BEYOND THE SHERRINGTON-KIRKPATRICK MODEL

C. DE DOMINICIS
Service de Physique Théorique, CEA Saclay,
F-91191 Gif sur Yvette, France

I. KONDOR
Bolyai College, Eötvös University,
H-1145 Amerikai út 96, Budapest, Hungary

T. TEMESVÁRI
Institute for Theoretical Physics, Eötvös University,
H-1088 Puskin utca 5-7, Budapest, Hungary

The state of art in spin glass field theory is reviewed. We start from an Edwards-Anderson-type model in finite dimensions, with finite but long range forces, construct the effective field theory that allows one to extract the long wavelength behaviour of the model, and set up an expansion scheme (the loop expansion) in the inverse range of the interaction. At the zeroth order we recover mean field theory. We evaluate systematic corrections to this around Parisi's replica symmetry broken solution. At the level of quadratic fluctuations we derive a set of coupled integral equations for the free propagators of the theory and show how they can be solved for short, intermediate and extreme long distances. To reveal the physical meaning of these results, we relate the various propagator components to overlaps of spin-spin correlation functions inside a single phase space valley resp. between different valleys. Next we calculate the first loop corrections to the theory above 8 dimensions, where we find that it maps back onto mean field theory, with basically temperature independent renormalization of the coupling constants, thereby demonstrating that Parisi's mean field theory is, at least perturbatively, stable against finite range corrections. In the range between six and eight dimensions various physical quantities pick up nontrivial temperature dependences which can, however, still be determined exactly. Upon approaching the upper critical dimension ($d = 6$) of the model, scaling which is badly violated in Parisi's mean field theory is gradually restored. Below 6 dimensions one should apply renormalization group methods. Unfortunately, the structure of RG is not completely understood in spin glass theory. Nevertheless, the first corrections in $6 - d$ to e.g. the exponent of the order parameter can still be calculated, moreover exponentiation to this power can be checked at the next order. The theory is, however, plagued by infrared divergences due to the presence of zero modes and soft modes. Systematic methods (like those developed in the O(n) model) to handle these infrared singularities are not yet available in spin glass theory.

1 Introduction

Parisi's mean field theory (MFT) [1] is generally accepted as the correct solution of the Sherrington-Kirkpatrick (SK) problem [2]. The crucial technical assumption that renders the SK model soluble is that every spin interacts with infinitely many neighbours. Such a situation could arise e.g. in a system in infinitely high spatial dimensions $d \to \infty$ or in a finite dimensional system with an infinitely long ranged interaction. While in some extensions of the theory (especially in applications outside physics, like combinatorial optimization or neural networks) the assumption of high connectivity is well justified, in real physical systems that live in low dimensions and have finite range forces MFT can serve, at best, as a zeroth approximation.

In order to go beyond the SK model and to approach a more realistic situation, one can set up an expansion scheme either in the inverse dimensionality ($1/d$ expansion) or in the inverse interaction radius (loop expansion). These will work, however, only if the phase space structure of the finite dimensional, finite ranged system is similar to that of the fully connected mean field model.

The physical picture underlying Parisi's solution is of the system having a rugged free-energy surface, with many, hierarchically organized equilibrium states in the frozen phase. However, a general consensus on whether such a structure can survive in real, finite dimensional, finite ranged models has never been reached. To the variety of approaches in this decade-long debate, ranging from scaling [3,4] and phenomenological renormalization [5] to large scale simulations [6-13] and to the first perturbative steps within the $1/d$ [14] and the loop expansions [15,16], respectively, there have recently been added the exact methods of mathematical physics [17-21]. With this, a fundamental problem, namely that of the definition of an equilibrium state in a random system, has been brought into focus.

The results to be reviewed in this paper have been obtained under the assumption that the "many valley structure" of MFT remains relevant for a finite dimensional, finite ranged system. We use the formalism of replica field theory [1], calculate free propagators and set out to determine the first loop corrections. All our results will be confined to the Ising spin glass just below its freezing temperature.

The technical difficulties we encounter are considerable, so we can fully compute the first corrections to the order parameter and to the excitation spectrum only in high ($d > 6$) dimensions. Near $d = 6$, the upper critical dimension, we have only partial results. We also notice severe infrared (IR) problems here. These are manifestations of the extreme sensitivity of the system to changes in distant regions: the unusually strong IR singularities in our propagators may well tell, in the language of replica field theory, the same story as the mathematical results concerning the chaotic response to changes on the boundary [20]. In this context we find it remarkable that the worst IR powers are displayed by those propagator components that also describe the chaotic response to infinitesimal variations in the control parameters [22,23].

If these IR problems turn out to be unsurmountable they will destroy replica symmetry breaking (RSB) field theory from within. If, on the other hand, recent efforts by Parisi and coworkers [24] and also by ourselves [25] succeed in saving the theory through formally exact statements like Ward identities etc., we will find it hard to believe that the theory is completely devoid of physical meaning.

We interpret the goal of this paper in a very restricted sense: apart from citing the results by Georges, Mézard and Yedidia [14] in a qualitative manner we shall not be able to cover the method of $1/d$ expansion, nor the important set of papers by Parisi and coworkers on the finite size corrections to MFT [26], although both of these approaches might find their well deserved place under the title of this paper. What we will try to give an account of is the construction of the loop expansion for spin glasses below the freezing temperature. Within the space available we will have to be fairly sketchy on this, too. We will not provide proofs but will only give some hints as to how the results we display can be derived. Concerning the results themselves, especially those for the propagators, we will try to be rather more exhaustive, however, and collect the various propagators, their limiting forms, etc. that have appeared over the years scattered in a series of papers [27-30,24]. We will also try to demonstrate their use for the calculation of short range corrections in a few cases.

The plan of the paper is the following. In Sec. 2 we briefly review the elements of the field theoretic formulation of the problem. In Sec. 3 we give a sketchy account of Parisi's MFT with the purpose of fixing notation. We take the first step beyond MFT in Sec. 4: we spell out the set of equations for the free propagators and, for the sake of a first orientation, solve them under the (wrong) assumption of replica symmetry, thereby displaying the famous de Almeida-Thouless (AT) instability [31], and also the two characteristic "mass scales" inherent in the system. In Sec. 5 we abandon the assumption of replica symmetry but confine our attention to the "high-momentum" region near the upper cutoff where the equations for the propagators can still be solved by a simple iteration. The results obtained in this region will be used later when calculating the first loop corrections in high dimensions ($d > 8$). In Sec. 6 we investigate the propagators in the near infrared region, i.e. for momenta around the "large mass" (which vanishes like the square root of the reduced temperature near the spin glass transition) and much larger than the "small mass" (vanishing like the first power of the reduced temperature). We show that in this region the complicated set of (ten) integral equations for the propagators reduces to a set of linear algebraic equations, which can then be solved by elementary means. We turn to the general study of the formidable set of integral equations for the propagators in Sec. 7. Their solution is broken down into two steps. First, by exploiting the residual (ultrametric) symmetry of the system, we indicate how the problem of the inversion of a general ultrametric matrix can be reduced to the inversion of a much simpler object that we call the kernel. Next, in the special case of the Hess matrix we show that in the vicinity of the transition temperature it has a very simple kernel which can be inverted with relative ease and yields the propagators in closed form. The singularities of the propagators determine the excitation spectrum of the system, and show that Parisi's solution is marginally stable. From the complicated exact expressions for the propagators we extract the limiting forms, valid in the far infrared or extreme long wavelength limit (where momenta are comparable to or smaller than the small mass scale) in Sec. 8. The propagator components are related to some combinations of correlation functions. This relationship is established in Sec. 9 where the physical meaning of some of the results obtained so far is also analysed. We step beyond the analysis of Gaussian fluctuations in Sec. 10 where we derive the first loop corrections to the order parameter and to the excitation spectrum above 8 dimensions. By absorbing the loop corrections into the coupling constants we map back the theory onto MFT and thereby show that Parisi's solution is, at least in high dimensions, perturbatively stable. However, the renormalized quartic coupling blows up as one approaches $d = 8$ from above. Therefore in the range $6 < d < 8$ one has to rearrange the loop expansion. This is done is Sec. 11. At and below $d = 6$ the loop expansion is bound to break down completely. Although the structure of the renormalization group, the usual remedy in such a situation, is not known in spin glass theory, we show in Sec. 12 that some information can still be extracted from the logarithmic singularities appearing here. In particular, we show that to first order in $\varepsilon = 6 - d$ the critical exponent β can be computed in agreement with the values of other critical exponents known from above T_c [32-34], moreover, this power can be checked to exponentiate at the next order in ε. Sec. 13 concludes the paper with a brief summary.

2 The elements of replica field theory

Our starting point is a standard Edwards-Anderson-like [35] model for N Ising spins in d dimensions, with a long but finite-ranged interaction:

$$\mathcal{H} = -\sum_{(i,j)} \frac{J_{ij}}{\sqrt{z}} f\left(\frac{|\mathbf{r}_i - \mathbf{r}_j|}{\rho a}\right) s_i s_j \quad . \tag{1}$$

In Eq. 1 the external field has been set to zero, the summation is over the pairs (i,j) of lattice sites, $z = \rho^d$ is the number of spins within the interaction radius measured in units of the lattice spacing a. $f(x)$ is a smooth positive function which takes the value 1 for $x \le 1$, and decays to zero sufficiently fast for $x > 1$, thereby cutting off the interaction around $r \sim \rho a$. J_{ij} are independent, Gaussian distributed random variables with mean zero and variance Δ^2. We will be interested in the long distance behaviour of the model (1), for which the details of $f(x)$ are to a large extent immaterial, so we can choose f according to convenience.

The long wavelength properties near T_c can be extracted by studying the associated effective Lagrangean of replica field theory [36] which, for an appropriate choice of f, works out to have the form:

$$\begin{aligned} \mathcal{L} \;=\; & -\frac{1}{4}\sum_{\alpha,\beta}\sum_{\mathbf{p}} \left((pa\rho)^2 - 2\tau\right) |\Phi^{\alpha\beta}_{\mathbf{p}}|^2 + \frac{w}{6\sqrt{N}} \sum_{\alpha,\beta,\gamma}\sum_{\mathbf{p_i}} \Phi^{\alpha\beta}_{\mathbf{p_1}}\Phi^{\beta\gamma}_{\mathbf{p_2}}\Phi^{\gamma\alpha}_{\mathbf{p_3}} + \\ & + \frac{u}{12N}\sum_{\alpha,\beta}\sum_{\mathbf{p_i}} \Phi^{\alpha\beta}_{\mathbf{p_1}}\Phi^{\alpha\beta}_{\mathbf{p_2}}\Phi^{\alpha\beta}_{\mathbf{p_3}}\Phi^{\alpha\beta}_{\mathbf{p_4}} + \ldots, \end{aligned} \tag{2}$$

where the wavevector summations are restricted by "momentum conservation" $\sum_i \mathbf{p_i} = 0$ and by an UV cutoff at $p \sim 1/\rho a$. τ is the reduced temperature measured relative to the mean field value $T_c^{MF} = \Delta$ of the critical temperature:

$$\tau = \frac{T_c^{MF} - T}{T_c^{MF}} \quad . \tag{3}$$

τ will be assumed small throughout this paper.

The fields $\Phi^{\alpha\beta}$ are symmetric in the replica indices $\alpha, \beta = 1, 2, \ldots, n$ and $\Phi^{\alpha\alpha} \equiv 0$. In the truncated Lagrangean (2), first proposed by Parisi [1] and used by several authors since, we have kept only that particular quartic term which is responsible for replica symmetry breaking on the mean field level. The numerical values of the bare coupling constants w and u work out to be 1 in the case of the Ising spin glass. By a slight generalization of the model we wish to regard these bare couplings as essentially free (positive) parameters, partly for book-keeping purposes, but also because they pick up short range corrections anyhow.

Now we split the field into an equilibrium and a fluctuating part as

$$\Phi^{\alpha\beta}_{\mathbf{p}} = \sqrt{N} q_{\alpha\beta}\delta^{Kr}_{\mathbf{p},0} + \Psi^{\alpha\beta}_{\mathbf{p}}$$

with

$$\begin{array}{rclrcl} q_{\alpha\beta} & = & q_{\beta\alpha}, & q_{\alpha\alpha} & = & 0 \\ \Psi^{\alpha\beta} & = & \Psi^{\beta\alpha}, & \Psi^{\alpha\alpha} & = & 0 \quad . \end{array} \tag{4}$$

The Lagrangean then splits into four terms:

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \mathcal{L}^{(3)} + \mathcal{L}^{(4)} \tag{5}$$

defined as follows:

$$\mathcal{L}^{(0)} = N\left[\frac{\tau}{2}\sum_{\alpha,\beta} q_{\alpha\beta}^2 + \frac{w}{6}\sum_{\alpha,\beta,\gamma} q^{\alpha\beta}q^{\beta\gamma}q^{\gamma\alpha} + \frac{u}{12}\sum_{\alpha,\beta} q_{\alpha\beta}^4\right], \tag{6}$$

$$\mathcal{L}^{(1)} = \sqrt{N}\sum_{\alpha,\beta}\Psi^{\alpha\beta}_{\mathbf{p}=0}\left[\tau q_{\alpha\beta} + \frac{w}{2}(q^2)_{\alpha\beta} + \frac{u}{3}q^3_{\alpha\beta}\right], \tag{7}$$

$$\mathcal{L}^{(2)} = -\frac{1}{2}\sum_{\alpha<\beta,\gamma<\delta}\sum_{\mathbf{p}}\Psi^{\alpha\beta}_{\mathbf{p}}\left(\tilde{G}^{-1}(\mathbf{p})\right)_{\alpha\beta,\gamma\delta}\Psi^{\gamma\delta}_{-\mathbf{p}} \quad , \tag{8}$$

where $\tilde{G}^{-1}$ is the inverse of the free propagator:

$$\begin{aligned} \left(\tilde{G}^{-1}(\mathbf{p})\right)_{\alpha\beta,\gamma\delta} &= (p^2\rho^2a^2 - 2\tau - 2uq^2_{\alpha\beta})(\delta^{Kr}_{\alpha\gamma}\delta^{Kr}_{\beta\delta} + \delta^{Kr}_{\alpha\delta}\delta^{Kr}_{\beta\gamma}) - \\ &\quad -w(\delta^{Kr}_{\alpha\gamma}q_{\beta\delta} + \delta^{Kr}_{\alpha\delta}q_{\beta\gamma} + \delta^{Kr}_{\beta\gamma}q_{\alpha\delta} + \delta^{Kr}_{\beta\delta}q_{\alpha\gamma}) \quad , \end{aligned} \tag{9}$$

$$\mathcal{L}^{(3)} = \frac{1}{\sqrt{N}}\left\{\frac{w}{6}\sum_{\alpha,\beta,\gamma}\sum_{\mathbf{p_i}}\Psi^{\alpha\beta}_{\mathbf{p_1}}\Psi^{\beta\gamma}_{\mathbf{p_2}}\Psi^{\gamma\alpha}_{\mathbf{p_3}} + \frac{u}{3}\sum_{\alpha,\beta}q_{\alpha\beta}\sum_{\mathbf{p_i}}\Psi^{\alpha\beta}_{\mathbf{p_1}}\Psi^{\alpha\beta}_{\mathbf{p_2}}\Psi^{\alpha\beta}_{\mathbf{p_3}}\right\} \tag{10}$$

and finally

$$\mathcal{L}^{(4)} = \frac{u}{12N}\sum_{\alpha,\beta}\sum_{\mathbf{p_i}}\Psi^{\alpha\beta}_{\mathbf{p_1}}\Psi^{\alpha\beta}_{\mathbf{p_2}}\Psi^{\alpha\beta}_{\mathbf{p_3}}\Psi^{\alpha\beta}_{\mathbf{p_4}} \quad . \tag{11}$$

We shall regard $\mathcal{L}^{(2)}$ as the bare Lagrangean and

$$\mathcal{L}^{(I)} = \mathcal{L}^{(1)} + \mathcal{L}^{(3)} + \mathcal{L}^{(4)}$$

as the interaction.

Statistical averages are calculated with the weight $\sim \mathrm{e}^{\mathcal{L}}$. For example, the expectation value of the fluctuation to first order in $\mathcal{L}^{(I)}$ works out as

$$< \Psi^{\alpha\beta}_{\mathbf{p}} > = \frac{\int[\mathrm{d}\Psi]\Psi^{\alpha\beta}_{\mathbf{p}}\mathrm{e}^{\mathcal{L}}}{\int[\mathrm{d}\Psi]\mathrm{e}^{\mathcal{L}}} = \frac{\int[\mathrm{d}\Psi]\Psi^{\alpha\beta}_{\mathbf{p}}\mathrm{e}^{\mathcal{L}^{(2)}}(1+\mathcal{L}^{(I)}+\ldots)}{\int[\mathrm{d}\Psi]\mathrm{e}^{\mathcal{L}^{(2)}}(1+\mathcal{L}^{(I)}+\ldots)} =< \Psi^{\alpha\beta}_{\mathbf{p}}\mathcal{L}^{(I)} >_{(0)} + \ldots \tag{12}$$

where $< \ldots >_{(0)}$ is the average with the weight $\mathrm{e}^{\mathcal{L}^{(2)}}$ and $< \Psi^{\alpha\beta}_{\mathbf{p}} >_{(0)} = 0$ has been used.

By a Wick decomposition Eq. 12 can further be written as:

$$
\begin{aligned}
< \Psi_{\mathbf{p}}^{\alpha\beta} > &= \sqrt{N}\delta_{\mathbf{p},0}^{Kr} \sum_{\alpha',\beta'} \widetilde{G}_{\alpha\beta,\alpha'\beta'}(\mathbf{p}=0) \Bigg\{ \eta_{\alpha',\beta'} + \frac{w}{2N} \sum_{\mathbf{p}} \sum_{\gamma' \neq \alpha'\beta'} \widetilde{G}_{\alpha'\gamma',\beta'\gamma'}(\mathbf{p}) + \\
&\quad + \frac{u}{N} \sum_{\mathbf{p}} q_{\alpha'\beta'} \widetilde{G}_{\alpha'\beta',\alpha'\beta'}(\mathbf{p}) \Bigg\} + \dots \quad . \qquad (13)
\end{aligned}
$$

The quantity η in Eq. 13 is

$$
\eta_{\alpha'\beta'} = \frac{1}{2}\left[2\tau q_{\alpha'\beta'} + w(q^2)_{\alpha'\beta'} + \frac{2u}{3} q^3_{\alpha'\beta'}\right]. \qquad (14)
$$

The expectation value of the fluctuation must vanish, so we have to impose the condition

$$
\eta_{\alpha\beta} + \frac{w}{2N} \sum_{\mathbf{p}} \sum_{\gamma \neq \alpha,\beta} \widetilde{G}_{\alpha\gamma,\beta\gamma}(\mathbf{p}) + \frac{u}{N} \sum_{\mathbf{p}} \widetilde{G}_{\alpha\beta,\alpha\beta}(\mathbf{p}) q_{\alpha\beta} + \dots = 0 \quad . \qquad (15)
$$

The propagator $\widetilde{G}$, as defined in Eq. 9, depends on the exact value of the order parameter, so (15) is, in principle, a self-consistent equation for $q_{\alpha\beta}$. For a long-ranged interaction ($z \gg 1$), however, the loop corrections are small, as can be seen by changing the scale of the momentum so as to set the UV cutoff at $|\mathbf{p}| = 1$:

$$
\frac{1}{N} \sum_{|\mathbf{p}|<1/\rho a} \dots = \frac{a^d}{(2\pi)^d} \int_{|\mathbf{p}|<1/\rho a} \mathrm{d}^d\mathbf{p} \dots = \frac{1}{z} \frac{1}{(2\pi)^d} \int_{|\mathbf{p}|<1} \mathrm{d}^d\mathbf{p} \dots \quad . \qquad (16)
$$

If we call $G(\mathbf{p})$ the propagator $\widetilde{G}$ with ρa absorbed into the momentum

$$
\begin{aligned}
\left(G^{-1}(\mathbf{p})\right)_{\alpha\beta,\gamma\delta} &= (p^2 - 2\tau - 2uq^2_{\alpha\beta})(\delta^{Kr}_{\alpha\gamma}\delta^{Kr}_{\beta\delta} + \delta^{Kr}_{\alpha\delta}\delta^{Kr}_{\beta\gamma}) - \\
&- w(\delta^{Kr}_{\alpha\gamma} q_{\beta\delta} + \delta^{Kr}_{\alpha\delta} q_{\beta\gamma} + \delta^{Kr}_{\beta\gamma} q_{\alpha\delta} + \delta^{Kr}_{\beta\delta} q_{\alpha\gamma}) \equiv (p^2 \underline{\underline{\mathbf{1}}} + \underline{\underline{\mathbf{M}}}^{(0)})_{\alpha\beta,\gamma\delta} \quad (17)
\end{aligned}
$$

then the equation of state (15) reads as

$$
\begin{aligned}
&2\tau q_{\alpha\beta} + w(q^2)_{\alpha\beta} + \frac{2u}{3} q^3_{\alpha\beta} + \\
&+ \frac{1}{z} \frac{1}{(2\pi)^d} \int_{|\mathbf{p}|<1} \mathrm{d}^d\mathbf{p} \left[w \sum_{\gamma \neq \alpha,\beta} G_{\alpha\gamma,\beta\gamma}(\mathbf{p}) + 2uq_{\alpha\beta} G_{\alpha\beta,\alpha\beta}(\mathbf{p}) \right] + \dots = 0 \qquad (18)
\end{aligned}
$$

where the corrections are of $\mathcal{O}(1/z^2)$.

Other physical quantities work out similarly, so by expanding in $\mathcal{L}^{(I)}$ we generate a series in powers of $1/z$ with G as the bare propagator. Before going into the analysis of the loop corrections, however, we have to solve the problem at zeroth order.

3 Mean field

Mean field theory is recovered in the present framework by letting the range of interaction go to infinity, $z \to \infty$. In this limit all fluctuations vanish, $\Psi_{\mathbf{p}} \equiv 0$, and the free energy of the system is just given by the constant in the Lagrangean:

$$F = -T \lim_{n \to 0} \frac{1}{n} \mathcal{L}^{(0)}(q_{\alpha\beta}) \tag{19}$$

evaluated at the solution of

$$2\tau q_{\alpha\beta} + w(q^2)_{\alpha\beta} + \frac{2u}{3} q^3_{\alpha\beta} = 0 \quad . \tag{20}$$

Because of the replica limit, $n \to 0$, involved in the formalism, one has to solve Eq. 20 in the space of 0×0 matrices which implies that one is able to parametrize the matrix $q_{\alpha\beta}$ in such a way as to permit the limit $n \to 0$ to be taken. The obvious Ansatz, due to Sherrington and Kirkpatrick [2], is to assume that $q_{\alpha\beta}$ does not depend on the replica indices, i.e.

$$q^{SK}_{\alpha\beta} = q(1 - \delta^{Kr}_{\alpha\beta}) \quad .$$

In the limit $n \to 0$ Eq. 20 would then become

$$2\tau q - 2wq^2 + \frac{2u}{3} q^3 = 0 \quad . \tag{20'}$$

However, the assumption of replica symmetry led to paradoxical results at low temperatures, and was subsequently shown to be unstable against replica symmetry breaking fluctuations by de Almeida and Thouless [31].

After some unsuccessful attempts by various groups, the solution which, at least within the context of mean field theory, is now generally accepted was found by Parisi [1]. Parisi's solution is based on a hierarchical replica symmetry breaking pattern which reflects the ultrametric organization of equilibrium states in the long range spin glass [37].

We do not need to go into the details of the Parisi solution here, so we discuss it only to the extent necessary for fixing notation.

We call the sizes of the Parisi blocks p_r, $r = 1, 2, \ldots R$ with R, the number of RSB steps, going to infinity at the end. For the sake of uniformity of notation it is convenient to add $p_0 \equiv n$ and $p_{R+1} \equiv 1$ to the two ends of the series p_r. The value of the order parameter on the r^{th} level of hierarchy will be called q_r, $r = 1, 2, \ldots R$. Upon analytic continuation in n the series p_r becomes monotonically increasing.

A useful concept we shall make frequent use of in the following is that of the overlap between replica indices: $\alpha \cap \beta$ is essentially the inverse of $q_{\alpha\beta}$, that is

$$\alpha \cap \beta = r \quad \text{if the corresponding} \quad q_{\alpha\beta} = q_r \quad . \tag{21}$$

Accordingly, the allowed values for $\alpha \cap \beta$ range from $\alpha \cap \beta = 0$ (corresponding to the outermost region in Parisi's pattern) to $\alpha \cap \beta = R$ (in the innermost block). By extension, we add $\alpha \cap \beta = R + 1$, corresponding to the diagonal, $\alpha = \beta$.

The overlap $\alpha \cap \beta$ is a kind of hierarchical codistance between replicas α and β. The metric generated by the overlaps is ultrametric: whichever way we choose three replicas α, β, γ, either all three of their overlaps are the same $\alpha \cap \beta = \alpha \cap \gamma = \beta \cap \gamma$, or one (say $\alpha \cap \beta$) is larger than the other two, but then these are equal ($\alpha \cap \beta > \alpha \cap \gamma = \beta \cap \gamma$).

It is evident that any quantity f constructed of the q's and depending on two replica indices (like $(q^2)_{\alpha\beta} = \sum_\gamma q_{\alpha\gamma} q_{\gamma\beta}$, e.g.) depends only on their overlap: $f_{\alpha\beta} = f(\alpha \cap \beta)$. Furthermore, any quantity $f_{\alpha\beta\gamma}$ depending on three replicas depends only on the three overlaps, and since of these at most two can be different, $f_{\alpha\beta\gamma}$ is, in fact, a function of only two variables, e.g. $\alpha \cap \beta$ and the larger of the other two:

$$f_{\alpha\beta\gamma} = f(\alpha \cap \beta, \max\{\alpha \cap \gamma, \beta \cap \gamma\}) \quad . \tag{22}$$

In the following we will also have to deal with quantities depending on two pairs of replicas, like the propagator $G_{\alpha\beta,\gamma\delta}$, for example. Ultrametrics implies that of the six possible overlaps between α, β, γ, δ at most three can be different, which corresponds to the elementary geometric fact that a tetrahedron built of equilateral and isosceles faces can only have three different edges. For these 4-replica quantities we proposed the following parametrization

$$G_{\alpha\beta,\gamma\delta} = G^{\alpha\cap\beta,\gamma\cap\delta}_{\max\{\alpha\cap\gamma,\alpha\cap\delta\},\max\{\beta\cap\gamma,\beta\cap\delta\}} \tag{23}$$

in [28]. This parametrization is redundant in that, according to what has just been said, of the four variables displayed in Eq. 23 at least two must coincide, on the other hand it has the merit of being symmetric.

We shall keep to this parametrization throughout most of the paper, in order to ensure consistency with previously published material. In Sec. 7, however, the redundancy would mask an important property, so there we will make use of the following observation: The only case when the two lower indices in Eq. 23 carry independent information is when the two upper indices coincide *and* both lower indices are larger than the common value of the upper ones. In all other cases the smaller of the two lower indices is a dummy variable which we may drop and keep only $\max\{\alpha \cap \gamma, \alpha \cap \delta, \beta \cap \gamma, \beta \cap \delta\}$ as the sole lower index.

A frequently used symbol will be

$$\delta_r = p_r - p_{r+1} \quad . \tag{24}$$

In the limit $R \to \infty$, q_r goes over into a continuous, monotonically increasing function $q(x)$, which turns out to have a breakpoint x_1, beyond which it is constant. Although for large R the precise choice of the block sizes is largely immaterial, it is convenient to arrange the p_r's so that they fill the interval $(0, x_1)$ with δ_r, $r = 0, 1, 2, \ldots, R-1$, becoming infinitesimal, of $\mathcal{O}(1/R)$, and the last one $\delta_R = p_R - p_{R+1} = p_R - 1 \to (x_1 - 1)$ staying finite.

Returning now to the equation of state (20), under the Parisi parametrization it becomes:

$$2\tau q_r + w\left[\sum_{t=0}^{r} \delta_t q_t^2 - p_{r+1} q_r^2 + 2q_r \sum_{t=r+1}^{R} \delta_t q_t\right] + \frac{2u}{3} q_r^3 = 0 \quad , \tag{25}$$

or, for $R \to \infty$:

$$2\tau q(x) - w\left[\int_0^x \mathrm{d}t q^2(t) + xq^2(x) + 2q(x)\int_x^1 \mathrm{d}t q(t)\right] + \frac{2u}{3}q^3(x) = 0 \quad . \tag{26}$$

This is easily solved by repeated differentiation yielding the well-known result [1]

$$q(x) \quad = \quad \begin{cases} \dfrac{w}{2u}x, & x < x_1 \\ \\ \dfrac{w}{2u}x_1 \equiv q_1, & x_1 < x < 1 \end{cases} \quad , \tag{27}$$

with the breakpoint x_1 to be determined from the condition on q_1

$$\tau - wq_1 + uq_1^2 = 0 \quad . \tag{28}$$

To leading order in τ, $q_1 = \tau/w$ which could have been read off from Eq. 26 directly, by noting that for $x_1 \sim q_1 \sim \tau$ all the terms in Eq. 26 are of $\mathcal{O}(\tau^3)$ except the first and the one coming from the upper end of the $\int_x^1$ integral. This type of approximation, consisting in dropping all replica integrals and keeping only the contribution coming from the vicinity of $x = 1$, will be used frequently in the following and will be referred to as the "innermost block approximation".

The solution given in Eq. 27 is valid as long as $\tau > 0$ ($T < T_c$). With $T \to T_c$, $q(x) \to 0$ and one enters the paramagnetic phase.

The physical meaning of the order parameter $q(x)$ depending on a continuous variable had remained a mystery until Parisi showed [38] that the derivative of its inverse

$$\frac{\mathrm{d}x}{\mathrm{d}q} = P(q) \tag{29}$$

is nothing but the probability distribution of the magnetic overlaps q_{ab} between the equilibrium states a and b of the system:

$$P(q) \quad = \quad \overline{< \delta(q - q_{ab}) >} \tag{30}$$

$$q_{ab} \quad = \quad \frac{1}{N}\sum_i < s_i >_a < s_i >_b \quad . \tag{31}$$

(In Eq. 30 $< \ldots >$ is the thermal average and the overbar is the average over the random couplings.)

4 Instability of the replica symmetric solution and mass scales

In order to go one step beyond mean field theory, we turn now to the study of the quadratic part $\mathcal{L}^{(2)}$ of the Lagrangean which describes free (Gaussian) fluctuations of the order parameter. The main task is to diagonalize the quadratic form in Eq. 8, i.e. to invert (17) and obtain the free propagator G. The spectrum of free fluctuations (the singularities of G) will also provide a stability test for the mean field solution found above. Multiplying Eq. 17 by G we get the following set of equations for the propagator components:

$$(p^2-2\tau-2uq^2_{\gamma\delta})G_{\alpha\beta,\gamma\delta}-w\sum_{\mu\neq\gamma,\delta}q_{\mu\delta}G_{\alpha\beta,\gamma\mu}-w\sum_{\mu\neq\gamma,\delta}q_{\mu\gamma}G_{\alpha\beta,\mu\delta}=\delta^{Kr}_{\alpha\gamma}\delta^{Kr}_{\beta\delta}\quad , \qquad \alpha<\beta,\quad \gamma<\delta\quad . \tag{32}$$

For a first orientation, let us work out the solution in the replica symmetric case. Then $G_{\alpha\beta,\gamma\delta}$ will have only three different components

$$\begin{array}{llll} G_{\alpha\beta,\alpha\beta}=G_1 & , & \alpha\neq\beta & \\ G_{\alpha\beta,\alpha\gamma}=G_2 & , & \alpha,\beta,\gamma & \text{all different,} \\ G_{\alpha\beta,\gamma\delta}=G_3 & , & \alpha,\beta,\gamma,\delta & \text{all different,} \end{array} \tag{33}$$

satisfying the simple set of equations

$$\begin{aligned} \left(p^2-2wq-\frac{4u}{3}q^2\right)G_1+4wqG_2 &= 1 \\ \left(p^2-\frac{4u}{3}q^2\right)G_2-wqG_1+3wqG_3 &= 0 \\ \left(p^2+6wq-\frac{4u}{3}q^2\right)G_3-4wqG_2 &= 0 \end{aligned} \tag{34}$$

where we have used Eq. 20' and put $n=0$. The solutions to (34), first written up by Pytte and Rudnick [39],

$$\begin{aligned} G_1 &= \frac{1}{p^2-\frac{4u}{3}q^2}\left(1+\frac{2wq}{p^2+2wq-\frac{4u}{3}q^2}+\frac{4w^2q^2}{(p^2+2wq-\frac{4u}{3}q^2)^2}\right) \\ G_2 &= \frac{1}{p^2-\frac{4u}{3}q^2}\left(\frac{1}{2}\frac{2wq}{p^2+2wq-\frac{4u}{3}q^2}+\frac{4w^2q^2}{(p^2+2wq-\frac{4u}{3}q^2)^2}\right) \\ G_3 &= \frac{1}{p^2-\frac{4u}{3}q^2}\frac{4w^2q^2}{(p^2+2wq-\frac{4u}{3}q^2)^2} \end{aligned} \tag{35}$$

display the AT instability [31] in that the first factors have a pole on the positive p^2 axis. Note that the instability is due entirely to the quartic interaction: if we had set $u=0$ in the Lagrangean (2), we would have found an acceptable pole structure in G, at least on the level of Gaussian approximation. Also note that there are two "mass-scales" displayed by Eq. 35: the unstable mode has a "mass" of order τ (the small mass), while the stable mass is $\mathcal{O}(\tau^{1/2})$ (the "large" mass).

5 The large momentum behaviour

When the assumption of replica symmetry is abandoned the solution of Eq. 32 immediately becomes very difficult. It is, therefore, important to realize that there are instances when we do not need to get seriously involved with the intricacies of the symmetry broken solution.

In sufficiently high dimensions the loop corrections to the various physical quantities depend upon the short distance behaviour of the propagators only, even in the vicinity of T_c, and for such large values of the wavevector (around the upper cutoff, i.e. much larger than either of the characteristic masses) one should be able to get a solution to Eq. 32 by simply expanding G for large p^2, without having to introduce any particular parametrization for the order parameter.

We have to keep in mind, of course, that on passing from the original model (1) to the effective Lagrangean (2) we have dropped all microscopic details, so we can certainly not expect to retrieve the precise short distance behaviour from the large-p expansion of Eq. 32. The qualitative behaviour will, however, be correct and the results will later allow us to illustrate some important points about how the renormalization of the coupling constants takes place in high dimensions, and also about the characteristic dimensions of the model.

In order to implement the program sketched above, we need to decompose (32) so as to separate terms containing propagator components of the first, second, and third kind ($G_{\alpha\beta,\alpha\beta}$, $G_{\alpha\beta,\alpha\gamma}$, and $G_{\alpha\beta,\gamma\delta}$ with $\alpha, \beta, \gamma, \delta$ all different, respectively). Dividing through by $(p^2 - 2\tau - 2uq^2_{\alpha\beta})$ we also wish to write Eq. 32 in the form of a set of Dyson equations. Thus we have:

$$G_{\alpha\beta,\alpha\beta} = G^{(0)}_{\alpha\beta,\alpha\beta} + w \sum_{\omega\neq\alpha,\beta} G^{(0)}_{\alpha\beta,\alpha\beta} q_{\omega\beta} G_{\alpha\beta,\alpha\omega} + w \sum_{\omega\neq\alpha,\beta} G^{(0)}_{\alpha\beta,\alpha\beta} q_{\omega\alpha} G_{\alpha\beta,\omega\beta} \tag{36}$$

$$\begin{aligned} G_{\alpha\beta,\alpha\gamma} = \; & wG^{(0)}_{\alpha\gamma,\alpha\gamma} q_{\beta\gamma} G_{\alpha\beta,\alpha\beta} + w \sum_{\omega\neq\alpha,\beta,\gamma} G^{(0)}_{\alpha\gamma,\alpha\gamma} q_{\omega\gamma} G_{\alpha\beta,\alpha\omega} + \\ & + w \sum_{\omega\neq\alpha,\beta,\gamma} G^{(0)}_{\alpha\gamma,\alpha\gamma} q_{\omega\alpha} G_{\alpha\beta,\omega\gamma} + wG^{(0)}_{\alpha\gamma,\alpha\gamma} q_{\alpha\beta} G_{\alpha\beta,\beta\gamma} \end{aligned} \tag{37}$$

$$\begin{aligned} G_{\alpha\beta,\gamma\delta} = \; & wG^{(0)}_{\gamma\delta,\gamma\delta} \left(q_{\alpha\delta} G_{\alpha\beta,\alpha\gamma} + q_{\beta\delta} G_{\alpha\beta,\beta\gamma} + q_{\alpha\gamma} G_{\alpha\beta,\alpha\delta} + q_{\beta\gamma} G_{\alpha\beta,\beta\delta} \right) + \\ & + w \sum_{\omega\neq\alpha,\beta,\gamma,\delta} G^{(0)}_{\gamma\delta,\gamma\delta} q_{\omega\delta} G_{\alpha\beta,\gamma\omega} + w \sum_{\omega\neq\alpha,\beta,\gamma,\delta} G^{(0)}_{\gamma\delta,\gamma\delta} q_{\omega\gamma} G_{\alpha\beta,\omega\delta} \end{aligned} \tag{38}$$

where

$$G^{(0)}_{\alpha\beta,\alpha\beta} = \frac{1}{p^2 - 2\tau - 2uq^2_{\alpha\beta}} \quad . \tag{39}$$

These equations are solved iteratively by noticing that for "large p^2" (i.e. for $p^2 \gg \tau, q$) the only term that survives is $G^{(0)}$ in Eq. 36, so

$$G_{\alpha\beta,\alpha\beta} = \frac{1}{p^2} + \cdots . \tag{40}$$

This is now substituted into Eq. 37 to yield

$$G_{\alpha\beta,\alpha\gamma} = \frac{wq_{\beta\gamma}}{p^4} + \cdots , \tag{41}$$

and so forth. The results one obtains after the first few iterations are the following:

$$G_{\alpha\beta,\alpha\beta} = \frac{1}{p^2} + \frac{2\tau + 2uq^2_{\alpha\beta}}{p^4} + \frac{1}{p^6} \left\{ (2\tau + 2uq^2_{\alpha\beta})^2 + w^2 \left((q^2)_{\alpha\alpha} + (q^2)_{\beta\beta} - 2q^2_{\alpha\beta} \right) \right\} +$$

$$
\begin{aligned}
&+\frac{1}{p^8}\Big\{(2\tau+2uq_{\alpha\beta}^2)^3+6w^2\tau\left((q^2)_{\alpha\alpha}+(q^2)_{\beta\beta}-2q_{\alpha\beta}^2\right)+\\
&+4uw^2\sum_\omega q_{\alpha\omega}^2q_{\omega\beta}^2++4uw^2q_{\alpha\beta}^2\left((q^2)_{\alpha\alpha}+(q^2)_{\beta\beta}-2q_{\alpha\beta}^2\right)+\\
&+w^3\left((q^3)_{\alpha\alpha}+(q^3)_{\beta\beta}-2q_{\alpha\beta}(q^2)_{\alpha\beta}\right)\Big\}+\cdots \qquad (42)
\end{aligned}
$$

$$
\begin{aligned}
G_{\alpha\gamma,\beta\gamma} &= \frac{wq_{\alpha\beta}}{p^4}+\frac{1}{p^6}\left\{4w\tau q_{\alpha\beta}+w^2(q^2)_{\alpha\beta}+2uwq_{\alpha\beta}(q_{\alpha\gamma}^2+q_{\beta\gamma}^2)\right\}+\\
&+\frac{1}{p^8}\Big\{12w\tau^2q_{\alpha\beta}+6\tau w^2(q^2)_{\alpha\beta}+w^3\Big(3q_{\alpha\beta}(q^2)_{\gamma\gamma}-4q_{\alpha\beta}(q_{\alpha\gamma}^2+q_{\beta\gamma}^2)+\\
&+(q^3)_{\alpha\beta}+q_{\alpha\gamma}(q^2)_{\beta\gamma}+q_{\beta\gamma}(q^2)_{\alpha\gamma}\Big)+12uw\tau q_{\alpha\beta}(q_{\alpha\gamma}^2+q_{\beta\gamma}^2)+\\
&+2uw^2\left((q^2)_{\alpha\beta}(q_{\alpha\gamma}^2+q_{\beta\gamma}^2)+q_{\alpha\beta}^2q_{\alpha\gamma}q_{\beta\gamma}+\sum_\omega q_{\gamma\omega}^2q_{\alpha\omega}q_{\beta\omega}\right)+\\
&+4u^2wq_{\alpha\beta}(q_{\alpha\gamma}^4+q_{\beta\gamma}^4+q_{\alpha\gamma}^2q_{\beta\gamma}^2)\Big\}+\cdots \qquad (43)\\
G_{\alpha\beta,\gamma\delta} &= \frac{2w^2}{p^6}(q_{\alpha\delta}q_{\beta\gamma}+q_{\alpha\gamma}q_{\beta\delta})+\cdots \quad . \qquad (44)
\end{aligned}
$$

We will use these expressions in Sec. 10 to calculate short range corrections to the equation of state and to the propagators in high ($d>8$) dimensions.

6 The near infrared region

Now we wish to probe deeper into the structure of the propagator and investigate its behaviour in the long, but not extremely long wavelength region. More precisely, what we mean is that the momentum may be comparable to the large mass but it remains much larger than the small mass scale:

$$
p^2 \gg \tau^2 \qquad (45)
$$

We call this region the near infrared region.

In order to solve the problem for momenta that are comparable to the large mass ($p^2\sim\tau$) we have to construct the equations for the various components of the propagator explicitly, by writing out Eq. 32 under the parametrization (23) and letting R go to infinity at the end. The set of integral equations obtained this way was first published in [28]; we display them here for the sake of completeness.

$$
\begin{aligned}
&\left(p^2-2\tau-2uq^2(x)\right)G_{1,1}^{x,x}+2w\int_0^x \mathrm{d}t q(t)G_{1,x}^{x,t}+\\
&\quad+2wxq(x)G_{1,x}^{x,x}+2w\int_x^1 \mathrm{d}t q(t)G_{1,t}^{x,x}+2wq(x)\int_x^1 \mathrm{d}t G_{1,x}^{x,t}=1 \quad , \qquad (46)
\end{aligned}
$$

$$
\begin{aligned}
&\left(p^2+uq^2(z)+uq_1^2-2uq^2(x)\right)G_{z,1}^{x,x}+w\int_0^x \mathrm{d}t q(t)G_{z,x}^{x,t}+\\
&\quad+w\int_x^1 \mathrm{d}t q(t)G_{z,t}^{x,x}-wq_1G_{z,1}^{x,x}+w\int_x^z \mathrm{d}t q(t)G_{1,t}^{x,x}+
\end{aligned}
$$

$$+wq(x)\left\{xG^{x,x}_{z,x}+\int_x^1 \mathrm{d}tG^{x,t}_{z,x}+zG^{x,z}_{z,x}+\int_z^1 \mathrm{d}tG^{x,z}_{t,x}-G^{x,z}_{1,x}\right\}+w\int_0^x \mathrm{d}tq(t)G^{x,t}_{1,x}+$$
$$+wq(z)\left\{\int_z^1 \mathrm{d}tG^{x,x}_{1,t}-G^{x,x}_{1,1}\right\}+wq(x)\left\{xG^{x,x}_{1,x}+\int_x^1 \mathrm{d}tG^{x,t}_{1,x}\right\}=0\ ,\ x\leq z<1, \quad (47)$$

$$\left(p^2+uq_1^2-uq^2(y)\right)G^{x,y}_{1,x}+w\int_0^y \mathrm{d}tq(t)\left(G^{x,t}_{1,x}+G^{x,t}_{y,x}\right)+w\int_y^1 \mathrm{d}tq(t)G^{x,y}_{t,x}+$$
$$+wq(y)\left\{yG^{x,y}_{y,x}+\int_y^1 \mathrm{d}tG^{x,t}_{y,x}+\int_y^1 \mathrm{d}tG^{x,t}_{1,x}\right\}+$$
$$+wq(x)\left\{xG^{x,x}_{y,x}+\int_x^1 \mathrm{d}tG^{x,x}_{y,t}+xG^{x,x}_{1,x}+\int_x^1 \mathrm{d}tG^{x,x}_{1,t}-G^{x,x}_{y,1}-G^{x,x}_{1,1}\right\}-$$
$$-wq_1G^{x,y}_{1,x}=0\quad,\qquad x\leq y<1\quad, \qquad (48)$$

$$\left(p^2+uq_1^2-uq^2(y)\right)G^{x,y}_{1,x}+w\int_0^y \mathrm{d}tq(t)\left(G^{x,t}_{1,x}+G^{x,t}_{y,y}\right)+w\int_y^x \mathrm{d}tq(t)G^{x,y}_{t,t}+$$
$$+w\int_x^1 \mathrm{d}tq(t)G^{x,y}_{t,x}+wq(y)\left\{\int_y^1 G^{x,t}_{y,y}+yG^{x,y}_{y,y}+\int_y^1 \mathrm{d}tG^{x,t}_{1,x}+\right.$$
$$\left.+xG^{x,x}_{1,x}+\int_x^1 \mathrm{d}tG^{x,x}_{1,t}-G^{x,x}_{1,1}\right\}+$$
$$+wq(x)\left\{xG^{x,y}_{x,x}+\int_x^1 \mathrm{d}tG^{x,y}_{x,t}-G^{x,y}_{x,1}\right\}-wq_1G^{x,y}_{1,x}=0\quad,\quad y\leq x<1\quad, \qquad (49)$$

$$\frac{1}{2}p^2G^{x,y}_{z,z}+w\int_0^y \mathrm{d}tq(t)G^{x,t}_{z,z}+wq(y)\int_y^1 \mathrm{d}tG^{x,t}_{z,z}+wq(z)\left\{zG^{x,z}_{z,z}+\int_z^x \mathrm{d}tG^{x,z}_{t,t}+\right.$$
$$\left.+xG^{x,z}_{x,x}+2\int_x^1 \mathrm{d}tG^{x,z}_{t,x}-2G^{x,z}_{1,x}\right\}=0\quad,\quad z\leq x,y<1\quad, \qquad (50)$$

$$\frac{1}{2}p^2G^{x,y}_{z,x}+w\int_0^y \mathrm{d}tq(t)G^{x,t}_{z,x}+wq(y)\int_y^1 \mathrm{d}tG^{x,t}_{z,x}+wq(z)\left\{zG^{x,z}_{z,x}+\int_z^1 \mathrm{d}tG^{x,z}_{t,x}-G^{x,z}_{1,x}\right\}+$$
$$+wq(x)\left\{xG^{x,x}_{z,x}+\int_x^1 \mathrm{d}tG^{x,x}_{z,t}-G^{x,x}_{z,1}\right\}=0\quad,\qquad x\leq z\leq y<1\quad, \qquad (51)$$

$$\left(p^2+uq^2(z)-uq^2(y)\right)G^{x,y}_{z,z}+w\int_0^y \mathrm{d}tq(t)\left(G^{x,t}_{y,y}+G^{x,t}_{z,z}\right)+w\int_y^z \mathrm{d}tq(t)G^{x,y}_{t,t}+$$
$$+wq(y)\left\{\int_y^1 \mathrm{d}tG^{x,t}_{y,y}+yG^{x,y}_{y,y}+\int_y^1 \mathrm{d}tG^{x,t}_{z,z}+zG^{x,z}_{z,z}+\right.$$
$$\left.+\int_z^x \mathrm{d}tG^{x,z}_{t,t}+xG^{x,z}_{x,x}+2\int_x^1 \mathrm{d}tG^{x,z}_{t,x}-2G^{x,z}_{1,x}\right\}+$$
$$+wq(z)\left\{xG^{x,y}_{x,x}+\int_z^x \mathrm{d}tG^{x,y}_{t,t}+2\int_x^1 \mathrm{d}tG^{x,y}_{t,x}-2G^{x,y}_{1,x}\right\}=0\ ,\ y\leq z\leq x<1\ , \quad (52)$$

$$\left(p^2+uq^2(z)-uq^2(y)\right)G^{x,y}_{z,x}+w\int_0^y \mathrm{d}tq(t)\left(G^{x,t}_{z,x}+G^{x,t}_{y,x}\right)+w\int_y^z \mathrm{d}tq(t)G^{x,y}_{t,x}+$$

$$+wq(y)\left\{yG^{x,y}_{y,x}+\int_y^1 \mathrm{d}tG^{x,t}_{y,x}+\int_z^1 \mathrm{d}t\left(G^{x,t}_{z,x}+G^{x,z}_{t,x}\right)+zG^{x,z}_{z,x}+\right.$$
$$\left.+\int_y^z \mathrm{d}tG^{x,t}_{z,x}-G^{x,z}_{1,x}\right\}+wq(x)\left\{xG^{x,x}_{y,x}+\int_x^1 \mathrm{d}tG^{x,x}_{y,t}+xG^{x,x}_{z,x}+\int_x^1 \mathrm{d}tG^{x,x}_{z,t}-\right.$$
$$\left.-G^{x,x}_{y,1}-G^{x,x}_{z,1}\right\}+wq(z)\left\{\int_z^1 \mathrm{d}tG^{x,y}_{t,x}-G^{x,y}_{1,x}\right\}=0 \quad , \qquad x\leq y\leq z<1 \quad , \tag{53}$$

$$\left(p^2+uq^2(z)-uq^2(y)\right)G^{x,y}_{z,x}+w\int_0^y \mathrm{d}tq(t)\left(G^{x,t}_{y,y}+G^{x,t}_{z,x}\right)+w\int_y^x \mathrm{d}tq(t)G^{x,y}_{t,t}+$$
$$+w\int_x^z \mathrm{d}tq(t)G^{x,y}_{t,x}+wq(y)\left\{\int_y^1 \mathrm{d}tG^{x,t}_{y,y}+yG^{x,y}_{y,y}+\int_y^1 \mathrm{d}tG^{x,t}_{z,x}+zG^{x,z}_{z,x}+\right.$$
$$\left.+\int_z^1 \mathrm{d}tG^{x,z}_{t,x}+xG^{x,x}_{z,x}+\int_x^1 \mathrm{d}tG^{x,x}_{z,t}-G^{x,z}_{1,x}-G^{x,x}_{z,1}\right\}+$$
$$+wq(x)\left\{xG^{x,y}_{x,x}+\int_x^1 \mathrm{d}tG^{x,y}_{x,t}-G^{x,y}_{x,1}\right\}+$$
$$+wq(z)\left\{\int_z^1 \mathrm{d}tG^{x,y}_{t,x}-G^{x,y}_{1,x}\right\}=0 \quad , \qquad y\leq x\leq z<1 \quad , \tag{54}$$

$$\left(p^2+uq^2(z_1)+uq^2(z_2)-2uq^2(x)\right)G^{x,x}_{z_1,z_2}+$$
$$+A(z_1,z_2)+A(z_2,z_1)=0 \quad , \quad x\leq z_1,z_2<1 \quad , \tag{55}$$

where

$$A(z_1,z_2)=w\int_0^x \mathrm{d}tq(t)G^{x,t}_{z_1,x}+w\int_x^{z_2} \mathrm{d}tq(t)G^{x,x}_{z_1,t}+$$
$$+wq(z_2)\left\{\int_{z_2}^1 \mathrm{d}tG^{x,x}_{z_1,t}-G^{x,x}_{z_1,1}\right\}+wq(x)\left\{xG^{x,x}_{z_1,x}+\int_x^1 G^{x,t}_{z_1,x}+z_1G^{x,z_1}_{z_1,x}+\right.$$
$$\left.+\int_{z_1}^1 \mathrm{d}tG^{x,z_1}_{t,x}-G^{x,z_1}_{1,x}\right\}, \quad x\leq z_1,z_2<1 \quad . \tag{56}$$

In Eqs. 46−56, the mean field equations (27), (28) have been repeatedly used.

Note that Eqs. 47, 48 and 49 can formally be obtained from Eqs. 55, 53 and 54, respectively, (putting $z_1=z$, $z_2=1$ in Eq. 55, and $z=1$ in Eq. 53 and 54) which is why we did not write them up explicitly in [28]. The reason for which we display them here is twofold. Firstly, it is not immediately obvious why Eq. 55 should contain Eq. 47, etc., since, although $G^{x,y}_{z_1,z_2}$ is, in general, a continuous function of its variables, this continuity does not apply when either of the lower variables takes the value 1. (Whenever this happens we have coinciding replica indices, and there is no reason to expect e.g. $G_{\alpha\beta,\gamma\delta}$, with $\alpha,\beta,\gamma,\delta$ all different, to be the same as $G_{\alpha\beta,\alpha\delta}$. In fact, they are different, and there is indeed a jump in $G^{x,y}_{z_1,z_2}$ at $z_1=1$.)

Nevertheless, the equation one derives for $G^{x,y}_{z_1,z_2}$ does go over into that for $G^{x,y}_{z,1}$ when $z_1=z$, $z_2=1$.

The other reason for us to display Eqs. 47, 48 and 49 here explicitly is that the "innermost block approximation" we are just about to apply to Eqs. 46−56 does not commute with the limit $z\to 1$.

Let us now assume that (45) holds. We can than neglect all the $\mathcal{O}(\tau^2)$ terms in (46)-(56) which means dropping all the uq^2 terms, but also the terms in xqG or $q\int G$, except for the contributions coming from the upper end of the replica integrals (the innermost block approximation). For example, the penultimate term in Eq. 46 becomes

$$2w\int_x^1 \mathrm{d}t q(t) G^{x,x}_{1,t} \to 2wq_1 G^{x,x}_{1,x_1}$$

where we have used the fact that (apart from the jump exactly at $z_1, z_2 = 1$ in the lower indices) whenever an overlap variable goes beyond the breakpoint x_1 it gets stuck to it.

The simplification gained by dropping the $\mathcal{O}(\tau^2)$ terms is tremendous: the set of integral equations (46)-(55) reduces to the following set of linear algebraic equations:

$$(p^2 - 2wq_1)G^{x,x}_{1,1} + 2wq_1 G^{x,x}_{1,x_1} + 2wq(x)G^{x,x_1}_{1,x} = 1 \quad , \tag{46'}$$

$$\begin{aligned} p^2 G^{x,x}_{z,1} &+ wq_1\left(G^{x,x}_{z,x_1} - G^{x,x}_{z,1}\right) + wq(x)\left(G^{x,x_1}_{z,x} + G^{x,x_1}_{1,x} + G^{x,z}_{x_1,x} - G^{x,z}_{1,x}\right) + \\ &+ wq(z)\left(G^{x,x}_{1,x_1} - G^{x,x}_{1,1}\right) = 0, \quad 1 > z \geq x \quad , \end{aligned} \tag{47'}$$

$$\begin{aligned} p^2 G^{x,y}_{1,x} &+ wq_1\left(G^{x,y}_{x_1,x} - G^{x,y}_{1,x}\right) + wq(y)\left(G^{x,x_1}_{y,x} + G^{x,x_1}_{1,x}\right) + wq(x)\left(G^{x,x}_{y,x_1} - G^{x,x}_{y,1} + \right. \\ &\left. + G^{x,x}_{1,x_1} - G^{x,x}_{1,1}\right) = 0, \quad x \leq y < 1 \quad , \end{aligned} \tag{48'}$$

$$\begin{aligned} p^2 G^{x,y}_{1,x} &+ wq_1\left(G^{x,y}_{x_1,x} - G^{x,y}_{1,x}\right) + wq(x)\left(G^{x,y}_{x,x_1} - G^{x,y}_{x,1}\right) + wq(y)\left(G^{x,x_1}_{y,y} + G^{x,x_1}_{1,x} + \right. \\ &\left. + G^{x,x}_{1,x_1} - G^{x,x}_{1,1}\right) = 0, \quad y \leq x < 1 \quad , \end{aligned} \tag{49'}$$

$$p^2 G^{x,y}_{z,z} + 2wq(y)G^{x,x_1}_{z,z} + 4wq(z)\left(G^{x,z}_{x_1,x} - G^{x,z}_{1,x}\right) = 0 \quad z \leq x, y < 1 \quad , \tag{50'}$$

$$\begin{aligned} p^2 G^{x,y}_{z,x} &+ 2wq(y)G^{x,x_1}_{z,x} + 2wq(z)\left(G^{x,z}_{x_1,x} - G^{x,z}_{1,x}\right) + \\ &+ 2wq(x)\left(G^{x,x}_{z,x_1} - G^{x,x}_{z,1}\right) = 0, \quad x \leq z \leq y < 1 \quad , \end{aligned} \tag{51'}$$

$$\begin{aligned} p^2 G^{x,y}_{z,z} &+ wq(y)\left(G^{x,x_1}_{y,y} + G^{x,x_1}_{z,z} + 2G^{x,z}_{x_1,x} - 2G^{x,z}_{1,x}\right) + \\ &+ 2wq(z)\left(G^{x,y}_{x_1,x} - G^{x,y}_{1,x}\right) = 0, \quad y \leq z \leq x < 1 \quad , \end{aligned} \tag{52'}$$

$$\begin{aligned} p^2 G^{x,y}_{z,x} &+ wq(y)\left(G^{x,x_1}_{y,x} + G^{x,x_1}_{z,x} + G^{x,z}_{x_1,x} - G^{x,z}_{1,x}\right) + wq(x)\left(G^{x,x}_{y,x_1} + G^{x,x}_{z,x_1} - \right. \\ &\left. - G^{x,x}_{y,1} - G^{x,x}_{z,1}\right) + wq(z)\left(G^{x,y}_{x_1,x} - G^{x,y}_{1,x}\right) = 0, \quad x \leq y \leq z < 1 \quad , \end{aligned} \tag{53'}$$

$$\begin{aligned} p^2 G^{x,y}_{z,x} &+ wq(y)\left(G^{x,x_1}_{y,y} + G^{x,x_1}_{z,x} + G^{x,z}_{x_1,x} - G^{x,z}_{1,x} + G^{x,x}_{z,x_1} - G^{x,x}_{z,1}\right) + \\ &+ w\left(q(x) + q(z)\right)\left(G^{x,y}_{x,x_1} - G^{x,y}_{x,1}\right) = 0, \quad y \leq x \leq z < 1 \quad , \end{aligned} \tag{54'}$$

$$\begin{aligned} p^2 G^{x,x}_{z_1,z_2} &+ wq(z_1)\left(G^{x,x}_{z_2,x_1} - G^{x,x}_{z_2,1}\right) + wq(z_2)\left(G^{x,x}_{z_1,x_1} - G^{x,x}_{z_1,1}\right) + \\ &+ wq(x)\left(G^{x,x_1}_{z_1,x} + G^{x,z_1}_{x_1,x} - G^{x,z_1}_{1,x} + G^{x,x_1}_{z_2,x} + \right. \\ &\left. + G^{x,z_2}_{x_1,x} - G^{x,z_2}_{1,x}\right) = 0, \quad x \leq z_1, z_2 < 1 \quad . \end{aligned} \tag{55'}$$

In Eqs. 46'−55' we have set $\tau = wq_1$, which is the correct stationary condition at this order.

It is evident from Eqs. 46'−55' that in the given approximation each component of G is $\frac{1}{p^2}$ times a function of $\frac{wq_1}{p^2}$, $\frac{wq(x)}{p^2}$, $\frac{wq(y)}{p^2}$ and $\frac{wq(z)}{p^2}$:

$$G = \frac{1}{p^2} g(\frac{wq}{p^2}) \quad , \quad p^2 \gg \tau^2 \tag{57}$$

where q stands for q_1, $q(x)$, $q(y)$, $q(z)$, according to the component in question.

The solution of Eqs. 46'−55' is particularly simple if all overlap variables are greater than x_1. The propagators in this limit will be seen to describe fluctuations in a single phase space valley. Then the set (46')− (55') collapses to 3 simple equations with the solutions:

$$\begin{aligned} G_{1,1}^{x_1,x_1} &= \frac{1}{p^2}\left(1 + \frac{2wq_1}{p^2+2wq_1} + \frac{4w^2q_1^2}{(p^2+2wq_1)^2}\right) \\ G_{1,x_1}^{x_1,x_1} &= \frac{1}{p^2}\left(\frac{1}{2}\frac{2wq_1}{p^2+2wq_1} + \frac{4w^2q_1^2}{(p^2+2wq_1)^2}\right) \\ G_{x_1,x_1}^{x_1,x_1} &= \frac{1}{p^2}\frac{4w^2q_1^2}{(p^2+2wq_1)^2} \quad . \end{aligned} \tag{58}$$

Comparing these with the replica symmetric propagators given in Eq. 35, we see that to the order regarded here (i.e. neglecting τ^2 terms), they are the same. The difference is, of course, that Eq. 58 is supposed to be valid only for $p^2 \gg \tau^2$, so for the time being we keep away from the region where the instability showed up.

Two particularly illuminating combinations are the transverse and the longitudinal propagators

$$\begin{aligned} G_\perp &= G_{1,1}^{x_1,x_1} - 2G_{1,x_1}^{x_1,x_1} + G_{x_1,x_1}^{x_1,x_1} = \frac{1}{p^2} \quad , \\ G_{\|} &= G_{1,1}^{x_1,x_1} - 4G_{1,x_1}^{x_1,x_1} + 3G_{x_1,x_1}^{x_1,x_1} = \frac{1}{p^2+2wq_1} \quad , \end{aligned} \tag{59}$$

which are precisely of the form of the Gaussian propagators in a massless phase.

Now we display the complete set of propagators valid for generic overlaps in the near infrared limit $p^2 \gg \tau^2$:

$$\begin{aligned} G_{1,1}^{x,x} &= \frac{1}{p^2}\left\{1 + \frac{2wq_1}{p^2} + \frac{2w^2q_1^2}{p^4} - \frac{2w^2q^2(x)}{p^4}\frac{p^4+8wq_1p^2+8w^2q_1^2}{(p^2+2wq_1)^2} + \right. \\ & \left. + \frac{8w^4q^4(x)}{p^4(p^2+2wq_1)^2}\right\} \end{aligned} \tag{60}$$

$$G_{1,z}^{x,x} = \frac{1}{p^2}\left\{\left(1+\frac{2wq_1}{p^2}\right)\frac{wq(z)}{p^2} - 4\frac{wq_1p^2+w^2q_1^2}{(p^2+2wq_1)^2}\frac{w^2q^2(x)}{p^4} - \frac{8w^3}{p^4}\frac{q^2(x)q(z)}{p^2+2wq_1} + \right.$$

$$+\frac{4w^4}{p^4}\frac{2q^4(x)+q^2(x)q^2(z)}{(p^2+2wq_1)^2}\Bigg\}, \quad x\le z<1 \tag{61}$$

$$G^{x,y}_{1,x} = G^{y,x}_{1,y}=\frac{wq(x)}{p^4}\Bigg\{1+\frac{2wq_1}{p^2}-4\frac{wq_1p^2+w^2q_1^2}{(p^2+2wq_1)^2}\frac{wq(y)}{p^2}-\frac{4w^2}{p^2}\frac{q^2(x)+q^2(y)}{p^2+2wq_1}+$$
$$+\frac{4w^3}{p^2}\frac{q^3(y)+2q(y)q^2(x)}{(p^2+2wq_1)^2}\Bigg\}, \quad x\le y<1 \tag{62}$$

$$G^{x,y}_{1,x} = \frac{wq(y)}{p^4}\Bigg\{1+\frac{2wq_1}{p^2}-4\frac{wq_1p^2+w^2q_1^2}{(p^2+2wq_1)^2}\frac{wq(x)}{p^2}-\frac{4w^2}{p^2}\frac{q^2(y)+q^2(x)}{p^2+2wq_1}+$$
$$+\frac{4w^3}{p^2}\frac{q^3(x)+2q(x)q^2(y)}{(p^2+2wq_1)^2}\Bigg\}, \quad y\le x<1 \tag{63}$$

$$G^{x,y}_{z,z} = 4\frac{w^2q^2(z)}{p^6}\frac{(p^2+2wq_1-2wq(x))(p^2+2wq_1-2wq(y))}{(p^2+2wq_1)^2}, \quad z\le x,y<1 \tag{64}$$

$$G^{x,y}_{z,x} = \frac{4wq(x)}{p^6}(p^2+2wq_1-2wq(y))\left(\frac{wq(z)}{p^2+2wq_1}-\frac{w^2q^2(x)+w^2q^2(z)}{(p^2+2wq_1)^2}\right), \quad x\le z\le y<1 \tag{65}$$

$$G^{x,y}_{z,x} = \frac{2wq(x)}{p^6}\Bigg[wq(y)+wq(z)-2w^2\frac{q^2(x)+2q(y)q(z)+q^2(y)}{p^2+2wq_1}+$$
$$+2w^3\frac{q^3(y)+2q^2(x)q(y)+q(y)q^2(z)}{(p^2+2wq_1)^2}\Bigg], \quad x\le y\le z<1 \tag{66}$$

$$G^{x,y}_{z,x} = \frac{2wq(y)}{p^6}\Bigg[wq(x)+wq(z)-2w^2\frac{q^2(y)+2q(x)q(z)+q^2(x)}{p^2+2wq_1}+$$
$$+2w^3\frac{q^3(x)+2q^2(y)q(x)+q(x)q^2(z)}{(p^2+2wq_1)^2}\Bigg], \quad y\le x\le z<1 \tag{67}$$

$$G^{x,y}_{z,z} = G^{y,x}_{z,y}=\frac{4wq(y)}{p^6}(p^2+2wq_1-2wq(x))\left(\frac{wq(z)}{p^2+2wq_1}-\right.$$
$$\left.-\frac{w^2q^2(y)+w^2q^2(z)}{(p^2+2wq_1)^2}\right), \quad y\le z\le x<1 \tag{68}$$

$$G^{x,x}_{z_1,z_2} = \frac{2}{p^6}\Bigg[w^2q^2(x)+w^2q(z_1)q(z_2)-\frac{4w^2q^2(x)(q(z_1)+q(z_2))}{p^2+2wq_1}+$$
$$+\frac{2w^4q^2(x)(q^2(z_1)+2q^2(x)+q^2(z_2))}{(p^2+2wq_1)^2}\Bigg], \quad x\le z_1,z_2<1 \quad . \tag{69}$$

These formulae can be verified by a direct substitution into Eqs. 46'−55'.

Note the simple excitation spectrum displayed by Eqs. 58−69: the propagator components have two poles, one at $p^2 = -2wq_1$, the other at $p^2 = 0$. When we work out the complete expressions for the propagators below, we shall see that these singularities form, in fact, two continuous bands which, having a bandwidth of the order of the small mass, cannot, however, be resolved in the near infrared. In addition to these cuts, we shall also find a pole at $p^2 = 0$.

The above formulae, derived under the only assumption of $p^2 \gg \tau^2$ should also cover the region near the cutoff. Indeed, the large p expansion of Eqs. 60−69 reproduces Eqs. 42, 43 and 44 provided the coefficients of $1/p^2$, $1/p^4$, etc. there are evaluated in the Parisi parametrization to leading order in τ. Eqs. 60−69 will be used when calculating loop corrections for $6 < d < 8$.

7 The complete solution for the propagators

The expressions derived in the previous Section are valid in the range where the momentum is much larger than the smaller mass scale involved in the problem. If we tried to continue these formulae into the exceedingly long wavelength limit, i.e. to momenta around the small mass scale, $p^2 \sim \tau^2$, or even much smaller, $p^2 \ll \tau^2$, we would encounter truly unmanageable infrared singularities, with the propagator components blowing up like p^{-6}. In order to uncover the true behaviour in this extremely long wavelength limit, which we will call the far infrared region, we have to return to the original set of Eqs. 46−56 for the propagators. Unfortunately, we have not been able to devise any approximation scheme (analogous to the "large p expansion" or the "innermost block approximation" working near the upper cutoff and in the near infrared, respectively) that would directly give the propagators for the far infrared. Therefore we have to face the task of solving the complete set of Eqs. 46−56.

This task may, at first sight, appear utterly hopeless. Nevertheless, it turned out to be possible to find a solution to the set (46)−(56) in closed form (even in a more general setting, with an external magnetic field acting on the system) which was published by two of us in [28], with a minor error corrected in [40].

In retrospect, it is clear that the solvability of the problem depends upon the combination of two independent factors. One of them is the residual symmetry which allows one to reduce the inversion of not only the Hessian, but also of any ultrametric matrix to the inversion of a much simpler object we called the kernel in [40,41]. The other is that close to the transition point this kernel turns out to be very simple. These two factors deserve a separate discussion in their own right which is what we present in the two subsections below.

7.1 The inversion problem of a generic ultrametrix matrix

For the sake of the discussion in this subsection we shall assume that both the replica number $n \equiv p_0$ and the block sizes $p_1, p_2, \ldots, p_R$, featuring in the Parisi construction are positive integers such that p_{i+1} is a divisor of p_i, $i = 0, 1, 2, \ldots, R-1$. This will allow us to stay within the limits of well established mathematics throughout this subsection. When coming to the application of the results in the next subsection, we shall have to consider, of course, the replica limit $n \to 0$, along with the p_i's to be continued into the interval $[0,1]$ and $R \to \infty$.

The free energy F in the MF approximation is a functional of the order parameters $q_{\alpha\beta}$, with $q_{\alpha\beta} = q_{\beta\alpha}$ and $q_{\alpha\alpha} = 0$, $\alpha, \beta = 1, 2, \ldots, n$. Since none of the n replicas is distinguished, F must depend only on permutation invariant combinations of the $q_{\alpha\beta}$'s, such as those in Eq. 6, for example. This means that F is invariant under the action of the elements Π of the permutation group S_n: $F = F(q_{\alpha\beta}) = F(q_{\Pi(\alpha),\Pi(\beta)})$.

The physical value of the free energy is obtained from the functional F by evaluating it at a saddle point where $\dfrac{\partial F}{\partial q_{\alpha,\beta}} = 0$. The solution of the saddle point equations is a point in the $\frac{1}{2}n(n-1)$ dimensional space spanned by the independent components of $q_{\alpha\beta}$, thus the order parameter $q_{\alpha\beta}$, which is usually spoken of as a matrix, is, in fact, a *vector*. We do not want to introduce a separate notation for this vector, but propose to think of it as a column vector, whose components are the elements (above the diagonal) of $q_{\alpha\beta}$ listed in some fixed order.

The symmetry of F under permutations does not imply that all the solutions of the saddle point equations must respect this symmetry. In fact, we know from de Almeida and Thouless [31] that at low temperatures the replica symmetric saddle point becomes unstable against fluctuations that break permutation symmetry. Symmetry breaking gives rise, like at any other phase transition, to the reduction of the original symmetry group to one of its subgroups. Parisi's Ansatz [1] is, in fact, nothing but a concrete choice for this residual symmetry, one that, as it turned out later [37], embodies the physical assumption of the existence of many equilibrium states with an ultrametric organization. For this reason we proposed the name ultrametric group for this particular residual symmetry in [41].

The vector pointing to the Parisi saddle point is invariant under the action of the ultrametric group. The set of $\frac{1}{2}n(n-1) \times \frac{1}{2}n(n-1)$ matrices that, acting on the Parisi $q_{\alpha\beta}$, perform the permutations belonging to the ultrametric group (and thus do not change the form of $q_{\alpha\beta}$) constitute an (in general, reducible) representation of the group.

Now consider the $\frac{1}{2}n(n-1) \times \frac{1}{2}n(n-1)$ dimensional matrices that commute with all the matrices belonging to the previous set, i.e. with the matrices representing the elements of the ultrametric group. These are the matrices that we called ultrametric matrices in [41] where we also gave a detailed description of their structure. If we transform our coordinate system in replica space such as to decompose the representation of the group into its irreducible components, we find that, by virtue of the Wigner- Eckart theorem, in this new representation all ultrametric matrices will be block diagonal.

The Hessian or stability matrix (essentially the inverse propagator) is an important example of an ultrametric matrix. Constructing the irreducible representations of the residual symmetry group one can therefore make significant progress towards the diagonalization and inversion of the Hessian or of any other ultrametric matrix.

The approach indicated above is *the* standard one. An alternative that may involve seemingly ad hoc steps but leads to perfectly identical results is to directly construct the orthogonal subspaces which are closed under the action of any ultrametric matrix and obtain the block diagonal form by transforming the Hessian to this new basis. This is what we did in [42], exploiting the consequences of this block diagonalization fully in [41]. (The decomposition of the reducible representation mentioned above into irreducible parts has since been constructed by Bántay and Zala [43].)

A most detailed account of our approach can be found in [41]. The same results were somewhat later reproduced via purely algebraic means by De Dominicis, Carlucci and Temesvári [44] by using an efficient tool called the replica Fourier transform (RFT).

Whatever the procedure, the central result one obtains is the following. In the new basis the Hessian breaks up into a string of $(R+1)\times(R+1)$ blocks followed by a string of 1×1 "blocks" along the diagonal. The $(R+1)\times(R+1)$ blocks are called, for reasons of little interest, longitudinal-anomalous (LA), and they are labelled by an index $k = 0,1,2,\ldots,R+1$. The sector where the 1×1 blocks appear, i.e. where the transformation actually diagonalizes the Hessian, is called the replicon (R) sector.

Let us call our generic ultrametric matrix (in the original coordinate system) $\underline{\underline{\mathbf{M}}}$ and its inverse $\underline{\underline{\mathbf{G}}}$. Their components can be parametrized as in Eq. 23. Let us further call the corresponding matrices in the new representation $\underline{\underline{\hat{\mathbf{M}}}}$ and $\underline{\underline{\hat{\mathbf{G}}}}$, respectively. As shown in [41,42], the diagonal elements of $\underline{\underline{\hat{\mathbf{M}}}}$ in the replicon sector are given by:

$$\hat{M}^{r,r}_{k,l} = \sum_{s=k}^{R+1} p_s \sum_{t=l}^{R+1} p_t (M^{r,r}_{t,s} - M^{r,r}_{t-1,s} - M^{r,r}_{t,s-1} + M^{r,r}_{t-1,s-1}) \quad , \tag{70}$$

where the discrete indices r, k, l needed to label these matrix elements take on the values $r = 0,1,\ldots,R$, and $k,l = r+1, r+2,\ldots,R+1$, respectively. Similar formulae hold, of course, for $\underline{\underline{\hat{\mathbf{G}}}}$:

$$\hat{G}^{r,r}_{k,l} = \sum_{s=k}^{R+1} p_s \sum_{t=l}^{R+1} p_t (G^{r,r}_{t,s} - G^{r,r}_{t-1,s} - G^{r,r}_{t,s-1} + G^{r,r}_{t-1,s-1}) \quad . \tag{71}$$

The combinations appearing in Eqs. 70 and 71 can alternatively be regarded as (double) RFT's [44]; in that context, to conform to earlier notation, it is useful to denote them as

$$\begin{aligned} \hat{M}^{r,r}_{k,l} &\equiv K^{r,r}_{k,l} \quad , \\ \hat{G}^{r,r}_{k,l} &\equiv F^{r,r}_{k,l} \quad , \quad r+1 \le k,l \le R+1. \end{aligned} \tag{72}$$

The idea of a transform sharing the convolution property with ordinary Fourier transforms was first proposed, for the continuum limit, by Mézard and Parisi [45] in the context of random manifolds. It was later recognized by Parisi and Sourlas [46] as a Fourier transform within the p-adic number approach (again limited to $R\to\infty$, $n\to 0$). In this context Parisi and Sourlas could derive, for the replicon sector, the relationship $F^{r,r}_{k,l}\cdot K^{r,r}_{k,l} = 1$ which in our geometric (or group theoretic) approach follows from the fact that $\underline{\underline{\mathbf{M}}}\cdot\underline{\underline{\mathbf{G}}} = \underline{\underline{\mathbf{1}}}$ and that these matrices are diagonal in that sector. The extension of the RFT to the discrete case and to the LA sector through the use of generalized Parisi boxes $p_t^{(r,s)}$, as described below, was made in [44]. Finally, the connection to standard Fourier transforms over a group was given in [47].

Turning to the longitudinal-anomalous sector, there we have the $(R+1)\times(R+1)$ diagonal blocks labelled by the index $k=0,1,\ldots,R+1$. The matrix elements work out to be [a]

$$\hat{M}_k^{r,s} = \Lambda_k(r)\delta^{Kr}_{r,s} + \frac{1}{4}K_k^{r,s}\delta_s^{(k-1)} \quad , \qquad r,s=0,1,\ldots,R \quad , \tag{73}$$

where $\Lambda_k(r)$ is shorthand for

$$\Lambda_k(r) = \begin{cases} \hat{M}^{r,r}_{k,r+1} \quad , & k > r+1 \\ \hat{M}^{r,r}_{r+1,r+1} \quad , & k \le r+1 \end{cases} \tag{74}$$

$$\delta_s^{(l)} = p_s^{(l)} - p_{s+1}^{(l)} \quad , \qquad l=0,1,\ldots,R+1 \quad , \quad s=0,1,\ldots,R \tag{75}$$

and

$$p_s^{(l)} = \begin{cases} p_s \quad , & s \le l \\ 2p_s \quad , & s > l \end{cases} \quad . \tag{76}$$

We shall refer to the objects K and F as the kernel or RFT of $\underline{\underline{\mathbf{M}}}$ and $\underline{\underline{\mathbf{G}}}$, respectively. $K_k^{r,s}$ is given in terms of the original matrix elements as

$$K_k^{r,s} = \sum_{t=k}^{R+1} p_t^{(r,s)}(M_t^{r,s} - M_{t-1}^{r,s}) \quad . \tag{77}$$

It is here that we make use of the observation made below Eq. 23, and keep only the larger of the two lower indices of $\underline{\underline{\mathbf{M}}}$. Formula (77) could not have been cast into the simple form above had we kept to the original, redundant parametrization.

The weight $p_t^{(r,s)}$ is defined as

$$p_t^{(r,s)} = \begin{cases} p_t, & t \le r \le s \\ 2p_t & r < t \le s \\ 4p_t & r \le s < t \end{cases} \quad . \tag{78}$$

The corresponding formulas for $\hat{G}_k^{r,s}$ are

$$\hat{G}_k^{r,s} = \frac{1}{\Lambda_k(r)}\delta^{Kr}_{r,s} + \frac{1}{4}F_k^{r,s}\delta_s^{(k-1)}, \tag{79}$$

with the kernel of $\underline{\underline{\mathbf{G}}}$ given by

$$F_k^{r,s} = \sum_{t=k}^{R+1} p_t^{(r,s)}(G_t^{r,s} - G_{t-1}^{r,s}) \quad . \tag{80}$$

[a] In order to make contact with earlier notation, we note that the quantity $\hat{M}^{r,r}_{k,l}$ we use here was called $\lambda(r;k,l)$ in [41]; $\hat{M}_k^{r,s}$ here is related to $M^{(k)}_{r,s}$ in [41] by a similarity transformation, and the relationship between the quantity $\Delta_r^{(k)}$ in [41] and $\delta_r^{(k-1)}$ here is $\Delta_r^{(k)} = \frac{1}{2}\delta_r^{(k-1)}$. The kernel is, however, invariant, so it is exactly the same as the quantity called $K_k(r,s)$ in [41], and the Dyson equation (81) also remains the same.

Now the equation $\underline{\underline{\mathbf{G}}} \cdot \underline{\underline{\mathbf{M}}} = \underline{\underline{\mathbf{1}}}$ expressed in terms of the diagonal blocks reads

$$\sum_{t=0}^{R} (\delta^{Kr}_{r,t} \frac{1}{\Lambda_k(t)} + \frac{1}{4} F_k^{r,t} \delta_t^{(k-1)})(\delta^{Kr}_{t,s} \Lambda_k(t) + \frac{1}{4} K_k^{t,s} \delta_s^{(k-1)}) = \delta^{Kr}_{r,s} \quad ,$$

or

$$F_k^{r,s} = -\frac{1}{\Lambda_k(r)} K_k^{r,s} \frac{1}{\Lambda_k(s)} - \frac{1}{\Lambda_k(s)} \sum_{t=0}^{R} F_k^{r,t} \frac{\delta_t^{(k-1)}}{4} K_k^{t,s} \quad . \tag{81}$$

If we regard $\underline{\underline{\mathbf{M}}}$ as the self-energy matrix and $\underline{\underline{\mathbf{G}}}$ the propagator then Eq. 81 is just the Dyson equation connecting their respective kernels or replica Fourier transforms. This Dyson equation was first obtained in [40] for the continuum limit and in [41] for the discrete case.

Given the matrix $\underline{\underline{\mathbf{M}}}$ to be inverted, $K_k^{r,s}$ can be computed using Eq. 77. Eq. 81 is then a set of matrix equations in the unknown $F_k^{t,s}$ (one for each value of k). Suppose we are able to solve it. Next we have to invert the relation (80) to get $\underline{\underline{\mathbf{G}}}$ itself in the LA sector, and invert Eq. 71 to get it in the R sector. One of the great merits of the RFT approach is that these inversion formulae work out to be fairly transparent. The inverse of Eq. 80 is

$${}_AG_t^{r,s} = \sum_{k=0}^{t} \frac{1}{p_k^{(r,s)}} (F_k^{r,s} - F_{k+1}^{r,s}) \quad , \tag{82}$$

that of the double transform in Eq. 71 is

$${}_RG_{u,v}^{r,r} = \sum_{k=r+1}^{u} \frac{1}{p_k} \sum_{l=r+1}^{v} \frac{1}{p_l} (F_{k,l}^{r,r} - F_{k+1,l}^{r,r} - F_{k,l+1}^{r,r} + F_{k+1,l+1}^{r,r}) \quad . \tag{83}$$

As for ${}_AG_{u,v}^{r,r}$ we find

$${}_AG_{u,v}^{r,r} = {}_AG_u^{r,r} + {}_AG_v^{r,r} - {}_AG_r^{r,r} \quad , \tag{83'}$$

whence the full propagator is obtained as

$$\begin{aligned} G_t^{r,s} &= {}_AG_t^{r,s} \quad , \\ G_{u,v}^{r,r} &= {}_RG_{u,v}^{r,r} + {}_AG_{u,v}^{r,r} \quad . \end{aligned} \tag{84}$$

Now we proceed to apply the above formalism to the special case of inverting (17) in order to determine the free propagator.

7.2 The exact solutions for the Gaussian propagators near the transition temperature

Let us now take the inverse bare propagator for the ultrametric matrix $\underline{\underline{\mathbf{M}}}$ of the previous subsection. As seen from Eq. 17, its components are

$$\begin{aligned} p^2 + M_{R+1,R+1}^{r,r} &= p^2 - 2\tau - 2uq_r^2, & r &= 0,1,\ldots,R \\ M_{R+1}^{r,s} &= -wq(\min\{r,s\}), & r,s &= 0,1,\ldots,R \\ M_{R+1,t}^{r,r} &= -wq_t, & r<t &= 1,2,\ldots,R \quad . \end{aligned} \tag{85}$$

In the truncated model (2) all other components are zero.

Now we substitute these into Eq. 70 and take the continuum limit, i.e. perform the analytic continuation in the p's into the interval (0,1) and let $R \to \infty$, according to Parisi's prescription [1]. In the new representation we get for the diagonal components in the replicon sector

$$\begin{aligned} \hat{M}^{x,x}_{t,s} &\equiv K^{x,x}_{t,s} = \left(F^{x,x}_{t,s}\right)^{-1} = p^2 + \lambda^{(0)}(x;s,t) \equiv \\ &\equiv p^2 + uq^2(s) + uq^2(t) - 2uq^2(x), \qquad 0 \le x < s,t \le 1, \end{aligned} \tag{86}$$

where we have also used the equation of state (26). Since in the replicon sector $\underline{\underline{\hat{\mathbf{M}}}}$ is diagonal, the $\lambda^{(0)}$ appearing in Eq. 86 are nothing but the eigenvalues of the Hessian matrix $\underline{\underline{\mathbf{M}}}^{(0)}$ defined in Eq. 17. These eigenvalues fill a continuous band with lower band edge $\lambda^{(0)}(x;x,x) = 0$ and upper band edge $\lambda^{(0)}(0;1,1) = 2uq_1^2 = \frac{2u}{w^2}\tau^2 + \cdots$

$$0 \le \lambda^{(0)}(x;s,t) \le \frac{2u}{w^2}\tau^2. \tag{87}$$

The LA kernel corresponding to the simple ultrametric matrix (85) is obtained from Eq. 77 as

$$K^{s,t}_k = -4wq(\min(\{s,t\}), \qquad 0 \le s,t \le 1 \quad . \tag{88}$$

Note that this is independent of the lower variable k. (If we had also kept the $\mathrm{Tr}(\phi^4)$ invariant in Eq. 2, as in [40], we would have a k-dependent kernel.)

In the continuum limit Eq. 81 then becomes

$$F^{x,y}_k = \frac{4wq(\min\{x,y\})}{\Lambda^{(0)}_k(x)\Lambda^{(0)}_k(y)} - \frac{2w}{\Lambda^{(0)}_k(y)}\int_0^1 \Delta^{(k)}_t q(\min\{y,t\})F^{x,t}_k, \qquad 0 \le k,x,y \le 1, \tag{89}$$

where

$$\Delta^{(k)}_t = \left(\frac{1}{2}\theta(k-t) + \frac{k}{2}\delta(k-t) + \theta(t-k)\right)\mathrm{d}t \tag{90}$$

is the continuum limit of $-\frac{1}{2}\delta^{(k-1)}_t$ and [b]

$$\Lambda^{(0)}_k(x) = \begin{cases} p^2 + \lambda^{(0)}(x;x,k) & , \quad x \le k \\ p^2 & , \quad x > k. \end{cases} \tag{91}$$

The kernel in Eq. 88 has a remarkable property, namely that it depends only on one (here, the smaller) of its upper variables. This is, in fact, the other crucial factor (in addition to the symmetry analysed in the previous subsection) that allows one to obtain the bare

[b] To avoid confusion we note that Eq. 90 is also the continuum limit of the *negative* of a quantity which was, unfortunately, denoted by the same symbol $\Delta^{(k)}_t$ in [41].

propagator in closed form. Indeed, for a kernel depending only on the smaller of its upper variables the solution to Eq. 89 becomes a product of two factors, one depending on x, the other on y.

We write it in the following form

$$F_k^{x,y} = \frac{4}{W_k} \frac{\Phi_k^+(\min\{x,y\})}{\Lambda_k^{(0)}(\min\{x,y\})} \frac{\Phi_k^-(\max\{x,y\})}{\Lambda_k^{(0)}(\max\{x,y\})} \tag{92}$$

where $\Phi_k^\pm$ are two independent solutions of

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{w\dot{q}(x)} \frac{\mathrm{d}}{\mathrm{d}x} \Phi_k^\pm(x) = -\frac{\Phi_k^\pm(x)}{2\Lambda_k^{(0)}(x)} \tag{93}$$

and the Wronskian

$$W_k = \frac{\dot{\Phi}_k^+(x)\Phi_k^-(x) - \Phi_k^+(x)\dot{\Phi}_k^-(x)}{w\dot{q}(x)} \tag{94}$$

is independent of x. (The dot means derivative w.r.t. the argument).

For the details of the derivation of $\Phi_k^\pm$ we refer the reader to [28,40], here we content ourselves with simply giving the results for $F_k^{x,y}$. In order to simplify the notation slightly we introduce

$$\hat{p}^2 \equiv \frac{u}{w^2} p^2 = \frac{x_1}{2wq_1} p^2. \tag{95}$$

With Eqs. 86 and 91 $\Lambda_k^{(0)}(x)$ now becomes

$$\Lambda_k^{(0)}(x) = \begin{cases} p^2 \left(1 + \dfrac{k^2 - x^2}{4\hat{p}^2}\right) & , \quad x \le k \\ p^2 & , \quad x > k. \end{cases} \tag{96}$$

$F_k^{x,y}$ will be expressed in terms of the solutions $\mathcal{G}_1$ and $\mathcal{G}_2$ of the Gegenbauer equation

$$(1 - \xi^2)\ddot{\mathcal{G}} = 2\mathcal{G} \tag{97}$$

belonging to the initial conditions

$$\begin{array}{lll} \mathcal{G}_1 \equiv C(\xi) & , \quad C(0) = 1 & , \quad \dot{C}(0) = 0 \\ \mathcal{G}_2 \equiv S(\xi) & , \quad S(0) = 0 & , \quad \dot{S}(0) = 1. \end{array} \tag{98}$$

The variable ξ is related to the original overlap variable x by

$$\xi = \frac{x}{2\hat{p}} \left(1 + \frac{k^2}{4\hat{p}^2}\right)^{-\frac{1}{2}}, \tag{99}$$

and we will frequently use abbreviations like $C(\xi) \equiv C_x$ or $C_k \equiv C_{x=k}$ etc. .

We now have

$$p^2 F_k^{x,y} = \frac{2}{\hat{p}} \cdot \frac{1}{1+\dfrac{k^2-x^2}{4\hat{p}^2}} \cdot \frac{1}{1+\dfrac{k^2-y^2}{4\hat{p}^2}} \cdot \frac{A_k}{A_k\sigma_k + B_k\gamma_k} \cdot \frac{S_x}{S_k} \cdot$$

$$\cdot \left[(A_k\sigma_k + B_k\gamma_k)C_y - \left(2\sqrt{1+\frac{k^2}{4\hat{p}^2}} \left(\sigma_k + \frac{k}{2\hat{p}}\gamma_k\right) C_k + \gamma_k \dot{C}_k \right) S_y \right], x \le y \le k \le x_1, \quad (100)$$

$$p^2 F_k^{x,y} = \frac{2}{\hat{p}} \cdot \frac{1}{1+\dfrac{k^2-x^2}{4\hat{p}^2}} \cdot \frac{A_k}{A_k\sigma_k + B_k\gamma_k} \frac{S_x}{S_k} \cdot \gamma_y \quad , \qquad x \le k \le y \le x_1, \quad (101)$$

$$p^2 F_k^{x,y} = \frac{2}{\hat{p}} \cdot \frac{A_k \cosh\left(\dfrac{x-k}{\hat{p}}\right) + B_k \sinh\left(\dfrac{x-k}{\hat{p}}\right)}{A_k\sigma_k + B_k\gamma_k} \cdot \gamma_y \quad , \quad k \le x \le y \le x_1, \quad (102)$$

where the notations

$$A_k = 2\sqrt{1+\frac{k^2}{4\hat{p}^2}}\, S_k \quad , \qquad (103)$$

$$B_k = \dot{S}_k + \frac{k}{\hat{p}}\sqrt{1+\frac{k^2}{4\hat{p}^2}}\, S_k \quad , \qquad (104)$$

$$\gamma_x = \cosh\left(\frac{x_1-x}{\hat{p}}\right) + \frac{1-x_1}{\hat{p}} \sinh\left(\frac{x_1-x}{\hat{p}}\right), \qquad (105)$$

$$\sigma_x = \sinh\left(\frac{x_1-x}{\hat{p}}\right) + \frac{1-x_1}{\hat{p}} \cosh\left(\frac{x_1-x}{\hat{p}}\right) \qquad (106)$$

have been introduced.

$F_k^{x,y}$ is a continuous function of x, y and k. For $y < x$, Eqs. 100–102 apply with x and y interchanged. Whenever either x, y or k goes beyond x_1, $F_k^{x,y}$ becomes a constant in that variable.

In the above notation the Wronskian in Eq. 94 works out to be

$$W_k = A_k\sigma_k + B_k\gamma_k \quad . \qquad (107)$$

It can be shown that the roots of

$$W_k(p^2 = -\lambda) = 0 \qquad (108)$$

give the LA eigenvalues of the Hessian $\underline{\underline{\mathbf{M}}}^{(0)}$.

Writing up Eq. 107 explicitly with the help of the definitions (103)-(106) we can show that Eq. 108 has infinitely many positive roots for any given $0 \le k \le x_1$. These can be labelled by a discrete index as $\lambda_m(k)$, $m = 0, 1, 2\ldots$. The eigenvalues belonging to $m = 0$ are of the order τ and, with k varying between 0 and x_1, they form a continuous band:

$$2\tau\left(1 - \frac{u}{3w^2}\tau + \cdots\right) < \lambda_0(k) < 2\tau\left(1 + \frac{u}{3w^2}\tau + \cdots\right). \qquad (109)$$

All the other eigenvalues are of $\mathcal{O}(\tau^2)$. For a given k they fall off like $\sim \frac{\tau^2}{m^2}$, while for a given m they form a continuous band of width of $\mathcal{O}(\tau^2)$ as k is varied. The bands corresponding to $m = 1, 2, \ldots$ partially overlap and even the largest $(\lambda_1(x_1))$ of them is smaller than the upper edge of the replicon band given in Eq. 87. Therefore the small, $\mathcal{O}(\tau^2)$ eigenvalues can be regarded as forming a single continuous band spanning the range $[0, \frac{2u}{w^2}\tau^2]$.

Replica symmetry breaking resolves part of the degeneracy of the AT eigenvalues: the large eigenvalue is split and smeared out into the narrow band (109), while the small (negative) eigenvalue, in addition to being split, also gets shifted, so that the smallest eigenvalue is now zero: RSB has cured the AT instability.

It is clear that these continuous bands of eigenvalues will show up as narrow branch cuts in the propagators. In the near infrared limit, with squared momenta much larger than the bandwidths, these cuts appeared as simple poles. We have now learnt the precise structure of the singularities so we are prepared to go into the far infrared limit. Before turning to this we write up the various propagator components with the help of the (continuous forms) of the inversion formulas given in Eqs. 82, 83 and 83'.

The various components of the propagator will be displayed with the LA and R contributions added up.

For the "first propagator" (two pairs of replica indices coinciding) we have:

$$\begin{aligned} G_{1,1}^{x,x} = \; & - \left(\int_0^x + \frac{1}{2}\int_x^{x_1}\right) \frac{\mathrm{d}k}{k}\frac{\partial}{\partial k} F_k^{x,x} + \frac{1}{2}F_{x_1}^{x,x} + \int_x^{x_1} \frac{\mathrm{d}k}{k}\frac{\partial}{\partial k}\int_x^{x_1}\frac{\mathrm{d}l}{l}\frac{\partial}{\partial l} F_{k,l}^{x,x} - \\ & - \int_x^{x_1}\left(\frac{\mathrm{d}k}{k}\frac{\partial}{\partial k}F_{k,x_1}^{x,x} + \frac{\mathrm{d}l}{l}\frac{\partial}{\partial l}F_{x_1,l}^{x,x}\right) + F_{x_1,x_1}^{x,x} \quad . \end{aligned} \tag{110}$$

For the "second propagator" (one pair of replica indices coinciding) we have two different formulae according to the relative values of the variables:

$$G_{1,x}^{x,y} = - \left(\int_0^x + \frac{1}{2}\int_x^y + \frac{1}{4}\int_y^{x_1}\right)\frac{\mathrm{d}k}{k}\frac{\partial}{\partial k}F_k^{x,y} + \frac{1}{4}F_{x_1}^{x,y} \quad , \qquad x \le y < 1, \tag{111}$$

$$\begin{aligned} G_{1,t}^{x,x} = \; & - \left(\int_0^x + \frac{1}{2}\int_x^t + \frac{1}{4}\int_t^{x_1}\right)\frac{\mathrm{d}k}{k}\frac{\partial}{\partial k}F_k^{x,x} + \frac{1}{4}F_{x_1}^{x,x} + \int_x^{x_1}\frac{\mathrm{d}k}{k}\frac{\partial}{\partial k}\int_x^t\frac{\mathrm{d}l}{l}\frac{\partial}{\partial l}F_{k,l}^{x,x} - \\ & - \int_x^t \frac{\mathrm{d}k}{k}\frac{\partial}{\partial k}F_{k,x_1}^{x,x} \quad , \qquad x \le t < 1. \end{aligned} \tag{112}$$

Finally, for the "third propagator" (all replica indices different) we find four different expressions, depending again on the order of the variables:

$$G_t^{x,y} = G_{x,t}^{x,y} = G_{y,t}^{y,x} = -\left(\int_0^x + \frac{1}{2}\int_x^y + \frac{1}{4}\int_y^t\right)\frac{\mathrm{d}k}{k}\frac{\partial}{\partial k}F_k^{x,y} \quad , \; x \le y \le t < 1, \tag{113}$$

$$G_t^{x,y} = G_{x,t}^{x,y} = G_{t,t}^{y,x} = -\left(\int_0^x + \frac{1}{2}\int_x^t\right)\frac{\mathrm{d}k}{k}\frac{\partial}{\partial k}F_k^{x,y} \quad , \qquad x \le t \le y < 1, \tag{114}$$

$$G_t^{x,y} = G_{t,t}^{x,y} = G_{t,t}^{y,x} = -\int_0^t \frac{\mathrm{d}k}{k}\frac{\partial}{\partial k}F_k^{x,y} \quad , \qquad t \le x \le y < 1, \tag{115}$$

$$G_{s,t}^{x,x} = -\left(\int_0^x + \frac{1}{2}\int_x^s + \frac{1}{4}\int_s^t\right)\frac{\mathrm{d}k}{k}\frac{\partial}{\partial k}F_k^{x,x} +$$

$$+\int_x^s \frac{\mathrm{d}k}{k}\frac{\partial}{\partial k}\int_x^t \frac{\mathrm{d}l}{l}\frac{\partial}{\partial l}F_{k,l}^{x,x} \quad , \qquad x \le s,t < 1. \tag{116}$$

In Eqs. 110−116 $F_k^{x,y}$ is the LA kernel given by Eqs. 100−102 and $F_{k,l}^{x,x}$ is the R kernel defined in Eq. 86.

The various propagator components are continuous functions of all overlap variables, except when either or both lower indices take on the value 1 corresponding to coinciding replica indices.

8 The far infrared region

The formulae derived in the previous section give a complete solution for the propagators of the Ising spin glass near the transition temperature and in zero field. Expanding them in the limit $p^2 \gg uq_1^2$ one can easily recover the results, derived by different means in Sec. 6, for the *near infrared* region. To extract information concerning the *far infrared* ($p^2 \sim uq_1^2$ or $p^2 \ll uq_1^2$) is considerably harder. The main difficulty consists in the extreme richness of behaviour shown by the various propagator components. Indeed, depending on the relative magnitude of the overlap variables we have such a large number of different analytic forms that any attempt to display an exhaustive set of results would be quite illusory. What we are able to do is to merely give a sample of the results which will illustrate the types of behaviours one encounters in the extreme long wavelength limit.

Let us start with the simplest propagators, with all overlap variables above the break-point (the "single-valley propagators"). With some effort one finds from the complete formulae in Sec. 7 that for $p^2 \ll uq_1^2$, i.e. far below the upper edge of the small mass band, these components are given by the following simple expressions:

$$G_{1,1}^{x_1,x_1} \approx \frac{3}{p^2} \quad , \tag{117}$$

$$G_{1,x_1}^{x_1,x_1} \approx \frac{3}{2p^2} \quad , \tag{118}$$

$$G_{x_1,x_1}^{x_1,x_1} \approx \frac{1}{p^2} \quad . \tag{119}$$

The remarkable fact about Eqs. 117-119 is that they are the same as the $p^2 \to 0$ limits of the formulae in Eq. 58 which were derived in the *opposite* limit, $p^2 \gg uq_1^2$. Although there are some (explicitly known but complicated) corrections around $p^2 \sim uq_1^2$, the order of magnitude of the propagators is still $\sim 1/p^2$, which means that the formulae given in Eq. 58 can be regarded as a good representation of the single-valley propagators in the entire momentum range. This was first pointed out in [30], but the p^{-2} like IR behaviour of the single valley propagators had been know already from [27].

We are not aware of any *a priori* reason why any propagator component should behave the same way both in the near and in the far infrared (and, indeed, no such continuity is observed in the other components), so we believe this coincidence carries a physical message. We shall return to this point in the next section.

From the continuity of the single-valley propagators one would tend to infer the same for the transverse and the longitudinal combinations given in Eq. 59. This is certainly true for the transverse component since, as shown already in [28], $G_\perp = 1/p^2$ is an exact result within the Gaussian approximation. As for the longitudinal component, its behaviour in the far infrared is less clear. The point is that the leading terms cancel in the particular combination $G_{||}$ and next to leading terms are not controlled reliably by the truncated model that we are using here. Nevertheless, it is safe to say that the longitudinal propagator is less singular than $1/p^2$ for $p \to 0$.

It is interesting to note that in some important respects (masslessness, the ratios 3:$\frac{3}{2}$:1 between the three propagator components, weaker singularity in $G_{||}$) the qualitative behaviour of the single-valley propagators of Eqs. 117$-$119 is rather similar to that of the correlation functions predicted by the droplet theory [4,5], despite the obvious conflict between the underlying physical pictures.

Let us now turn to the other propagator components. For the diagonal propagator $G_{1,1}^{x,x}$ we find:

$$G_{1,1}^{x,x} \approx \frac{u}{w^2}\left(\frac{1}{x\hat{p}^3} - \frac{1}{2x^2\hat{p}^2}\right), \qquad 0 < x < x_1, \quad p \to 0 \quad , \tag{120}$$

where $\hat{p}^2$ is the rescaled momentum introduced in Eq. 95. Although $G_{1,1}^{x,x}$ is evidently continuous at $x = x_1$, Eq. 120 does not match Eq. 117. The apparent contradiction is resolved by noting that the limit $p \to 0$ is nonuniform: near $x = x_1^-$, $G_{1,1}^{x,x}$ depends on combinations like $(x_1 - x)/p$ and $(x_1 - x)^{1/2}/p$. The limit $p \to 0$ is nonuniform also around $x = 0$, so for a fixed value of $x \in (0,1)$ Eq. 120 will hold only if the wavenumber is sufficiently small to satisfy both $p \ll x$ and $p \ll (x_1 - x)$. Similar remarks apply in all the cases that follow.

For the propagator component $G_{1,z}^{x,x}$, $x < z$, we find

$$G_{1,z}^{x,x} = \frac{u}{w^2}\left(\frac{1}{x\hat{p}^3} - \frac{1}{2x^2\hat{p}^2}\right) \quad \text{if} \quad 0 < x < z \le x_1 \quad , \quad p \to 0. \tag{121}$$

Again, in Eq. 121 $(z - x)$ is understood to be large compared with p.

The component $G_{1,x}^{x,y}$, $0 < x \le y \le x_1$ tends to a constant

$$G_{1,x}^{x,y} \to -\frac{4u}{w^2}\frac{1}{x_1(x_1^2 - x^2)}\frac{S(x/x_1)}{S(1)} \quad \text{if } 0 < x < y = x_1 \quad , \quad p \to 0 \quad . \tag{122}$$

The divergence of this constant for $x \to x_1 - 0$ is a signal of the IR divergence of the limit $x = y = x_1$, as given by Eq. 118. For $x < y < x_1$ and $p \to 0$, $G_{1,x}^{x,y}$ tends to another constant which is too complicated to be recorded here.

The last item in this category is $G_{1,x}^{x,x}$. For $0 < x < x_1$ we find

$$G_{1,x}^{x,x} = \frac{u}{w^2}\left(\frac{1}{x\hat{p}^3} - \frac{5}{2}\frac{1}{x^2\hat{p}^2}\right) \quad , \quad p \to 0. \tag{123}$$

Now we list a number of results for the "third kind" of propagator components with neither of their lower indices equal to 1.

Let us start with $G^{x,x}_{x,x}$. For a fixed $x \in (0, x_1)$ and $p \to 0$ we find:

$$G^{x,x}_{x,x} = \frac{u}{w^2}\left(\frac{1}{x\hat{p}^3} - \frac{7}{2}\frac{1}{x^2\hat{p}^2}\right) \quad , \quad 0 < x < x_1 \quad , \quad p \to 0. \tag{124}$$

Next we consider $G^{x,y}_{x,x}$, $0 < x \le y$. For $0 < x < y \le x_1$ we find a qualitatively new type of behaviour:

$$G^{x,y}_{x,x} = \frac{2u}{w^2}\left(\frac{1}{x\hat{p}^2} - \frac{7}{2}\frac{1}{x^2\hat{p}}\right)\exp\left[-\frac{x_1 - x}{\hat{p}}\right] \quad , \quad 0 < x < y = x_1 \quad , \quad p \to 0, \tag{125}$$

$$G^{x,y}_{x,x} = \frac{u}{w^2}\left(\frac{1}{x\hat{p}^3} - \frac{7}{2}\frac{1}{x^2\hat{p}^2}\right)\exp\left[-\frac{y - x}{\hat{p}}\right] \quad , \quad 0 < x < y < x_1 \quad , \quad p \to 0. \tag{126}$$

In the long wavelength limit, $p \to 0$, Eqs. 125 and 126 become δ-like distributions. Distribution-like components were first identified among the spin glass propagators by two of us [48], see also [49]. Independently, Ferrero and Parisi put them to use in their recent analysis of the IR divergences in spin glasses [24].

Returning to the list of propagators we find that the component $G^{x,y}_{x,y}$, $0 < x < y \le x_1$, is equal to the constant in Eq. 122 for $0 < x < y = x_1$, and

$$G^{x,y}_{x,y} = -\frac{8u}{w^2}\frac{1}{y^2(y^2 - x^2)}\frac{S(x/y)}{S(1)} \quad , \quad 0 < x < y < x_1 \quad , \quad p \to 0. \tag{127}$$

Furthermore,

$$G^{x,x}_{x,y} = \frac{u}{w^2}\left(\frac{1}{x\hat{p}^3} - \frac{5}{2}\frac{1}{x^2\hat{p}^2}\right) \quad , \quad 0 < x < y \le x_1 \quad , \quad p \to 0, \tag{128}$$

$$G^{x,x}_{y,y} = \frac{u}{w^2}\left(\frac{2}{y\hat{p}^2} - \frac{7}{y^2\hat{p}}\right)\exp\left[-2\frac{x_1 - y}{\hat{p}}\right] \quad , \quad y < x = x_1 \quad , \quad p \to 0, \tag{129}$$

$$G^{x,x}_{y,y} = \frac{u}{w^2}\left(\frac{1}{y\hat{p}^3} - \frac{7}{2}\frac{1}{y^2\hat{p}^2}\right)\exp\left[-2\frac{x - y}{\hat{p}}\right] \quad , \quad y < x < x_1 \quad , \quad p \to 0. \tag{130}$$

The remaining components depend on three different overlap variables:

$$G^{x,x}_{y,z} = \frac{u}{w^2}\left(\frac{1}{x\hat{p}^3} - \frac{1}{2}\frac{1}{x^2\hat{p}^2}\right) \quad , \quad 0 < x < y, z \le x_1 \quad , \quad p \to 0, \tag{131}$$

$$G^{x,y}_{x,z} = -\frac{4u}{w^2}\frac{1}{x_1(x_1^2 - x^2)}\frac{S(x/x_1)}{S(1)} \quad , \quad 0 < x < y = z = x_1 \quad , \quad p \to 0, \tag{132}$$

$$G^{x,y}_{x,z} = -\frac{16u}{w^2}\frac{\hat{p}}{z^2(z^2 - x^2)}\frac{S(x/z)}{S(1)}\exp\left[-\frac{x_1 - z}{\hat{p}}\right], \; 0 < x < z < y = x_1, \; p \to 0, \tag{133}$$

$$G^{x,y}_{x,z} = -\frac{8u}{w^2}\frac{1}{z^2(z^2 - x^2)}\frac{S(x/z)}{S(1)}\exp\left[-\frac{y - z}{\hat{p}}\right], \; 0 < x < z \le y < x_1 \; , p \to 0, \tag{134}$$

$$G^{x,y}_{z,z} = \frac{u}{w^2}\left(\frac{2}{z\hat{p}^2} - \frac{7}{z^2\hat{p}}\right)\exp\left[-2\frac{x_1 - z}{\hat{p}}\right] \quad , \quad z < x = y = x_1 \quad , \quad p \to 0, \tag{135}$$

$$G^{x,y}_{z,z} = \frac{u}{w^2}\left(\frac{2}{z\hat{p}^2} - \frac{7}{z^2\hat{p}}\right)\exp\left[-\frac{x_1 + x - 2z}{\hat{p}}\right] \quad , \quad z < x < y = x_1 \quad , \quad p \to 0, \tag{136}$$

$$G^{x,y}_{z,z} = \frac{u}{w^2}\left(\frac{1}{z\hat{p}^3} - \frac{7}{2}\frac{1}{z^2\hat{p}^2}\right)\exp\left[-\frac{y + x - 2z}{\hat{p}}\right] \; , \; z < x < y < x_1 \; , \; p \to 0, \tag{137}$$

and finally $G^{x,y}_{x,z}$ goes to different constants as $p \to 0$, for $x < y = z = x_1$, $x < y < z = x_1$, and $x < y < z < x_1$, respectively.

The components with one or two upper indices equal to zero are special. If the two upper indices are different and the smaller is zero, then the propagator component in question vanishes indentically. If the two upper indices coincide and vanish, then the leading infrared behaviour in the diagonal component is given by

$$G^{0,0}_{1,1} = \frac{\pi}{4}\frac{u}{w^2}\frac{1}{\hat{p}^4} \quad , \quad p \to 0. \tag{138}$$

As long as the lower indices s, t are larger than zero, the off-diagonal components $G^{0,0}_{1,t}$ and $G^{0,0}_{s,t}$ are the same as Eq. 138; for $t \to 0$ (or s and/or $t \to 0$) they vanish again.

The strong, p^{-4} like singularity in a correlation function related to Eq. 138 was first noticed by Sompolinsky and Zippelius [50]. Several results concerning the far infrared region were published by us [27–30], by Goltsev [51], and by Ferrero and Parisi [24].

The compilation of Eqs. 117–138 above gives more details than any of these papers, nevertheless it is far from exhaustive. In a propagator component depending on, say, two overlap variables we may have, e.g., that one of them is of the order of x_1, while the other is much less than even x_1^2, thereby defining a new mass scale. Depending on the value of the momentum relative to these new mass regions we may have further details in the IR behaviour. With the exception of Eq. 138 the results listed in this section have been derived under the assumption that all overlap variables are of the same order of magnitude as x_1 and the momentum is much smaller than x_1^2.

9 The physical meaning of the propagators

In addition to their role as building blocks of the interacting field theory, the Gaussian propagators have also a direct physical meaning which we would like to discuss now. This will also allow us to give a physical interpretation to some of the results we have found so far.

The propagators are related, as in any field theory, to some correlation functions. These correlation functions reflect the underlying assumption we have adopted about the structure of phase space, namely that it splits into a large number of equilibrium states with a hierarchical organization.

The simplest correlation function one can define is the overlap of the spin-spin correlation function $< s_i s_j >$ in valley a with the same in valley b:

$$C_{ab}(\mathbf{r}) = \frac{1}{N}\sum_i < s_i s_j >_a < s_i s_j >_b = q^2_{ab} + \frac{1}{N}\sum_{\mathbf{p}} \mathrm{e}^{-\mathrm{i}\mathbf{p}\mathbf{r}} C_{ab}(\mathbf{p}) \tag{139}$$

where the distance $\mathbf{r} = \mathbf{r}_i - \mathbf{r}_j$ between the sites is kept fixed as we sum over the lattice points i, and q_{ab} is the overlap between states a, b, Eq. 31.

In principle, $C_{ab}(\mathbf{r})$ as defined in Eq. 139 could depend on the concrete realization of the random couplings J_{ij} and also on the two states a and b. Following the considerations of Parisi [38] and also those of Mézard and Virasoro [52] one is led to the conclusion, however, that $C_{ab}(\mathbf{r})$ is self-averaging (i.e. independent of the sample J_{ij} for a large system) and depends on a and b only through the overlap q_{ab}. Moreover, the Fourier transform

$$C_{ab}(\mathbf{p}) = \frac{1}{N}\sum_{ij}\exp[\mathrm{i}(\mathbf{r}_i - \mathbf{r}_j)\mathbf{p}] < s_i s_j >_a < s_i s_j >_b - N q_{ab}^2 \delta_{\mathbf{p},0}^{Kr}$$

can be calculated in the Gaussian approximation via the replica formalism and turns out to be nothing but the diagonal component of the propagator:

$$C_{ab}(\mathbf{p}) = G_{1,1}^{x,x}(\mathbf{p}) \tag{140}$$

where the relationship between a, b and x is given by the inverse of Parisi's order paramater function: $x = x(q_{ab})$. The derivation of Eq. 140 follows the same steps that led Parisi to the identification of $\frac{\mathrm{d}x}{\mathrm{d}q}$ as the probability distribution of overlaps [38], and can therefore be omitted here. The essential ingredient is the assumption about the states having the property of clustering.

Correlation functions involving three or four states can be treated similarly:

$$C_{abc}(\mathbf{r}) = \frac{1}{N}\sum_i < s_i s_j >_a < s_i >_b < s_j >_c = q_{ab} q_{ac} + \frac{1}{N}\sum_{\mathbf{p}} \mathrm{e}^{-\mathrm{i}\mathbf{p}\mathbf{r}} C_{abc}(\mathbf{p}) \tag{141}$$

and

$$C_{abcd}(\mathbf{r}) = \frac{1}{N}\sum_i < s_i >_a < s_i >_b < s_j >_c < s_j >_d = q_{ab} q_{cd} + \frac{1}{N}\sum_{\mathbf{p}} \mathrm{e}^{-\mathrm{i}\mathbf{p}\mathbf{r}} C_{abcd}(\mathbf{p}) \tag{142}$$

are also self-averaging and depend on the states abc, resp. $abcd$ only through the overlaps. The Fourier transforms are given by the off-diagonal components of the propagators as

$$C_{abc}(\mathbf{p}) = G_{1,z}^{x,y}(\mathbf{p}) \tag{143}$$

with $x = x(q_{ab})$, $y = x(q_{ac})$, $z = \max\{x(q_{ba}), x(q_{bc})\}$, and

$$C_{abcd}(\mathbf{p}) = G_{z_1,z_2}^{x,y}(\mathbf{p}) \tag{144}$$

with $x = x(q_{ab})$, $y = x(q_{cd})$, $z_1 = \max\{x(q_{ac}), x(q_{ad})\}$, $z_2 = \max\{x(q_{bc}), x(q_{bd})\}$.

Eqs. 140, 143 and 144, first written up in [30], give a direct physical meaning to the various propagator components. We see now that the complicated behaviour of the propagators reflects the structure of correlation overlaps inside a state and also between different states.

In particular, we can see that the propagator components with all overlaps above $x_1 = x(q_{\max})$ correspond indeed to correlation functions inside a single state, as anticipated in Sec. 6. In the near infrared region we found for these propagators the simple forms in Eq. 58, and the transverse and longitudinal combinations given in Eq. 59. These clearly suggest the idea of a massless phase with a transverse correlation function falling off like

$1/r^{d-2}$ in real space and an exponentially decaying longitudinal correlation function with a characteristic length

$$\xi = \frac{1}{\sqrt{2wq_1}} \approx \frac{1}{\sqrt{2\tau}} \quad . \tag{145}$$

This reveals the meaning of the large mass: it is the inverse coherence length of the longitudinal fluctuations in a single state.

In order for this interpretation to be consistent, the single-state propagators should not develop stronger infrared singularities in the far infrared region than the behaviour we have just seen in the near infrared. Recalling Eqs. 117, 118 and 119, valid for momenta far below the upper edge of the small mass band, we see that this is precisely what happens: the leading terms $3/p^2$, $3/2p^2$ and $1/p^2$ one finds for the single state propagators in the far infrared coincide with the $p \to 0$ limit of the formulae (58), derived in the near infrared. It seems therefore that the long-distance behaviour of the single-state propagators is relatively mild: although the phase is massless, the infrared behaviour does not appear to be more violent than in the Heisenberg model, for example. In addition, as shown in [42], the result $1/p^2$ for the transverse single-state propagator is, in fact, exact within the Gaussian approximation; it holds not only in the truncated model or near T_c, but throughout the ordered phase.

For a complete characterization of the correlations in the system we have to study also the interstate overlaps of correlation functions, with some or all replica overlaps going below x_1. The behaviour one finds in this case depends radically on the momentum range considered. For momenta in the near infrared the correlation functions display a reasonable degree of complexity: p scales with the large mass, and the overall scaling power is $1/p^2$, as shown by Eqs. 60−69. This means that the qualitative features of the correlation overlaps between two states with a given overlap $q < q_{\max}$ are similar to those inside a single state, except on the extremely long wavelength scales. The characteristic length beyond which qualitative changes occur is $\xi' \sim 1/\tau$, corresponding to the upper edge of the small mass band. This gives a meaning to the small mass scale.

As we see from the results in the previous section, for wavenumbers in the range of the small mass or below, the correlation overlaps show very strong infrared divergences, going up to $1/p^4$. This violent IR behaviour poses a formidable challenge to spin glass theory.

10 The first-loop corrections above 8 dimensions

In this section we apply the previous results for the calculation of the first short-range corrections to the equation of state and to the mass spectrum. The treatment follows our previous works [15,16] closely. For the time being we assume that the dimensionality is sufficiently high for the ordinary loop-expansion to work: as we shall see shortly, this means $d > 8$.

Our starting point is Eq. 18 where we have to remember that $q_{\alpha\beta}$ is, in principle, the exact order parameter, but τ is the reduced temperature measured relative to the mean field transition temperature. Let τ_c be the critical value of τ where the exact $q_{\alpha\beta}$ vanishes. Now we can write up Eq. 18 at τ_c with the help of Eqs. 40 and 41 which are valid for $q_{\alpha\beta} \to 0$, and subtract the resulting equation from (18). We get:

$$2tq_{\alpha\beta} + w(q^2)_{\alpha\beta} + \frac{2u}{3}q^3_{\alpha\beta} + \tag{146}$$
$$+\frac{1}{z}\frac{1}{(2\pi)^d}\int_{|\mathbf{p}|<1} \mathrm{d}^d p \left[w \sum_{\gamma\neq\alpha,\beta} \left(G_{\alpha\gamma,\gamma\beta}(\mathbf{p}) - \frac{wq_{\alpha\beta}}{p^4} \right) + 2uq_{\alpha\beta}\left(G_{\alpha\beta,\alpha\beta}(\mathbf{p}) - \frac{1}{p^2} \right) \right] = 0 \ ,$$

where $t = \tau - \tau_c$ is the "true" reduced temperature. The critical value of τ works out to be

$$\tau_c = \frac{1}{z}(w^2 I_4 - uI_2) \quad , \tag{147}$$

where

$$I_l = \int_{|\mathbf{p}|<1} \frac{\mathrm{d}^d p}{(2\pi)^d}\frac{1}{p^l} \quad , \quad l = 2, 4, 6, \ldots \quad . \tag{148}$$

(For $l \geq d$ an IR cutoff at $p > \sqrt{2t}$ is understood in the integrals I_l.)

Since τ_c is of $\mathcal{O}(1/z)$ we can replace τ by t in the propagators appearing in the loop terms. These propagators satisfy the set of equations (36)−(39), with τ replaced everywhere by t. In the dimensionality range considered here the dominant contributions to the loop integrals come from the neighbourhood of the upper cutoff. In this momentum range the "large-p expansion" results of Eqs. 42−44 are valid. As the mean field part of the equation of state goes up to $\mathcal{O}(q^3)$, we have to stop at the same order also in the loop terms. This means we have to go to $\mathcal{O}(q^3)$ in $G_{\alpha\gamma,\beta\gamma}$ in Eq. 43, but have to stop at $\mathcal{O}(q^2)$ in $G_{\alpha\beta,\alpha\beta}$, Eq. 42, which is already multiplied by $q_{\alpha\beta}$. Substituting these approximate propagators into Eq. 146, performing the summation over the replica index γ and collecting the coefficients of $q_{\alpha\beta}$, $(q^2)_{\alpha\beta}$, and $q^3_{\alpha\beta}$, we end up with an equation which is of the same form as the mean field equation of state with t, w and u replaced by some new, renormalized coupling constants $\widetilde{t}$, $\widetilde{w}$ and $\widetilde{u}$. The mapping works out as follows:

$$\begin{aligned} \widetilde{t} &= t\left(1 - \frac{1}{z}[4w^2 I_6 + 12w^2 t I_8 - 2uI_4 - 4utI_6]\right) \\ \widetilde{w} &= w\left(1 - \frac{1}{z}[2w^2 I_6 + 12w^2 t I_8]\right) \\ \widetilde{u} &= u\left(1 - \frac{1}{z}[12w^2 I_6 - \frac{12w^4}{u} I_8 - 6uI_4]\right) \quad . \end{aligned} \tag{149}$$

In addition to the above, some new type of couplings are also generated by the loop terms. These would be corrections to those quartic couplings which we discarded already at the very beginning. Since they enter only subleading terms in all quantities which we are interested in, we can safely ignore them.

With this we have mapped back our loop-corrected theory onto MFT. The order parameter will be of the same form as Parisi's order parameter function given in Eqs. 27 and 28, with τ replaced by $\widetilde{t}$ and w, u replaced by $\widetilde{w}$, $\widetilde{u}$, respectively. A more detailed discussion of the effects of these replacements can be found in [15,16], here we merely note that the slope $\widetilde{w}/2\widetilde{u}$ of $q(x)$ decreases, while the breakpoint x_1 increases with decreasing dimensionality. This means that replica symmetry breaking effects are enhanced by the loop corrections, in complete agreement with the conclusions drawn by Georges, Mézard and Yedidia from the $1/d$ expansion [14].

We can arrive at the same conclusion by considering the excitation spectrum of the system. In order to be able to calculate the renormalized masses we need first the corrections to the self-energy matrix. At the first loop level these are typically bilinear combinations of the various propagator components, integrated over the loop momentum. For our present purposes we may disregard "wave function renormalization effects" and evaluate the self-energies for zero external momentum. For $d > 8$ the propagators entering the calculation can again be replaced by their "large-p" approximations from Eqs. 42−44 where we can now stop at $\mathcal{O}(q^2)$. Performing all the necessary replica summations and momentum integrations we find at the end that the self-energies to $\mathcal{O}(1/z)$ are precisely of the same form as at the zero loop level, Eq. 85, with the replacement of τ, w and u by the renormalized coupling $\widetilde{t}$, $\widetilde{w}$ and $\widetilde{u}$, respectively. This shows the full consistency of our scheme and guarantees that important qualitative features of the spectrum (e.g. gaplessness) are not violated. The renormalized masses themselves can then be obtained from their mean-field counterparts, with the replacement of the bare couplings by the tilded ones. Considering Eq. 109 we can then see that the centre of the large mass band will change very little but, due to the renormalization of the u coupling, the bandwidth will grow considerably as the dimension d decreases. The same is true for the upper edge of the small mass band. These are further signals of growing RSB effects (without RSB the spectrum would consist of two points).

We can sum up these findings in the following way: Parisi's RSB field theory has proved to be perturbatively stable above $d = 8$ dimensions. The loop corrections do not change the qualitative features of the theory, they just shift the constants by a small amount, and, at least to the low order regarded here, they even enhance RSB effects.

As we approach $d = 8$ from above, however, the mapping set up in Eq. 149 becomes singular: the integral I_8 (cut off at the lower end at $p = \sqrt{2t}$) blows up. This causes the most serious problem in $\widetilde{u}$ where I_8 appears alone, without any additional t factor multiplying it. As it stands, $\widetilde{u}$ simply does not make sense below $d = 8$: for $t \to 0$ it diverges like $t^{(d-8)/2}$. The next section is devoted to the resolution of this paradox.

11 Between 6 and 8 dimensions

The formal reason for the mapping (149) becoming singular is easy to understand: whereas for $d > 8$ the loop integral I_8 is dominated by the contribution of momenta around the UV cutoff, for $d < 8$ the main contribution comes from the lower end where the momentum is of the same order of magnitude as the large mass. Then the large-p expansion is not a good representation of the propagators any more. In particular, as the p^{-8} singularity comes from the expansion of $G_{\alpha\gamma,\gamma\beta}$, we have to treat this component more carefully. (The other propagator, $G_{\alpha\beta,\alpha\beta}$, in the u-loop term in (146) does not cause problems for $d > 6$, so we can continue to use the large-p expansion result Eq. 42 for it.)

Working out the replica summation in Eq. 146 we get in the continuum limit:

$$\sum_{\gamma \neq \alpha,\beta} G_{\alpha\gamma,\gamma\beta} = -\int_0^x \mathrm{d}y G_{x,1}^{y,y} - x G_{x,1}^{x,x} - 2\int_x^1 \mathrm{d}y G_{x,1}^{x,y} \quad , \quad \alpha \cap \beta = x \quad . \tag{150}$$

Considering that $x \leq x_1 \sim t \ll 1$, it is clear that the largest term in Eq. 150 comes from the upper end of the last integral, so we can again use the "innermost block approximation" here:

$$\sum_{\gamma \neq \alpha,\beta} G_{\alpha\gamma,\gamma\beta} \approx -2G_{x,1}^{x,x_1} \quad . \tag{151}$$

Now one can check that for $d > 6$ the contribution of the far infrared region to the integral of Eq. 151 over the momentum is still negligible, so we can use Eq. 62, valid in the near infrared, for the propagator in Eq. 151. Then substituting Eqs. 62 (taken at $y = x_1$) and 42 into Eq. 146 and collecting the coefficients of $q(x)$ and $q^3(x)$, respectively, we arrive finally at a new set of renormalized coupling constants which are, in fact, very similar to those in Eq. 149. The only difference is that $\widetilde{u}$ is now given by

$$\widetilde{u} = u + 12\frac{w^4}{z}\int \frac{\mathrm{d}^d p}{(2\pi)^d}\frac{1}{p^4(p^2+2t)^2} \quad , \tag{152}$$

where $2wq_1$ in the denominator of the integrand has been replaced by its zeroth order value $2t$. In the dimension range $6 < d < 8$ regarded here the integral in Eq. 152 is well-defined both for small and for large momenta, so to leading order in t we can send the upper cutoff to infinity. The effective coupling $\widetilde{u}$ then becomes

$$\widetilde{u} = u + \mathrm{const} \cdot \frac{t^{(d-8)/2}}{z} \quad . \tag{153}$$

for fixed t and z very large (i.e. for temperatures not very close to T_c and for very long range forces) the shift in the quartic coupling is small, and we are back to the previous situation: the loop-corrections do not significantly alter the mean field results. In the opposite limit, however, where z, albeit large, is kept fixed and the temperature is allowed to go arbitrarily close to T_c, the bare u becomes negligible compared with what was supposed to be a small correction. It is clear that under these circumstances the ordinary loop-expansion breaks down.

Standard power counting arguments tell us that the upper critical dimension of the model defined in Eq. 2 is $d_c = 6$. It would therefore be perfectly normal to find an infrared breakdown of the perturbation expansion in six dimensions. Why we should have IR problems already in $d = 8$, however, demands explanation.

To understand the origin of the problem, we have to realize that the term that blows up in 8 dimension is a one-loop contribution to the 4-point function at zero external momenta. It is evident that this quantity (the "box graph") should be singular at the critical point in $d = 8$.

It should be emphasized that this effect is not a spin glass peculiarity. Many-point functions at exceptional momenta (where power counting arguments fail) can blow up at

T_c in high dimensions in any theory. For example, the one-loop correction to the sixth derivative of the thermodynamic potential with respect to the magnetization in an ordinary φ^4 theory (the triangle graph) will be singular at T_c in $d = 6$, i.e. *above* the upper critical dimension of that model. This six-point function is related to a higher order nonlinear susceptibility, which is normally of little interest, so the problem is rarely mentioned. The triangle graph (and all the higher polygons that have similar IR singularities in higher and higher even dimensions) enter, however, as insertions also in some of the high order loop-corrections to physical quantities such as the two-point function (i.e. the inverse linear susceptibility) which are regularly discussed and claimed to be free of IR problems above the upper critical dimension. One may wonder whether these IR-singular many-point insertions do not make also the 2-point function singular. They do not: the singularity in the many-point functions occurs only at exceptional (zero external) momenta, so when these graphs appear as insertions in higher order diagrams and finite momenta flow through them their singularity will be suppressed by the loop integrals. Therefore, the isolated IR singularities appearing in the many-point functions do not proliferate in higher orders and do not destroy the loop-expansion for the quantities which one is usually interested in.

The situation in spin glass field theory is quite similar to the above, except that here the many-point functions do appear directly in the coefficients of the Taylor expansion of the order parameter in the variable x. In particular, the 4-point function appears in the x^3 term in the equation of state, hence in the slope of $q(x)$, so if we want to calculate $q(x)$ to linear order we have to learn how to handle this difficulty.

Since above 6 dimensions the singularity will not proliferate in higher orders, it is evident what we have to do. We have to absorb this isolated singular term into the mean field part of our equation and treat the rest as perturbation. This way we can save the loop-expansion between 6 and 8 dimensions in a reorganized form.

The starting point in this new expansion is an effective mean field theory which is exactly of the same structure as what we have discussed so far but with the bare coupling u replaced by $\widetilde{u}$ everywhere. Since the remaining loop-corrections do not qualitatively modify the results, in the following we restrict ourselves to the discussion of this effective MFT. Moreover, we will be interested only in the fluctuation dominated regime where

$$\frac{t^{(d-8)/2}}{z} \gg 1 \quad , \tag{154}$$

so the bare u can even be completely omitted.

The nontrivial renormalization of the four-point coupling has a profound consequence for the structure of the theory. To fully appreciate its significance we have to go back for a moment to the original MFT. As first stressed by Fisher and Sompolinsky [53], scaling is badly violated in Parisi's MFT: not only the so called hyperscaling laws break down, but also those that do not explicitly contain the dimension d, such as e.g. $\beta\delta = 1 - \alpha/2 + \gamma/2$. As a matter of fact, it is not even possible to unambiguously assign a critical exponent to some physical quantities. For example, the maximum of the order parameter function scales as $q_1 \sim t$, but Sompolinsky's susceptibility anomaly [54] $\Delta = \frac{1}{T}\left(q_1 - \int_0^1 \mathrm{d}x\, q(x)\right)$ which is a particularly useful measure of ordering in the presence of an external magnetic field goes as $\Delta \sim t^2$. The existence of two "mass scales", i.e. two characteristic lengths, one diverging as $t^{-1/2}$ the other as t^{-1}, is another example of this ambiguity. Now the most important effect of the replacement $u \to \widetilde{u}$ below 8 dimensions is that upon approaching the upper critical dimension $d = 6$ from above scaling gets gradually restored [53], [15,16].

Indeed, consider e.g. the slope $c = w/2\widetilde{u}$ of $q(x)$. With $\widetilde{u} \sim t^{d/2-4}/z$, valid in the fluctuation dominated regime, the slope becomes $c \sim zt^{4-d/2}$. For $d \to 6^+$ c is therefore $\sim t$. On the other hand, the plateau of $q(x)$ is independent of $\widetilde{u}$, so it remains $q_1 \sim t$. As for the breakpoint $x_1 = 2\widetilde{u}t/w^2$, it becomes $\sim t^{d/2-3}/z$. Note that for $d \to 6^+$ the breakpoint becomes *independent* of the temperature (but remains small, of the order of $1/z$). What all this amounts to is that upon approaching $d = 6$ from above the order parameter function becomes of the form

$$q(x) = t^{\beta} f(x) \quad , \tag{155}$$

where $\beta = 1$, and $f(x)$ is independent of temperature. The form (155) is necessary for all the meaningful combinations one can form from $q(x)$ to scale with the *unique* exponent β.

Similar remarks apply for the mass scales. The center of the band of the large eigenvalues remains $2t$, but as shown by Eq. 109 the band width $4t^2\widetilde{u}/3w^2$ varies with d as $\sim t^{d/2-2}/z$ and becomes $\sim t/z$ as $d \to 6^+$. The same is true for the upper edge $2\widetilde{u}t^2/w^2$ of the band of small eigenvalues. This means that instead of two mass scales having different temperature exponents, we have a single exponent $\nu = 1/2$ for $t \to 6^+$. Note however that the separation of scales still remains in the form of a numerical difference: the small eigenvalues and the band width of the large eigenvalues is down by a factor $1/z$ compared with the center of the large eigenvalues.

To make contact with the work of Green, Moore and Bray [55] (who were, in fact, the first to note the role of the renormalized four-point coupling in a particular instance), we consider now the effect of an external field h on the order parameter $q(x)$. Although we have avoided the problem of the field in this paper, it is easy to show that the replacement $u \to \widetilde{u}$ will work also in the presence of the field. Therefore the well-known results for the field dependence of $q(x)$ can be readily taken over from ordinary MFT, and one can see, in particular, that the AT line [31] will be given by $h_{AT}^2 = 4\widetilde{u}t^3/3w^3$. The standard mean field result $h_{AT}^2 \sim t^3$ will then be replaced by $h_{AT}^2 \sim t^{d/2-1}/z$ for $6 < d < 8$, becoming $h_{AT}^2 \sim t^2/z$ as $d \to 6^+$. This coincides with the result found by Green et al. [55] which they obtained from the zero of the replicon self-energy in a one-loop calculation in the disordered phase. The agreement between these two independent calculations is a testimony for the consistency of the effective MFT. Green et al. also suggested that the shifted AT exponent might be exact. This must indeed be true for all the exponents predicted by the effective MFT, because, as argued above, the singularity found in the four point function will not proliferate and the higher loop corrections will not be more singular then the one-loop contribution worked out here.

When we were talking about the limit $d \to 6^+$ we always meant to remain slightly above $d = 6$. In exactly six dimensions the other corrections, disregarded in the effective MFT also become singular and produce the usual logarithms appearing at the upper critical dimension. The little we know about the range $d < 6$ is the subject of the next section.

12 The first step below 6 dimensions

As we cross $d = 6$ infrared divergences start to plague the whole perturbation expansion. Now they do proliferate: every order will be more singular then the previous one. In this situation one normally turns to renormalization group (RG) methods to reorganize the perturbation expansion. The expansion in the distance ε from the upper critical dimension proved particularly successful in the context of ordinary critical phenomena. The application of field theoretic RG methods to spin glasses was pioneered by Harris, Lubensky and Chen [32]. Building on [33], Green calculated spin glass critical exponents to $\mathcal{O}(\varepsilon^3)$ in [34]. The structure of the RG in spin glasses is poorly understood, however. No systematic RG analysis has ever been made in the condensed phase, and the negative result of Bray and Roberts [56] who failed to locate a stable fixed point in a field below $d = 6$ remains an enigma to resolve.

Neither do we have a magic key to the riddle. What we are able to do here is to take just the very first step toward an ε-expansion in the ordered phase of spin glasses, without the solid basis of an RG analysis. The findings in the previous section provide some important clues.

The first of these is that the bare quartic coupling u can be dropped altogether – the cubic theory builds up its own quartic coupling at the first loop level. This was already observed by Bray and Moore [57] who, working around a replica symmetric starting point, run into an AT-like instability, however. Secondly, with the restoration of scaling the breakpoint x_1 of the order parameter becomes small, of $\mathcal{O}(1/z)$. If there exists an ε-expansion at all, this should translate into $x_1 = \mathcal{O}(\varepsilon)$ below $d = 6$. But if x_1 is small, so is the whole region where RSB occurs, hence to leading order we can apply the innermost block approximation in all replica summation. Thirdly, the upper edge of the small mass band must also become $\mathcal{O}(\varepsilon)$, so to leading order we can always use the near infrared propagators.

With all these simplifications the calculation of some of the critical exponents to $\mathcal{O}(\varepsilon)$ becomes quite feasible, for the exponent β which we present here as an illustration, it becomes almost trivial.

In order to obtain β, it is sufficient to calculate q_1. Let us regard the equation of state as a polynomial in x. Then a little reflection shows that the maximum of $q(x)$ is determined by the linear terms in x. (The slope at $x = 0$ would be determined by the x^3 terms.)

Where do we get linear terms in Eq. 146? The first term is such. In the second term we have

$$(q^2)_{\alpha\beta} = -\int_0^x \mathrm{d}y q^2(y) - xq^2(x) - 2q(x)\int_x^1 \mathrm{d}y q(y) \quad , \quad \alpha \cap \beta = x \quad , \tag{156}$$

which provides a linear term coming from the last integral

$$(q^2)_{\alpha\beta} \to -(2 - x_1)q_1 q(x) + \mathcal{O}(x^3) \quad . \tag{157}$$

With the bare u omitted the only term we are left with is the w-loop. Using the innermost block approximation (151) for the replica summation, and selecting the linear term from the near infrared approximation (62) for the propagator appearing here we have

$$\sum_{\gamma\neq\alpha,\beta}\left(G_{\alpha\gamma,\gamma\beta}-\frac{wq_{\alpha\beta}}{p^4}\right)=-\frac{4wtq(x)}{p^2(p^2+2t)^2}+\mathcal{O}(x^3)\quad, \tag{158}$$

where, in view of this loop term being of $\mathcal{O}(\varepsilon)$, we used the zeroth order result $wq_1=t$.

Collecting Eqs. 156−158 and evaluating the integral of Eq. 158 in $d=6$ for $t\to 0$, we find that the equation of state to first order in x is:

$$\left(2t-2wq_1+wx_1q_1-\frac{4w^2K_6}{z}\ln\sqrt{2t}\right)q(x)+\mathcal{O}(x^3)=0\quad, \tag{159}$$

where $K_6=\frac{1}{64\pi^3}$.

Writing $wq_1=At^\beta=(1+A^{(1)}+\cdots)t(1+\beta^{(1)}\ln t+\cdots)$ where $A^{(1)}$ and $\beta^{(1)}$ are the $\mathcal{O}(\varepsilon)$ corrections to the amplitude A and to the critical exponent β, respectively, we finally obtain

$$\beta^{(1)}=\frac{w^2K_6}{z}\quad. \tag{160}$$

From [33,34] we know that at the fixed point of the cubic coupling the combination in Eq. 160 is $\varepsilon/2$. Thus we get

$$\beta=1+\frac{\varepsilon}{2}+\mathcal{O}(\varepsilon^2)$$

which is in accord with the exponents calculated at T_c [34].

If the interpretation of the IR logarithm coming from the loop-integral as a correction to β is correct then at the next order the coefficient of $\ln^2 t$ must be half of the coefficient of the first order log. We have evaluated the two-loop correction to the equation of state to $\ln^2 t$ order and found that the above condition of exponentiation is fulfilled, indeed, provided (160) is chosen $\varepsilon/2$, its fixed point value.

Although this provides an important check on our first order calculation and raises the hope that the ε-expansion can be consistently carried through also below T_c, we have to point out that at the second order we find not only the $\ln^2 t$, $\ln t$ terms necessary for scaling, but also infrared divergences like p^{-6}. In higher orders we will clearly find even stronger infrared powers, due to the presence of zero modes and soft modes in the system. Though we do notice some cancellations, a systematic method (similar to the exploitation of rotational symmetry through the use of Ward identities in the O(n) model, or the use of the small masses as an infrared cutoff) is still to be found in spin glass theory. Work is in progress in that direction.

13 Summary

Let us briefly recapitulate the main points in the paper. Having set up the field theoretic formalism, we devoted a long discussion to the free, quadratic fluctuations about an equilibrium solution with Parisi's ultrametric symmetry. We showed that for momenta higher than any characteristic mass, i.e. for wavelengths shorter than any characteristic distance in the system, the propagators describing these fluctuations can be obtained even without having to specify the concrete symmetry breaking pattern.

In the long but not extremely long wavelength limit, that we called the near infrared, we obtained simple approximate forms (rational fractions with poles at zero and at the large mass) for the propagators. We also found that in this region the propagators show a simple and explicit functional dependence on the order parameter: they are polynomials, going up to the quartic order, in $q(x)$.

We also displayed the group theoretic and algebraic techniques one needs to block- diagonalize any ultrametric matrix. The additional simplification occuring in the kernel of these block- diagonal forms in the case of the Gaussian propagators near T_c then allowed us to obtain exact, closed expressions from which the far infrared behaviour could be determined. The dependence of the propagators on $q(x)$ is much more complicated in this region than in the near infrared, in particular, the high IR powers we found for extremely small wavenumbers are intimately related to $q(x)$ being a linear function for small x.

The physical meaning of these results could be understood on the basis of the relationships we established between the propagators and some intra- and intervalley overlaps of spin-spin correlation functions. In particular, we learned that transverse fluctuations are long-ranged also inside a single phase, while the longitudinal fluctuations have a finite coherence length, given by the inverse large mass. Overlaps of correlations between different phase space valleys were found to start to qualitatively deviate from those inside a single phase around a distance given by the inverse small mass. Most remarkably, inside a single phase the extreme long wavelength behaviour was found to be a smooth continuation of the behaviour on the intermediate, near infrared scales, whereas the overlap between correlation functions in two distant valleys shows a markedly different, more singular behaviour in the far infrared than in the near infrared.

Having understood the structure of correlations in the various regimes we proceeded to apply our propagators in the calculation of the first loop corrections. We showed that for $d > 8$ the theory maps back onto MFT with small, numerical shifts in the coupling constants, which demonstrated the stability of Parisi's MFT against short range corrections, at least to the low order investigated here.

For $6 < d < 8$ we had to reorganize the loop expansion in order to incorporate a divergent correction to the quartic coupling. This led us to an effective MFT with exactly calculable but dimension dependent critical exponents, and enabled us to follow the development of the theory with decreasing dimensionality towards a form reached at $d = 6^+$, with scaling restored but a nontrivial replica symmetry breaking pattern preserved.

Entering the range $d < 6$ we noted that both the breakpoint x_1 and the small masses become of the order of ε . This simplified the calculation tremendously, and allowed us to readily identify the term that yields the $\mathcal{O}(\varepsilon)$ correction to the critical exponent of the order parameter, and even to check exponentiation at the next order.

We concluded the paper by pointing out the challenge posed by the appearence of infrared singularities in the expansion and by hinting at the methods by which this challenge may be met in the future.

Acknowledgments

During the course of the long-term research project reviewed in this paper we benefitted from interaction with a large number of colleagues, of whom special thanks are due to A. Bray, E. Brezin, B. Derrida, S. Franz, Z. Horváth, C. Itzykson, M. Mézard, M. A. Moore, H. Orland, G. Parisi, N. Sourlas, A. P. Young, J. Zinn-Justin and J.-B. Zuber. This work has been supported by the French Ministry of Foreign Affairs, contract NGE3T0/023, by the Hungarian National Science Fund OTKA, Grant No. T 019422, and by the European Research Network on "Statistical Physics of Collective Behaviour in Disordered Systems and Information Processing", contract CHRX-CT92-0063.

References

1. G. Parisi, *Phys. Rev. Lett.* **43**, 1754 (1979); *J. Phys.* A: *Math. Gen.* **13**, L115 (1980); **13**, 1101 (1980); **13**, 1887 (1980).
2. D. Sherrington and S. Kirkpatrick, *Phys. Rev. Lett.* **35**, 1792 (1975).
3. W.L. Mc Millan, *Phys. Rev.* B **29**, 4026 (1984); **30**, 476 (1984).
4. D.S. Fisher and D. Huse, *Phys. Rev. Lett.* **56**, 1601 (1986); *Phys. Rev.* B **38**, 386 (1988).
5. A.J. Bray and M.A. Moore, in *Proceedings of the Heidelberg Colloquium on Glassy Dynamics* (Lecture Notes in Physics 275), eds. J.L. van Hemmen and I. Morgenstern (Springer, Heidelberg, 1986).
6. A.T. Ogielski and I. Morgenstern, *Phys. Rev. Lett.* **54**, 928 (1985).
7. R.N. Bhatt and A.P. Young, *Phys. Rev. Lett.* **54**, 924 (1985).
8. R.N. Bhatt and A.P. Young, *Phys. Rev.* B **37**, 5606 (1988).
9. J.D. Reger, R.N. Bhatt and A.P. Young, *Phys. Rev. Lett.* **64**, 1859 (1990).
10. S. Caracciolo, G. Parisi, S. Patarnello and N. Sourlas, *Europhys. Lett.* **11**, 783 (1990).
11. N. Kawashima and A.P. Young, *Phys. Rev.* B **53**, R484 (1996).
12. E. Marinari, G. Parisi, J.J. Ruiz-Lorenzo and F. Ritort, *Phys. Rev. Lett.* **76**, 843 (1996).
13. E. Marinari, G. Parisi and J.J. Ruiz-Lorenzo, this volume.
14. A. Georges, M. Mézard and J.S. Yedidia, *Phys. Rev. Lett.* **64**, 2937 (1990).
15. C. De Dominicis, I. Kondor and T. Temesvári, *J. Phys.* A: *Math. Gen.* **24**, L301 (1991).
16. I. Kondor, C. De Dominicis and T. Temesvári, *Physica* A **185**, 295 (1992).
17. C.M. Newman and D.L. Stein, *Phys. Rev. Lett.* **72**, 2286 (1994); *J. Stat. Phys.* **82**, 1113 (1996); *Phys. Rev. Lett.* **76**, 515 (1996); *Phys. Rev. Lett.* **76**, 4821 (1996).
18. D.L. Stein and C.M. Newman, *Phys. Rev.* E **51**, 5228 (1995).
19. M. Aizenman and J. Wehr, *Commun. Math. Phys.* **130**, 489 (1990).
20. C.M. Newman and D.L. Stein, *Phys. Rev.* B **46**, 973 (1992).
21. F. Guerra, *Int. J. Mod. Phys.* B **10**, 1675 (1996).
22. I. Kondor, *J. Phys.* A: *Math. Gen.* **22**, L163 (1989).
23. I. Kondor and A. Végső, to be published.
24. M.E. Ferrero and G. Parisi, *J. Phys.* A: *Math. Gen.* **29**, 3795 (1996).

25. T. Temesvári, I. Kondor and C. De Dominicis, to be published.
26. G. Parisi, F. Ritort and F. Slanina, *J. Phys.* A: *Math. Gen.* **26**, 247 (1993); **26**, 3775 (1993).
27. C. De Dominicis and I. Kondor, *J. Physique Lett.* **45**, L205 (1984).
28. C. De Dominicis and I. Kondor, *J. Physique Lett.* **46**, L1037 (1985).
29. C. De Dominicis and I. Kondor, in *Lecture Notes in Physics* vol. **216**, ed. L. Garrido (Springer, Heidelberg, 1985).
30. T. Temesvári, I. Kondor and C. De Dominicis, *J. Phys.* A: *Math. Gen.* **21**, L1145 (1988).
31. J.R.L. de Almeida and D.J. Thouless, *J. Phys.* A: *Math. Gen.* **11**, 983 (1978).
32. A.B. Harris, T.C. Lubensky and J.H. Chen, *Phys. Rev. Lett.* **36**, 415 (1976).
33. O.F. de Alcantara Bonfim, J.E. Kirkham and A.J. Mc Kane, *J. Phys.* A: *Math. Gen.* **14**, 2391 (1981).
34. J.E. Green, *J. Phys.* A: *Math. Gen.* **17**, L43 (1985).
35. S.F. Edwards and P.W. Anderson, *J. Phys.* F: *Metal. Phys.* **5**, 965 (1975).
36. See e.g. K. Binder and A.P. Young, *Rev. Mod. Phys.* **58**, 801 (1986).
37. M. Mézard, G. Parisi, N. Sourlas, G. Toulouse and M.A. Virasoro, *Phys. Rev. Lett.* **52**, 1156 (1984); *J. Physique* **45**, 843 (1984).
38. G. Parisi, *Phys. Rev. Lett.* **50**, 1946 (1983).
39. G. Pytte and J. Rudnick, *Phys. Rev.* B **19**, 3603 (1979).
40. C. De Dominicis, I. Kondor and T. Temesvári, *J. Phys.* I *France* **4**, 1287 (1994).
41. T. Temesvári, C. De Dominicis and I. Kondor, *J. Phys.* A: *Math. Gen.* **27**, 7569 (1994).
42. I. Kondor and C. De Dominicis, *Europhys. Lett.* **2**, 617 (1986).
43. P. Bántay and G. Zala, to be published.
44. C. De Dominicis, D.M. Carlucci and T. Temesvári, *J. Phys.* I *France* **7**, 105 (1997).
45. M. Mézard and G. Parisi, *J. Phys.* I *France* **1**, 809 (1991).
46. G. Parisi and N. Sourlas, unpublished.
47. D.M. Carlucci, thesis, Pisa, March 1997; D.M. Carlucci and C. De Dominicis, C.R.A.S. **325**, (1997).
48. C. De Dominicis and I. Kondor, in *Neural Networks and Spin Glasses* (Proc. Porto Alegre 1989), eds. W.K. Theumann and R. Köberle (World Scientific, Singapore, 1990).
49. I. Kondor, in *Festkörperprobleme (Advances in Solid State Physics)*, Volume XXV, 225, ed. P. Grosse (Vieweg, Braunschweig, 1985).
50. H. Sompolinsky and A. Zippelius, *Phys. Rev. Lett.* **50**, 1297 (1983).
51. A.V. Goltsev, *J. Phys.* C: *Solid State Phys.* **17**, L241 (1984); **19**, L215 (1986).
52. M. Mézard and M.A. Virasoro, *J. Physique* **46**, 1293 (1985).
53. D.S. Fisher and H. Sompolinsky, *Phys. Rev. Lett.* **54**, 1063 (1985).
54. H. Sompolinsky, *Phys. Rev. Lett.* **47**, 935 (1981).
55. J.E. Green, M.A. Moore and A.J. Bray, *J. Phys.* C: *Solid State Phys.* **16**, L815 (1983).
56. A.J. Bray and S.A. Roberts, *J. Phys.* C: *Solid State Phys.* **13**, 5405 (1980).
57. A.J. Bray and M.A. Moore, *J. Phys.* C: *Solid State Phys.* **12**, 79 (1979).