

$$\begin{aligned}\langle h(i_1)^2\, h(i_2)^2\rangle^c &= \langle h(i_1)^2\, h(i_2)^2\rangle - 2\langle h\rangle\, \langle h(i_1)\, h(i_2)^2\rangle - 2\langle h\rangle\, \langle h(i_1)^2\, h(i_2)\rangle \\ &\quad - \langle h(i_1)^2\rangle\, \langle h(i_2)^2\rangle - 2\langle h(i_1)\, h(i_2)\rangle^2 + 2\langle h(i_1)^2\rangle\, \langle h\rangle^2 \\ &\quad + 2\langle h\rangle^2\, \langle h(i_2)^2\rangle + 8\langle h\rangle^2\, \langle h(i_1)\, h(i_2)\rangle - 6\langle h\rangle^4, \qquad \text{(A.3}a\text{)} \\ &= [\langle h(i_1)\, h(i_2)\rangle - \langle h\rangle^2]\, [-2\langle h(i_1)\, h(i_2)\rangle + 6\langle h\rangle^2] \qquad \text{(A.3}b\text{)}\end{aligned}$$

where we have used $h(i)^2 = 1$. When i_1 and i_2 are far apart we may set

$$-2\langle h(i_1)\, h(i_2)\rangle + 6\langle h\rangle^2 = 4\langle h\rangle^2,$$

whence

$$R_2 = 4U_0^2 \langle h(i_1)\, h(i_2)\rangle^c. \tag{A.4}$$

To generalize the discussion of R_n to arbitrary $n \geqslant 2$ we imagine carrying out the following steps analogous to equation (A.3). First, R_n is expanded in terms of correlation functions. In some terms there will occur factors of $h(i)^2$ and these are set equal to unity. Each correlation function now remaining is expressed in terms of cumulants as in equation (2.5*a*). Note that now for each k, the factor $h(i_k)$ occurs either twice in different cumulants or not at all (that is, if it occurred twice in the same cumulant and the relation $h(i_k)^2 = 1$ was used). There are some terms in which, for some values of k, $h(i_k)$ does not

occur. These terms must cancel each other since they do not have the cluster property with respect to the index i_k, whereas R_n certainly does have that property. The arguments given following equation (2.19) show that the dominant contribution to R_n comes from terms having the minimum number of separation variables. Thus we set

$$R_n = q_n \prod_{m=1}^{n} \langle h(i_m)\rangle \langle \prod_{l=1}^{n} h(i_l)\rangle^c. \tag{A.5}$$

The coefficient q_n is the coefficient of $\langle h(i_1)\, h(i_2) \ldots h(i_n)\rangle^c \langle h(i)\rangle^n$ in the expansion of R_n in terms of cumulant averages. Thus we may write

$$q_n = \langle h\rangle^{-n} \frac{\partial R_n}{\partial \Lambda_n}\bigg|_{\Lambda_2 = \Lambda_3 = \ldots = \Lambda_n = 0} \tag{A.6}$$

where $\Lambda_n \equiv \langle h(i_1)\, h(i_2) \ldots h(i_n)\rangle^c$. Since Λ_n is linearly related to $\theta_n \equiv \langle h(i_1)\, h(i_2) \ldots h(i_n)\rangle$, we may write equation (A.6) as

$$q_n = \langle h\rangle^{-n} \frac{\partial R_n}{\partial \theta_n}\bigg|_{\theta_k = \langle h\rangle^k}. \tag{A.7}$$

What we have to do now is to count the terms in the expansion of R_n as products of correlation functions which contain one or more θ_n. (The term in θ_n^2 is weighted with an extra factor of 2 according to equation (A.7).) In the expansion of a cumulant in terms of correlation functions, the terms consisting of the product of r correlation functions are weighted by a factor $(-1)^{r-1}(r-1)!$. We now express q_n in terms of the combinatorial coefficients S_{nm}, the Stirling numbers of the second kind (Abramowitz and Stegun 1964). The S_{nm} are the number of ways of distributing n objects into m boxes such that no box is empty. Hence

$$q_n = \sum_{m=1}^{n} 2^n S_{nm} (-1)^m m!. \tag{A.8}$$

The term with $m = 1$ corresponds to the term in Λ_n^2, there being 2^{n-1} ways of realizing this term. The other terms, each of which can be realized in 2^n ways, correspond to terms linear in Λ_n in which one factor of $h(i_k)$ for $k = 1, 2, \ldots, n$ is distributed over m cumulants. Abramowitz and Stegun (1964) give

$$q_n = (-2)^n \tag{A.9}$$

so that

$$R_n \approx (-2U_0)^n \langle \prod_{m=1}^{n} h\langle i_m)\rangle^c. \tag{A.10}$$

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