Off-equilibrium confined dynamics in a glassy system with level-crossing states

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We study analytically the dynamics of a generalized *p*-spin model, starting with a thermalized initial condition. The model presents birth and death of states, hence the dynamics (even starting at equilibrium) may go out of equilibrium when the temperature is varied. We give a full description of this constrained out-of-equilibrium behavior and we clarify the connection to the thermodynamics by computing (subdominant) TAP states, constrained to the starting equilibrium configuration.

I. INTRODUCTION

Many interesting physical systems live for very long times out of equilibrium, and, in this regime, they display highly nontrivial behaviors which are still to be understood (e.g., rejuvenation and memory effects in spin glasses). In general, these systems fall out of equilibrium when some external parameter is changed. For example, fragile glass-forming liquids undergo a dramatic slowing down of their relaxational dynamics when the temperature is dropped below the glass transition temperature.[1] This effect is sharpened in certain mean-field models where, at a critical temperature T_d, a transition occurs from an equilibrium kind of dynamics to an off-equilibrium aging one.[2] The phenomenon is ubiquitous and can be found also in very different fields: e.g., in local search algorithms for solving hard optimization problems the time complexity may become extremely large by varying a macroscopic parameter.[3] A better understanding of the mechanisms leading to the dramatic slowing down in out-of-equilibrium dynamics is a subject of broad interest and wide applicability.

In describing the dynamical slowing down (and possible eventual arrest) the common view suggests that at a low temperature a huge number of metastable states appears (with energies higher than the equilibrium one, E_{eq}), making relaxation to equilibrium very slow, and even impossible, if interactions are long ranged and metastable states lifetimes diverge in the thermodynamic limit. This picture has been verified by solving the out-of-equilibrium Langevin dynamics of a particularly simple mean-field model, the so-called fully connected spherical *p*-spin model, whose Hamiltonian is[4]

$$\mathcal{H}\{\underline{\sigma}\} = -\sum_{1 \leq i_1 < \ldots < i_p \leq N} J_{i_1 \cdots i_p} \sigma_{i_1} \cdots \sigma_{i_p}, \qquad (1)$$

where the N spins σ_i are continuous variables subject to the spherical constraint $\sum_i \sigma_i^2 = N$ and the couplings are independent and identically distributed random variables with zero mean and variance $p!/(2N^{p-1})$. In this model (hereafter $p \geq 3$) if we consider a *quench*, that is, if we choose an initial configuration of high energy and let the system relax at a fixed value of the temperature $T < T_d$ (T_d being the dynamic transition temperature), the asymptotic dynamics remains

trapped at the energy level of the highest and most numerous metastable states, the so-called *threshold states*. Time-translation invariance and the dynamic fluctuation-dissipation relation are violated and aging is observed in correlation and response functions.[5]

These features are intriguing and experimentally relevant, since aging behavior has been observed in many disordered systems. Nevertheless, in order to compare with more realistic situations it is extremely useful to understand the dynamical behavior during a cooling. In this case the system relaxes while the temperature is slowly decreased from an initial high value to a final temperature below T_d.

In the single-*p*-spin model, defined by the Hamiltonian (1), changing the temperature during the dynamics has no effect on the asymptotic states approached by the relaxing system.[6] This can be easily understood by considering the structure of metastable states. Thanks to a particular symmetry, at any temperature $T \leq T_d$ the ordering (in free energy) of the metastable states is the same they have (in energy) at zero temperature.[7] As a consequence the metastable threshold states, those where the out-of-equilibrium dynamics converges to, are the same at any temperature below T_d. However, for this very reason, the single-*p*-spin model can be considered as pathological. In more realistic systems, when the temperature (or any other external parameter) varies, metastable states may appear or disappear and their ordering is no longer preserved. For such systems, many questions on the dynamical behavior are still open: For example, it is not clear what is the asymptotic state of the off-equilibrium dynamics if the temperature varies during the relaxation and whether such an asymptotic state can be computed directly from the thermodynamical measure (i.e., statically), without solving the dynamical equations. In this paper we try to give some answers to the above points.

A last comment on the relevance of the present work concerns the difference between a cooling and a quench. The common experience (exploited by the simulated annealing method) tells us that, in the same amount of finite time, the cooling is able to reach lower energies than the quench. But what happens in the large times limit? In the single-*p*-spin model the answer is simple. Any cooling scheme, included the quench, converges to threshold states, which are T independent. On the contrary, if the threshold states varied with temperature, the answer was unclear and we show analyti-

cally that the asymptotic state may depend on the cooling scheme.

The rest of the paper is organized as follows. Section II describes the model we want to study and summarizes what was already known about it; in Sec. III we write and solve the equation for the out-of-equilibrium dynamics, starting from a thermalized configuration; in Sec. IV we report the results of the computation of the number of TAP states, constrained to a fixed distance from a reference state (more details are given in the Appendix) and we compare these results with the solution of the dynamical equations. Finally in Sec. V we summarize our results and give some future perspectives.

II. THE MODEL

We focus our attention on a modified version of the p-spin Hamiltonian, the so-called multi-p-spin model, in which spins do interact in r-uples with r taking more than one value:

$$\mathcal{H}\{\underline{\sigma}\} = -\sum_r c_r \sum_{1 \leq i_1 < \ldots < i_r \leq N} J_{i_1 \cdots i_r} \sigma_{i_1} \cdots \sigma_{i_r}. \quad (2)$$

Using the overline for the disorder average, we have that

$$\overline{\mathcal{H}\{\underline{\sigma}\}\mathcal{H}\{\underline{\tau}\}} = \frac{1}{2}\sum_r c_r^2 q^r \equiv f(q), \quad (3)$$

where $q \equiv \Sigma_i \sigma_i \tau_i / N$ is the overlap among $\underline{\sigma}$ and $\underline{\tau}$. The single-p-spin model corresponds to $f(q) = q^p/2$.

The choice of the Hamiltonian (2) is motivated by the request of an exactly solvable dynamics, for which we need continuous variables interacting in a fully connected fashion. Unlike the single-p-spin case, in the multi-p-spin model there is level crossing of metastable states by varying the temperature.[8]

From a statical point of view, this model is characterized by the presence of a large number of metastable states $\mathcal{N}(f) \sim \exp[N\Sigma(f)]$. The so-called *complexity* Σ is an increasing function of the free-energy f which is zero at the lower band edge f_0 and maximal for a certain value $f = f_{max}$. For high temperatures the Gibbs measure is dominated by the paramagnetic state ($m_i = 0$), while for $T < T_d$ (T_d being the dynamical critical temperature), metastable states start to play a relevant role, much in the same way as for the single-p-spin model. In this region the thermodynamic equilibrium is given by a class of metastable "equilibrium" states with finite complexity (see, e.g., Ref. 9), the global free energy of the system thus bearing a contribution from this state-related entropy, i.e., $-T \ln Z = F = f_{eq} - T\Sigma(f_{eq})$. Lowering the temperature still more, the complexity of the equilibrium states decreases until a point where it becomes zero and the lower band-edge states, nonexponential in number, become dominant. The temperature where this occurs, T_s, is the static transition temperature for this model, as can be seen also by a direct computation of the partition function with the replica method. The interpretation of this transition as an "entropy crisis" for metastable states is particularly relevant when comparing this model with real systems; indeed, fragile

glasses do exhibit in this respect a very similar phenomenology.

The structure of metastable states can be investigated in more detail by considering the TAP approach, where mean-field equations can be formulated for the local magnetizations m_i (at a fixed disorder realization), and stable solutions of these equations identified as states of the system. Recently, some novel intriguing features of this formalism have emerged, according to which metastable states can either satisfy a supersymmetry between fermionic and bosonic integration variables,[10] or break it.[11] Supersymmetric (SS) states are very robust to external perturbations, while supersymmetry breaking (SSB) ones are extremely fragile, and even a small perturbation can dramatically change their number and global structure.[12] Interestingly, the multi-p-spin model addressed in this paper, contrary to the single-p case, exhibits states of both classes[13] and allows a comparative study of their role. In particular, states in the range $[f_0, f_{th}]$ are SS, while states with $f \in [f_{th}, f_{max}]$ are SSB. The free-energy level f_{th} separating the SS from the SSB region, that we shall call *threshold energy*, also plays a relevant role in the dynamical behavior of this system.

Another important feature of metastable states, which is more relevant for the questions we want to address, is their behavior with changing the temperature. For the single-p-spin spherical model, as anticipated above, states can be transposed in temperature and their energy ordering does not change. There is no birth or death of states with varying the temperature, or, in other words, a TAP solution at zero temperature persists when the temperature is turned on until $T = T_d$. In the multi-p-spin model this is not the case. To investigate this point more explicitly, we need a method to "pin" out a state and follow it while varying the temperature. This can be done by resorting to a dynamical analysis.

III. THE DYNAMICS

Given the Hamiltonian (2) with a generic correlator $f(q)$, it is possible to write the equations for the Langevin dynamics at temperature $T = 1/\beta$ as

$$\frac{\partial \sigma_i(t)}{\partial t} = -\frac{\partial \mathcal{H}\{\underline{\sigma}\}}{\partial \sigma_i} + \eta(t), \quad (4)$$

where $\eta(t)$ is a thermal Gaussian noise with zero mean and variance

$$\langle \eta(t) \eta(t') \rangle = \frac{2}{\beta} \delta(t - t'). \quad (5)$$

Given the initial conditions this equation can be solved exactly using the method of the generating functional.[14] If the initial condition is drawn from a Boltzmann distribution with inverse temperature β', self-consistent equations for the correlation function $C(t,t') = \langle \sigma_i(t)\sigma_i(t') \rangle$ and the response function $R(t,t') = \partial \sigma_i(t)/\partial h_i(t')$ read

$$\frac{\partial C(t,t')}{\partial t} = -\mu(t)C(t,t') + \int_0^{t'} ds f'[C(t,s)]R(t',s)$$

$$+ \int_0^t ds R(t,s)f''[C(t,s)]C(s,t')$$

$$+ \beta' f'[C(t,0)]C(t',0), \qquad (6)$$

$$\frac{\partial R(t,t')}{\partial t} = -\mu(t)R(t,t') + \int_0^{t'} ds f''[C(t,s)]R(t,s)R(s,t'), \qquad (7)$$

where $\mu(t)$ is a Lagrange multiplier implementing the spherical constraint on the spins and obeys the dynamical equation

$$\mu(t) = \int_0^t ds f'[C(t,s)]R(t,s) + \int_0^t ds R(t,s)f''[C(t,s)]C(s,t)$$

$$+ \frac{1}{\beta} + \beta' f'[C(t,0)]C(t,0). \qquad (8)$$

The most studied case is the one where initial conditions are chosen at random ($\beta'=0$), and the system starts exploring the configuration space from a high-energy configuration. In this context, for example, the first analytical complete treatment of aging behavior has been carried out for the single-p-spin model.[5]

From our point of view, however, the most interesting situation is another one. If we choose the initial condition $\underline{\sigma}(t=0)$ to belong to a given metastable state, then we can let the system evolve and check whether the state is stable and well defined (in which case we expect an equilibriumlike relaxation dynamics inside the state) or loses stability (exhibiting off-equilibrium behavior). To this aim,[15] we may choose an initial condition thermalized at temperature $T' = 1/\beta'$, i.e.,

$$P\{\underline{\sigma}(0)\} = \frac{1}{Z} \exp[-\beta'\mathcal{H}\{\underline{\sigma}(0)\}]. \qquad (9)$$

Indeed, since for $T_s < T' \leqslant T_d$ the Boltzmann measure is dominated by a class of metastable states, the distribution (9) naturally picks out a configuration $\underline{\sigma}(0)$ which belongs to one of such states. Besides, since the class of dominating states varies with the temperature, we can use T' to select the kind of state (i.e., energy, complexity, and self-overlap) we want the system to start in.

Summarizing, the dynamics that we are considering, described by Eqs. (4)–(9), involves two distinct temperatures. The first one, T, controls the thermal noise and therefore represents the temperature at which the dynamical evolution takes place. The second one, T' is used to force the system to start into a given metastable state, and to select its properties. We now analyze the dynamical behavior of the system by tuning these two parameters.

A. The quench

The case $\beta'=0$ corresponds to random initial conditions, that is to a *quench*. In this case the system undergoes a dy-

namical transition at a critical temperature $T=T_d$ where the relaxation time diverges (much in the same way as in the single-p-spin model). For $T < T_d$ the dynamics exhibits aging and asymptotically reaches the threshold states, which have energy density E_{th} (corresponding to free-energy density f_{th}) and self-overlap $q_{th}=q_m$, where $q_m(T)$ is the solution to the marginality condition

$$f''(q_m)(1-q_m)^2 = T^2. \qquad (10)$$

A similar behavior occurs for any $T' > T_d$.

We note that the dynamics following a quench always converges to the edge of the SS region, despite that a larger number of SSB states are present at higher energy densities ($E_{th} < E < E_{max}$). At least two explanations are possible: (i) SSB states are "invisible" for the dynamics we have solved; (ii) SSB states are marginally unstable (they have a finite number of zero modes in the thermodynamic limit) and they are unable to trap the system during the relaxation.[11]

B. The case $T_s < T' \leqslant T_d$

When $T_s < T' \leqslant T_d$, the situation is more complex: as described above the thermodynamic equilibrium is no longer given by the paramagnetic state but rather by a set of metastable states with energy density $E \in [E_0, E_{th}]$. Thus, the initial configuration belongs to one of such states. For $T=T'$ the system undergoes an equilibrium dynamics in the state where it was at the starting time. For $T < T'$ the initial condition is out of equilibrium and, according to the value of T, different dynamical behaviors can be observed. In particular, a critical temperature $T_{ag}(T')$ exists, such that

(i) For $T_{ag}(T') < T < T' < T_d$ the system follows, at large times, an equilibrium relaxation dynamics. Equations (6) and (7) can be easily solved exploiting time-translational invariance and the fluctuation-dissipation relation between correlation and response. The asymptotic regime is then fully described by the two parameters

$$q_1 = \lim_{(t-t')\to\infty} \lim_{t'\to\infty} C(t,t') \quad \text{and} \quad \tilde{p} = \lim_{t\to\infty} C(t,0), \qquad (11)$$

which turn out to be different from zero,[15] similarly to the single-p-spin model.[6] The physical interpretation is clear: The system has been prepared inside an equilibrium state at temperature T'; at temperature T this state still exists, even if with slightly different features, and the system dynamically relaxes into it. In this view, q_1 identifies the self-overlap of the state at temperature T, while $\tilde{p}$ measures the overlap between the equilibrium state at T', where the initial configuration is placed, and the *same* state transposed at temperature T.

(ii) For $T < T_{ag}(T') < T_d$ the dynamics remains out of equilibrium even for large times, showing aging and violation of the time-translation invariance. Equations for the correlation and the response functions can be written using the same scaling ansatz as the single-p-spin model.[5,15] For asymptotic but close times $[t' \to \infty, \ t-t' = \mathcal{O}(1)]$, time-translation invariance is recovered and the parameter q_1 can be defined as in Eq. (11). For asymptotic and well separated times $[t' \to \infty, t/t' = \mathcal{O}(1)]$, the correlation function scales as
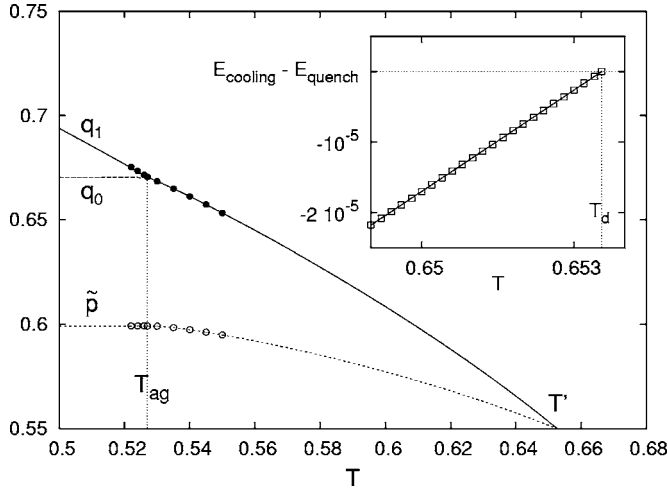
FIG. 1. The dynamical parameters q_1, q_0, and $\tilde{p}$ (curves); the mutual overlap q_{12} and the self-overlap, obtained from the TAP complexity computation (points). Here $f(q)=q^3/2+(0.45)^2q^4/2$. Relevant temperature values are: $T_d=0.6543$, $T'=0.653$, and $T_{ag}(0.653)=0.527$. Inset: difference between asymptotic energies in a quench and a cooling (see text).

$C(t,t')=\mathcal{C}(\lambda)$, with $\lambda \equiv t'/t \in [0,1]$ and $\mathcal{C}(1)=q_1$ (the same scaling holding for the response). This regime defines another asymptotic parameter $q_0=\lim_{\lambda\to 0}\mathcal{C}(\lambda)$. In this temperature region, the asymptotic limit is fully described in terms of the three parameters q_1, q_0, and $\tilde{p}$, together with the so-called fluctuation-dissipation ratio $x_{dyn}\equiv TR(t,t')/\partial_{t'}C(t,t')$ measuring the violation of the fluctuation-dissipation relation. The explicit equations for these quantities read[23]

$$1=\beta^2 f''(q_1)(1-q_1)^2,$$

$$\frac{q_1}{\beta(1-q_1)}=\beta f'(q_1)(1-q_1)+\beta x[q_1 f'(q_1)-q_0 f'(q_0)]$$
$$+\beta'\tilde{p}f'(\tilde{p}),$$

$$\frac{\tilde{p}}{\beta(1-q_1)}=\beta x\tilde{p}[f'(q_1)-f'(q_0)]+\beta' f'(\tilde{p}),$$

$$\frac{q_0}{\beta(1-q_1)}=\beta f'(q_0)(1-q_1)+\beta x f'(q_0)(q_1-q_0)$$
$$+\beta x q_0[f'(q_1)-f'(q_0)]+\beta'\tilde{p}f'(\tilde{p}). \quad (12)$$

The parameters q_0, q_1, and $\tilde{p}$ are plotted in Fig. 1 for $f(q)=q^3/2+(0.45)^2q^4/2$ (the same correlator used in Ref. 13). T' has been chosen very close to T_d in order to have a large T_{ag} value.

The first of equations (12) coincides with the marginality condition obeyed by threshold states and defining $q_m(T)$. However, the asymptotic dynamical energy $E_{dyn}=\lim_{t\to\infty}E(t)$ that we obtain is different (and lower) from the threshold energy E_{th}, indicating that the asymptotic dynamics takes place in a marginal manifold below the threshold states one. This feature also holds when $T'=T_d$, which is relevant for understanding the behavior of a cooling in this system.

Indeed an infinitely slow cooling is able to reach thermal equilibrium at any temperature above and at T_d:[24] thus for temperatures T below T_d an infinitely slow cooling is roughly equivalent to a dynamics starting thermalized at $T'=T_d$. Our result then indicates that for this system the asymptotic states reached with a cooling are lower than those reached with a quench. Actually the difference between these asymptotic states is really very tiny (see inset of Fig. 1). To our knowledge this is the first analytically solvable model showing up a dependence of the asymptotic dynamical states on the cooling schedule.

Note also that the solution of Eqs. (12) has $q_0 \neq 0$. This means that the system never decorrelates completely. We are observing aging together with a strong dynamic ergodicity breaking, contrary to the weak-ergodicity breaking scenario analyzed for this kind of models so far.[2] As long as $\tilde{p}>0$ the initial condition is not forgotten by the aging system and thus the initial condition acts like a magnetic field, inducing $q_0>0$.

The change in the dynamical behavior at $T=T_{ag}$ can be better understood by noticing that $q_1>q_m$ as long as $T>T_{ag}$, and $q_1=q_m$ for $T\leq T_{ag}$. The simplest interpretation is that states dominating the Gibbs measure at T' are stable for $T>T_{ag}(T')$, but at $T_{ag}(T')$ they become marginal, forming a manifold where the system ages on. For $T<T_{ag}(T')$ these states become unstable and the system keeps aging in a nearby critical manifold with $q_1=q_m$ and $q_0<q_1$.

This interpretation also tells us that in the multi-p-spin model, contrary to the single p spin, it is not possible to follow any state from $T=0$ to a finite temperature, or vice versa, because some states lose stability as the temperature is varied and there may be birth or death of states with temperature. Moreover, changing the temperature the complexity Σ varies along the dynamical trajectory, implying that there is a mixing of states. The variation in Σ is very tiny: e.g., for $f(q)=q^3/2+(0.45)^2q^4/2$ is $\mathcal{O}(10^{-5}-10^{-6})$ and for this reason it would be hardly visible in a numerical simulation.[16]

IV. THE CONSTRAINED COMPLEXITY

To confirm the interpretation given above and to better understand the dynamical behavior, we can look more in details at the structure of the metastable states in the region where the asymptotic dynamics occurs. In particular, we can consider the following static quantity. Given a reference state, that will be appropriately chosen as the state to which the initial configuration of the dynamics belongs, we compute the number of metastable states of given free-energy density that have fixed mutual overlap with it. Using the index 1 for the reference state and 2 for the metastable states we wish to count, we compute $\mathcal{N}(q_{12},f_2|f_1)\sim\exp[N\Sigma(q_{12},f_2|f_1)]$, that is the number of states of free-energy density f_2 that have mutual overlap q_{12} with a reference state of free-energy density f_1. Temperatures are not written explicitly, but it is assumed that $T_1=T'$ and $T_2=T$.

This computation can be performed in the TAP approach,[17] where metastable states are identified with local minima of the mean-field energy functional $F_{TAP}\{m\}=H\{m\}-1/(2\beta)\ln(1-q)-\frac{\beta}{2}[f(1)-f(q)-(1-q)f'(q)]$, where q

$\equiv N^{-1}\Sigma_i m_i^2$. In this case, the number of states reads

$$\mathcal{N}(q_{12}, f_2 | f_1)$$

$$= \int \prod_i (dm_i^{(1)} dm_i^{(2)} \delta(\partial_i F_{\mathrm{TAP}}\{\underline{m}^{(1)}\}) \delta(\partial_i F_{\mathrm{TAP}}\{\underline{m}^{(2)}\}))$$

$$\times |\det \widehat{H}\{\underline{m}^{(1)}\}||\det \widehat{H}\{\underline{m}^{(2)}\}| \delta(F_{\mathrm{TAP}}\{\underline{m}^{(1)}\} - N f_1)$$

$$\times \delta(F_{\mathrm{TAP}}\{\underline{m}^{(2)}\} - N f_2) \delta(\underline{m}^{(1)} \cdot \underline{m}^{(2)} - N q_{12}) \mathcal{N}(f_1)^{-1}.$$

$$\tag{13}$$

In this expression the first two δ functions ensure that $\underline{m}^{(1)}$ and $\underline{m}^{(2)}$ are stationary points of F_{TAP}, $(\widehat{H}_{ij}\{\underline{m}\} \equiv \partial_i \partial_j F_{\mathrm{TAP}}\{\underline{m}\}$ being the Hessian in the appropriate normalization factor), and imply that we are looking at solutions of the mean-field equations (i.e., states). The other δ functions fix, respectively, the free-energy densities and the mutual overlap of the solutions we wish to count. Note also that we have divided by the number of metastable states with free energy f_1 in order to get the number of states of free energy f_2 with overlap q_{12} with a given reference state of kind 1 (otherwise we would have gotten the number of pairs).

The entropy related to (13), also called *constrained complexity*, can be computed using standard techniques[18,19] within the annealed approximation, which is in general adequate to treat large free energies $f_1, f_2 \sim f_{th}$ (see the Appendix). Alternatively, the entropy can be computed as the Legendre transform of a constrained thermodynamic free energy.[15,20] The two results coincide within numerical precision.

Let us now use this constrained entropy to investigate the structure of the phase space sampled by the asymptotic dynamics. To this end, let us fix $f_1 = f_{eq}(\beta')$ and $f_2 = f_{dyn}(\beta) \equiv E_{dyn}(\beta) - TS(E_{dyn})$. That is, the reference state is chosen as the state where the dynamics has been started in (an equilibrium state at temperature T'), while the states to be counted have energy density equal to the asymptotic dynamical energy. The behavior of Σ as a function of q_{12} is displayed in Fig. 2. We see that two different situations occur above and below T_{ag}.

For $T > T_{ag}(T')$ the constrained complexity is positive and decreasing with increasing q_{12}, (as discussed for the single-p-spin model in Ref. 18), it becomes negative at some value of the mutual overlap and touches back the zero axis for $q_{12} = \tilde{p}$ (see the lowest curve in the inset of Fig. 2), with $\tilde{p}$ given by the dynamical equations (12). The interpretation is straightforward. For small overlaps we are counting states in a very large manifold, and we thus find many of them. As q_{12} decreases, this manifold becomes smaller and, consequently, the number of counted states decreases until it becomes zero (negative complexity). However, if we still increase the overlap, looking closer to the reference state, at some point we are bound to find the state itself. This is signaled by the zero value of the complexity at $q_{12} = \tilde{p}$, which therefore represents the overlap between the reference state and the same state evolved at temperature T, consistently with the dynamical interpretation. In this point one also has $q^{(2)} \equiv \underline{m}^{(2)} \cdot \underline{m}^{(2)} / N = q_1$, with q_1 given again by the dynamical equations. Please
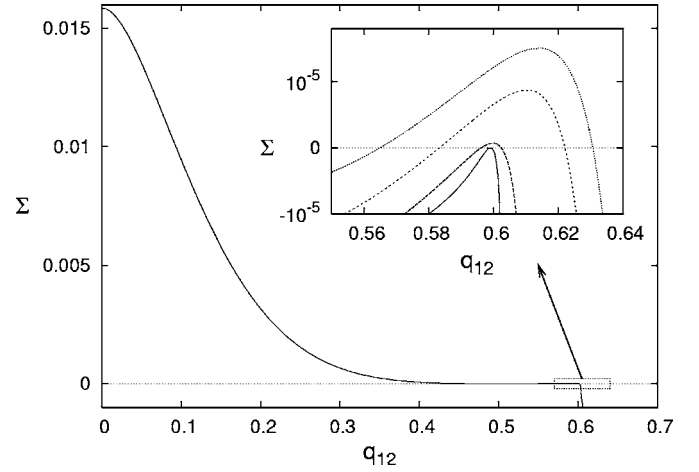

FIG. 2. Constrained complexity as a function of the mutual overlap for $T = 0.53$ and $T' = 0.653$ ($T_{ag} = 0.527$). Inset: the behavior of the secondary peak for different temperatures, from bottom to top, $T = 0.53$, $T = 0.5$, $T = 0.4$, and $T = 0.35$.

note that $q^{(2)} = q_1$ is not the typical value for TAP states at temperature T and free-energy f_{dyn}; so the dynamics is restricted to a set of subdominant states, that can be selected by constraining the TAP measure as in (13). The interpretation is straightforward: the T' equilibrium state has evolved in a slightly modified state at temperature T, which has overlap $\tilde{p}$ with the original one. This is the only state (i.e., $\Sigma = 0$) that we count at temperature T when fixing $q_{12} = \tilde{p}$. Note that this state is stable (by computing the replicon) and it is "isolated," that is the Σ curve is negative in the (q_{12}, f_2) plane around the point $(\tilde{p}, f_{dyn})$.

For $T < T_{ag}(T')$ the secondary peak of the constrained complexity becomes positive (see inset of Fig. 2) and the $\Sigma > 0$ region in the (q_{12}, f_2) plane (with f_1 fixed) is shown in Fig. 3. The T' equilibrium state opens up at $T = T_{ag}(T')$ and a nontrivial structure of metastable states appears close to where the dynamics is taking place; these states are responsible for the aging behavior, but it is still unclear which are the thermodynamical parameters of the dynamical asymptotic states. This is the main question when trying to describe
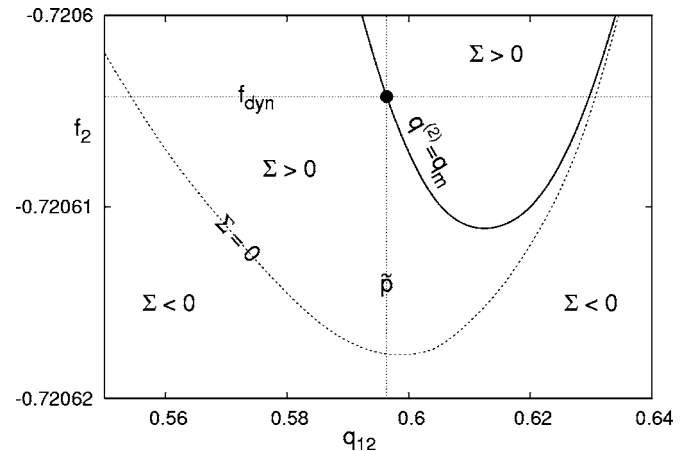

FIG. 3. Constrained complexity in the (q_{12}, f_2) plane, with $f_1 = f_{eq}(T' = 0.653)$ and $T = 0.35$.

the dynamical behavior in terms of static observations.

We know that for $T < T_{ag}$ the dynamics is taking place on a marginal manifold, so we can fix $q^{(2)} = q_m(T)$: states with this self-overlap are found along the full line in Fig. 3. All the points along this line are possible candidates for the asymptote of the dynamics, but understanding which one is actually chosen during system relaxation is a difficult task.

Consistently with the dynamical computation the point $(\tilde{p}, f_{dyn})$ is always on the line. Moreover, at this point, $x_{st} = x_{dyn}$ holds, where $x_{st}(q_{12}, f_2) \equiv T \partial_f \Sigma(q_{12}, f, f_1)|_{f=f_2}$ and x_{dyn} is the dynamical fluctuation-dissipation ratio.

It seems that at least one dynamically computed quantity must be plugged in the static computation to predict the asymptotic states: this can be equivalently $q_{12} = \tilde{p}$ or $x_{st} = x_{dyn}$ (for a quench the computation is easier: Starting with a random configuration one has $q_{12} = 0$ by definition). It would be very useful to find an extremizing principle to select, among all the candidate TAP states, those which are actually reached by the constrained out-of-equilibrium dynamics.

V. SUMMARY AND PERSPECTIVES

The spherical multi-p-spin model defined by the Hamiltonian (2) has the nice properties of being exactly solvable (thanks to its continuous variables), while showing nontrivial dependence on temperature of its states (birth, death, and level crossing). These features makes the model more realistic than other mean-field models and a perfect candidate for studying glassy relaxation under variations of temperature.

We have performed such a study finding several interesting analytical results. (i) The relaxation at any temperature converges to TAP states, satisfying the supersymmetry between fermionic and bosonic integration variables. In order to understand whether TAP states breaking the supersymmetry are relevant for finite times dynamics, the method described in Ref. 21 could be applied to the present model. (ii) Energies reached by a cooling are lower than those reached by a quench. This result is based on the assumption than an infinitely slow cooling equilibrates at any temperature $T \geq T_d$, which needs to be improved. (iii) The solution to the dynamical equations is consistent with the constrained complexity of TAP states computed thermodynamically: Starting from a thermalized configuration and lowering the temperature, the system starts aging when the state it belongs to becomes marginally unstable. For lower temperatures, states where aging is taking place cannot be predicted solely from the constrained complexity; a new extremizing principle is needed in order to make the connection between static and dynamic computations.

APPENDIX: THE CONSTRAINED COMPLEXITY VIA THE TAP APPROACH

Starting from Eq. (13), the computation of the constrained complexity can be performed with standard techniques. The annealed computation is the simpler one, since it involves averaging directly the number of solutions over the quenched disorder, rather than its logarithm. Replicas are therefore not needed.

To proceed, we introduce bosonic representations for the δ functions appearing in Eq. (13), and fermionic representations for the two determinants (the modulus can be safely disregarded for this model, since one can show that minima dominate over saddles in the relevant free-energy density range[22]). In this way, we get

$$\mathcal{N}(q_{12}, f_2, \beta | f_1, \beta')$$

$$= \frac{\int D[\underline{m}^{(1)}, \underline{m}^{(2)}, \underline{x}^{(1)}, \underline{x}^{(2)}, \underline{\bar{\psi}}^{(1)}, \underline{\psi}^{(1)}, \underline{\bar{\psi}}^{(2)}, \underline{\psi}^{(2)}, u_1, u_2, w] e^{S_{tot}}}{\mathcal{N}(f_1, \beta')}$$

(A1)

with

$$S_{tot} = \beta' S(\underline{m}^{(1)}, \underline{x}^{(1)}, \underline{\bar{\psi}}^{(1)}, \underline{\psi}^{(1)}, u_1; f_1, \beta')$$
$$+ \beta S(\underline{m}^{(2)}, \underline{x}^{(2)}, \underline{\bar{\psi}}^{(2)}, \underline{\psi}^{(2)}, u_2; f_2, \beta)$$
$$+ w(Nq_{12} - m^{(1)} \cdot m^{(2)}),$$ (A2)

$$S(\underline{m}, \underline{x}, \underline{\bar{\psi}}, \underline{\psi}, u; f, \beta) = \sum_i x_i \partial_i F_{\text{TAP}}(\underline{m}, \beta)$$
$$+ \sum_{ij} \bar{\psi}_i \partial_i \partial_j F_{\text{TAP}}(\underline{m}, \beta) \psi_j$$
$$+ u(F_{\text{TAP}}(\underline{m}, \beta) - Nf),$$ (A3)

and

$$F_{\text{TAP}}(\underline{m}, \beta) = H\{\underline{m}\} - 1/(2\beta) \ln(1 - q)$$
$$- \frac{\beta}{2} [f(1) - f(q) - (1 - q) f'(q)],$$ (A4)

where $\underline{x}^{(1,2)}$ are the Lagrange multipliers enforcing the TAP equations, $u_{1,2}$ are those enforcing the free-energy constraint, w is the one for the mutual overlap constraint, and $\underline{\psi}^{(1,2)}$, $\underline{\bar{\psi}}^{(1,2)}$ are the Grassman variables used to represent the determinants. As usual, ∂_i is the shorthand notation for ∂/∂_{m_i}.

After averaging over the disorder the numerator and the denominator of expression (A1), consistently with the annealed approximation, site-dependent variables can be integrated out, leaving an effective action which only depends on global variables:

$$\overline{\mathcal{N}} = \int D[\Omega, \omega, u_1, u_2, w]$$

$$\times \exp[NS_{eff}(\Omega, \omega, u_1, u_2, w; q_{12}, f_2, \beta, f_1, \beta') - N\Sigma(f_1, \beta')], \tag{A5}$$

where $\Omega = [q_1, q_2, B_1, B_2, R_1, R_2, B_{12}, B_{21}]$, defined by ($a, b = 1, 2$)

$$q_a = (\underline{m}^{(a)} \cdot \underline{m}^{(a)})/N, \quad B_a = (\underline{m}^{(a)} \cdot \underline{x}^{(a)})/N,$$

$$R_a = (\overline{\underline{\psi}}^{(a)} \cdot \underline{\psi}^{(a)})/N, \quad B_{ab} = (\underline{m}^{(a)} \cdot \underline{x}^{(b)})/N, \tag{A6}$$

and $\omega = [\lambda_1, \lambda_2, b_1, b_2, r_1, r_2, b_{12}, b_{21}]$ are the corresponding Lagrange multipliers.

The explicit expression for the effective action is the following:

$$S_{eff} = \frac{1}{2} \ln z - \frac{1}{2} \ln \Xi + \ln[2\beta' g'(q_1) + r_1]$$

$$+ \ln[2\beta g'(q_2) + r_2] + \beta' u_1[g(q_1) - f_1]$$

$$+ \beta u_2[g(q_2) - f_2] + \frac{\beta'^2}{2}(B_1^2 - R_1^2)f''(q_1)$$

$$+ \frac{\beta^2}{2}(B_2^2 - R_2^2)f''(q_2) + \frac{\beta'^2}{2}u_1^2 f(q_1) + \frac{\beta^2}{2}u_2^2 f(q_2)$$

$$+ \beta\beta' B_{12}B_{21}f''(q_{12}) + \beta\beta' u_1 u_2 f(q_{12}) - r_1 R_1 - r_2 R_2$$

$$- b_1 B_1 - b_2 B_2 - b_{12}B_{12} - b_{21}B_{21}$$

$$- \lambda_1 q_1 - \lambda_2 q_2 - wq_{12}, \tag{A7}$$

with

$$z = (\beta\beta')^2 \{f'(q_1)f'(q_2) - [f'(q_{12})]^2\},$$

$$\Xi = (l_{11} - 2z\lambda_1)(l_{22} - 2z\lambda_2) - (l_{12} - zw)^2,$$

$$g(q) = -1/(2\beta)\ln(1 - q) - \frac{\beta}{2}[f(1) - f(q) - (1 - q)f'(q)],$$

$$l_{11} = \beta^2 f'(q_2)d_{11}^2 - 2\beta\beta' f'(q_{12})d_{11}d_{21} + \beta'^2 f'(q_1)d_{21}^2,$$

$$l_{12} = \beta^2 f'(q_2)d_{11}d_{12} - 2\beta\beta' f'(q_{12})(d_{11}d_{22} + d_{21}d_{12})$$
$$+ \beta'^2 f'(q_1)d_{21}d_{22},$$

$$l_{22} = \beta^2 f'(q_2)d_{12}^2 - 2\beta\beta' f'(q_{12})d_{12}d_{22} + \beta'^2 f'(q_1)d_{22}^2,$$

$$d_{11} = 2\beta' A(q_1) + b_1 + \beta'^2 u_1 f'(q_1),$$

$$d_{12} = b_{12} + \beta\beta' u_2 f'(q_{12}), \quad d_{21} = b_{21} + \beta\beta' u_1 f'(q_{12}),$$

$$d_{22} = 2\beta A(q_2) + b_2 + \beta^2 u_2 f'(q_2).$$

The effective action (A7) has to be extremized with respect to all the integration variables in order to obtain the constrained complexity as $\Sigma(q_{12}, f_2, \beta, f_1, \beta') = S_{eff}^{extr}(q_{12}, f_2, \beta, f_1, \beta') - \Sigma(f_1, \beta')$.

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[23] Equations such as (12) first appeared in Ref. 15. However, in that paper there was a mistake in one such equation and the off-equilibrium dynamics was not solved.

[24] Convergence to equilibrium at T_d is a subtle point: there are no $\mathcal{O}(N)$ barriers (states are marginal), and so the equilibration time is less than exponential in N (we expect it to grow polynomially with N).