Out of Equilibrium dynamics in Spin-Glasses and other Glassy Systems

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We review recent theoretical progress on glassy dynamics, with special emphasis on the importance and universality of the *aging regime*, which is relevant to many experimental situations. The three main subjects which we address are: (i) Phenomenological models of aging (coarsening, trap models), (ii) Analytical results for the low-temperature dynamics of mean-field models (corresponding to the mode-coupling equations); and (iii) Simple non-disordered models with glassy dynamics. We discuss the interrelation between these approaches, and also with previous work in the field. Several open problems are underlined – in particular the precise relation between mean-field like (or mode-coupling) descriptions and finite dimensional problems.

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1 Introduction

Glassy systems are characterized by the fact that their relaxation time becomes exceedingly long for low temperatures, so long that these systems are never in equilibrium on laboratory (or even geological) time scales. Notwithstanding, most theories of spin glasses and other disordered systems have first aimed at describing the putative equilibrium state of these systems. [1–6] In doing so, many difficulties and surprises have emerged – most notably the intricate structure of the Parisi solution of the Sherrington-Kirkpatrick (SK) [2] mean-field model for spin-glasses [7]. Despite some early attempts [8,9], phenomenological and analytical descriptions of the non-equilibrium phenomena in disordered systems have only recently appeared, which we shall review below. These *dynamical* approaches have been developed mostly because of the accumulating body of experimental data on *aging* [10–15], which is a striking experimental consequence of the fact that these systems are out of equilibrium even on macroscopic time scales. This aging regime is not the most general out of equilibrium situation: a certain degree of *universality* emerges in the non-equilibrium properties. While usual equilibrium dynamics is stationary, i.e. invariant under time translations, the aging regime presents a kind of 'covariance': after transients have decayed, the dynamical evolution of an old system of age t_w is described by the same equations as that of a younger system of age $t_w/2$, up to a rescaling of time.

The change of focus from equilibrium to non equilibrium situations also has the interesting consequence of unveiling strong analogies between disordered systems such as spin-glasses, and other types of glasses where disorder is a priori absent, such as fragile glasses. These analogies are both phenomenological and formal: many experimental facts are in close correspondance (for example, aging phenomena were first studied in detail by Struick on polymer glasses [10]), but also, the structure of the mean-field equations used to describe non-equilibrium spin-glasses are almost identical to the Mode Coupling Theory (MCT) of supercooled liquids [16,17]. There is thus a strong feeling that the two types of systems should be deeply connected [18–20], and there have been several attempts in the past few years to establish some precise bridges, which we shall review in this paper.

The scope of the present paper is mainly descriptive: we focus on general ideas and concepts rather than on more technical aspects. We refer the reader to the relevant papers for more details.

1.1 Experiments: History dependence and Aging

The simplest way to see that spin-glasses below the phase transition temperature T_g are not in equilibrium even after times of the order of hours (or more) is the following: the sample is quenched rapidly (under zero magnetic field) from high temperatures $T \gg T_g$ to the working temperature $T_1 < T_g$ which is reached, by convention, at time $t = 0$. Then a very small oscillating field is applied to measure the a.c. susceptibility χ of the sample at a certain frequency ω. What is observed is a slow continuous decrease of the amplitude of χ as a function of the time t_w elapsed since the sample reached the temperature T_1, which is called an *aging effect.* [11–15] In other words, χ is a function of both frequency *and* time: $\chi(\omega, t_w)$. The response of the system to a perturbation thus depends on the thermal history.

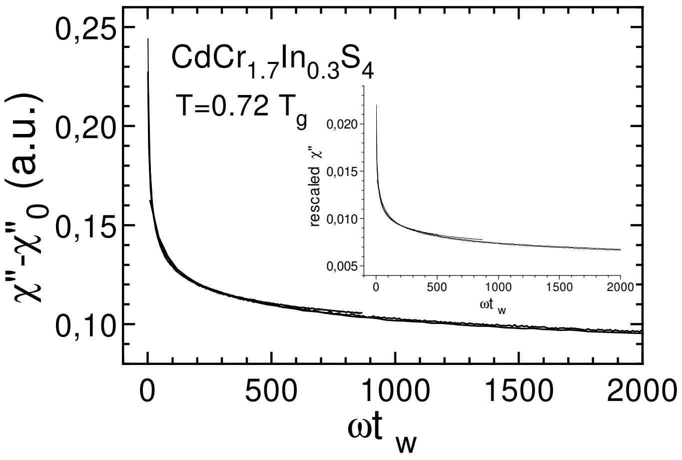

Figure 1: The amplitude of the out of phase magnetic susceptibility $\chi''(\omega, t_w)$ vs ωt_w for the insulating spin-glass $CdCr_{1.7}In_{0.3}S_4$ (from Ref. [14]). The frequencies are $\omega = 0.01, 0.03, 0.1, 1.$ Hz and t_w is the total time elapsed after the quench. In the inset, a more refined scaling for $\chi''(\omega, t_w)$ as discussed in Ref.[14].

To a good approximation, the shape of $\chi(\omega, t_w)$ can be parametrized as follows (see Fig.1):

$$\chi(\omega, t_w) = A\ (\omega t_w)^{-b} + \chi_{\rm ST}(\omega) \qquad\qquad (\omega t_w > 1) \tag{1}$$

where A is a temperature dependent amplitude and b an exponent that moves in the range $0.1 \to 0.4$. The important points of the above parametrization are:

- The response function is the sum of a *stationary part* $\chi_{\rm ST}(\omega)$ which is independent of the age of the system t_w, and of an *aging* (or non-stationary) contribution, which decreases with time. $\chi_{\rm ST}(\omega)$ behaves as ω^a with a small exponent a (sometimes called α) [12,13], or perhaps as $\log \omega$. For systems in equilibrium, the time dependent (aging) contribution disappears.

- The aging contribution can be described with a function of the scaling variable [13,14] ωt_w. In general, the susceptibility of a system with a single relaxation time τ is a function of $\omega\tau$. Hence, the above scaling form means that the effective relaxation time of the system is of the order of its age t_w itself. (See [14] for a discussion of the inset in Fig. 1 and a more detailed discussion of an alternative description of the $\chi''_{\rm AG}$ data.)

Another set of experiments which basically carry the same information is those of the so-called 'Thermo-Remanent Magnetisation' (TRM) relaxation [13,14]. The system is cooled under a small magnetic field H, which is left from $t = 0$ (the time of the quench) to $t = t_w$, and then suddenly switched off. The subsequent relaxation of the magnetisation M can be decomposed as [21,14]

$$M(t_w + \tau, t_w) = M_{\rm ST}(\tau) + M_{\rm AG}(t_w + \tau, t_w) \qquad\qquad M_{\rm ST}(\tau) \equiv \lim_{t_w \to \infty} M(t_w + \tau, t_w)\ , \tag{2}$$

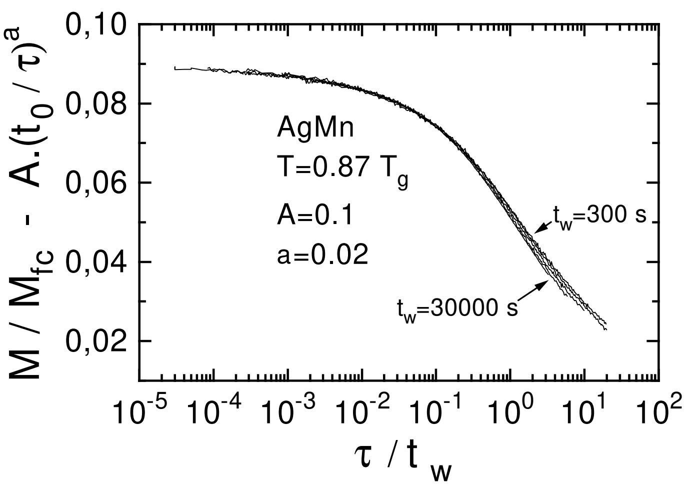

Figure 2: The aging part of the thermoremanent magnetization $M_{\rm AG}(t_w+\tau,t_w)$ (normalized by the zero field cooled value $M_{\rm fc}$) vs. τ/t_w for $AgMn_{2.6}$ in a $\log_{10}$ scale (from Ref. [14]). The sample was cooled in a 0.1 Oe field from above the glass transition $T_g=10.4K$ to a subcritical temperature $T=9K$. It waited for $t_w=300,1000,3000,10000,30000$ sec. under the field that was suddenly switched off at t_w. The decaying magnetization was recorded during all subsequent times $\tau+t_w$.

where again, there is a 'fast' [b] stationary contribution $M_{\rm ST}(\tau)$ which is independent of t_w, and an aging part $M_{\rm AG}(t_w+\tau,t_w)$ which to a good approximation (see Fig. 2 and Section 3) is a function of the ratio $\tau/(\tau+t_w)$. This again suggests that the effective relaxation time of the system is of the order of its age. Actually, within the linear response approximation, $\chi(\omega,t_w)$ and $M(t_w+\tau,t_w)$ are essentially Fourier transform of each other. More precisely, introducing the response function $R(t,t')$, one has

$$M(t_w+\tau,t_w)=H\int_0^{t_w} dt'\ R(t_w+\tau,t')\ , \qquad \chi(\omega,t_w)=\int_0^{t_w} dt'\ R(t_w,t')e^{i\omega(t'-t_w)}\ . \quad (3)$$

TRM relaxation in spin-glasses [13] and stress relaxation [10,22], electric polarisation [23,24] or specific heat [25] relaxation in many very different glassy materials show – rather remarkably – similar features, with a fast initial drop at small times τ, followed by a slow decrease of the signal on time scales of the order of the waiting time t_w. The same picture also pertains – on much smaller time scales – to numerical simulations of the response function of the three- [26,27] and four- dimensional [28] Edwards-Anderson and in mean-field [29–31] spin-glass models.

Since the response function depends on some aspects of the thermal history, it is interesting to consider more complicated experimental protocoles such as different cooling rates from high temperatures to T_1, temperature cycles [32,33] between two temperatures T_1 and T_2, or even field-cycling [34]. The detailed discussion of these situations is beyond the scope of the present paper. However, it is interesting to notice that while the a.c. susceptibility

[b] It is fast in the sense that experimentally, the major part of $M_{\rm ST}(\tau)$ has decayed to zero after the first second. However, $M_{\rm ST}(\tau)$ only decays as $(t_0/\tau)^a$ with a small: see after Eq. (1) above.

$\chi(\omega, t_w)$ depends extremely weakly on the cooling rate in spin-glasses, there are experimental systems (e.g. dipolar glasses [24]) for which this dependence is large. We shall come back to this point in Section 2.2.

Finally, let us mention that aging can also be seen in *correlation functions* (rather than response functions). In equilibrium, these two quantities are related by the well-known Fluctuation-Dissipation Theorem (FDT), which is, as we shall discuss in Section 1.3, not necessarily valid in out of equilibrium situations: in general, correlation and response do not contain the same information.

From an experimental point of view, correlations are obtained from time dependent noise spectrum measurements [35,36] $S(\omega, t_w)$, which are rather more difficult than response measurements [c]. From a numerical point of view, however, it is very easy to compute the time dependent correlation function $C(t_w + \tau, t_w)$, which for a spin system is defined as $C(t_w + \tau, t_w) = \frac{1}{N}\sum_i S_i(t_w + \tau)S_i(t_w)$. The behaviour of $C(t_w + \tau, t_w)$, obtained on relatively short time scales from simulations for finite dimensional [26–28] and mean-field [29–31] spin-glass models, reveals aging below the spin-glass transition (for reviews, see [37,38]), again qualitatively described by a shape similar to Eq. (2), with M's replaced by C's.

1.2 Think in the two times plane : time sectors.

Out of equilibrium situations have long been considered as untrustworthy. What the experiments tell us (and what the theories below will confirm) is that provided one abandons the idea that the correlation or response functions should be 'time translational invariant' (TTI), one can make sense of the experimental data by explicitly keeping the dependence on the two times: $C(t_w + \tau, t_w) \neq C(\tau)$. Equivalently, the Fourier transform of these quantitites will not be functions of the frequency ω only, but of both ω and the time since the quench t_w.

More precisely, for any physical system there are a priori two other time scales, one of them is microscopic (and will be noted t_o) and determines for example, the single spin-flip time. The other time scale is the equilibration time $t_{\rm ERG}$, which, for a finite size system, will always be finite (albeit often astronomical). The regime in which one expects to observe some 'universal' features (independent, for example, of the details of microscopic dynamics) is the regime where:

$$t_o \ll t_w + \tau \ll t_{\rm ERG} \quad \text{and} \quad t_o \ll t_w \ll t_{\rm ERG} \,. \tag{4}$$

This does not require any particular relation between τ and t_w. For $t_{\rm ERG} = \infty$, one may in general expect (and one indeed finds in some models) that the functional form of – say – the correlation depends on the way $t_w + \tau$ and t_w are taken to infinity.

The simplest example to see this is the case of ferromagnetic domain growth. After a quench from high temperatures to a non-zero temperature below the Curie point, a pattern of domains of positive and negative magnetizations starts coarsening. The typical domain size $\xi(t_w)$ at time t_w diverges as a power law (or possibly more slowly in the presence of

[c] This requires to perform many independent quenches where the magnetic noise is recorded for different ages t_w and then averaged over the different quenches.

impurities – see Section 2.2). The ergodic time $t_{\rm ERG}$ is then the time at which the size of the domains is that of the sample. We shall consider the thermodynamic limit in which $t_{\rm ERG} = \infty$. Within an approximate (large n) theory of coarsening, the correlation function for large times is indeed found to be [39]

$$C(t_w+\tau, t_w) = C_{\rm ST}(\tau) + C_{\rm AG}\left(\frac{\xi(t_w)}{\xi(t_w+\tau)}\right) \ . \tag{5}$$

The first term describes the fast relaxation of the spins within each domain, and has the same form as it would have in equilibrium, when there is only one infinite domain. Its limit[d]:

$$q_{\rm EA} \equiv \lim_{\tau\to\infty} \lim_{t_w\to\infty} C(t_w+\tau, t_w) \tag{6}$$

is an example for this simple model of the Edwards-Anderson parameter, a quantity that plays an important role in glassy dynamics (in this case it is simply the magnetization squared). The second term of Eq. (5) describes the relaxation of the system due to the motion of domain walls and it manifestly depends on the waiting time.

As t_w goes to infinity, one will be probing two distinct regimes, depending on whether one takes $t_w \to \infty$ with τ finite ('stationary' regime) or $\tau, t_w \to \infty$ with $\xi(t_w)/\xi(t_w+\tau) < 1$ ('aging' or coarsening regime). In terms of the correlation function, the stationary and aging regimes are simply defined as the regimes of (large) times in which $C(t_w+\tau, t_w) > q_{\rm EA}$ and $C(t_w+\tau, t_w) < q_{\rm EA}$, respectively.

These considerations can be translated in Fourier space as follows. Defining

$$\hat{C}(\omega, t_w) = \int_0^{t_w} d\tau \; C(t_w, t_w-\tau)\; e^{i\omega\tau} \ , \tag{7}$$

one finds that $\hat{C}(\omega, t_w)$ does not only depend upon ω (as it would in equilibrium situations) but also upon t_w. For example, for the correlation of the form (5), one obtains, in the limit $\omega t_w \gg 1$:

$$\hat{C}(\omega, t_w) = \hat{C}_{\rm ST}(\omega) + \frac{1}{\omega}\, \mathcal{C}\left(\frac{\xi(t_w)\,\omega}{\xi'(t_w)}\right) \tag{8}$$

where $\mathcal{C}$ is a certain function related to $C_{\rm AG}$. In the simplest case in which ξ grows as a power law, one finds $\xi(t_w)\omega/\xi'(t_w) = \omega t_w$.

More generally, one can envisage the possibility that different physical mechanisms act on different large-time sectors, defined as

$$\frac{\tau}{t_0} = O(1) \ , \qquad \frac{h_1(t_w)}{h_1(t_w+\tau)} = O(1) \ , \qquad \frac{h_2(t_w)}{h_2(t_w+\tau)} = O(1) \ , \qquad \text{etc.} \tag{9}$$

where the different functions h_i (no longer necessarily related to domain sizes) are monotonously increasing functions which grow differently, in such a way that:

$$0 < \frac{h_i(t_w)}{h_i(t_w+\tau)} < 1 \quad \Rightarrow \quad \frac{h_j(t_w)}{h_j(t_w+\tau)} = 1 \quad \text{for } \; i<j \tag{10}$$

[d] Note the crucial ordering of the two limits.

Notice that these time sectors correspond to asymptotically distinct relative 'epochs' in the sense that if t_1, t_2 belong to the domain defined by h_i and t_2, t_3 to the one defined by h_j with $j > i$, then t_1, t_3 also belong to the sector by h_j.

The correlation function can be, for example, the sum of terms of the type (5), each with a different scaling function h_i replacing ξ. Because the different scalings vary in time-sectors that do not overlap, such a function cannot be reexpressed in terms of a simpler scaling form valid for all large times.

A simple example for the $h_i(t)$ is

$$h_i(t) = \exp\left(\frac{t^{1-\mu_i}}{(1-\mu_i)t_o^{1-\mu_i}}\right) \tag{11}$$

with $0 \leq \mu_i \leq 1$. In this case the i^{th} scaling form corresponds to $\tau \sim t_o^{1-\mu_i} t_w^{\mu_i}$. In particular, $\mu = 0$ yields the time-translational invariant form and $\mu = 1$ the τ/t_w scaling variable which is independent of the microscopic time t_0.

The main message of this Section is that once one abandons time-translational invariance, as one should in systems that never equilibrate, the two-time correlation (or response) function in the long-times limit may have a rich structure including multiple-scaling forms like (9). It will turn out that a rather general classification of the asymptotic behaviour in the two-time plane can be made using only the monotonicity property of the correlation and simple group theory (see Section 3.4).

The simple example of coarsening also illustrates the fact that in order to decide whether a system can be considered to be in equilibrium, *one-time* quantities (such as the energy, magnetization, etc) can be misleading. Indeed, at any finite time the excess energy density of a coarsening ferromagnet is proportional to the total domain surface divided by the volume, a quantity which soon becomes very small. If one were to judge the degree of equilibration by only measuring the excess energy density, one would wrongly conclude that the system equilibrates rather rapidly. On the other hand, two time quantities (such as the correlation function) reveal very clearly that the system is still out of equilibrium even at long times.

1.3 Fluctuation-Dissipation Relations

As was mentioned before, in a system at equilibrium, the response to an external magnetic field and the autocorrelation functions are related through the fluctuation-dissipation theorem (FDT). This is true in general for the response to a field h conjugate to any observable O and the corresponding autocorrelation $C_{\rm O}(t_w+\tau, t_w) \equiv \langle O(t_w+\tau) O(t_w)\rangle$. In equilibrium:

$$R_{\rm O}(t_w+\tau, t_w) \equiv \left.\frac{\delta \langle O(t_w+\tau)\rangle}{\delta h(t_w)}\right|_{h=0} = R_{\rm O,EQ}(\tau) = -\frac{1}{T}\frac{\partial C_{\rm O,EQ}(\tau)}{\partial \tau} \tag{12}$$

or, introducing the integrated response $\tilde{\chi}_{\rm O}(t_w+\tau, t_w) = \int_{t_w}^{t_w+\tau} R_{\rm O}(t_w+\tau, t')dt'$:

$$\tilde{\chi}_{\rm O}(t_w+\tau, t_w) = \tilde{\chi}_{\rm O,EQ}(\tau) = \frac{1}{T}(C_{\rm O,EQ}(0) - C_{\rm O,EQ}(\tau)) \; . \tag{13}$$

If O is the energy, one obtains the relation between energy fluctuations and specific heat, if O is the magnetization, one finds a relation between the (time dependent) field induced magnetisation and the magnetic noise correlations, etc.

In order to study the relation between 'fluctuations' and 'dissipation' in out-of-equilibrium systems, one has to think in terms of two-time correlation and response functions. Let us then consider a given t_w and make a parametric plot of $\tilde{\chi}_{\mathrm{O}}(\tau + t_w, t_w)$ *vs.* $C_{\mathrm{O}}(\tau + t_w, t_w)$ when τ varies. One then takes a larger t_w and repeats the plot, and so on. If the system equilibrates after a finite time t_{ERG}, one obtains, when $t_w \gg t_{\mathrm{ERG}}$, a limiting $\tilde{\chi}_{\mathrm{O}}$ *vs.* C_{O} curve which is a straight line with slope $-1/T$: this is the FDT.

Consider instead what happens in the example of domain growth in an infinite size system. Within a large n treatment of the problem [40], one obtains for the χ *vs.* C curves a family of curves shown in Fig. 3. For large t_w, the curves approach a *broken line*: one with slope $-1/T$ for values of the correlation larger than q_{EA} (*i.e.* for times in the stationary regime defined in the previous Section), and one with zero slope for values of the correlation smaller than q_{EA} (the aging regime). This simple example illustrates that (despite the fact that the dynamics is very slow when $t_w \to \infty$) the system cannot be thought of as in a 'quasi-equilibrium' state, for which concepts from equilibrium are more or less valid: there is always a regime in which FDT is strongly violated. (It would be interesting to confirm these results within more realistic models of coarsening, and also, obviously, experimentally and numerically.) We also see that the Edwards-Anderson parameter plays a role for the response functions.

Similar results are found analytically in the mean-field spin-glass systems which we shall review below (see Fig. 4 and Section 3). The main difference is that the aging part of the curve (i.e. $C < q_{\mathrm{EA}}$) has a non-zero slope.[41–45] This is quite important, since it means that the integral of the response function over the aging regime gives a *non-zero* contribution.

In the more realistic $3D$ Edwards-Anderson model, the form of the $\tilde{\chi}$ *vs.* C curves which are obtained numerically (at least for the computer times accessible at present) are actually remarkably similar [46] to the mean-field prediction for the corresponding mean-field model [43]. The same is true of some very recent numerical simulations of binary soft-sphere mixtures [47].

The violation of FDT can be parametrized by introducing a violation factor $X_{\mathrm{O}}(t, t')$ defined as

$$R_{\mathrm{O}}(t, t') \equiv \frac{X_{\mathrm{O}}(t, t')}{T} \frac{\partial C_{\mathrm{O}}(t, t')}{\partial t'} . \tag{14}$$

Note that we are differentiating with respect to the *smallest* time t'. In analytic studies of mean-field systems, one can furthermore show that for large times X_{O} depends on t, t' only through the value of the correlation function: $X_{\mathrm{O}}(t, t') = X[C_{\mathrm{O}}(t, t')]$. In particular, when $C_{\mathrm{O}} > q_{\mathrm{EA,O}}$, $X_{\mathrm{O}} = 1$, and the FDT is recovered.

It turns out [48] that the 'effective temperature'

$$T_{\mathrm{O}}^{\mathrm{EFF}}(t_w + \tau, t_w) \equiv \frac{T}{X_{\mathrm{O}}(t_w + \tau, t_w)} \tag{15}$$

is precisely the temperature which would be read on a thermometer with response time τ (or frequency $\omega \sim 1/\tau$) when connected to the observable O at time t_w. A 'fast' thermometer

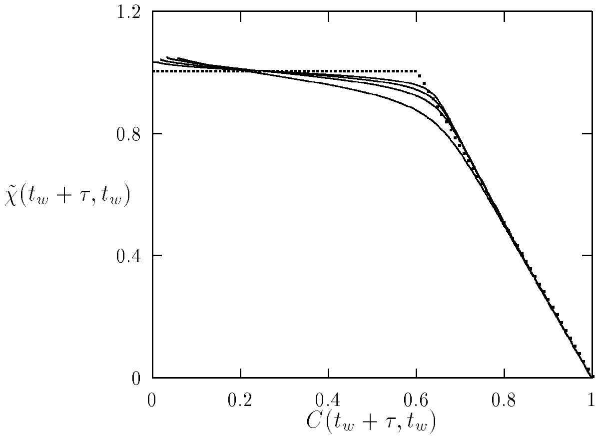

Figure 3: The susceptibility $\tilde{\chi}(t_w+\tau,t_w) \equiv \int_{t_w}^{t_w+\tau} ds\, R(t_w+\tau,s)$ vs the correlation $C(t_w+\tau,t_w)$ for domain growth in a n-component vector ferromagnet at $T<T_g$, obtained from an analytical treatment of the large n limit. (From Ref.[48].)

of response time $\tau \ll t_w$ will then probe the stationary regime for which $X_o = 1$ and thus measure the heat-bath temperature. This is the reason why glasses, although still out of equilibrium after many hours, feel as cold as the room they are in.

1.4 Edwards-Anderson parameter, weak-ergodicity breaking and clonation

The situation we have described in the last two subsections is one in which equilibrium is not achieved, in the sense that configurations are not visited with a probability given by the Gibbs-Boltzmann weight. However, this is not the main point: none of the above results can be explained within a *strong ergodicity breaking* scenario where the system falls into a very long-lived metastable state, and achieves fast equilibration within such restricted sector of phase-space. A mixture of hydrogen and oxygen, or a crystal of diamond are systems which are metastable, but for all dynamical purposes in equilibrium. As Feynman puts it: 'Equilibrium is when all fast things have happened, and slow things not yet'. The out of equilibrium situations which are of interest to us are those where, in a sense, 'things keep happening on all time scales'.

More precisely, in a spin-glass (or a coarsening problem) in the low temperature phase the spins do, on average, remember for some time their orientation, which leads to a non-zero Edwards-Anderson parameter $q_{\rm EA} = \lim_{\tau\to\infty} \lim_{t_w\to\infty} C(t_w+\tau,t_w)$. However, if the waiting time is *finite*, the system is able to escape arbitrarily far from the configuration it had reached at t_w, leading to

$$\lim_{t_w\to\infty} \lim_{\tau\to\infty} C(t_w+\tau,t_w) = 0 \tag{16}$$

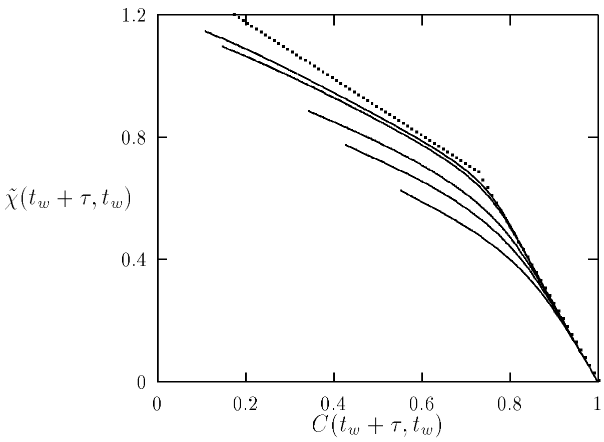

Figure 4: The susceptibility $\tilde\chi$ vs the correlation C for the 'discontinuous' mean-field spin glass model of Section 3 ($p=3$) at $T<T_g$. In this case the FDT violation for $C<q_{\rm EA}\sim 0.76$ is given by $-X/T$, with $X=(1-q_{\rm EA})/q_{EA}\sim 0.76$. More complicated situations are described in Section 3. (From Ref.[48].)

even in the low temperature phase where $q_{\rm EA}>0$. This situation was called 'weak-ergodicity breaking' in Refs.[19,41,43].

It is important to note that the fact that a system undergoes weak (as opposed to strong) ergodicity breaking for infinite times does not mean that stable states do not exist in phase-space. The dynamical behaviour depends on the choice of initial conditions. The example of ferromagnetic coarsening is again eloquent in this respect. There exist two true equilibrium states, and the relaxation from an initial condition close to one of them (e.g. all spins up) is fast and shows no aging. But in the relaxation from a random initial condition (e.g. a typical configuration at a temperature above T_c), the system remains forever (in the thermodynamic limit) undecided as to which state it will go to.

Another interesting question is the following: suppose that at time t_w one 'duplicates' the spin system and evolves subsequently the two copies using two independent thermal baths. (This process was called 'clonation' in Ref.[30].) Will the two copies stay 'close together' or conversely will they evolve independently, forgetting their common breed? This is measured by the 'overlap' function $Q(t_w+\tau,t_w+\tau)$ defined as [50]

$$Q(t_w+\tau,t_w+\tau)=\frac{1}{N}\sum_{i=1}^{N}S_i^{(1)}(t_w+\tau)S_i^{(2)}(t_w+\tau) \qquad S_i^{(1)}(t_w)\equiv S_i^{(2)}(t_w)\ , \tag{17}$$

where the superscripts $^{(1,2)}$ refer to the two copies. We shall see below that the limit

$$\lim_{t_w\to\infty}\lim_{\tau\to\infty}Q(t_w+\tau,t_w+\tau)=Q_\infty \tag{18}$$

can be zero or non zero, even if $q_{\rm EA}$ is non zero. This may serve to distinguish different types of aging dynamics [51,40,30,31].

If the limit is taken in reversed order (i.e., $t_w \to \infty$ first), the overlap function contains the same information as the correlation function. More precisely [51]:

$$\lim_{t_w \to \infty} Q(t_w + \tau, t_w + \tau) \equiv C_{\rm ST}(2\tau) \ . \tag{19}$$

2 Phenomenological models of aging

In order to account for the above experimental results, one has as usual the choice between some physically motivated, but phenomenological, pictures or some rather more precise microscopic models, in a limit in which they are analytically tractable. Both approaches are actually complementary, and shed light on each others' limitations. We shall start by reviewing the phenomenological pictures of aging, based either on domain growth arguments, or on models of random walks in phase space.

2.1 Coarsening in non disordered systems

As we noted in the previous Section, the simplest model where aging occurs is the ferromagnetic Ising model suddenly quenched below its Curie point temperature. The initial configuration is random, and it orders progressively through domain growth. Depending on the class of microscopic dynamics [39], the typical size ξ of the domains grows as $t^{1/2}$ ('non-conserved' case) or $t^{1/3}$ ('conserved' case). The 'age' of the system is thus directly encoded in the spatial correlation functions. The two-time correlation function $C(t_w + \tau, t_w) = \frac{1}{N} \sum_i S_i(t_w + \tau) S_i(t_w)$ can be calculated exactly in some cases (e.g. the Ising model in one dimension, or the large-n 'spherical' model), or using some approximations (for a review see Ref.[39]). One finds an expression as Eq.(5)

$$C(t_w + \tau, t_w) = C_{\rm ST}(\tau) + C_{\rm AG}\left(\frac{\xi(t_w)}{\xi(t_w + \tau)}\right) \tag{20}$$

where $C_{\rm AG}$ decays as a (non trivial) power-law for large arguments. $C_{\rm AG}(1+u)$ is in general also singular around $u = 0$, its behaviour is characterised by an exponent [45] b and, in the 'non-conserved case', one can argue that [52,53], $C_{\rm AG}(1) - C_{\rm AG}(1+u) \propto \sqrt{u}$. Correspondingly, the aging part of the a.c. susceptibility decays as in Eq. (1), with $b = 1/2$.

The overlap function $Q(t_w + \tau, t_w + \tau)$ can also be estimated in this simple coarsening situation. Within standard approximate treatments of coarsening [40,51], one finds that the quantity Q_∞ defined in Eq.(18) is non zero, meaning that the two copies follow each others' footsteps (within finite times) in their evolution towards equilibrium. Note also that within the large n approximation, the FDT violating factor X goes to zero at infinite times, as ξ^{-1}. It would be interesting to know whether this is a more general property of coarsening dynamics.

It is interesting to remark that in the presence of a small external magnetic field H, aging is 'interrupted' after a finite time $t_{\rm ERG}(H)$, since one of the two phases is favoured by the field. For example, in the case of the spherical model, one finds [54] $t_{\rm ERG}(H) \propto H^{-2}$. This behaviour is expected on general grounds: it corresponds to the time beyond which the curvature induced driving force is superseded by the field induced driving force.

2.2 Coarsening in disordered systems I: Random ferromagnet or random field

Let us now consider the case of a disordered ferromagnet in dimension larger than 2, where random local magnetic fields or random local couplings are present. If the disorder is sufficiently weak (for example, if a small fraction of the ferromagnetic bonds are removed), the ground state of the system still has long-range ferromagnetic order, and the description in terms of the growth of ordered domains is valid. However, due to the presence of disorder, the domain *walls* will tend to be pinned by local inhomogeneities. The problem of domain walls in disordered environments has been the focus of intense study in the recent years [55]. In many aspects, this problem is close to the spin-glass problem, with a large number of metastable states (although of course the 'spin-glass' nature of the problem only concerns the small fraction of spins which belong to the domain walls). The dynamics of a given section of a domain wall proceeds by thermally activated hops between different favourable configurations. Ordered domains thus grow on average, but at a much reduced rate compared to the pure ferromagnet described above.

A generally accepted description is as follows [56]: the typical pinning energy scale of a domain wall of linear size R grows as ΥR^θ, where θ is an exponent which depends on the problem (random field/random bond) and the dimension of space. A simple scaling argument[e] then suggests that the time needed for a domain to reach a certain size R is given by

$$\tau(R) \propto t_o \exp\left(\frac{\Upsilon R^\theta}{kT}\right) . \tag{21}$$

After a certain time t, the typical size of the domains is thus expected to be given by

$$\xi(t) \propto \left(\frac{kT \log\left(\frac{t}{t_o}\right)}{\Upsilon}\right)^{\frac{1}{\theta}} , \tag{22}$$

provided the corresponding pinning energy $\Upsilon \xi(t)^\theta$ is large compared to kT. (In the other limit, the pinning energy is negligible, and one recovers the growth law which we discussed in the previous paragraph.) This logarithmic growth of domain sizes has been rather carefully checked numerically [59] in $D = 3$ ($T \neq 0$).

Apart from the fact that domains grow very slowly, one expects that the picture prevailing in the pure ferromagnetic case is not drastically modified. In other words, the correlation function should still be given by Eq. (5), but with a logarithmically growing $\xi(t)$. This was confirmed numerically on the Ising model with random fields, for $D = 1$ [53] and $D = 3$ [60]. In $D = 1$, the motion of a 'domain wall' (a point) is given by Sinai's diffusion law, i.e.

[e]This argument assumes that barrier heights between metastable states behave (as a function of R) similarly to the pinning energy of each state [57,58]

$\xi(t) \propto \log^2(t)$ (at least for times smaller than a certain temperature dependent t_{ERG}, which diverges when $T \to 0$). The aging part of the two-time correlation function can indeed be satisfactorily rescaled when plotted versus $\xi(t_w)/\xi(t_w+\tau)$.

It is important to notice that the above scenario, where the system tries to reach a well defined (temperature independent) state, but is slowed down due to pinning by impurities, leads to large cooling rate effects. This is because the crossover energy $\Upsilon\xi(t)^\theta \simeq kT$ will be reached at later times if the cooling rate is slower [60]. The size of the domains when the working temperature T_1 is reached will thus be larger, the smaller the cooling rate, and this will affect many physical observables, such as the energy.

Let us finally point out that disorder is actually not necessary to obtain logarithmic in time (rather than power-law) growth of the domain size. This was first shown in Ref.[61] for a pure Ising model with next-nearest neighbour couplings. Actually, if the domain walls are below their roughening temperature, the dynamics proceeds via the nucleation of terraces[62], and also leads to logarithmic domain growth.

2.3 Coarsening in disordered systems II: Spin glasses and droplets

It is not obvious whether the above coarsening description also applies to spin glasses, because of the non-conventional nature of their order parameter. However, Fisher and Huse [63] have argued that for spin-glasses in finite dimension and for any given temperature below the spin-glass transition, there are only two 'pure states' (spin-reversed from each other) which have to be considered, which can conventionally be called 'up' and 'down', very much like in the Mattis model [64]. This assumption gives some physical content to a scaling description of the spin glass phase, first advocated in [65,66]. It allowed several authors [63,67,68] to develop a rather complete phenomenological picture of spin-glasses in low dimensions, where the spin-glass is considered as a 'disguised ferromagnet' (with however the important difference that the two pure states are not stable when the temperature is changed – see below). This is at variance with the mean-field picture emerging from Parisi's solution [7] of the Sherrington-Kirkpatrick model [2], where many (non trivially related) pure states coexist [4,69]

If this 'two-state' picture is retained, the dynamics of the system can again be described in terms of growing and coalescing compact domains [67], which have also been called 'droplets' [63] in this context. The presence of disorder presumably pins the domain walls, leading again to a logarithmic growth of the droplets, and thus to a two-time correlation very similar to the random-bond or random field case described above. In particular, the aging part of the correlation function should be a function of[f] $\xi(t_w)/\xi(t_w+\tau)$.

One striking property of experimental spin-glasses, however, is the very weak dependence of its physical properties on the cooling rate [70]. For example, the asymptotic value of the a.c. susceptibility, $\chi(\omega, t_w \to \infty)$, is nearly independent of the cooling rate. This is at first sight surprising in a scenario of activated domain growth, as we emphasized above. But if one argues that the spin-glass phase is 'chaotic' [66,63,67], i.e. that the two pure states towards which the system evolves are extremely fragile to temperature changes, it becomes

[f] Fisher and Huse actually postulated that the scaling variable would rather read $\xi(\tau)/\xi(t_w)$.

obvious that the dynamics of the system at temperatures greater than T_1 is useless to bring the system closer to its equilibrium at the working temperature T_1. In a first approximation, the configuration reached by the system at temperature $T_1 + \delta T$ is as remote from the 'true' equilibrium state at temperature T_1 than a high temperature configuration. Hence the cooling rate may indeed have a negligible effect.

How does the two-state picture compare with aging experiments or numerical simulations, in the case of a $3-d$ Ising spin-glass? The detailed discussion of this point is beyond the scope of the present paper and the conclusions still controversial [34,15,71,38], in particular the existence of a transition in a magnetic field [72–75]. One should however mention the following results:

—— Both experimentally and numerically, the aging part of the correlation and response functions follow a scaling which is systematically closer to τ/t_w than the expected $\xi(t_w)/\xi(t_w+\tau)$ (or even $\xi(\tau)/\xi(t_w)$) if the barrier heights scaled as R^θ, e.g. $\xi(t_w) = \log(t_w/t_o)^{1/\theta}$. The point is that such a scaling would place the curves of Fig. 2 in the reversed order: the young ones would be *below* the old ones in a τ/t_w plot.

However, as suggested by Rieger [37], the two state picture with barrier heights scaling as $\Upsilon \log R$ (i.e. $\theta \to 0$) would lead to an algebraic domain growth law (as postulated by Koper and Hilhorst [67]): $\xi(t) \propto t^\alpha$ with $\alpha = kT/\Upsilon$ [27,38], and thus in turn to a τ/t_w scaling.

—— A direct numerical indication of a growing length scale was searched for in [76,27]. A possibility is to study the following correlation function

$$G(\vec{r},t) = \frac{1}{N}\sum_i \langle S_i^{(1)}(t) S_i^{(2)}(t) S_{i+\vec{r}}^{(1)}(t) S_{i+\vec{r}}^{(2)}(t)\rangle \ , \tag{23}$$

where $S_i^{(1)}$ and $S_i^{(1)}$ which has the following intuitive meaning: knowing that the copies 1 and 2 are in the same state at site i (resp. opposite states), what is the probability that they are still in the same state (resp. still in opposite states) a distance $\vec{r}$ apart ? Numerically, $G(\vec{r},t)$ is seen (in 3 dimensions) to be of the form [77]:

$$G(\vec{r},t) = \frac{1}{r^\zeta} g\left(\frac{r}{\xi(t)}\right) \qquad \xi(t) \propto t^\alpha \tag{24}$$

which indeed suggests the presence of a growing scale $\xi(t)$ in the dynamics. The conventional droplet picture predicts (apart from a logarithmic, rather than power-law, growth of $\xi(t)$) that $\zeta = 0$, since the equilibrium state should be unique up to a global sign change, whereas simulations by Marinari *et al.* [77,38] suggest that $\zeta > 0$, just as in the equilibrium situation obtained with replica field theory [78] in dimensions smaller than 6.

—— The temperature cycling experiments [13] where the system is cooled to temperature $T_1 < T_g$, then to temperature $T_2 < T_1$, and finally back to temperature T_1, show a very striking conjunction of 'rejuvenation' (when the temperature is decreased) *and* memory (when the system is heated back). The coexistence of these two effects is rather awkward to interpret within the droplet picture [13,79]. In the same spirit, Weissmann et al. [36] have argued that the 'second noise spectrum' of spin-glasses does not conform with what could be expected from a simple 'two-state' picture.

In summary: even if the 'disguised ferromagnet' picture did provide a correct description of the equilibrium properties of low-dimensional spin-glasses, it is not obvious that this description is sufficient to account for the *out-of-equilibrium* properties. One reason is that – precisely because of the chaotic nature of the equilibrium phases – the system will not only nucleate domains of the nominal equilibrium state, but probably also 'phases' corresponding to nearby temperatures, which will thus contribute to the non-stationary part of the dynamics.

2.4 Stranded in phase space: the 'trap' model

'Phase-space' models are another very useful class of phenomenological models for the dynamics of complex systems, and have been advocated by very many authors over the years[80,85]. The dynamics of the whole system is summarized in the motion of a single point evolving within a complicated energy landscape in configuration space. From a general point of view, one expects this energy landscape to be made of 'valleys', or 'traps' (within which all configurations are mutually accessible in a short time) separated by 'barriers', which the system can only overcome by thermal activation. A coarse-grained representation of the problem can thus be given in terms of states $\alpha, \beta, \gamma ...$ between which the system wanders. The dynamics of the system is thus described by a master equation for the probability to find the system in the state α:

$$\frac{\partial P_\alpha}{\partial t} = -\sum_\beta W_{\alpha\to\beta} P_\alpha + \sum_\beta W_{\beta\to\alpha} P_\beta \ . \tag{25}$$

The choice of the hopping rates $W_{\alpha\to\beta}$ then encodes the statistics of the barrier heights and the geometry of the phase space. It is rather arbitrary apart from the constraint of detailed balance. For example, one can organize the 'traps' on a hierarchical tree, and choose $W_{\alpha\to\beta}$ to only depend on the distance between α and β along the tree. This has led to several 'ultrametric' diffusion models [80], with many interesting results, including, in some cases, aging effects.[87–89]

- The one-level tree.

The simplest of these models,[49] for which the appearance of aging has a particularly clear interpretation, is when the hopping rate only depends on the starting state: $W_{\alpha\to\beta} = (N\tau_\alpha)^{-1}$, where N is the total number of states. (The final state β is thus independent of the initial state; the process starts anew at each jump.) This corresponds to the picture drawn on the left of Fig. 5. The trapping times τ_α are of the form $t_o \exp(B_\alpha/(kT))$, where B_α is the energy barrier 'surrounding' state α. Within this description, the *equilibrium* measure P_α^{eq} (if it exists) is simply proportional to τ_α. In order to reproduce the correct Bolzmann equilibrium, one should thus identify B_α with the free-energy of the state[g] f_α. Mean-field

[g]The relation between barrier heights and energy depths is however not obvious in general; for recent work on this subject, see Ref. [57]

models of spin-glasses [90] or replica treatment of randomly pinned manifold suggest that the distribution of the metastable states' free energies f_α is exponential [90,4,91,92]

$$\rho(f_\alpha) \propto \exp\left(-\frac{x|f_\alpha|}{kT}\right) \tag{26}$$

with a certain parameter $x \leq 1$ in the glassy phase, $x = T/T_g$ in the Random Energy Model[93,94]. The appearance of this exponential tail for 'deep' states can be understood on general grounds, and is related to the so-called 'extreme value statistics' (for a more precise discussion, see Refs. [95,96]). The corresponding distribution of trapping times [h] is then easily found to be

$$\rho(\tau)d\tau = \rho(f)df \longrightarrow \rho(\tau) \propto_{\tau \gg t_o} \frac{t_o^x}{\tau^{1+x}} \; . \tag{27}$$

Let us now introduce the quantity $\Pi(t_w+\tau, t_w)$ defined as the probability that the system has not changed trap between time t_w and time $t_w+\tau$. This quantity is found to be very different depending on whether x is larger or smaller than 1. In the former case, $\lim_{t_w\to\infty}\Pi(t_w+\tau,t_w)$ is well defined, and found to be proportional to $(t_o/\tau)^{x-1}$. For $x<1$, however, one finds that $\Pi(t_w+\tau,t_w)$ 'ages', and is given by:

$$\Pi(t_w+\tau,t_w) = \frac{\sin(\pi x)}{\pi}\int_{\frac{\tau}{\tau+t_w}}^{1} du(1-u)^{x-1}u^{-x} \qquad (t_w \gg t_o) \; . \tag{28}$$

In physical terms, this means that after a waiting time t_w, the only states which have an appreciable probability are those with a trapping time of the order of t_w itself. This reflects the fact that the distribution of trapping times $\rho(\tau)$ becomes so broad when $x<1$ (the average trapping time becomes infinite), that the sum of all the trapping events $\tau_1+\tau_2+...+\tau_N$ is always dominated by its largest term, which is thus of the order of the experimental time itself[49]. This dominance of a few very important events is a characteristic feature of Lévy statistics [98]. On the other hand, when $x>1$, the trapping times are all of order t_o.

Let us now define the overlap $q_{\alpha\beta}$ between states as $q_{\alpha\alpha} = q_{\text{EA}}$ and $q_{\alpha\neq\beta} = q_0$ (more general choices will be discussed below). The self-overlap q_{EA} is smaller than 1 in general; this reflects the fact that many microscopic configurations are mutually accessible within times of order t_o; this contributes to the equilibrium part of the correlation function which the one-tree level cannot describe. The spin-spin correlation function, averaged over the disorder [i] is thus given by

$$C_{\text{AG}}(t_w+\tau,t_w) = q_{\text{EA}}\Pi(t_w+\tau,t_w) + q_0(1-\Pi(t_w+\tau,t_w)) \; . \tag{29}$$

Note in particular that $\lim_{\tau\to\infty}\lim_{t_w\to\infty} C_{\text{AG}}(t_w+\tau,t_w) = q_{\text{EA}}$ when $x<1$, but that $\lim_{t\to\infty}\lim_{t_w\to\infty} C_{\text{AG}}(t_w+\tau,t_w) = q_0$ for $x>1$. Within this model, $x=1$ thus corresponds to a true glass transition.

[h] Generalisation to other distributions of f_α has been considered in Ref.[97]

[i] i.e. over the distribution of f_α. Note that in this model, the correlation function is not self-averaging, precisely for the same reason as for the static overlap distribution $P(q)$ in mean-field spin-glasses [69]

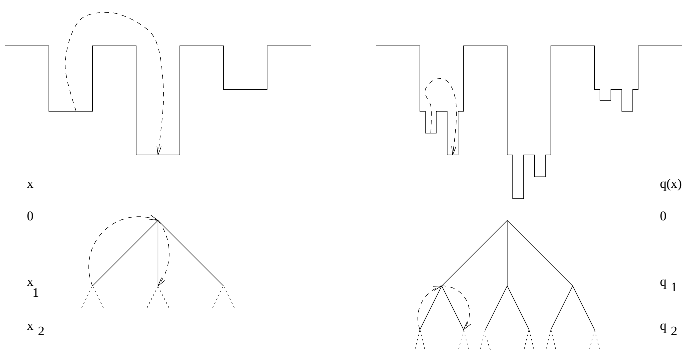

Figure 5: Schematic phase-space landscape of a one level tree and of a multi-level tree.

Equation (28) leads to the following asymptotic behaviour for $C_{\rm AG}(t_w+\tau,t_w)$ [79]:

$$C_{\rm AG}(t_w+\tau,t_w) \simeq q_{\rm EA} - \frac{\sin(\pi x)(q_{\rm EA}-q_0)}{\pi(1-x)}\left(\frac{\tau}{t_w}\right)^{1-x} \qquad (\tau \ll t_w) \tag{30}$$

$$\simeq q_0 + \frac{\sin(\pi x)(q_{\rm EA}-q_0)}{\pi x}\left(\frac{t_w}{\tau}\right)^{x} \qquad (\tau \gg t_w) \tag{31}$$

Hence, both the 'short time' ($t_o \ll \tau \ll t_w$) and 'long time' regimes are described by power-laws, much like in the simple coarsening models described in Sections 2.2 and 2.3.

The above model can be endowed with magnetic properties by assigning to each state a certain magnetisation m_α, and modifying the hopping rates in the presence of a field to recover the correct equilibrium weights [79]. One then finds that the generalised form of the FDT (Eq. (14)) holds, with:

$$X(t_w+\tau,t_w) = 1-\zeta+\zeta\frac{t_w}{t_w+\tau} \tag{32}$$

where ζ is a free parameter of the model, restricted to the interval $[0,1]$. For $\zeta=0$, the thermoremanent magnetisation is simply proportional to $C(t_w+\tau,t_w)$.

Experiments on the thermoremanent magnetisation or the a.c. susceptibility show that both the 'short time' and 'long time' regimes can be fitted by power-laws (see Fig. 7 in Ref.[[14]]). The exponent x which comes out of these two fits is however different: $x=1-b\simeq 0.6-0.9$ (depending on temperature) from the $\tau \ll t_w$ region, and $x\simeq 0.1-0.3$ from the $\tau \gg t_w$ region [79,14].

One can also define the overlap function $Q(t_w+\tau,t_w+\tau)$ within such a model, and show that it is simply related to C through $Q(t_w+\tau,t_w+\tau)=C(t_w+2\tau,t_w)$. One thus finds, in particular, $Q_\infty=q_0$: two copies of the same system decorrelate completely from each other [51]. Note however that this would not be true if the 'traps' were organized along a low dimensional 'path' in phase space.

• The multi-level tree.

An interesting generalisation of this 'one-level' trap model is to consider a hierarchical organisation of traps within traps, and to relate the overlap $q_{\alpha\beta}$ between two states to their distance along the tree (see the figure on the right of Fig. 5). The hopping rate $W_{\alpha\to\beta}$ is still taken to be independent of the final state for a given distance along the tree, or for a given overlap $q = q_{\alpha,\beta}$

$$W_{\alpha\to\beta} \propto \frac{1}{\tau_{\alpha,q}} . \tag{33}$$

As reviewed in Refs [1,91,99], the interpretation of the static full replica symmetry breaking solution suggests that the distribution of free-energies of the states at a certain level of the tree is still exponential, but with a parameter x which now depends on the overlap between these states (and is the inverse of the Parisi function $q(x)$)

$$\rho_q(f) \propto \exp\left(-\frac{x(q)(\tilde{f}-f)}{kT}\right) , \tag{34}$$

where $\tilde{f}$ is the free-energy of the 'ancestor' state, itself distributed exponentially with a parameter $x(q-dq)$, etc. Assuming again that the barrier between the states is related to the free-energy depth as pictured in Fig. 5, one is led to surmise that the trapping time $\tau_{\alpha,q}$ is still distributed as a power-law, but with a q dependent exponent:

$$\rho_q(\tau) \propto_{\tau\gg t_o} \frac{t_o^{x(q)}}{\tau^{1+x(q)}} . \tag{35}$$

Note that $x(q)$ is an increasing function of q, which means that the smaller the overlap between states, the broader the distribution of time scales. In other words, 'fast' processes are deep down the tree. The correlation function is now determined from [j]

$$C(t_w+\tau,t_w) = \sum_{j=0}^{M} q_j[\Pi_j(t_w+\tau,t_w) - \Pi_{j+1}(t_w+\tau,t_w)] = \sum_{j=0}^{M}[q_j - q_{j-1}]\Pi_j(t_w+\tau,t_w) , \tag{36}$$

where j labels the level of the tree from top to bottom, and $\Pi_j(t_w+\tau,t_w)$ is the probability that no jump beyond the j^{th} level of the tree has occured between t_w and $t_w+\tau$. All the levels $j > M^*$ such that $x(q_j)$ is larger than 1 are equilibrated on microscopic ($\sim t_o$) times. This means that the corresponding Π_j are zero as soon as $\tau \gg t_o$. This part of the tree thus contributes to the *stationary* dynamics, while the levels $j \leq M^*$ contribute to the *aging* part

$$C(t_w+\tau,t_w) = \sum_{j=0}^{M^*} q_j[\Pi_j(t_w+\tau,t_w) - \Pi_{j+1}(t_w+\tau,t_w)] + C_{\mathrm{ST}}(\tau) . \tag{37}$$

It is easy to show [79] that the short time ($\tau \ll t_w$) decay of the aging part of $C(t_w+\tau,t_w)$ behaves as in Eq. (30), with $x \to x(q_{M^*})$, while the long time ($\tau \gg t_w$) decay is decribed

[j] With the convention $\Pi_{M+1} \equiv 0$, and $q_{-1} = 0$.

by Eq. (31) with $x \to x(q_0)$. Not surprisingly, the short time decay is mostly sensitive to the fastest part of the tree $j = M^*$, while the long time decay is governed by the slowest processes $j = 0$. Note that the experimental data is indeed such that the long time $x(q_0)$ is smaller than the short time exponent $x(q_{M^*})$.

The presence of levels such that $x \simeq 1$ is very interesting from the point of view of '$1/f$'-noise, for the following reason. If $x > 1$, the corresponding contribution to the a.c. susceptibility is stationary and behaves as $\chi''_{\rm ST}(\omega) \propto (\omega t_o)^{x-1}$. The noise spectrum is thus given by

$$S(\omega) = \frac{2T}{\pi}\frac{\chi''(\omega)}{\omega} \propto \omega^{x-2} \, . \tag{38}$$

So the leading contribution to the low frequency noise coming from these equilibrated processes with $x > 1$ comes from $x \simeq 1$ and scales as $1/\omega$. On the other hand, if $x < 1$, the contribution to the a.c. susceptibility is aging and behaves as $\chi''(\omega, t_w) \propto (\omega t_w)^{x-1}$. This contribution is thus decaying with time, but more and more slowly as x approaches 1 from below. Among these aging modes, those which are the slowest to disappear correspond to $x = 1^-$, leading again to a $1/\omega$ dependance. Within this picture, $1/f$ noise appears rather naturally; furthermore one expects that this noise should generically exhibit some non-stationary contributions.

Another interesting aspect of the dynamics on a 'multi-level' tree is its response to temperature cycling. It was suggested in Refs[13,33] that the negative temperature cyclings – which reveal both 'rejuvenation' in the intermediate, low temperature period, followed by a perfect 'memory' of the a.c. susceptibility – point towards a hierarchical picture of phase space, where finer details are progressively revealed as the temperature is lowered. Using the fact that the whole curve $x(q)$ decreases when the temperature is lowered, it is easy to account for this phenomenon within the multi-level tree model [79].

- Energy or 'entropy' barriers ?

Before discussing the possible relation between this hierarchical picture and real-space, droplet like descriptions, one should emphasize that in the trap models discussed above, aging is induced by the presence of energy barriers, the crossing of which becomes slower and slower as the temperature is decreased. Aging is nevertheless also present in models where there is no barrier crossing at all. It was already noticed [100] that the SK model at $T = 0$ has a slow decrease in energy and never reaches a stable configuration. This same situation can be seen more clearly in the following model: let us consider the case [84] where the hopping rate $W_{\alpha\to\beta}$ is equal to zero if the state energy f_β is larger than f_α, and equal to $W_0 = (Nt_o)^{-1}$ otherwise. This rule actually corresponds to the Glauber dynamics at zero temperature. In the limit $N \to \infty$, the system never reaches the ground state: there always exists states of lower energy towards which the system can evolve – the number of these 'escape directions' however becomes smaller and smaller as time increases. One can show that *independently* of the distribution of energies $\rho(f)$ (but provided that these energies are independent), the correlation function in this model is given by [84]

$$C(t_w+\tau,t_w)=(q_{\rm EA}-q_0)\frac{t_w}{t_w+\tau}+q_0\ . \tag{39}$$

Note that the decay of $C(t_w+\tau,t_w)$ is regular when $\tau\to 0$, in contrast with the above trap model and, as we shall see, with the generic mean-field situation. Slow dynamics in this model can thus be attributed to 'entropic barriers', i.e. the fact that paths leading to smaller energies become more and more scarce as time increases. A similar scenario holds in the 'Backgammon model' introduced by Ritort [101] and studied in detail in Ref. [[102]]. It is reasonable to expect that both energy and entropy barriers should contribute to non-equilibrium dynamics in real systems.

- Speculations about trees and clusters.

What could be the interpretation of Eq. (36) in real space ? Clearly, large q's should be related to small clusters of reversed spins, corresponding to 'fast' processes, while small q's correspond to large clusters, or 'slow' processes. Let us suppose that the overlap between two configurations can be written as a sum of contributions from 'clusters' of different linear scales ℓ:

$$q_{\alpha,\beta}=\frac{1}{L^d}\sum_{\ell=1}^{L}\sum_{i_\ell=1}^{(\frac{L}{\ell})^d}q_\ell\Theta^\ell_{\alpha,\beta}(i_\ell)\ , \tag{40}$$

where L is the size of the sample, q_ℓ is the incremental contribution to q of the clusters of size ℓ; for fractal clusters [k] of dimension d_f one expects $q_\ell\propto\ell^{d_f}$. $\Theta^\ell_{\alpha,\beta}(i_\ell)$ is equal to zero if the states α and β differ by the reversal of the cluster of size ℓ in the 'cell' labeled i_ℓ and equal to one otherwise. Following the same speculative vein, one can write the two-time correlation function as

$$C(t_w+\tau,t_w)=\sum_{\ell=1}^{L}\frac{q_\ell}{\ell^d}\ \Pi_\ell(t_w+\tau,t_w) \tag{41}$$

where $\Pi_\ell(t_w+\tau,t_w)$ is the average (over all the 'cells' of size ℓ) fraction of clusters which have not flipped between t_w and $t_w+\tau$. The disorder average is no longer needed here since there is a *space* average over many independent clusters.

Assuming that the barrier heights are distributed exponentially with a scale dependent parameter $x(\ell)$, we find that the correlation function will behave much as above in Eqs.(30,31). The parameter $x(\ell)$ actually fixes the relation between energy scales $T/x(\ell)$ and length scales (see Refs.[[99,92]]). Taking $x(\ell)=T/(\Upsilon\ell^\theta)$ (as suggested by scaling arguments or replica calculations on the problem of pinned manifolds [104,99,92]), one finds that there exists a characteristic length scale ℓ^* such that $x(\ell^*)=1$, separating small length scales $\ell<\ell^*$ – for which equilibrium is reached – from large length scales where aging takes

[k] The idea of fractal clusters of spins in spin-glasses dates back to Ref [[103]].

place. In other words,

$$C(t_w+\tau,t_w) = \sum_{\ell=\ell^*}^{L} \frac{q_\ell}{\ell^d}\, \Pi_\ell(t_w+\tau,t_w) + C_{\rm ST}(\tau) \tag{42}$$

with $C_{\rm ST}(\tau) = \sum_{\ell=1}^{\ell^*} \frac{q_\ell}{\ell^d}\Pi_\ell(\tau)$. In particular, one has:

$$q_{\rm EA} \equiv \lim_{\tau\to\infty}\lim_{t_w\to\infty} C(t_w+\tau,t_w) = \sum_{\ell=\ell^*}^{L} \frac{q_\ell}{\ell^d}\,. \tag{43}$$

Assuming that[63] $\Upsilon \propto (T_g - T)^\omega$, one thus finds $q_{\rm EA} \propto (T_g - T)^\beta$, with $\beta = ((d-d_f)\omega)/\theta$. Interestingly, as the temperature is decreased, there is an infinite sequence of 'glass transitions', where all the length scales (in decreasing order) are progressively driven out of equilibrium.

- Conclusion

Although there is still a lot of work to do to clarify the above picture and make it consistent, the idea of modelling the dynamics of a complex system through the motion of a point 'particle' in a random potential is fruitful, and actually used in many different contexts (structural glasses, protein folding, etc.[85]). The aim of the present Section was to show that such models can naturally lead to aging. Actually, the next Sections, devoted to an analytical study of some mean-field models of spin glasses (which can also be seen as models of diffusion in a random potential) will share many similarities *and* important differences with the above discussion.

3 Mean-field models of aging: analytical results

It took several years to realize that mean-field models of spin-glasses, endowed with a suitable relaxational dynamics (usually Langevin, though Glauber is also possible), actually do capture some aging phenomena in the glassy phase[49,41,29]. They thus provide a set of microscopical models where glassy dynamics and aging effects can be studied analytically. This Section will summarize the main results obtained in the recent years on these mean-field models, which have been most valuable in clarifying some of the basic theoretical issues described in the Introduction. The relation between these models and finite dimensional systems is still very much a matter of debate; we shall however postpone this discussion to Section 3.8 and the Conclusion.

The basic simplification occuring in mean-field models is that, after averaging over the disorder and making the number of spins large ($N\to\infty$), one obtains a set of *closed* equations for the two-time correlation and response functions, from which the energy and magnetization can also be calculated. As we discuss below, these equations imply the existence of a critical temperature T_c, below which aging effects appear, and the FDT is violated. On the other hand above T_c, the same equations allow for a TTI solution, compatible with FDT.

Most mean-field dynamics studied so far have focused on models which belong to the following family of spin glass Hamiltonians, describing the interactions of N continuous spins ϕ_i, $i = 1...N$:

$$E(\{\phi\}) = \sum_{r=1}^{\infty} F_r \sum_{i_1 < i_2, ... < i_{r+1}} J_{i_1, i_2, ..., i_{r+1}} \phi_{i_1} ... \phi_{i_{r+1}} \tag{44}$$

where the $J_{i_1, i_2, ..., i_{r+1}}$ are random Gaussian variables with variance N^{-r-1}. Different choices of F_r lead to different models.

A quartic spin weight term can be added in order to make a 'soft' version of Ising spins, or one can consider a spherical version, i.e. $\sum_i \phi_i^2 = N$. A popular choice is the p-spin spherical model [105] defined by $F_r = g\delta_{r+1,p}$, plus a spherical constraint. In what follows we shall mainly use as an example the spherical (or Gaussian) versions. The same model (44) can also be seen as the potential energy of a point particle in a random potential, where the ϕ_i's are the coordinate of the particle's position in a N dimensional space [106]. In this case, the energy is a Gaussian random potential, the correlations of which are related to the F_r's as

$$\overline{E(\{\phi\})E(\{\phi'\})} = N\mathcal{V}\left(\frac{1}{N}\sum_i \phi_i \phi'_i\right) \qquad \mathcal{V}(x) = \sum_{r=1}^{\infty} \frac{F_r^2}{(r+1)!} x^{r+1} \tag{45}$$

where the overline means an average over the quenched disorder.

We shall in the following consider the dynamics to be modeled by a Langevin equation:

$$\frac{d\phi_i}{dt} = -\mu(t)\phi_i - \frac{\partial E}{\partial \phi_i} + \eta_i(t) + h_i(t) \tag{46}$$

where the white noises η_i are mutually independent and of variance $2T$, and $h_i(t)$ is a time dependent external field. The 'mass' term $\mu(t)$ is incorporated in order to enforce the spherical constraint each time, or to model the presence of an harmonic potential in the case of a particle in a random potential, but may be set to zero in other cases.

The correlations and response are defined as:

$$C(t,t') = \frac{1}{N}\sum_{i=1}^{N} \overline{\langle \phi_i(t)\phi_i(t')\rangle}\,, \qquad R(t,t') = \frac{1}{N}\sum_{i=1}^{N} \left.\overline{\frac{\delta\langle\phi_i(t)\rangle}{\delta h_i(t')}}\right|_{h_i=0} \tag{47}$$

where the braces mean average over the thermal noises η. There exist by now well established methods in order to obtain the equations of motion for these systems in the large N limit (time being kept finite). The best known is to introduce a dynamical field theory partition function, average over disorder the partition function by using the fact that it is normalised [107], and compute it for large N by a saddle point method. This is the route which was followed originally by Sompolinsky and Zippelius [8,9]. An alternative derivation uses the cavity method. We refer the reader to the original papers or to more recent textbooks [4,6], and rather focus on the solution of these equations. Starting the dynamics at time $t = 0$ from a random configuration (chosen with a uniform distribution in configuration space, corresponding to a quench from infinite temperatures) the dynamical equations for the spherical or the Gaussian case are found to be

$$\begin{aligned}
\frac{\partial C(t,t')}{\partial t} &= -\mu(t)\, C(t,t') + 2T\, R(t',t) \\
&\quad + \int_0^{t'} ds\; D(t,s)\; R(t',s) + \int_0^{t} ds\; \Sigma(t,s)\; C(s,t')\;, && (48)\\
\frac{\partial R(t,t')}{\partial t} &= -\mu(t)\; R(t,t') + \delta(t-t') + \int_{t'}^{t} ds\; \Sigma(t,s)\; R(s,t')\;, && (49)
\end{aligned}$$

where

$$\begin{aligned}
\Sigma(t,t') &\equiv R(t,t')\mathcal{V}''[C(t,t')]\;, \\
D(t,t') &\equiv \mathcal{V}'[C(t,t')]\;. && (50)
\end{aligned}$$

It is worth keeping in mind the limitations of the present approach. For example, the simple form for Σ, D in terms of C, R is a peculiarity of this class of models. More complicated forms in which $\Sigma(t,t'), D(t,t')$ are *functionals* of C, R (with integrals involving C and R at intermediate times) rather than ordinary functions are obtained, for example, in the case of the Sherrington-Kirkpatrick model. So long as the functional dependency is only on C, R, one may however expect that they can be treated with the same methods. On the other hand, as soon as one introduces a Hamiltonian with a finite number of neighbours per spin (even for mean-field-like Hamiltonian such as Bethe lattice, or random lattice systems), the dynamical equations do not close on the two point correlation and response. One must then introduce a whole hierarchy of k point correlations and responses, which has not been investigated yet.

3.1 Self-averageness and 'universality'

Equations (48), (49) and (50) are exact in the large N limit — for times that do not diverge with N. Furthermore, one can show using the same methods by which they are derived, that the correlations and responses are self-averaging with respect to both the thermal noise *and* the realization of disorder, again for times that do not diverge with N. Hence, we could well have omitted the braces and the overbar in Eq. (47). Non self-averageness in certain macroscopic quantities appears only for times that diverge with the system size, and in particular within the equilibrium Gibbs-Boltzmann measure.

The insensitivity of dynamics with respect to the realisation of disorder is probably intimately related to the fact, discussed in the next Section, that certain models *without* quenched disorder show very similar *out of equilibrium* behaviour, and hence can be studied by considering them as disordered. The correspondence between the random and non-random version might however break down for (divergent) times when non-self averaging features appear.

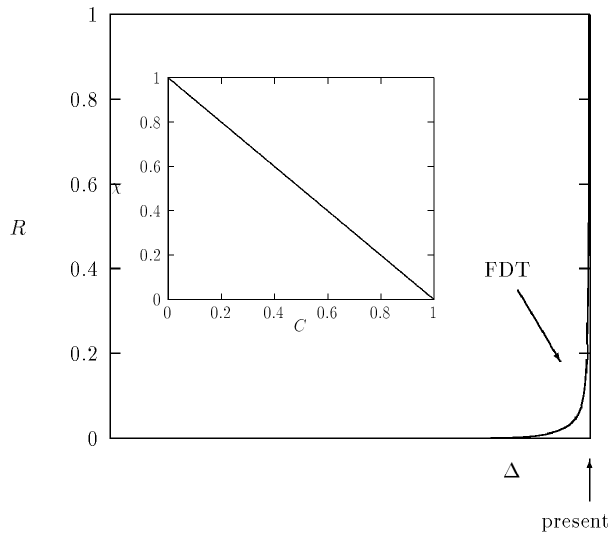

Figure 6: The response $R(t,t')$ versus $t'-t$ at high temperatures $T=1>T_c$ for four total times $t=t_1,t_2,t_3,t_4$. The curves were obtained from a numerical resolution of the dynamical equations (48), (49) and (50) in a $p=3$ spherical spin glass. All four curves merge into one, $R(t,t')=R(t-t')$. In the inset, $\tilde{\chi}(t,t')=\tilde{\chi}(t-t')$ vs. $C(t,t')=C(t-t')$, FDT is satisfied and the slope is just $-1/T$.

3.2 The high temperature phase: two types of spin glasses

One expects that the above spin glass models converge fast towards a paramagnetic equilibrium phase at high enough temperatures, where the system has no long-term memory and obeys TTI and FDT. This can be seen nicely from a numerical study of the dynamical equations at relatively short times, which will also be useful to identify the qualitative behaviour at low temperatures. In order to study the memory of the system, we plot the response functions $R(t,t')$ versus $t'-t$ at different values of time t, $t=t_1,t_2,t_3,t_4$. (These curves were obtained numerically from the model model defined in Eq.(44) with only $F_2\neq 0$. A simple numerical procedure consists in discretizing time evenly and iterating the dynamical equations which are causal. With some extrapolation procedure on the mesh of the grid one can reach safely times of order 1000 [44].)

Figure 6 shows that the response at high temperatures does not extend to the distant past and that it depends only on time differences. The system achieves equilibrium after a transient, and eventually forgets the origin of times. Actually, the correlation function is also TTI. Finally, the $\tilde{\chi}$ vs. C plot (see Section 1.3) is a straight line of slope $-1/T$ for all values of C (see the inset in Fig. 6) showing that FDT is satisfied.

Turning now to the analytical treatment of the equation, it is easy to show that if TTI holds (i.e. if the two unknown functions of two variables $C(t_w+\tau,t_w)$ and $R(t_w+\tau,t_w)$ actually depend only on τ), then the second equation (49) is a consequence of the first (48) provided the FDT is satisfied. We are thus left with a single function $C(\tau)$ in Eq. (48), and the kernels Σ and D are related by $dD/d\tau=-\Sigma/T$. Using the finite extent of the memory we can safely send the initial time to $-\infty$, to obtain:

$$\frac{dC(\tau)}{dt} = -\mu_\infty C(\tau) + \frac{1}{T}\int_{-\infty}^{\tau} d\tau' \, \mathcal{V}'[C(\tau-\tau')]\frac{\partial C(\tau')}{\partial \tau'} \,, \qquad C(\tau=0)=1 \tag{51}$$

where

$$\mu_\infty = \mu(t\to\infty) - \mathcal{V}'[1]/T \,. \tag{52}$$

The above equation is valid as long as $C(\tau)$ decays to zero in the long τ limit.

Equation (51) is basically the general 'schematic' Mode-Coupling equation for the density correlations in a supercooled liquid above the dynamical transition temperature introduced by Leutheusser [16], Götze and others [17] as a model for the ideal glass transition. The only difference lies in the fact that the Mode-Coupling equations also possess an 'inertial' term $\partial_\tau^2 C(\tau)$. This coincidence will be further discussed in Section 4.4.

The behaviour of the solution to these equations when one lowers the temperature depends upon the structure of the disorder, i.e. upon the function $\mathcal{V}$. It turns out that there are two broad classes of mean-field spin glasses, characterized by rather different behaviours. The point is that as one lowers the temperature, there appears a critical point T_c at which the decay behaviour of $C(\tau)$ shows a marked change. For 'discontinuous' models, $C(\tau)$ does no longer decay to zero, but rather to a finite value $q_{\rm EA} > 0$ (which is also called the non ergodicity parameter f in the context of glasses). For 'continuous' models, $C(\tau)$ still decays to $q_{\rm EA} = 0$ at T_c but with critical slowing down. Below T_c, $q_{\rm EA}$ grows continuously from zero [l].

- 'Discontinuous' models

The simplest prototype, which we discuss here, is the pure spherical p-spin model with $F_r = g\delta_{r+1,p}$. Another example is provided by the problem of a particle in a random potential with short range correlations, i.e. when $\mathcal{V}(x)$ decays sufficiently fast for large x's.

The solution, studied in Refs.[17,108,109] behaves as follows. Above the critical temperature T_c, the correlation $C(\tau)$ decays to zero at large τ. Slightly above T_c, the correlation already starts developing a plateau at $C \sim q_{\rm EA}$ before eventually decaying to zero, as shown in Fig. 7. The length of the plateau increases as a power law of $T-T_c$ when the temperature gets closer to T_c. The details of how $C(\tau)$ first decays towards the plateau and then departs from it has been one of the most studied aspect of the MCT in the context of glass forming liquids, since these features can be directly tested experimentally using a variety of techniques. One finds that:

$$\begin{array}{lcll} C(\tau) & \sim & q_{\rm EA} + c_a \tau^{-a} & C \gtrsim q_{\rm EA} \,, \\ C(\tau) & \sim & q_{\rm EA} - c_b \tau^{b} & C \lesssim q_{\rm EA} \,, \end{array} \tag{53}$$

where the exponents a, b are related by

$$\frac{\Gamma^2[1+b]}{\Gamma[1+2b]} = \frac{\Gamma^2[1-a]}{\Gamma[1-2a]} = \frac{T_c}{2}\,\frac{\mathcal{V}'''(q_{\rm EA})}{(\mathcal{V}''(q_{\rm EA}))^{3/2}} \,. \tag{54}$$

[l] These two types are also called, respectively, A and B in the context of the Mode Coupling theory [17]. One should keep in mind that we speak here of dynamical phase transitions, and this terminology is not related to the Ehrenfest classification of equilibrium phase transitions.

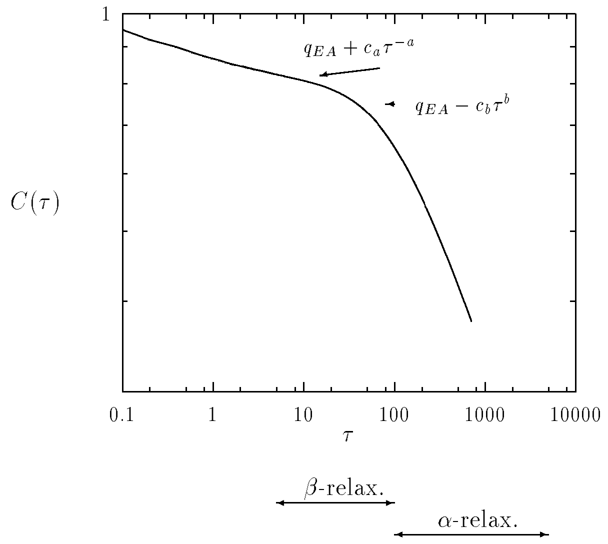

Figure 7: The correlation for the same model as in Fig. 6 at high temperatures close to the transition $T \gtrsim T_c$

Clearly, the equilibration time for a system that is quenched to $T > T_c$ is at least as large as the time of the plateau $\tau(T)$. A formulation in terms of time-differences (52) is thus valid only when the waiting time is much larger than the α-relaxation time $\tau(T)$. For times that are smaller than $\tau(T)$, one has to go back to the two-time equations — this will be the case throughout the low-temperature phase.

- 'Continuous' models

The class of 'continuous' spin glasses contains the more usual case of the SK model [8] (which is however not described by an equation of the type (51)). Among the systems we are discussing is the case of a particle in a long-range correlated random potential [110] ($\mathcal{V}(x)$ decaying as a power law), or some spin systems such as a mixture between $p = 2$ and $p = 4$ interactions [111]: $F_r = g\delta_{r+1,2} + g'\delta_{r+1,4}$.

Again there is a critical temperature T_c above which the correlation $C(\tau)$ decays to zero at large τ. The main difference with the previous case is the absence of the plateau structure around $q_{\rm EA}$ for T slightly above T_c, which is obviously related to the fact that in this case the Edwards-Anderson order parameter departs continuously from zero when one decreases the temperature through T_c.

3.3 Low temperatures: Weak long term memory and weak ergodicity breaking

Let us now discuss what happens below T_c. Again, we first show the numerical solution of the full causal dynamical equations (48), (49) and (50), and look at the same plot as

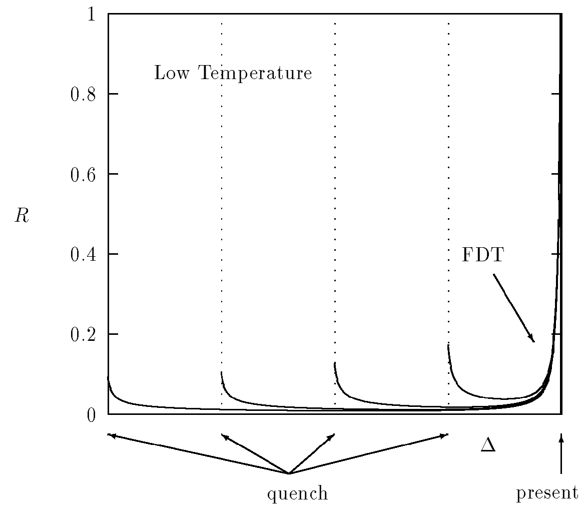

Figure 8: The response $R(t_i, t')$ in terms of $t' - t_i$ at a low temperature $T < T_c$ for the same model as in Fig.6. For each curve t_i is $t_1 = 100$, $t_2 = 200$, $t_3 = 300$, $t_4 = 400$, respectively. 'Quench' corresponds to $t' = 0$ while 'present' to $t' = t_i$.

before, namely the response functions $R(t, t')$ versus $t' - t$, for different values of time t, $t = t_1, t_2, t_3, t_4$. Figure 8 shows that:

– The system has a strong response to pertubations in the immediate past that is quite similar to the high temperature response.

– However, a long tail extending down to the quench time $t' = 0$ has now appeared.

The total area under the response curves, $\int_0^t ds\, R(t, s)$ approaches at large times a finite limit, which is equal to the linear susceptibility $\tilde{\chi}$ to a constant field. Part of this area is already given by the peak to the right of Fig. 8, which is the high-frequency, stationary contribution to the susceptibility. It turns out however that the area below the long-time tail also gives a non-zero contribution – the memory to the distant past is substantial and can never be neglected:

$$\forall t^* \qquad \lim_{t \to \infty} \int_{t-t^*}^{t} ds R(t, s) < \lim_{t \to \infty} \int_0^t ds R(t, s) \ . \tag{55}$$

This can be hinted from the simulations. As we shall see, it can also be derived from the dynamical equations.

At this stage it would seem that in order to solve for the correlations and responses at large times we need to know the complete solution at *all* times, because the memory kernels (which involve the response) have nontrivial contributions from all the past. If this were the case, the problem of finding an asymptotic solution for long times would be hopeless from the analytical point of view!

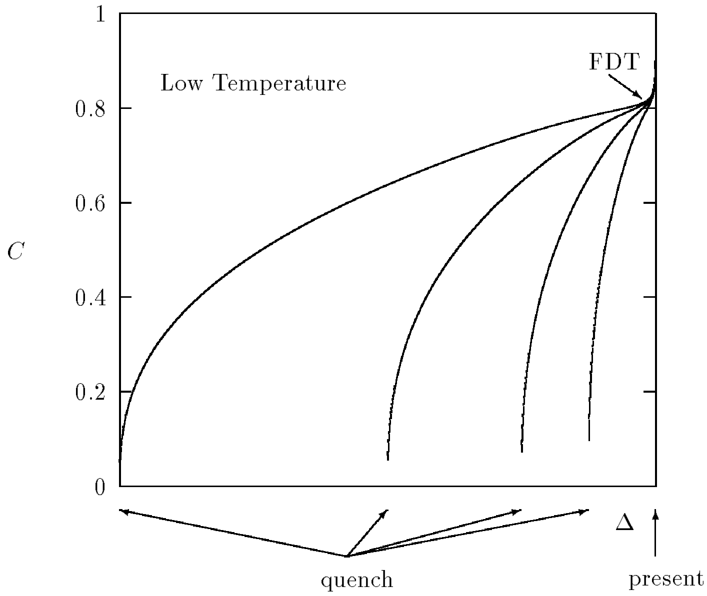

Figure 9: The correlation function $C(t_i, t')$ vs $t' - t_i$ at low temperatures for the same model as in Figs 6 and 8. From left to right $t_i = 400, 200, 100, 50$.

Fortunately, something remarkable happens within the models considered here: even though the area under the long-time tails of the response remains finite, the height of the tails themselves tend to zero as we consider larger times. More precisely, the integrated response at time t to a signal between the initial time $s = 0$ and any *finite* time $s = t^*$ tends to zero at large t:

$$\lim_{t \to \infty} \int_0^{t^*} ds\, R(t,s) = 0 \tag{56}$$

for any fixed t^*. The same can be said about the integral of $R \times G$ where G is any finite function, such as the memory kernel Σ. This behaviour was described as 'weak long term memory' in Ref. [11]: the memory tends to be 'weak' for any finite interval of the past, but is strong when integrated over the whole past [m]. This is close in spirit to the 'weak ergodicity breaking' property defined in Section (1.4): a perturbation lasting for any finite duration will eventually be forgotten. Therefore the long time dynamics will decouple from the initial (non universal, and out of control) transients.

The evolution of the two-time correlation function also has (at least) two distinct regimes, as shown in Figs. 9 and 10. For times such that $\tau = t - t'$ is small all curves merge and one has TTI. (These are times close to 'present' in Fig. 9 and to the left in Fig. 10.) The corresponding decay is 'fast' (see footnote[b] above) in the regime where C drops from 1 at equal times to the plateau value $q_{\rm EA}$, defined in Eq.(6). However, when C decays

[m] Not every system will satisfy this condition; on the contrary, systems that remember their initial transients are much harder to treat and how to deal with their dynamics is still a fully open problem.

below $q_{\rm EA}$ it does it in a manner which depends on both t and τ. This subsequent decay is 'slow'.

This suggests (and the analytical calculation later confirms) that one can perform, for both the correlation and response functions, a decomposition into a stationary and an aging part, similar to the one introduced above for the description of aging experiments (see Figs. 1 and 2):

$$\begin{aligned} R(t_w+\tau,t_w) &= R_{\rm AG}(t_w+\tau,t_w)+R_{\rm ST}(\tau)\ , \\ C(t_w+\tau,t_w) &= C_{\rm AG}(t_w+\tau,t_w)+C_{\rm ST}(\tau)\ . \end{aligned} \tag{57}$$

Defining as in Eq.(6) the Edwards-Anderson order parameter: $q_{\rm EA}=\lim_{\tau\to\infty}\lim_{t_w\to\infty} C(t_w+\tau,t_w)$, the stationary (and thus the aging) parts are defined by:

$$C_{\rm ST}(\tau)\equiv \lim_{t_w\to\infty} C(t_w+\tau,t_w)-q_{\rm EA} \qquad R_{\rm ST}(\tau)\equiv \lim_{t_w\to\infty} R(t_w+\tau,t_w) \tag{58}$$

It turns out that the stationary parts, which are by definition TTI, also satisfy the FDT, $R_{\rm ST}(\tau)=-\frac{1}{T}dC_{\rm ST}(\tau)/d\tau$. Finally, note that the above definitions imply:

$$\begin{array}{rclrcl} \lim_{t_w\to\infty} C_{\rm AG}(t_w+\tau,t_w) &=& q_{\rm EA} & \lim_{t_w\to\infty} R_{\rm AG}(t_w+\tau,t_w) &=& 0 \\ \lim_{\tau\to\infty} C_{\rm ST}(\tau) &=& 0 & C_{\rm ST}(0) &=& 1-q_{\rm EA} \\ \lim_{\tau\to\infty} R_{\rm ST}(\tau) &=& 0 & R_{\rm ST}(0) &=& 1 \end{array} \tag{59}$$

3.4 Low temperature solution of the dynamical equations

• General strategy.

An asymptotic solution to the dynamical equations was first found in Ref. [41] in the p-spin spherical model and then generalised to various situations [42–45,112,51]. It has been described in detail in several papers [43,45,51]. We restrict here to the essential assumptions and the main ideas allowing one to find the low temperature solution of the dynamical equations (48) and (49).

The solution strongly relies on the weak long term memory assumption. It is asymptotic in the sense that it holds only at large times t_w: the transient effects are much more complicated, and have not been studied yet. The method of solution we shall outline here only determines the time-dependences in the aging regime up to a reparametrization in time $t\to h(t)$. Thus, one obtains a family of solutions for the aging part, related to one another by time reparametrisations:

$$C_{\rm AG}(t,t')\to C_{\rm AG}(h(t),h(t')) \qquad R_{\rm AG}(t,t')\to\left(\frac{dh(t')}{dt'}\right) R_{\rm AG}(h(t),h(t'))\ . \tag{60}$$

(Note that the presence of the factor $dh(t')/dt'$ comes from the fact that it is the integral of R over time, rather than R itself, which is the physical quantity). The 'selection problem' of determining the actual function h chosen by the system is still an open one that requires the matching of the regimes of short and long time differences.

The starting step consists in proposing asymptotic forms for the aging part of the correlation function in the two-time plane valid for large $t_w + \tau$ and τ. As we mentioned in (1.2), a possible form for the correlation is:

$$C_{\rm AG}(t_w+\tau, t_w) \sim \sum_i \mathcal{C}_i \left[\frac{h_i(t_w)}{h_i(t_w+\tau)} \right] \tag{61}$$

where each term will vary in separate time sectors, defined by taking the times to infinity with $0 < h_i(t_w)/h_i(t_w+\tau) < 1$ (see Eq.(10)). The asymptotic form of the correlation is given then by the knowledge of $\mathcal{C}_i$ and h_i. This form is meant to represent the correlation only in the limit of large times, and it need not be unique. One may wonder whether it exhausts all possibilities.

This question can be answered with the following construction[43]. Consider the configurations at three large times $t_{min} \le t_{int} \le t_{max}$, and the corresponding correlations $C(t_{max}, t_{min})$, $C(t_{max}, t_{int})$ and $C(t_{int}, t_{min})$. Using the fact that the correlations decrease with time-separations, one can show that in the limit of large times the three correlations must be related by

$$C(t_{max}, t_{min}) = f[C(t_{max}, t_{int}), C(t_{int}, t_{min})] \tag{62}$$

where the function f defines the geometry of the triangles described by the trajectory of the system in phase space. Now, it is easy to see that f is an associative function, and one can classify all the possible forms of an associative, monotonical function [43] using elementary group theory.

This in turn leads to a classification of all the possible two-time scalings as follows. One considers the special (fixed point) values of the correlation $q_0, q_1, \dots, q_k \equiv q_{\rm EA}, q_{k+1} \equiv C(t,t)$ defined by $f(q_i, q_i) = q_i$. If the number of fixed points is finite, one can prove that the correlation can be written for large times as (61) with $q_i = \sum_{j<i} \mathcal{C}_j(1)$. The intermediate values of correlation between two fixed points constitute a 'correlation scale', within which only one term of (61) is non-constant. Triangles whose sides belong to different scales (e.g. $C(t_{max}, t_{int}) < q_i$, $C(t_{int}, t_{min}) > q_i$) are isosceles with $C(t_{max}, t_{min}) = \min[C(t_{max}, t_{int}), C(t_{int}, t_{min})]$: there is ultrametricity [113] between correlation scales. It may also happen that the fixed points q_i form a continuum within a certain range (q_a, q_b). In that case we have, for any two correlations within this range:

$$C(t_{max}, t_{min}) = \min[C(t_{max}, t_{int}), C(t_{int}, t_{min})] \qquad \text{as } \; t_{min} \to \infty \; . \tag{63}$$

Correspondingly, the correlation has to be represented in that case as a limit of a continuous sum of infinitely many scaling functions $\mathcal{C}_i$ with vanishing weight. Note that the above construction does not rely on any concrete model and is not restricted to mean-field.

It is interesting to see that this form of ultrametricity in the correlations appears in a very natural way within out of equilibrium dynamics. It is already present in the simplest form in the case of domain growth: if we consider three very large times such that $C(t_{max},t_{int}) > m^2$ and $C(t_{int},t_{min}) < m^2$ (where m is the magnetization), then $C(t_{max},t_{min}) = \min[C(t_{max},t_{int}), C(t_{int},t_{min})]$. Ultrametricity between correlations larger and smaller than m^2 expresses the fact that the time scale corresponding to relaxation due to domain wall motion becomes infinitely larger than the one corresponding to thermal fluctuations within each domain. It is however not clear that there are finite dimensional models for which a 'full' ultrametric form (63) holds for *any* two values of the correlations.

Having discussed the possible asymptotic forms of the correlations, the next step is to make an Ansatz for the response function. One can write without loss of generality:

$$R(t,t') = \frac{X(t,t')}{T} \frac{\partial C(t,t')}{\partial t'} \tag{64}$$

$(t \geq t')$. The Ansatz now consists in proposing that for large t, t', X only depends on time through C:

$$X(t,t') = X[C(t,t')] \tag{65}$$

or, in other words, that the parametric plots $\tilde{\chi}$ vs C in Section 1.3 converge to a limit curve as $t_w \to \infty$. The asymptotic form for the response is thus obtained from that of the correlation through the introduction of a certain function $X[C]$. At this point one substitutes the above Ansätze for the correlation and response functions into the dynamical equations. In this way one determines

i) $X[C]$ (which contains the information on aging of the response) and

ii) f, or, equivalently, the number of terms in (61) and the respective $\mathcal{C}_i$.

iii) One should in principle also obtain the h_i, but this requires to solve the selection problem discussed above.

The corresponding computations can be rather lengthy. We shall not detail them here, but rather present the results obtained when one applies this technique to the mean-field models considered above.

- The case of discontinuous models.

For low temperatures $T < T_c$ the correlation is given by Eq.(61) with only *one* aging sector (i.e. one function $h(t)$) plus the TTI scale. It reads

$$C(t_w+\tau, t_w) = C_{\text{ST}}(\tau) + C_{\text{AG}}\left(\frac{h(t_w)}{h(t_w+\tau)}\right) . \tag{66}$$

The memory properties of the system are controlled by $X[C]$, which takes a particularly simple form:

$$\begin{aligned} X[C] &= 1 \qquad \text{for} \quad C > q_{\text{EA}} \qquad (\text{FDT}) \\ X[C] &= X < 1 \qquad \text{for} \quad C < q_{\text{EA}} \,. \end{aligned} \tag{67}$$

X is a positive, temperature-dependent number, constant throughout the aging regime. The fact that X is constant is not assumed *a priori*, but comes out from the equations of motion. A possible explanation for this fact is found when interpreting $T^{\text{EFF}} = T/X$ as an 'aging temperature': it means that degrees of freedom of comparable frequencies are mutually thermalised [48]. Note that at the glass transition, the aging temperature is equal to the bath temperature. The ratio $T^{\rm EFF}/T$ *increases* with decreasing bath temperature; in particular, $T^{\rm EFF}$ remains non-zero at zero bath temperature.

The behaviour of the correlation around the plateau $q_{\rm EA}$ provides the low-temperature extension of the one already encountered at high temperatures. Taking the limit of large t_w *before* taking the limit of large τ one only explores the approach of the correlation to the plateau $q_{\rm EA}$. This decay is still given by a power law characterised by a temperature-dependent exponent a, as was the case for $T > T_c$. However, the subsequent departure from $q_{\rm EA}$ is t_w-dependent and characterized [45] by another temperature-dependent exponent b:

$$\begin{array}{rclc} C(t_w+\tau,t_w) & \sim & q_{\rm EA} + c_a \tau^{-a} & C \gtrsim q_{\rm EA} \\ C(t_w+\tau,t_w) & \sim & q_{\rm EA} - c_b \left(\frac{\tau}{\mathcal{T}_w}\right)^b & C \lesssim q_{\rm EA} \end{array} \tag{68}$$

where $\mathcal{T}_w$ is an effective waiting time, defined as $\mathcal{T}_w = h(t_w)/h'(t_w)$. (In the case in which $h(t)$ is a simple power law, one has $\mathcal{T}_w \propto t_w$.) The exponents a, b have precisely the same meaning as the exponents a, b defined in Section (1.1), and are in this case related by

$$X \frac{\Gamma^2[1+b]}{\Gamma[1+2b]} = \frac{\Gamma^2[1-a]}{\Gamma[1-2a]} = \frac{T}{2} \frac{\mathcal{V}'''(q_{\rm EA})}{(\mathcal{V}''(q_{\rm EA}))^{3/2}} \tag{69}$$

with $q_{\rm EA}$ given by Eq.(6).

- The case of continuous models.

As regards the temporal behaviour of the correlations in the stationary regime, the approach to the plateau at $q_{\rm EA}$ is also given by a power law with a temperature-dependent exponent a. The situation within the aging regime is however more complicated than in the case of discontinuous models. The aging part of the correlations satisfy the full ultrametric triangle relations (63):

$$f(C_1, C_2) = \min[C_1, C_2] \tag{70}$$

if at least one of C_1, C_2 is smaller than $q_{\rm EA}$. A representation like (68) for the escape from the plateau is not possible for these models unless one makes the exponent b waiting-time dependent.

For purely continuous models the function $X[C]$ is not a constant and, remarkably, coincides with the function $x(q)$ in the replica treatment of equilibrium[n]. This and other coincidences between out of equilibrium dynamics and statics of continuous models have escaped any kind of physical understanding so far. (For some recent work in this direction, see Ref. [31].)

[n]The dynamical $X[C]$ is much easier to obtain numerically than its static counterpart because one does not have to equilibrate. The numerical confirmation of the analytical form of $X[C]$ for the SK model is astonishingly good [114,115].

3.5 Generalization to several coupled modes — the case of spatial dependence

New physical insights appear when we consider the generalization to several coupled models, and, in particular, to mean-field cases in which there is also spatial dependence. In such cases we have to deal with several correlations $C_{k,k'}(t,t')$ and responses $R_{k,k'}(t,t')$, where the indices k,k' refer to different modes, for example they can represent spatial positions x_k or Fourier components.

The construction of an Ansatz proceeds as before[112]. One has to add now a prescription for the *long-times* relationship between different correlations. Choosing one particular correlation function $C_{0,0}$ as an effective 'clock', one may look for solutions of the form:

$$C_{k,k'}(t,t') = \mathcal{F}_{k,k'}[C_{0,0}(t,t')] \tag{71}$$

with $\mathcal{F}_{k,k'}$ to be determined. One also introduces the fluctuation-dissipation violation factors defined by

$$R_{k,k'}(t,t') = \frac{X_{k,k'}[C_{k,k'}(t,t')]}{T} \frac{\partial C_{k,k'}}{\partial t'} \,. \tag{72}$$

Interestingly, it turns out that the Ansatz closes with two extreme possibilities:

i) $X_{k,k'} \neq 0$ for $k \neq k'$, and $X_{k,k} = X_{k',k'}$ at equal times;

ii) $X_{k,k'} \to 0$ for $k \neq k'$, and $X_{k,k}, X_{k',k'}$ possibly different.

This can be understood as a property of partial thermalisation [48]: remembering that $T_k \equiv T/X_{k,k}$ is an effective temperature, in the case *i)* the subsystems have $T_k = T_{k'}$ at corresponding time scales, while in case *ii)* the subsystems have zero cross response and T_k may be different from $T_{k'}$.

A mean-field case with many modes is obtained when one studies the dynamics of a random manifold in a disordered medium within the Hartree approximation [112]. One has all the time scalings described so far for each mode k, plus dynamical scalings in terms of k and times. It turns out that a solution satisfying (71) and (72) appears naturally in that case.

3.6 Speculations on the 'effective' age function $h(t)$

Although the above solution describes some general features of the low temperature, aging regime of mean-field spin glass models, it is still, even in the simpler discontinuous case, incomplete. As mentioned above, the dynamical equations become, in the asymptotic ($t \to \infty$) limit, invariant under any monotonous time reparametrisation $t \to h(t)$. The function $h(t)$ can in principle only be determined through matching with the early ($\tau \sim t_o$) solution, or by a numerical solution of the two-time equations. For the spherical p-spin problem, the latter procedure suggests that $h(t)$ is close to a power-law [41], in other words that the aging part of the correlation function $C(t_w+\tau, t_w)$ is a function of the ratio τ/t_w. In principle, $h(t)$ could be any other function of time, for example:

$$h(t) = \exp\left[\frac{1}{1-\mu}\left(\frac{t}{t_o}\right)^{1-\mu}\right] \quad \text{or} \quad h(t) = \exp\left[\log^\nu(\frac{t}{t_o})\right] \tag{73}$$

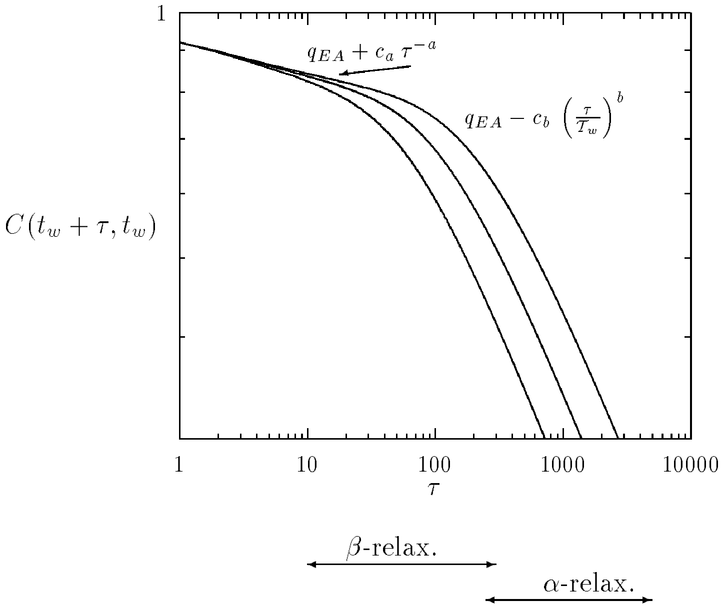

Figure 10: The correlation $C(t_w + \tau, t_w)$ as a function of τ at a low temperature $T < T_c$. t_w is $t_w = 50, 100, 200$, respectively.

in which case, the effective 'age' $\mathcal{T}_w$ appearing in, e.g. (68) is

$$\mathcal{T}_w \equiv \frac{h(t_w)}{h'(t_w)} = t_w^\mu t_o^{1-\mu} \quad \text{or} \quad \frac{t_w}{\nu \log^{\nu-1}(\frac{t_w}{t_o})} \,. \tag{74}$$

Except in the cases $\mu = 1$ or $\nu = 1$, where $h(t) = t/t_o$, these more complicated forms lead to an explicit dependence of the effective age $\mathcal{T}_w$ on the microscopic time scale t_o. (More precisely, when $\mu < 1$ (or $\nu > 1$), the effective age $\mathcal{T}_w$ is much smaller than t_w, a situation called 'subaging' in Ref.[[14]].)

The simplest scaling situation would be that the value of t_o becomes irrelevant on the experimental time scale where $t_w, \tau \gg t_o$. This would lead immediately to 'full' aging, $\mathcal{T}_w \propto t_w$. Naive scaling can however break down in some cases [116], which means that the value of t_o is important even in the $t_w \to \infty$ limit – in other words that the effective age $\mathcal{T}_w$ depends on t_o. Such is the case of systems with logarithmic domain growth *e.g.* the random field Ising models, where $\mathcal{T}_w \propto t_w \log(\frac{t_w}{t_o})$ ('superaging'). Signs that subaging may also happen in the mean-field models considered here can be found in [117]. We feel that the determination of h is one major unsolved issue in mean-field dynamics. Correspondingly, the same ambiguity remains from an experimental point of view: a detailed analysis of the TRM reveals small but systematic deviations from 'full' aging. A scaling function $h(t)$ of the above form, with $\mu = 0.97$ or $\nu \sim 2$, does a better job at fitting the experiments [14] (see Figs. 3.b and 3.c. in Ref.[[14]]). However, these deviations might alternatively be interpreted as being 'interrupted aging' (*i.e.* equilibration in a long but finite time) due to finite field [10,118] or finite size [49,119] effects.

At any rate, note that a pure τ/t_w behaviour would rule out the existence of many time sectors with full ultrametricity, and would leave a scaling as in the discontinuous models as the only possibility.

3.7 Out of equilibrium versus 'equilibrium on diverging time scales'

Dynamical studies of mean-field spin glass models in their low temperature phase are not new. They started more than 15 years ago with the work of Sompolinsky and Zippelius [8,9] and were subsequently applied to many other problems. The above description, which focuses onto out of equilibrium dynamics, has followed a very different route – which was only found recently [41]. However, as all these works address similar issues, a comment on their relationship is in order. For lack of space we shall not be able to present in any detail the former approaches, but these are by now well documented [8,9,3,4,6]. We rather want to stress the conceptual differences with the present approach.

Consider a mean-field spin glass model below T_c. The two times dynamical equations (48) and (49) are exact equations relating the correlation and response in the mean-field models. They describe the behaviour of the system at times that do not diverge with N. Even when speaking of 'long times' within this framework, one really means:

$$\lim_{t,t'\to\infty} \lim_{N\to\infty} \; . \tag{75}$$

Obviously for finite N the spin glass will equilibrate in a finite time. There exists an equilibration time $t_{\text{ERG}}(N)$, such that for t_w much larger than $t_{\text{ERG}}(N)$ the system is equilibrated, which means that the configurations are sampled with a frequency proportional to their Boltzmann-Gibbs weight. The correlation and response then become time translational invariant, and related by the FDT. At large N the equilibration time diverges.

The approach of Ref. [9] starts from the very same dynamical equations (48) and (49), but one assumes that *the size of the system is finite* and large. One also assumes that the initial time of the dynamics has been sent to $-\infty$ and that TTI holds. The corresponding construction relies on the hypothesis that there exists a strong hierarchical structure of time scales which all diverge with N. Calling t_x these time scales, where x is an index chosen for instance in $[0,1]$, the hierarchy means

$$\lim_{N\to\infty} t_x = \infty, \quad \lim_{N\to\infty} \frac{t_x}{t_y} = \infty \quad \text{if} \quad x < y \; . \tag{76}$$

The largest of these time scales thus corresponds to the equilibration time t_{ERG}.

This allows to produce a solution to the dynamical equations which exhibits a non-trivial dynamics within each diverging time scale t_x. This dynamics exhibits TTI, as assumed from the beginning, but it violates the FDT. This astute solution presents several formal similarities with the static solution of Parisi (and also with the out of equilibrium study), starting with the hierarchical structure, which acquires an appealing interpretation in terms of diverging time scales. However it suffers from several problems. At the level of the results first: the correlation function $C(\tau)$, at very large times $\tau > t_{\mathrm{ERG}}$, does not go to the correct Boltzmann Gibbs equilibrium value, known from the static studies. This is clearly inconsistent. A careful study of their derivation reveals two weak points.

First, there is an inconsistency in the hypotheses. If the initial time has been sent to $-\infty$, for a finite N system, then it is fully equilibrated, and then the dynamics is necessarily TTI *and* obeys FDT. Otherwise (if the waiting time is much smaller than t_{ERG}) there is no reason to assume either TTI or FDT.

Second, this approach uses the dynamical equations derived within the $N \to \infty$ theory, for a finite system where activated processes take place. Even though large N saddle-point approaches *can* be used to study activated processes that occur for large but finite N, there are subtleties related to the existence of multiple solutions. The multiplicity of solutions is related to the fact that the results at times greater than t_{ERG} should be non-self averaging, as the replica solution shows [120].

Because of these problems, which created many discussions [121,122,123,5], there have been various attempts in the litterature to try to amend this solution, while keeping most of its nice mathematical structure. One possibility, first suggested by Horner [122] and developed in Ref.[[123]], is to keep the cooling rate finite. This allows to send the volume to infinity while keeping a regularization time scale which is the inverse cooling rate, which is sent to infinity in the end. What happens once the cooling procedure is over is still not clear in this approach. On the other hand, Horner [124] proposed an alternative 'regularizing' procedure which consists in making the disorder time-dependent. In this way, aging disappears, the solution is TTI and manifestly out of equilibrium.

We believe that the present out of equilibrium dynamical approach, inspired by the experiments themselves, is very clear. It is seen to work consistently within the finite time, infinite N regime (75). The price to pay is that one has to abandon the postulate of TTI, and think in the two times plane. In a vague sense, there is also a regularization time which is involved, namely the age of the system.

Finally we want to point out that in all the approaches developed so far to study the spin glass phase at the mean-field level, there is a hierarchical (ultrametric) structure which is involved. This is true in the statics where it is hidden in Parisi's Ansatz. It is also true in the 'dynamics on diverging time scale' approach where it appears in the hypothesis of strong time hierarchy (76). In the out of equilibrium dynamical approach, the situations is rather more favourable in that ultrametricity can be proven with mild assumptions (see 3.4 above).

3.8 On the links between the static and dynamical approaches. Phase space geometry

In an infinite system, there is no reason for the out of equilibrium dynamics to be related to the static, equilibrium picture. Indeed as we saw, the dynamics refer to finite time scales (when $N \to \infty$), while the static properties are only recovered in the opposite, non-physical limit [o]. Yet it is instructive to compare the results of the two approaches. It allows to gain some intuition on the physical mechanism at the origin of aging. It further underlines the

[o]The divergence of t_{ERG} at large N is known to be of the type $t_{\mathrm{ERG}} \sim \exp(N^{\alpha})$.

physical difference between the low-temperature behaviour of the two classes of mean-field spin-glasses.

In continuous spin-glasses one gets the natural result that the dynamical transition temperature T_c coincides with the static critical temperature corresponding to the onset of a non zero $q_{\rm EA}$ and of replica symmetry breaking effects. Furthermore, the large time values of one-time intensive quantities such as the energy density $E(\infty)$ in the limit (75) coincide with the value $E_{\rm EQ}$ found at equilibrium within the static (replica) approach.

In discontinuous spin-glasses the results are more surprising: the static transition temperature T_s is lower than the dynamic one[19,122], $T_s < T_c$. In fact the static thermodynamic quantities computed within the Boltzmann Gibbs equilibrium are perfectly analytic in the neighborhood of the dynamical temperature T_c, and conversely nothing special happens at T_s in the dynamics (75). Furthermore, throughout the low-temperature phase, the out of equilibrium dynamical energy does not converge to the equilibrium one $E_{\rm EQ}$. In fact it never goes below a 'threshold level'[41] $E(\infty) \equiv E_{\rm THRES} > E_{\rm EQ}$.

- The energy landscape

A geometric explanation for this strange phenomenon can be found within the framework of the static mean-field equations of Thouless, Anderson and Palmer (TAP equations). In the static limit it is possible to write a free energy $F_{\rm TAP}(m_1, ..., m_N)$ in terms of local 'magnetization' variables which represent the average value of a spin on a large time-window (but the large N limit has to be taken first). The minima of this free energy correspond to various metastable states; it is known that their weighted sum gives back the correct equilibrium results[125].

It is far from obvious in general that the dynamical evolution of a system can be seen as the relaxation of a point in this free-energy landscape. Yet it is always possible to compute the dynamical properties, such as the energy, and to see in what region of the landscape the dynamics takes place. This actually provides interesting insights for discontinuous spin-glasses. The most complete discussion is available for the case of the spherical p-spin system. It turns out that the TAP states can be computed rather easily in this case. The reason is the absence of chaoticity: A TAP solution at temperature T_1 can be followed adiabatically when one changes the temperature to T_2, the only change is a global rescaling of the $(m_1, ..., m_N)$. Once the free energies of the solutions are ordered at a temperature T_1, this order is maintained at all temperatures: there is no crossing of the solutions in the temperature - free energy plane (see Fig. 11). Non-trivial TAP solutions exist in a wide range of temperatures, extending above the static transition temperature T_s, and even above the dynamic one T_c.

Comparing this landscape to the results of the dynamics, the present understanding of the situation is as follows[126,41,127]: above the dynamical temperature T_c, there is a coexistence of a paramagnetic state and a bunch of non trivial, but isolated, TAP states. When the dynamics starts from random initial conditions, the system thermalises within the paramagnetic state. However if one chooses carefully the initial conditions, one can let the system thermalize within one of the TAP states, which are separate ergodic components[127].

Between T_s and T_c, the paramagnetic state is fractured into exponentially (in N) many separate ergodic components, each of which has a higher free-energy compared to that of the paramagnetic state, by an amount which is exactly equal to their overall configurational entropy, called the 'complexity' [20,129]. Correspondingly, the simplest static approach (see below for more elaborate ones) does not notice this subtlety, and still describes the system as a simple paramagnet. In the dynamical approach, starting from random initial conditions, one finds that the energy actually remains above the threshold level which is the energy of the *highest* TAP state (see Fig. 11). Therefore in the limit (75) the system never reaches the energies where it would get forever trapped into one of the TAP states. This is the origin of weak ergodicity breaking in these models.

In the case of continuous spin-glasses the geometry of the TAP landscape looks superficially quite clear: while above the transition temperature $T_s = T_c$ the free-energy has only one minimum, it develops below T_c exponentially many states, which matches nicely with the Parisi picture. The identification of the low lying TAP minima, and of the pure states in Parisi's construction to the dynamical ergodic components seems inevitable. The puzzle of how to match this equilibrium picture with the out of equilibrium dynamics containing infinitely many time sectors is however completely open, as we already saw in the previous Section. Surprisingly, many purely static quantities involving different states (which are in principle mutually inaccessible) coincide with their dynamical counterparts. In particular, the large time values of one-time intensive quantities such as the energy density $E(\infty)$ in the limit (75) coincide with the values found at equilibrium within the static approach. This means that the set of two dynamical equations (48) and (49) contain all the information on the replica symmetry breaking solution of these systems at low temperatures (A careful numerical check of this point can be found in Refs.[[12,44,114]]). Therefore the study of these equations may provide an alternative route to a rigorous study of the spin glass phase.

- The geometrical description of mean-field aging.

The models we have been describing can be thought of as the motion of a point in a high-dimensional phase-space landscape (to the extent that we deal with soft spins). For discontinuous models, one finds that, surprisingly, the effect of temperature on the dynamics is very minor, provided that one stays within the low temperature phase. Indeed, the solution of the equations is regular at $T = 0$: the dynamics at zero-temperature is not very different from the one at finite temperature, $T < T_c$. This at once tells us that activated processes are not the main ingredient here. What is then the origin of aging in mean-field models?

At $T = 0$, we simply have gradient descent within the basin of attraction of some phase-space minimum, and we can discuss [128] the essence of the problem without having to postulate a 'free-energy landscape' with a dynamical meaning. Since the basins are high dimensional objects, a random starting condition will be in the thermodynamical limit practically on a border between basins, and hence will remain close to this border for all finite times without reaching a minimum. Now, the border is itself partitioned into basins of attraction of the stable points on it, which are the saddles with one negative eigenvalue.

The same argument as before tells us that, again, they will not reached in finite times: the system remains almost on the 'border of a border'. One can now iterate this argument, invoking borders within borders and saddles of higher (but smaller than $O(N)$) indices.

The conclusion is that high dimensional (phase-space) wells take long to fall into. In this sense, the above 'endless descent' scenario concerns equally the motion of a mean-field system within a basin of attraction and the 'fall' of a ferromagnet (in any spatial dimension) into one of its two 'wells'. Indeed, both ordinary coarsening and mean-field aging are reminiscent of the phase-space model described [84,101] in Section 2, in which aging is due to the fact that downhill directions are always present, although in decreasing number as time grows. What we have described so far is the fact that a macroscopically different phase takes an infinite time to grow in the absence of a driving field, a fact that when looked upon from the phase-space point of view is recognized to hold *even at the mean-field level.*

At finite temperature one cannot invoke a simple gradient descent, but one can still use the TAP free-energy to understand why the ($N = \infty$) system ages and never falls below the threshold level [41]. The above discussion can be reformulated as follows: the density of eigenvalues of the matrix of second derivatives in each TAP minimum is a shifted semicircle law. The smallest eigenvalue $\lambda_{\text{MIN}}(F)$ decreases as one considers minima with higher F, and becomes zero precisely for those with $F = F_{\text{THRES}}$. Above the threshold, the spectrum for the saddle points continues to shift, with now $\lambda_{\text{MIN}} < 0$, so that one encounters saddles with more and more negative eigenvalues. If one makes a cut of the free energy landscape at different values of F, one obtains disconnected 'islands' around each minimum for $F < F_{\text{THRES}}$. As one raises F just above F_{THRES} one crosses saddles with larger and larger number of negative eigenvalues. Each time, a separatrix develops and the set of mutually disconnected components becomes more and more connected. One can then picture the ($N = \infty$) aging system as falling in more and more disconnected space, hence moving more and more slowly – without ever quite stopping, since there are always directions where the free-energy decreases (while staying above threshold) [p].

Again, even at finite temperature this scenario is rather distinct from the 'trap' picture of Section 2: Because N is infinite (and the model is fully connected) the system never reaches the bottom of a trap from which it could only escape through thermal activation. For finite N, however, activated processes will begin to play a crucial role after a finite amount of time (see Section 5).

- The 'marginal stability' criterion.

Finally, we would like to mention the possibility of identifying the dynamical transition temperature from purely static computations. This is very useful since static calculations are generally easier than dynamical ones, in particular when one deals with discrete spin variables.

In continuous spin-glasses, the clearest mathematical characterization of the onset of the static transition is to consider two identical copies of the system with the same disorder, coupled through a small, but extensive, coupling term of strength proportional to g. For

[p] This is similar to the idea of 'percolation' in phase space, investigated in Refs. [82,83]

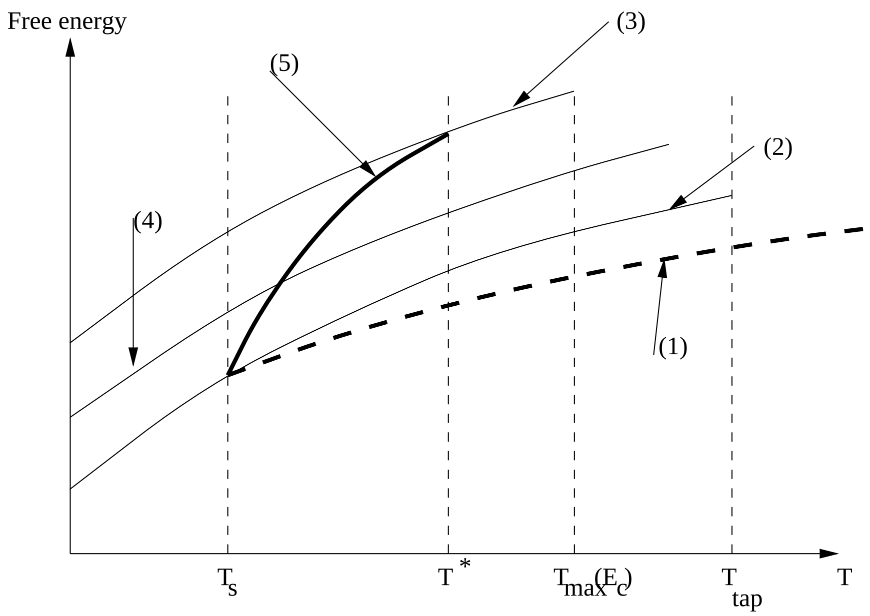

Figure 11: A sketch of the free energy of the TAP states in the spherical p-spin systems. Each TAP state like (2) or (4) can be followed adiabatically in temperature until it disappears. The line (1) is the static equilibrium free energy, taking into account the multiplicity of the TAP solutions. (5) shows the free energy of the TAP states giving the leading contribution to the static equilibrium partition function. The highest TAP states (3) are marginally stable and provide the threshold energy: In its dynamical evolution, the energy of a system starting from random initial conditions stays above this threshold.

instance in the Edwards-Anderson model, one can compute the partition function Z_2 for two spin systems s and σ, coupled through their local energy densities [130]:

$$H_2 = -\sum_{(ij)} J_{ij}(s_i s_j + \sigma_i \sigma_j) - g \sum_{(ij)} (J_{ij} s_i s_j)(J_{ij} \sigma_i \sigma_j) \tag{77}$$

The system is in its low temperature phase when the overlap between the two copies (defined as $-1/(\beta N)\partial \log(Z_2)/\partial g$) is discontinuous at $g \to 0$. When $g \to 0^+$ this overlap tends to the Edwards-Anderson order parameter, while when $g \to 0^-$, it is the smallest possible overlap between different states.

In the case of discontinuous spin-glass transitions this construction must be modified because of the existence of exponentially many TAP states. Roughly speaking the partition function can be approximated as

$$Z = \sum_{\alpha} e^{-\beta F_\alpha} = \int df e^{N(S_c(f) - \beta f)} \tag{78}$$

where α labels various TAP states, $f = F/N$ are the free energy densities, and $\exp(N\, S_c(f))$, usually called the complexity, is the number of TAP states at a given free energy density. The large degeneracy of TAP states shifts the dominant region of free energies in the integral (78). If one took two copies of the system with a small extensive attraction as in (77), the leading contribution to Z_2 should come from the case where the two spin systems are in the same state. However, this leads to $Z_2(g \to 0^+) \simeq \int df e^{N(S_c(f)-2\beta f)}$ which is dominated by a wrong saddle point in f. Therefore, in this case, one must be more clever and study instead m identical copies of the system in the formal limit $m \to 1$ (in which case, the correct saddle point in f will be recovered). The onset of a non-ergodic phase with many components (and thus the value of the dynamical transition temperature T_c), is signalled by the existence of a non-trivial limit for the order parameter, which is now the overlap of any two of these m copies in the carefully ordered limit $\lim_{m\to 1} \lim_{g \to 0^+} \lim_{N\to\infty}$. (In a replica language what one needs to compute is the free energy within the 'one step rsb' Ansatz, where the n replicas are grouped into n/m groups of m replicas, an then expand the resulting free energy at $n=0$ around the point $m=1$: $F = F_0 + (1-m)F_1$. One then studies whether the piece F_1 has a non trivial saddle point in the overlap.)

For technical reasons which we shall not explain here, this criterion is known under the name of 'marginal stability'. It has been used in many instances for discontinuous spin-glasses. An insightful interpretation due to Monasson [131] shows how this procedure amounts to calculate a partition function restricted to a subset of configurations, chosen by a random pinning field. A related approach [132] uses another technique to calculate a partition function over a subset of configurations, where one starts from a typical configuration at given energy called the 'pivot', and calculates the free energy $-1/\beta \log(Z_q)$ associated to all configurations at overlap q from the 'pivot'. T_c now appears as the temperature where this free-energy develops a local minimum for $q \neq 0$.

4 Glasses and spin-glasses without disorder

As emphasized above, the Mode-Coupling equations which have been used with some success in the recent years to describe supercooled liquids [17] formally coincide with the exact equations describing some mean-field spin-glasses, or the motion of a point particle in a random potential [18,133,134] (in large spatial dimension). This is a priori surprising in view of striking differences between the behaviour and the basic constitutive ingredients of spin-glasses on the one hand and structural glasses on the other hand. Let us mention a few rather obvious ones:

– In spin-glasses, there exists some quenched disorder (e.g. the random position of the spin carrying atoms in spin-glasses, which do not move with time) which is absent in glass forming liquids, where the disorder is, in a sense clarified below, 'self-induced'.

– The glass transition is a dynamical effect, basically defined as the temperature or density at which the relaxation times reaches the experimentally accessible order of magnitude of hours. The existence of a true transition at lower temperatures is very controversial. In spin-glasses, on the other hand, there exists some (experimental [75,15] and numerical [38]) evidence in favour of the existence of a second order phase transition (at least in zero magnetic field), with a power law divergence of the relaxation time and of the non-linear magnetic susceptibility.

– Another important difference is the existence of a crystalline phase in structural glasses, which can in principle be reached in a very slow annealing procedure. This has no counterpart in spin-glasses.

In spite of these important differences, there has been recently an increasing convergence between the two fields, both at the theoretical level and experimentally, because of the existence of an aging regime at low temperatures. This chapter will review some of these points of convergence.

4.1 Phenomenology of glasses: a few basic facts

Many very different glass formers exhibit surprisingly similar properties when approaching the glass transition. The glass transition temperature itself is a purely conventional (and somewhat anthropomorphic) temperature where the relaxation time reaches a value of the order of 10^3 seconds. However a common experimental feature is the stretching and shouldering of the relaxation (of, e.g. the density fluctuations) as the temperature is decreased [17,86] towards T_g. More precisely, the relaxation evolves from a simple Debye exponential at high temperatures (liquid) to a *two-step* process at lower temperature (supercooled liquid), where the correlation function first decays rather quickly to a 'plateau', and later departs from this plateau value on a much longer time scale $\tau(T)$. Correspondingly, the frequency dependent susceptibility $\chi''(\omega)$ evolves from a one peak, high frequency, structure to a two-peak structure when the temperature is decreased. The second, low frequency peak (called the 'α-peak') shifts to lower and lower frequencies $\omega_\alpha = 1/\tau(T)$ as T is lowered; the shape of the peak is furthermore strongly non-Debye, which reflects the fact that the time relaxation functions are non-exponential (and often fitted by stretched exponentials). The shape of the *minimum* lying in-between these two peaks has been the focus of an intense interest recently, essentially because one of the major predictions of MCT is that, around a certain temperature T_c,

$$\frac{\chi''(\omega)}{\chi''(\omega_{\min})} \propto \begin{cases} \left(\frac{\omega_{\min}}{\omega}\right)^b & \omega \ll \omega_{\min}\ , \\ \left(\frac{\omega}{\omega_{\min}}\right)^a & \omega \gg \omega_{\min}\ , \end{cases}$$

where a and b are two (positive) exponents related through Eq. (54). This behaviour reflects, in frequency space, the behaviour of the correlation function $C(\tau)$ represented in Fig. 7 (see Eq. (53)).

The relaxation time $\tau(T)$ grows extremely fast as the temperature is decreased, in general faster than the Arrhenius law $\exp(\Delta/T)$. Systems for which $\tau(T)$ are close to an Arrhenius behaviour are called 'strong', whereas systems for which the divergence is faster are called 'fragile' [86]. For the latter systems, a widely used description of the experimental data (based on dielectric measurements, viscosity measurements, etc.) is the Vogel-Fulcher law

$$\tau(T) \sim t_o e^{\frac{\Delta}{T-T_0}} \tag{79}$$

which suggests that $\tau(T)$ actually diverges when $T \to T_0$, i.e. that there is a true phase transition at $T = T_0$. This temperature furthermore appears to coincide with the temperature (called Kauzman temperature) at which the extrapolated excess entropy (as compared to the crystal) would vanish. However, the divergence of $\tau(T)$ is so rapid that $\tau(T)$ becomes larger than experimental time scales at the glass temperature T_g which is often appreciably larger than T_0. Is is thus difficult to claim the existence of a transition on the basis of the fit of $\tau(T)$ only (see however [136]). In particular, other functional forms, such as $\tau(T) \sim t_o \exp((\Delta/T)^2)$, also give reasonable fits of the data [137] – without invoking the existence of a critical temperature where $\tau(T)$ would diverge [138].

4.2 Discontinuous spin-glasses: a mean-field scenario for structural glasses

In order to understand better why some theoretical ideas emerging from spin-glass mean-field theory might also be relevant for structural glasses, it is important to keep in mind the fact that, as we already emphasized in Section 3, there exist two different classes of mean-field spin-glasses, continuous and discontinuous. While the most conventional – continuous – ones provide a good starting point for the description of real spin-glasses, with a second-order phase transition where the Edwards-Anderson order parameter vanishes, some discontinuous models are characterized by a *discontinuous* static phase transition at a temperature T_s, where the order parameter is finite just below T_s. An extreme example of this type of spin-glass ordering is provided by the Random Energy model [93], which has zero entropy density in the whole low temperature phase. There are in fact many other such examples (see Section (3.2)). As discussed there, the discontinuous models generally possess a rather peculiar dynamical behaviour, with a dynamical transition temperature T_c which is *higher* than the static one (and which coincides with the Mode-Coupling critical temperature). When approaching T_c from above, the relaxation time $\tau(T)$ diverges as a power law, but there is no singularity in the static thermodynamic quantities, which are analytic around T_c (see Fig. 12). Thermodynamic singularities, including a jump in the specific heat, only occur at the lower temperature, T_s. As explained above (Section 3.8), this behaviour originates from the fact that as soon as $T < T_c$, the system starts aging for ever in a slow descent towards a state with a free-energy extensively higher than the one of the equilibrium state. Note that between T_s and T_c, the number of such metastable states is exponentially large, the associated configurational entropy (the 'complexity') being exactly equal to the free energy difference between the metastable states and the equilibrium (paramagnetic) state.

Such a scenario can however only exist, strictly speaking, at the mean-field level, where nucleation barriers are infinite. In finite dimensional space, metastable states with a free energy density larger than that of the ground state have a finite lifetime, since the nucleation of large 'bubbles' of the ground state are always favoured. The gain in free energy for a bubble of size L scales as L^D, while the surface energy cost scales at most like L^{D-1} (if the interaction is short-ranged). Therefore the mean-field picture cannot survive as such in a finite dimensional system. However, one can argue that around the mean-field T_c the bubble nucleation will be a slow process, leading to a rapid increase of the terminal relaxation time $\tau(T)$, which only diverges when the static transition temperature T_s is reached. This picture was proposed nearly a decade ago in an insightful series of papers by Kirkpatrick, Thirumalai and Wolynes [18–20]: within this hypothesis T_c is a crossover temperature below

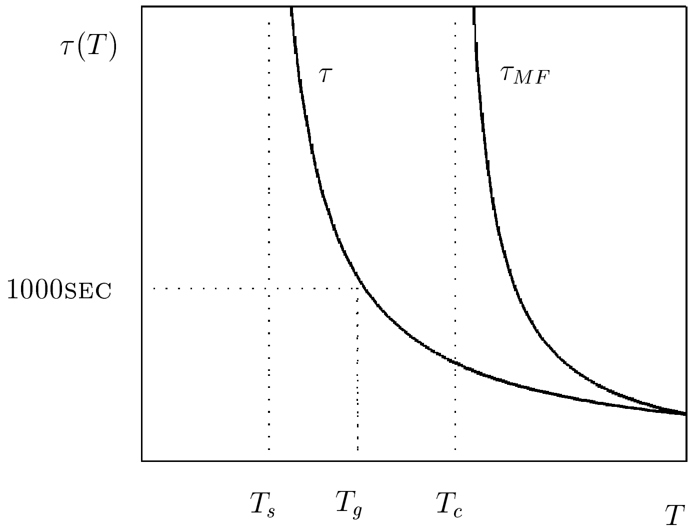

Figure 12: Relaxation time *vs* temperature in discontinuous spin-glasses. The right hand curve is the mean-field prediction, which gives a dynamical transition at a temperature T_c above the static transition temperature T_s. The left curve is a conjecture on the behaviour in finite dimensional systems: activated processes smear the dynamic transition. The relaxation time diverges only at the static temperature T_s, but becomes experimentally large already around the glass temperature T_g.

which activated processes become important [q] leading to a rapidly increasing (*à la* Voger-Fulcher [139]) relaxation time as the temperature is further reduced. The static ordering critical temperature T_s appears as a Vogel-Fulcher or Kauzman temperature (below which freezing is complete), while the experimental glass temperature T_g, where $\tau(T)$ reaches 10^3 seconds, lies somewhere between T_s and T_c (see Fig.12).

This rather appealing idea however suffers from important theoretical loopholes. First of all, it relies on a model with quenched disorder, absent in structural glasses. As discussed in the next paragraph, this might not be too serious as this disorder might well be 'self-induced' by the system itself. One would actually like to be sure that the above nucleation arguments are indeed correct for finite dimensional versions of discontinuous models (there exist a few numerical simulations for the Potts glass, p-spin spin-glass and 'frustrated percolation' models [140–142]). The subtlety comes from the fact that the nucleation is in the present case rather peculiar, since the nucleating phase cannot be the ground state (otherwise the system would be completely frozen after a finite time, and would loose the contribution of the complexity to the free-energy), but rather *another* metastable phase with exactly the same free-energy density – which makes it hard to understand why the bubbles should grow at all. The meaning of the 'entropic driving force' invoked in Ref.[20] is not very clear to us, and, surprisingly, little progress has been made to support this conjecture [139]. Another picture, somehow related to that of Ref.[143], is that the complexity induces a microphase

[q]Similar ideas can be found within the context of 'extended' MCT,[17] although the precise relation between the two pictures is not obvious to us.

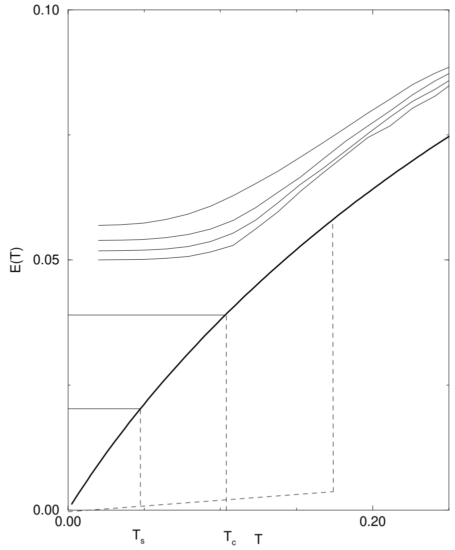

Figure 13: The energy of the LABS model with open boundary conditions vs temperature. The top curves are the results from a Monte-Carlo simulation of a $N = 401$ spin system with logartihmically decreasing cooling rates (from Ref.[158]). The other curves are derived analytically with the fiduciary random system. T_c is the dynamical transition temperature below which the fiduciary system freezes, and T_s is the static transition temperature which cannot be found from a Monte Carlo, but is accessible in principle from exact enumerations of systems with a much smaller size. The lowest curve, corresponding to the 'crystalline' state, has been found only in the case of periodic boundary conditions and for some special values of N.

separation into 'grains', each of a different metastable phase, with a certain temperature dependent size. The relaxation time would correspond to the time needed to a 'grain' to disappear or for a new grain to appear. In this sense, glass dynamics might have a lot in common with foams or microemulsions, as recently advocated in Ref.[[144]].

4.3 Self-Induced Quenched Disorder: Spin glasses without disorder

An important obstacle if one wants to convert the above picture valid for some spin-glasses into a theory for structural glasses is the meaning of the quenched disorder in the latter case. It turns out however that a series of recent works [145–153] has shown the existence of discontinuous spin-glass like behaviour in systems with frustration but *without quenched disorder*. These systems thus provide natural spin analogues of glass formers. Although their microscopic description does remain remote from that of structural glasses (in particular because they involve infinite range interactions), they provide at least an existence proof to the phenomenon of self-induced disorder, and their study is worth the effort, both for their intrinsic beauty and as a source of inspiration for modelling structural glasses. Furthermore, from an experimental point of view, Charge Density Wave systems (among others [154]) have recently been shown to behave very much like disordered systems [25], with however a very small density of impurities, suggesting that incommensurability effects alone (inducing some frustration) might be sufficient to generate 'self-induced disorder' [155].

Monte Carlo simulations show that the low-temperature dynamics is highly non-trivial in all cases. Although an analytical study of the dynamical features could in principle be done along the lines described in Section 3, in many of the recent papers the shortcut was taken of calculating some dynamical properties using the 'pseudo-statical' approaches we described in Section 3.7.

- Low autocorrelation binary sequences.

We shall first concentrate on some spin systems with frustration but without disorder, which contain long-range interactions. These systems exhibit the same behaviour as the discontinuous mean-field spin-glasses, namely a dynamical transition at a temperature T_c larger than the temperature T_s of the static transition. The first example is the problem of 'low autocorrelation binary sequences' (LABS). This is an old and important problem from communication theory [156] which was restated in physical terms by Bernasconi [157] as follows: take a one dimensional chain of Ising spins $\sigma_i = \pm 1,\ i = 1...N$. Compute the correlation function at distance k

$$C_k = \sum_{i,j=1}^{N} \sigma_i \sigma_j \delta_{j,i+k} \tag{80}$$

and define the energy function as

$$E(\{\sigma\}) = \frac{1}{2(N-1)} \sum_{k=1}^{N-1} C_k^2 \ . \tag{81}$$

The interest in communication theory is to find the low energy configurations. In fact the ground state of this energy function does provide a sequence of bits which minimizes the two point correlations, and this is also useful for building a pseudo random number generator. Two versions of this problem have actually been studied, differing in the choice of boundary conditions. Due to the infinite range of the interactions, they present significative differences. A first version studied in Refs.[145,147,158] has free boundary conditions, where the correlation function C_k is defined with the sum over the spin indices i going from 1 to $N-k$. Another choice, studied in [147] is that of periodic boundary conditions, where one can define C_k through a sum over the spin indices i going from 1 to $N-1$. As any other optimisation problem, this can be generalised to a finite temperature study by assigning to each sequence of spins a Boltzmann weight $P(\{\sigma\}) = \exp(-\beta E(\{\sigma\})/Z$.

Let us first discuss the case with free boundary conditions. It was shown by Monte Carlo simulations that there exists a finite temperature freezing region in a temperature range around $T \simeq 0.1$, with a weak cooling rate dependance of the low temperature energies (see Fig. 13). Computations spanning very long time scales have been perfomed at low temperature by using an efficient Monte Carlo algorithm [158–160], and reveal a clear aging effect, characterized by a τ/t_w scaling (see Fig. 14). The smearing of the transition and the cooling rate dependence might be a finite N effect. In this case, the presence of 'traps' in phase space with a broad distribution of trapping times [158] is rather convincingly observed for finite N (see Fig.15).

• A replica analysis for non-disordered problems.

The analytical study of the LABS model is in itself very interesting. Despite its simplicity, we know of no direct solution. A rather indirect, but illuminating, way of proceeding is to replace the non-disordered LABS model by a 'fiduciary' model with quenched disorder. The basic idea consists in considering the model at hand as one special sample of an ensemble of systems containining quenched disorder. In the case with free boundary conditions this is achieved as follows [145,147]: one defines a 'disordered' correlation function

$$C_k^d = \sum_{ij} {M^{(k)}}_{ij}\sigma_i\sigma_j \ , \tag{82}$$

where $M^{(k)}$ is a matrix with random elements, equal to 0 or 1, with the only constraint that $\sum_{ij} M_{ij}^{(k)} = N - k$. The original problem is a particular choice of $M^{(k)}$, where the only nonzero elements are on the k^{th} diagonal. The hope is that this particular case is a generic case, and this is actually not at all obvious. (For example, it would be nonsense to claim that a ferromagnet is a special instance of a spin-glass with $J_{ij} = \pm J$ couplings, where all J_{ij} happen to be equal to $+J$: the ferromagnet is just a very atypical sample.) There is in fact quite a bit of educated guesswork involved in the choice of the ensemble of disordered system, of which the original model is argued to be a generic member – see below. In the present case, the original model is extremely frustrated due to the long-range and conflicting nature of the interactions, two features which are indeed retained by the Hamiltonian defined using Eq. (82).

Now, the crucial remark is that if the model is indeed generic, its static properties can be obtained by means of the replica method, where the averaging is performed over the fictitious disorder. In the case at hand, the resulting free energy indeed turns out to be a good approximation of the original model in the high temperature, replica symmetric phase. This approximation actually corresponds to the one proposed by Golay [156] using different arguments; as seen from a high temperature expansion, this approximation is however not exact (but see next paragraph). Its main virtue is to predict the existence of a static phase transition at a temperature $T_s = 0.0476$, below which a breaking of replica symmetry of the discontinuous type appears (see Fig. 13). The low temperature phase is characterised by a residual entropy density which is linear in T, but small (less than 10^{-5} per spin at T_s). From a glass point of view this phase transition can be seen as the resolution of an entropy crisis appearing at an extrapolated Kauzman temperature which is very close to T_s. The prediction for the ground state energy density, $E_0/N \simeq 0.0202$, is compatible with a large N extrapolation of the ground state energies found by exact enumeration on small samples [161,157] $N \leq 48$. On the other hand, it does substantially differ from the apparent ground state energy extracted from Monte Carlo simulations, even after extrapolating to very small cooling rate (see Fig.13). In fact, similar discrepancies have been seen and studied in detail before on several disordered spin-glasses (for instance, the binary perceptron problem [162]). This is again reminiscent of the above discussion of discontinuous spin-glasses and of the existence of a dynamical transition at $T_c > T_s$, where the Langevin (or Monte Carlo) dynamics gets trapped in metastable states with high energies $E_{\text{THRES}} = E(T_c)$. In analogy

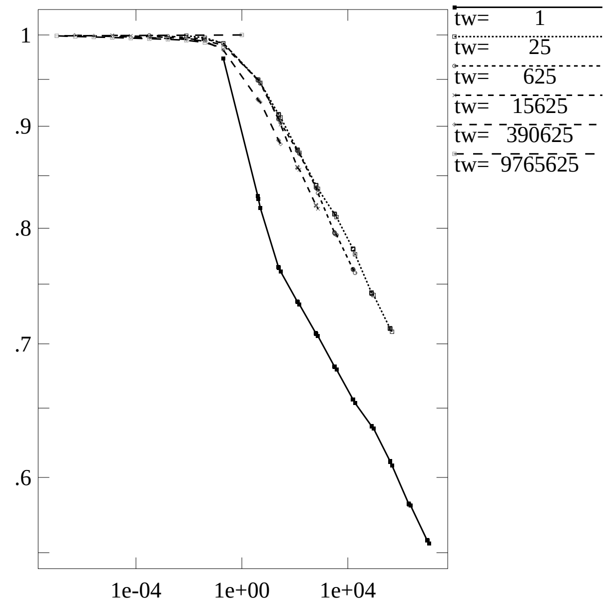

Figure 14: Two time correlation $C(t_w + \tau, t_w)$ obtained in an extensive Monte Carlo simulation of the LABS model of 400 spins with open boundary conditions (Ref. [158]), at a temperature $T = .075$, plotted versus τ/t_w. The data exhibits a clear aging effect.

with other discontinuous spin-glasses, one may thus expect that the dynamical transition where the energy freezes takes place at a temperature T_c fixed by the marginality condition (see Section 3.7), i.e the temperature where a replica symmetry broken solution first appears, with a breakpoint (in Parisi's Ansatz) equal to $m = 1$. This leads to [146] $T_c = 0.103$, in reasonable agreement with the Monte-Carlo data (see Fig. 13).

- A model with a 'crystalline' state.

In summary, we have shown that the LABS model with free boundary conditions provides an interesting example of a non disordered mean-field spin system, sharing many similarities with discontinuous spin glasses. However, the fiduciary Hamiltonian constructed using the disordered correlation Eq. (82) is only approximate, and does not, for example, give the exact free-energy in the high temperature phase. It turns out that the situation is under better control with *periodic* boundary conditions [147,148]. In this case, the model furthermore exhibits, in some sense, a 'crystal' phase.

The simplification comes from the fact that with periodic boundary conditions, the energy can be expressed as

$$E = \frac{1}{N^2} \sum_{q=1}^{N} |s(q)|^4 \ , \tag{83}$$

where we have introduced the Fourier series $\sigma(q) = \sum_j \sigma_j \exp(2i\pi qj/N)$. The problem

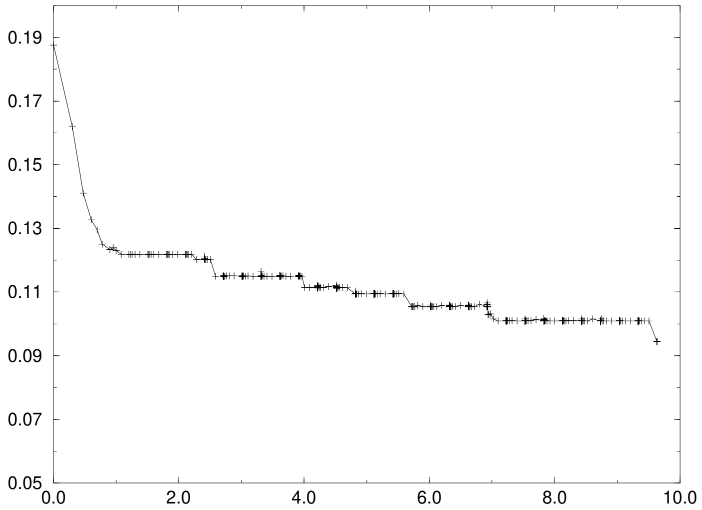

Figure 15: Energy per spin *vs* $\log_{10}$ (time) in a single Monte Carlo run for a $N = 400$ LABS spin system at $T = .075$ (From Ref.[158]).

remains non trivial since one has to remember that $s(q)$ is constrained to be the Fourier transform of a *binary* sequence. In this case a clever choice of the fiduciary disordered system was found in [147], which consists in substituting the usual Fourier components $\sigma(q)$ by a disordered version of the Fourier transform:

$$\sigma^d(q) = \sum_j U_{q,j}\sigma_j \tag{84}$$

where U is a kind of random unitary matrix [r]. One can again compute the equilibrium properties of the disordered problem with the replica method. One finds, similarly to the case with free boundary conditions, two transition temperatures T_s and T_c. However, the periodic LABS problem is in fact richer [147,163]:

– It can be shown that the disordered model has the same free energy as the original one *to all orders* in a high temperature expansion. The phase transitions, both dynamical and static, predicted by the replica solution of the disordered problem are in good agreement with the numerical simulations. The identity of the two problems in the low temperature phase, at the level of extensive thermodynamic quantitites, is however still a conjecture.

– It turns out that the periodic LABS model possesses a very special ground state for some particular values of N. When N is prime, clever number theoretic properties can be used to generate a sequence of spins with a finite energy E, and therefore a vanishingly small

[r] There is a subtlety in this construction, namely the fact that one wishes to introduce a 'disordered Fourier transform' with a matrix U which is unitary, but also satisfies $U_{-q,j} = U^*_{q,j}$, where * stands for complex conjugation. The proper construction through random orthogonal transformation is described in Ref.[147].

energy density E/N in the thermodynamic limit. (Actually for N prime of the form $4k+3$, k integer, the ground state energy is $E=1$, and for N prime of the form $N=4k+1$, is is found to be $E=5$). These special configurations are very difficult to find using Monte-Carlo like dynamics: the energy landscape has sometimes been described as a 'golf course' potential, in the sense that the ground state is an unexpected deep hole surrounded by rather unfavorable states. These ground states constitute the analogue of a crystalline state. If one simulates a LABS model with a 'magic N' (in the above sense), starting at zero temperature from the crystalline state, one finds that the low temperature specific heat is very small, and the energy has a discontinuous jump (first order transition) to the high temperature energy curve at a 'melting' temperature $T_m > T_c$ (a sketch is given in Fig. 13). No such crystalline state has been found in the free boundary case [161].

- Speculations on the 'fiduciary' disordered Hamiltonian strategy.

We have thus seen how some frustrated spin systems without disorder can be solved (approximately, or even exactly at least in the high T phase for the LABS with periodic boundary conditions), following a rather interesting strategy. This strategy consists in substituting the original problem by a 'fiduciary' one with quenched disorder, and solving the disordered system using, e.g., the replica method to obtain the static properties and information about the transitions. There is unfortunately no systematic method of choosing the fiduciary model so far. The above two examples, or other models which have been studied in a similar way [148,150], show the importance of symmetry considerations in the choice of the fiduciary disordered problem, and suggest as a criterion that this disordered model should be as 'close' as possible to the original one in the high temperature (liquid) phase. This strategy is reminiscent of the very fruitful approach to energy levels in complex nuclei through the study of fiduciary random Hamiltonians with the proper symmetries [167]. In our case we do not yet understand when such an approach may be successful or not, if it is only restricted to finite time dynamics or if it does apply to thermodynamical properties. In some cases (see [133,148,151] and below), the 'spin-glasses without disorder' explicitly involve pseudo random numbers in the sense that the spin couplings are deterministic, but very rapidly oscillating. This is much less obvious in the LABS model, especially with free boundaries. Finally, as discussed in the next subsection, self-consistent (Mode-Coupling) approximations of non disordered models often lead to equations which are exact for some adequately chosen disordered systems.

In a loose sense, one expects that the slow dynamics at low temperatures originates from some degrees of freedom which freeze and play the role of an effectively quenched disorder field for the other degree of freedom. The success of the present strategy might lie in the fact that the precise identification of these 'slow' variables is not necessary to understand the freezing transition in these models.

- Other mean-field spin-glass models without disorder.

There exist by now a few such examples of spin-glasses without disorder. Besides the 'fiduciary' Hamiltonian strategy, other approaches have been attempted, in particular some direct diagrammatic calculations – either static ones[149,151] (basically through a resummation of the high temperature expansion) or using dynamical perturbation theory[133,152]. Because of space limitation we cannot mention them all here. We should nevertheless point out the case of Josephson junction arrays because it may allow a direct experimental investigation of many of the ideas discussed here[151,152]. This model involves two sets of N spins, $\sigma_j = \exp(i\phi_j)$ (living on the rows of a 2-D array of Josephson junctions) and $\sigma'_k = \exp(i\phi'_k)$ (living on the columns of the same array) which are normalised two component vectors (U(1) spins). These are coupled through the Hamiltonian

$$H = -\frac{J}{\sqrt{N}} \sum_{j,k=1}^{N} \Big(\exp(i2\pi\alpha jk/N)\sigma_j^* \sigma'_k + c.c. \Big) \tag{85}$$

where c.c. means complex conjugate. This can be realised by using a stack of two mutually perpendicular sets of N parallel wires. There is supposed to be a Josephson junction at each crossing of an horizontal wire with a vertical one. The variable ϕ_j is the reference phase of the j^{th} horizontal wire, while ϕ'_k is the reference phase of the k^{th} vertical wire. The system lies in an external transverse field H, which induces a phase shift per unit area α proportional to H. The high temperature expansion has been resummed for $1/N \ll \alpha \ll 1$, both for the static theory[151] and for the two-time correlation and response function[152], leading *exactly* to the same equations as those describing the $p = 4$ spherical spin-glass which we have discussed in Section 3. Again, the system undergoes two phase transitions, a dynamical one at a temperature T_c larger than the static one T_s. These dynamical equations are also equivalent to a particular case of the 'schematic' mode coupling equations of supercooled liquids. Josephson junction arrays may thus provide an interesting experimental playground to test directly the predictions of these theoretical studies.

- Towards the description of the glass phase for interacting particles

As is often the case in statistical physics, it is technically easier to study a phenomenon (here the glass transition in spin systems without disorder) on the magnetic case. A natural problem is to try to extend the kind of ideas that we have seen at work on these spin analogues towards more realistic problems where a glass phase appears without any quenched disorder, like glass forming liquids.

One must thus describe interacting particles. The studies in this direction are still rather preliminary. Two routes have been recently explored. One is to consider interacting particles in large dimensional (D) spaces. In Ref.[149] n point-like particles interacting repulsively are constrained to move on the vertices of a D-dimensional hypercube. The model can be mapped onto a modified $O(n)$ matrix model. It has a sharp glassy transition of the kind described previously. Amusingly, for n multiple of four, an unproven conjecture of Hadamard gives the 'crystalline' ground states. On the same line, similar properties are found in a model[150] of hard spheres moving on the surface of a D-dimensional hypersphere.

Another study retains the three dimensional nature of the problem, and starts from a reasonable analytic approximation of the liquid phase given by the Hypernetted Chain

Approximation (HNC). It turns out that one can generalize this HNC description through the introduction of a replicated theory as described in Section 3.7. This approach [164] yields a good analytical prediction for the glass transition temperature or density of hard or soft spheres systems. A precise description of the low temperature phase along the same lines is however still missing.

4.4 p-spin models, Mode Coupling Theory of glasses, and its extension at low temperatures

The above arguments suggest that discontinuous mean-field spin-glasses, in spite of the presence of quenched disorder, can provide a good starting mean-field theory for structural glasses. Among the most striking convergence between the two subjects, already alluded to many times above, is the equivalence (in the high temperature phase) between the Mode Coupling equations for glasses and the dynamical mean-field equations for p-spin-glass models. This analogy was first noticed already long ago [18]. It has been useful technically, in particular by transposing the mathematical analysis of the Mode Coupling equations developed for glasses [17,16] to the study of the high temperature dynamics of p-spin systems [108] and manifolds in random media [109,110], e.g. the divergence of the relaxation time and the shouldering of the relaxation. Following similar developments in the field of fully developed turbulence [165,166], it has been realized more recently that the factorization property which is at the heart of the mode coupling approximation actually becomes *exact* for certain systems, which turn out either to contain quenched disorder [134] or some deterministic version of disordered systems, in the sense of having rapidly oscillating couplings which have statistical properties of disordered ones [133]. We shall first give a general flavour of why this is the case, and then turn to the implications of this general result to the low temperature extension of the Mode-Coupling equations, and its physical consequences.

- Mode coupling approximation and hidden disorder.

Both glass forming systems and turbulence can be described by some non-linear stochastic dynamical equation. In order to describe the essence of the mode coupling approximation and its relation with disordered systems, we shall explain it briefly on the simple case of a single scalar degree of freedom ϕ, with a Langevin dynamics

$$\frac{\partial \phi}{\partial t} = -\mu(t)\phi - \frac{g}{3!}\,\phi^3 + \eta \tag{86}$$

with initial condition $\phi(t=0)=0$. The thermal noise η is a Gaussian noise η with $\langle\, \eta(t)\,\rangle = 0$ and $\langle\, \eta(t)\,\eta(t')\,\rangle = 2T\,\delta(t-t')$ (in the following the brackets will always denote an average over the realisations of the Gaussian white noise η). The coupling constant g serves as a book-keeping parameter to set up a perturbative expansion. This expansion can either be well-behaved or ill-behaved depending – say – on the dimension of space. It is in any case rather useless when $g = O(1)$ if it cannot be resummed in one way or another. The mode coupling consists in a 'one-loop' self-consistent perturbation theory. This amounts to resumming a particular (infinite) set of terms in the perturbation expansion. In this way, non-trivial self-consistent equations are obtained, which enable one to peep into the strong coupling regime, through an approximation which is however not easily controlled.

Setting $R_0 = [\mu(t) + \frac{\partial}{\partial t}]^{-1}$, which gives $R_0(t,t') = \exp\left(-\int_{t'}^{t} d\tau\ \mu(\tau)\right)$, the perturbative expansion for $\phi(t)$ is easily written as:

$$\phi(t) = R_0 \otimes \eta - \frac{g}{3!}\, R_0 \otimes \{R_0 \otimes \eta \bullet R_0 \otimes \eta \bullet R_0 \otimes \eta\} + ... \tag{87}$$

where $\otimes$ means a time convolution: $(R_0 \otimes f)(t) = \int_0^t dt' R_0(t,t') f(t')$ and $\bullet$ is a simple product.

The lowest non trivial order perturbative expansion for the correlation function $C(t,t')$ and the response function $R(t,t')$ is easily written in terms of the kernels $\Sigma(t,t')$ and $D(t,t')$ through the Dyson equations:

$$R(t,t') \equiv R_0(t,t') + \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2\ R_0(t,t_1)\ \Sigma(t_1,t_2)\ R(t_2,t')\ , \tag{88}$$

$$C(t,t') \equiv \int_0^{t} dt_1 \int_0^{t'} dt_2\ R(t,t_1)\ D(t_1,t_2)\ R(t',t_2)\ . \tag{89}$$

The mode coupling approximation for this problem amounts to an approximation of the kernels $\Sigma(t,t')$ and $D(t,t')$ where one takes their values at order g^2 and substitutes in them the bare propagator R_0 and the bare correlation by their renormalised values. This gives the following self-consistent equations:

$$\begin{aligned} \Sigma(t,t') &= \frac{g^2}{2}\, C^2(t,t')\, R(t,t') \\ D(t,t') &= 2T\ \delta(t-t') + \frac{g^2}{6}\, [C(t,t')]^3 \ , \end{aligned} \tag{90}$$

This approximation neglects 'vertex renormalisation'.

The problem is of course to try to control this procedure. An important step in this direction is to identify a model for which the self-consistent equations are *exact*. The basic remark (first made by Kraichnan in the context of turbulence [166], where the analogous method is named direct interaction approximation) is that the diagrams retained by the mode coupling approximation are precisely those which survive if one considers the following disordered problem. First, one upgrades ϕ to an $N-$ 'colour' object ϕ_α, where $\alpha = \{1, 2, ..., N\}$. The equation of motion Eq. (86) is then generalized to:

$$\frac{\partial \phi_\alpha}{\partial t} = -\mu(t)\ \phi_\alpha - 4g \sum_{\beta<\gamma<\delta} J_{\alpha\beta\gamma\delta}\ \phi_b \phi_\gamma \phi_\delta + \eta_\alpha \tag{91}$$

with independent noises η_α. The couplings $J_{\alpha\beta\gamma\delta}$ are *quenched*, i.e. time-independent, Gaussian random variables of zero mean and variance $\overline{J^2_{\alpha\beta\gamma\delta}} = 1/N^3$. In the large N limit, the correlation: $C(t,t') \equiv \frac{1}{N}\sum_{\alpha=1}^{N} \overline{\langle \phi_\alpha(t)\phi_\alpha(t')\rangle}$ (where the overline denotes the average over the random couplings $J_{\alpha\beta\gamma\delta}$) and the response: $R(t,t') \equiv \frac{1}{N}\sum_{\alpha=1}^{N} \overline{\langle \frac{\partial \phi_\alpha(t)}{\partial \eta_\alpha(t')} \rangle}|_{\eta=0}$ precisely obey the mode coupling approximation equations [134], Eqs. (88) and (89).

• A particle in a random potential.

The same construction can be generalised [134] to an arbitrary nonlinearity $F(\phi)$ substituting the $\frac{g}{3!}\phi^3$ in Eq.(86). Therefore one finds that the general schematic mode coupling equations developed in the study of glass forming liquids can be derived exactly from the Langevin dynamics of N continuous spins ϕ_α, of the type:

$$\frac{\partial \phi_\alpha}{\partial t} = -\mu(t)\,\phi_\alpha - \frac{\delta V[\{\phi\}]}{\delta \phi_\alpha} + \eta_\alpha\,. \tag{92}$$

This disordered multispin Hamiltonian is precisely the generic mean-field problem (44) which we studied in Sect. 3, which can also be seen as describing a particle evolving in an $N \to \infty$ dimensional space in a quenched random potential $V[\{\phi\}]$.

This interpretation is rather appealing. Let us introduce the following highly simplified picture of a glass: the motion of a given particle can be thought of as taking place in a random potential created by its neighbours. Since the motion of the molecules is extremely slow at low temperatures, one can assume that this random potential has a static component, in the spirit of the 'self-induced quenched disorder' scenario which we discussed above [168]. In large dimension of space, one can establish the *exact* equations relating the two-time correlation function $C(t_w+\tau, t_w) = \langle \vec{r}(t_w+\tau)\cdot\vec{r}(t_w)\rangle$ (where $\vec{r}(t)$ is the position of the particle at time t), and the two-time response to an external force $R(t_w+\tau, t_w)$. For temperatures higher than T_c, we have seen that C and R are actually TTI, and furthermore that the FDT $R(\tau) = -\frac{1}{T}\Theta(\tau)\frac{\partial C(\tau)}{\partial \tau}$ is obeyed. As noted in Section 3.2, one can then eliminate $R(\tau)$ and find an equation for $C(\tau)$ which is precisely the schematic Mode-Coupling equation, with a kernel related to the correlation of the random potential.

Hence, the physical content of the (schematic) MCT is clear: it is a mean-field description of a single point in a static *quenched random* potential. The important point is thus that MCT implicitly assumes the presence of some *quenched disorder* which should rather, as discussed above, be 'self-induced' by the dynamics itself. In a sense it looks rather similar to the introduction of fiduciary models discussed before.

• Mode coupling at low temperatures

Coming back to the general equations relating C and R, one can now postulate that they are the correct generalisation of the schematic Mode Coupling equations for two time quantities, and investigate the 'glass' phase $T < T_c$. The results of Section 3 are thus directly applicable. In particular, the correlation and response function cease to be functions of τ only. More precisely, $C(t_w+\tau, t_w)$ can be written as the sum of a stationary contribution $C_{\rm ST}(\tau)$ which only depends on τ, and an aging part which depends on the ratio $t_w/(t_w+\tau)$, (or a generalisation thereof, see Section 3.6): $C_{\rm AG}(t_w/(t_w+\tau))$. The expected shape of the correlation function in the glass phase is thus given in Fig. 10.

The same decomposition holds for the response function, and the aging parts of C and R are related by an 'anomalous' FDT, where T is replaced by an effective temperature T/X, with $X \leq 1$.

In more physical terms, this means that for a finite waiting time t_w after the quench below T_c, one expects that the susceptibility $\chi(\omega, t_w)$ still exhibits two peaks: a high frequency β-peak very similar to the high temperature ($T > T_c$) one, and a low frequency α-peak which reaches a maximum at a frequency $\omega_\alpha \simeq 1/t_w$, which thus progressively disappears as $t_w \to \infty$. An interesting prediction of this low temperature extension of MCT is that the high frequency part of the aging α-peak behaves as $(\omega t_w)^{-b}$, while the low frequency 'foot' of the β-peak behaves as ω^a, with the following relation between a, b, and X (see (69)): $X\Gamma^2[1+b]/\Gamma[1+2b] = \Gamma^2[1-a]/\Gamma[1-2a]$. This equation generalizes the well known MCT relation (54) between a and b for $T > T_c$, for which $X \equiv 1$. It would be extremely interesting to try to test these predictions experimentally, taking care of the fact that the above picture is only valid insofar as t_w is small compared to the relaxation time $\tau(T)$, such that 'activated' effects might indeed be neglected.

5 Conclusion. Where do we stand ?

Let us now summarize some of the most important ideas developed in this review and discuss some open problems.

We have tried to show that aging effects are not spurious, irreproducible artifacts of non equilibrium situations, but rather an unescapable feature of systems characterized by a very large relaxation time, because time-translational invariance breaks down and the well known fluctuation-dissipation theorem has to be modified in a non trivial way. However, the asymptotic aging regime where all times are large reveals some universal properties; in particular, the 'effective' relaxation time becomes a time dependent notion and grows with the waiting time. The detailed investigation of these aging effects actually offer a unique way to probe the phase-space structure of complex systems. From that point of view, a system in equilibrium is 'dead'.

We have described several simple models where aging can be described in detail: coarsening models, 'trap' models and mean-field spin-glass models: (which turn out to give dynamical equations in exact correspondance with the 'Mode-Coupling' description of supercooled liquids). Coarsening leads to aging in the correlation function but *not* in the response function: the FDT is most strongly violated in the aging regime. This scenario can thus only include aging *in the response* of spin-glasses or polymer glasses as a transient effect even in an infinite system – quite apart from the fact that it is not obvious what would actually *grow* with time in these systems.

Mean-field spin-glasses provide an interesting testing field. One generically finds aging in the low temperature phase of these models, although two very different categories of systems emerge. In the discontinuous case (corresponding in the high temperature phase to 'model B' of MCT), a dynamical temperature transition is found *above* the equilibrium transition. Throughout the low-temperature phase the asymptotic energy-density arrived at after any type of cooling is higher than the equilibrium energy-density. The two-time plane breaks up into two sectors, which correspond to the stationary and aging dynamical regimes. On the contrary, for the continuous case (such as the Sherrington-Kirkpatrick model) static and dynamical transition temperatures coincide, as do the asymptotic out of equilibrium energy density and the equilibrium one. The two-time behaviour is much more complicated, reflecting in some way the subtleties of Parisi's ultrametric equilibrium solution.

Interestingly, if one views mean-field glassy dynamics as the dynamics of a point in a many dimensional rugged phase-space, one finds that the basic mechanism for aging is germain to that of simple coarsening: because of high dimensionality of phase-space the system starts near the border of a basin of attraction, and remains forever undecided about where to go.

It is of course crucial to know how these mean-field predictions are modified when one goes beyond mean-field and studies finite dimensional systems. From a theoretical point of view, the difficulty comes from the fact that activated processes, which are effectively absent in mean-field, come into play when the dimension is finite. For example, the infinitely long-lived metastable states in mean-field acquire a finite relaxation time in finite dimension, through bubble nucleation. The dynamical transition, which corresponds in the language of supercooled liquids to the Mode-Coupling critical temperature, is thus smeared out. Similarly, a particle in a random potential in finite dimension reaches a local minimum after a finite time, beyond which thermal activation starts playing an important role; contributions to aging of a somewhat different nature — such as those described by the 'trap' picture — then set in. The relation of these trap models to the aging dynamics of real spin-glasses or other disordered systems such as pinned vortex lines, dislocations, domain walls, or polymers is however still rather tentative.

For the same reason (absence of activated effects), mean-field models are not suited to describe cooling rate dependent effects, which can be very large in some disordered systems and in glass forming liquids.

Therefore, new theoretical ideas which would allow one to extend the previous approaches to finite dimensions (bearing in mind that activated effects cannot, in general, be accounted for within perturbative schemes) are clearly desirable. Returning to the question of aging in the response, the FDT-violation factor X in finite dimension is particularly interesting: this factor — that is related to effective temperatures in the system — is found to be non-trivial ($0 < X < 1$) in the disordered mean-field models; what is the situation in finite dimensions? Some numerical results[46,47] suggest that X does indeed remain non trivial in finite dimensions, while arguments based on coarsening pictures would suggest that asymptotically, $X = 0$. This might actually be a clear-cut dynamical distinction between droplet like coarsening pictures and mean-field like pictures of real spin-glasses, which surely deserves more efforts – as stated above, spin-glasses, after all, *do* exhibit aging in response functions.

We have also discussed rather at length the relation between glasses and disordered systems, noting that:

– some models without disorder behave very much like disordered systems and can actually be theoretically described as such, and

– some approximation schemes for systems without disorder, but strongly interacting, lead to dynamical equations (i.e. the Mode-Coupling equations) which correspond to the hidden assumption of the existence of some quenched disorder and are actually exactly the equations describing disordered mean-field spin-glasses.

Correspondingly, many of the ideas developed to describe aging in disordered systems are *de facto* also relevant for glasses. We have mentioned in particular how aging should manifest itself, within the Mode-Coupling scenario and for times smaller than the relaxation time as a waiting time dependent α-peak. A detailed analysis of these aging regimes should enable one to distinguish, again, between mean-field like descriptions and activated, 'trap' like, models; or perhaps understand how both mechanisms are blended.

Several questions still remain completely open. In particular, is there a general criterion allowing one to understand when a 'complicated' system can be described as disordered ? Is this description only viable at finite times, where the specificity of the system at hand has not yet had time to reveal itself ? A possibility is that this time scale can only be infinite for mean-field like models, such as the LABS model or the long-range Josephson array. Related to this is of course the lurking question of the very existence of a true glass transition in any short range system without disorder.

The question of the existence of a true phase transition is however, after all, not crucial to understand the out-of-equilibrium properties, as it is displayed in so many physical systems. In this respect, it is plausible that mean-field models provide in general a much better starting point for the finite time, dynamical properties of real systems, than for its long time, equilibrium properties.

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