Elusive Spin-Glass Phase in the Random Field Ising Model

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We show rigorously that the spin-glass susceptibility in the random field Ising model is always bounded by the ferromagnetic susceptibility, and therefore that no spin-glass phase can be present at equilibrium out of the ferromagnetic critical line. When the magnetization is, however, fixed to values smaller than the equilibrium value, a spin-glass phase can exist, as we show explicitly on the Bethe lattice.

Few disordered spin models have generated as much interest and studies as the random field Ising model (RFIM). Yet, despite four decades of efforts in mathematics and physics, the thermodynamic properties and the nature of the phase transitions still remain debated. Originally proposed by Larkin [1] for modeling the pinning of vortices in superconductors, the RFIM has grown to be used for modeling problems as diverse as (among others) diluted antiferromagnets in a homogeneous external field [2], binary liquids in porous media [3], Coulomb—or electron—glass [4], as well as systems near the metal-insulator transition [5]. The nonequilibrium behavior of the RFIM has been used to model the physics of hysteresis and avalanches [6], and the model is also popular in the study of complex systems, for instance, to model opinion dynamics [7]. The Hamiltonian of the RFIM reads

$$\mathcal{H} = -\sum_{\langle ij \rangle} J_{ij} S_i S_j + \sum_i h_i S_i, \tag{1}$$

where $J_{ij} > 0$ (usually $J_{ij} = 1$), the N Ising spins $S_i = \pm 1$ are placed at the vertices of a graph (usually a periodic lattice), and the $\{h_i\}$ are quenched random fields, usually having either Gaussian distribution with zero mean and variance H_R^2 or a bimodal distribution $h_i = \pm H_R$.

An important controversy concerning the lower critical dimension has been resolved using rigorous argument [8,9], and it is now known that the RFIM develops long-range order for $d > 2$. Another puzzle is associated with the failure of the so-called *dimensional reduction* property of the RFIM. Standard perturbation theory predicts to all orders that the critical behavior of the RFIM in dimension d is the same as that of the pure Ising model in $d - 2$ dimensions [10], a fact that violates rigorous results [9]. The reason for this failure is often related to the presence of multiple metastable states; consequently, the presence or the absence of a spin-glass (SG) phase in the RFIM has started to attract a lot of attention.

But is there a thermodynamic SG phase in the RFIM? Based on an extension of the RFIM to m-component vector spins and the large m expansion [11], it has been argued that in the phase diagram of the three-dimensional RFIM

the paramagnetic and ferromagnetic phases are separated by a SG phase in which the replica symmetry is broken, as in mean-field spin glasses [12]. Studies using perturbative replica field theory also show the presence of an intermediate SG phase below six dimensions [13], and others claimed that a nonperturbative effect would lead to a SG phase [14]. Note that a recent work [15] uses nonperturbative renormalization group to explain the failure of the dimensional reduction without the use of replica symmetry breaking. Even in mean-field models, the question has sparked debates: While in the fully connected setting of [16] no SG phase was observed, some works suggested the existence of a such a phase on the Bethe lattice [17], others do not [18], while some remained inconclusive [19]. In numerical studies, although in [20] many solutions to the so-called naive mean-field equations have been found close to the critical temperature, results from equilibrium [21] and out-of-equilibrium [22] simulations in finite dimension found no evidences of such a SG phase.

In this Letter, we consider this elusive SG phase in RFIM and show rigorously that the SG susceptibility is always upper bounded by the ferromagnetic susceptibility, for any lattice, any dimension, and any choice of fields. Consequently, there cannot be a SG phase out of the critical ferromagnetic point or line. Secondly, we revise the solution of the RFIM on the Bethe lattice and show that *only* when the magnetization is fixed to values smaller (in absolute value) than the equilibrium value a SG phase can exist. Note that the RFIM with fixed magnetization appears in many applications involving a mapping from a lattice gas, e.g., in the Coulomb glasses or in binary liquids.

A rigorous bound on the spin-glass susceptibility.—A commonly accepted definition of a SG phase is the divergence of the SG susceptibility defined as

$$\chi_{\text{SG}} \equiv \frac{1}{N} \sum_{i,j} (\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle)^2, \tag{2}$$

where $\langle \cdot \rangle$ is the thermal average. The susceptibility χ_{SG} is related to the experimentally measured nonlinear susceptibility [23]. In the replica symmetry breaking theory [12], the de Almeida–Thouless condition [24]—the smallest

eigenvalue of the corresponding Hessian being negative—implies the divergence of χ_{SG}. The study of χ_{SG} identifies the SG phase in theories with (static) replica symmetry breaking [25] and in the droplet model [26].

Let us now consider a RFIM on a fully connected topology with $J_{ij} \geq 0$, any other topology can be obtained by setting $J_{ij} = 0$ for all pairs of spins which are not nearest neighbors. We first prove that, for any value of the external fields and on any given sample, connected correlation functions for any pair of spins i, j satisfies

$$\langle S_i S_j \rangle_c \equiv \langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle \geq 0. \quad (3)$$

In order to do so, we proceed recursively, and show that if this holds for a system with N spins, then it holds for a system with $N + 1$ spins. When $N = 2$, we have straightforwardly $\langle S_1 S_2 \rangle_c = 8 \sinh(2\beta J_{12})/Z^2$, which is indeed non-negative as long as $J_{12} \geq 0$. Consider now a system with N spins S_i with $i = 1, \ldots, N$ such that $\forall$ i, $j \in [1, \ldots, N]$ Eq. (3) holds for *any choice* of the random fields. We now add a new spin S_{N+1} with couplings $J_{(N+1)i}$ and external magnetic field h_{N+1}. In the $N + 1$ spins system we denote $w_{\pm} = \mathbb{P}[S_{N+1} = \pm 1]$.

We now express the correlations in the systems of $N + 1$ spins in terms of the correlations in the system of N spins. The correlations involving the new spin S_{N+1} read

$$\langle S_{N+1} S_i \rangle_c^{(N+1)} = 2 w_+ w_- (\langle S_i \rangle_+^{(N)} - \langle S_i \rangle_-^{(N)}),$$

where the averages $\langle \cdot \rangle_{\pm}^{(N)}$ are computed in the N spins system whose Hamiltonian has been changed by the addition of the term $-\sum_i \pm J_{(N+1)i} S_i$, which is nothing but a change in the random fields. Given that the external fields in the measure $\langle \cdot \rangle_-^{(N)}$ are not greater than the corresponding fields in $\langle \cdot \rangle_+^{(N)}$, and that susceptibilities are non-negative in the N spins system by assumption, $\langle S_i \rangle_-^{(N)}$ is not greater than $\langle S_i \rangle_+^{(N)}$, and so we have $\langle S_{N+1} S_i \rangle_c^{(N+1)} \geq 0$.

The correlation $\langle S_i S_j \rangle_c^{(N+1)}$ (with $i, j \neq N + 1$) is

$$\langle S_i S_j \rangle_c^{(N+1)} = w_+ \langle S_i S_j \rangle_{c,+}^{(N)} + w_- \langle S_i S_j \rangle_{c,-}^{(N)} + w_+ w_- (\langle S_i \rangle_+^{(N)}$$
$$- \langle S_i \rangle_-^{(N)})(\langle S_j \rangle_+^{(N)} - \langle S_j \rangle_-^{(N)}).$$

By the initial assumption, correlations $\langle S_i S_j \rangle_{c,\pm}^{(N)}$ are both non-negative, and by the argument used above the last term is also non-negative, and so $\langle S_i S_j \rangle_c^{(N+1)} \geq 0$. This proves relation (3). In fact, what we have proven is a just particular case of the Fortuin, Kasteleyn, and Ginibre [27] inequality, which is well known in mathematical physics.

Our main point is that from Eq. (3) it directly follows that the SG susceptibility is upper bounded by the ferromagnetic one:

$$\chi_{SG} = \frac{1}{N} \sum_{i,j} \langle S_i S_j \rangle_c^2 \leq \frac{1}{N} \sum_{i,j} \langle S_i S_j \rangle_c = \chi_F. \quad (4)$$

This is true on any lattice and for any choice of the external fields, as long as the pairwise interactions are non-negative

(and hence not frustrated). Equation (4) implies, in particular, that χ_{SG} cannot diverge if χ_F stays finite. We can say even more: The fact that the correlation matrix has all non-negative elements, $C_{ij} = \langle S_i S_j \rangle_c \geq 0$, implies that among all possible susceptibilities the ferromagnetic one χ_F is always the largest one. This means that, in order to understand whether any kind of long-range order develops in a RFIM, it is enough to check whether the ferromagnetic susceptibility is diverging, and this is a great step of reductionism. The paramagnetic phase is defined by the nondivergence of χ_F; thus, clearly, there is no SG phase for $T > T_c$, where T_c is the ferromagnetic critical temperature. This statement allows us to reject many predictions in the literature: All scenarios where χ_{SG} diverges while χ_F is finite [11,13] are ruled out.

In the ferromagnetic phase, $T < T_c$, the χ_F would diverge because of the coexistence between the "up" and the "down" phases, with magnetization m^+ and m^-, respectively. Nevertheless, we can select one of these two states by using proper boundary conditions or by adding an infinitesimal field. In each of these states the ferromagnetic susceptibility is finite (they are not critical and the clustering property holds), and therefore the SG susceptibility is again finite. The ferromagnetic susceptibility truly diverges only exactly at a second order critical point T_c, where two new states are generated from the paramagnetic one. At this point the Hessian, which is the inverse of the correlation matrix, develops a zero mode whose eigenvector has all non-negative elements (thanks to $C_{ij} \geq 0$). In other words, the two new states generated by a second order transition will have different magnetizations. The susceptibility χ_F is thus diverging exactly at T_c, but leads only to a ferromagnetic long-range order below T_c.

It seems to us that the only scenario we are unable to exclude, for existence of a SG phase, is to have a dense set (e.g., in T) of ferromagnetic critical points. We have, however, no reason to believe that such an exotic scenario appears in the RFIM (nor actually in any other model that we know of), and thus we conclude that there is no SG transition in the RFIM.

RFIM with fixed magnetization on the Bethe lattice.—In order to go beyond the strong constraints of Eq. (4), we now consider a RFIM where the magnetization m is fixed to an arbitrary value. In this case, it is worth considering the free energy $f(m)$ as a function of the magnetization m.

If two states exist with different magnetization, $m^- < m^+$, and if one fixes the magnetization, $m \in (m^-, m^+)$, then in any finite dimension there is a phase separation between the m^- phase and the m^+ phase, with the appearance of (at least) one domain wall. The free energy $f(m)$ is thus concave and given by the Maxwell construction between m^+ and m^-.

Such arguments, however, do not apply when the RFIM is defined on a mean-field topology, e.g., on the Bethe lattice—a random graph with fixed degree c. Indeed, random lattices are such that the surface-to-volume ratio of

any subset of vertices does not decrease to zero when the subset is made larger (but still much smaller than the entire lattice). The shape of $f(m)$ thus does not need to be concave, and indeed develops the double-well shape typical of mean-field ferromagnets for $T < T_c$ (see Fig. 1). Mean-field geometries are important in many applications, such as statistical inference and combinatorial optimization. The RFIM with a fixed magnetization corresponds to the weighted graph partitioning where the presence of a SG phase is expected.

The precise determination of $f(m)$ for a given sample or even for the ensemble average is a nontrivial task. Ideally, one would like to compute the free energy in the presence of external field H_m chosen such that the equilibrium magnetization is exactly m and obtain $f(m) = f(H_m) + H_m m$. However, for a double-well shaped $f(m)$ with minima in m^- and m^+, magnetizations in the interval (m^-, m^+) are in principle unreachable, as the Legendre transform computes the convex envelope of $f(m)$.

We now propose an algorithm for computing $f(m)$ on the Bethe lattice even in these situations. Our approach is based on the Bethe-Peierls method, also known as the cavity method [28] or belief propagation algorithm [29]. For every directed link (ij) we define a cavity field $u_{i \to j}$ as the effective local magnetic field that spin j receives from spin i. The cavity fields must satisfy the following self-consistent equations:

$$u_{i \to j} = \frac{1}{\beta} \tanh^{-1} \left\{ \tanh(\beta J_{ij}) \tanh\left[\beta\left(H_m + h_i \right. \right. \right.$$
$$\left. \left. \left. + \sum_{k \in \partial i \backslash j} u_{k \to i} \right) \right] \right\}, \tag{5}$$

where the summation is over all neighbors of i except j. The external uniform field H_m must be chosen such as to fix the global magnetization to the desired value by

$$m = \frac{1}{N} \sum_i \tanh\left[\beta\left(H_m + h_i + \sum_{j \in \partial i} u_{j \to i} \right) \right]. \tag{6}$$

Solving Eqs. (5) and (6), the (extensive) free energy is

$$- \beta F = \sum_i \log\left\{ 2 \cosh\left[\beta\left(H_m + h_i + \sum_{j \in \partial i} u_{j \to i} \right) \right] \right\}$$
$$- \sum_{ij} \log\left\{ \sum_{s = \pm 1} e^{s \beta J_{ij}} 2 \cosh[\beta(u_{j \to i} + s u_{i \to j})] \right\}.$$

To fix the magnetization to a value m corresponding to the nonconvex part of the free energy $f(m)$, we solve Eqs. (5) and (6) by the following iterative scheme. (A) Set $t = 0$ and assign random values to $\{u_{i \to j}^{(0)}\}$ and H_m. (B) Repeat (i) compute $\{u_{i \to j}^{(t)}\}$ by Eq. (5), (ii) compute H_m solving Eq. (6) by the bisection method, (iii) increment t by 1, until convergence is met or a maximum number of iterations is reached. (C) If converged, compute the free energy $f(m)$ using fixed point cavity fields $\{u_{i \to j}^*\}$ and H_m^*.
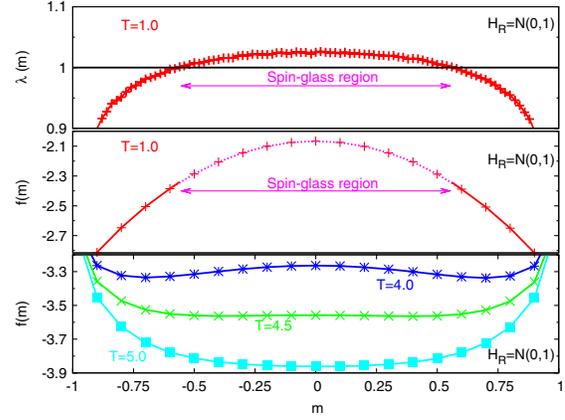
FIG. 1 (color online). Free energies $f(m)$ and the stability parameter $\lambda(m)$ versus the magnetization m on a Bethe lattice of degree $c = 6$ with Gaussian random fields of unit variance and zero mean. Bottom: We show $T = 5$, $T = 4.5$, and $T = 4.0$ (with $T_c = 4.66$); notice the appearance of two minima for $T < T_c$. Middle: A low-temperature case ($T = 1$). A spin-glass region appears when the magnetization is fixed to low enough values. Top: The corresponding stability parameter $\lambda(m)$ showing the spin-glass order for $|m| < 0.56$.

Note that the bound derived in Eq. (4) is valid for any external field, and therefore for any values of m such that $f(m)$ coincides with its convex envelope. This bound is, however, *not valid* for the values of m in the interval (m^-, m^+), and this is the place we should check for the appearance of a SG phase. There are several different methods for computing the SG instability (for a review see Appendix C in [30]), all of them generalizing the de Almeida–Thouless condition [24]. The numerically most precise one is to study the fate of a small perturbation to the cavity fields [31] that are evolving according to the following linear equations:

$$\delta u_{i \to j}^{(t+1)} = \frac{\partial u_{i \to j}^{(t+1)}}{\partial u_{k \to i}^{(t)}} \delta u_{k \to i}^{(t)}. \tag{7}$$

The divergence of the root mean square of the δu's signals a local instability and the appearance of a SG phase. In practice we measure the parameter λ, which is the rate of growth of the root mean square of the δu's.

In Fig. 1 we show a typical free energy $f(m)$ on a Bethe lattice and the stability parameter λ. Clearly, a SG phase is present for some of the nonequilibrium values of the magnetization. Figure 2 shows the phase diagram of the RFIM on a Bethe lattice. Note that on a cubic $3d$ lattice with Gaussian random fields, the transition in zero field is at $T_c \approx 4.5$ and for zero temperature at $H_c \approx 2.3$. Corresponding critical values on the Bethe lattice are larger, as expected for a mean-field approximation. Just as in the mean-field solution of [16], the ferromagnetic transition is of first order for the bimodal distribution of fields at low enough temperature [for $c \geq 4$, at zero temperature spinodal lines end in $H_{sp} = (c - 1)/2$ and $H_{sf} = c - 2$]. The SG region always appears at a smaller value of
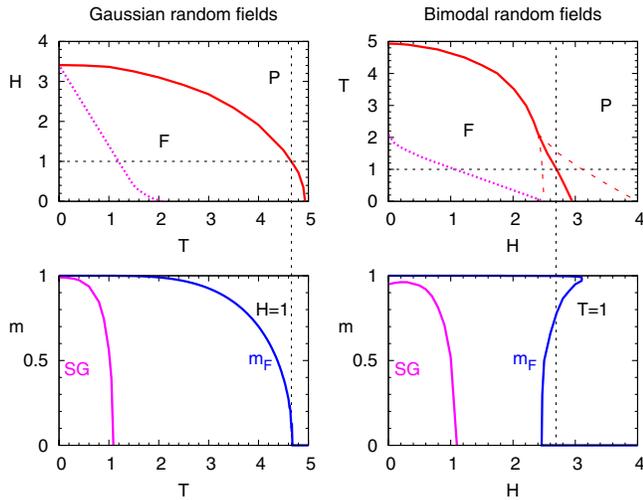
FIG. 2 (color online).   Phase diagram of the RFIM on the $c = 6$ Bethe lattice with Gaussian (left) and bimodal (right) random field. Top panels: Boundary between paramagnetic (P) and ferromagnetic (F) phases [solid (red) line]. In the low-T region the transition is first order for bimodal fields (red dashed lines are the spinodals). Below the purple dotted line a spin-glass (SG) phase exists for $m = 0$. Lower panels: Equilibrium magnetization m_F [darker (blue) line]. A SG phase exists only below the lighter (purple) line.

magnetization than the equilibrium one. At zero temperature the ferromagnetic critical point is also critical for the SG phase with $m = 0$. In the renormalization group approach this is the relevant fixed point, and the SG instabilities seen in the perturbative approach could perhaps be linked to this fact.

Conclusions.—We have shown that there is no spin-glass phase at equilibrium in the RFIM, closing a long-standing debate on the elusive spin-glass phase in this model. It is *only* if one fixes the magnetization to nonequilibrium values that a true SG phase can exist, as we showed explicitly on the Bethe lattice. In finite dimensional systems the existence of such a stable phase, although unlikely, remains open. This SG phase, or its vestige, although thermodynamically subdominant, may influence the dynamical behavior. This is particularly true at zero temperature where the many local energy minima get stabilized. Our rigorous result puts a large question mark on the field theoretical approaches that have led to erroneous conclusions and that are still widely used in studies of more complex disordered systems such as spin glasses.

[1]   A. I. Larkin, Sov. Phys. JETP **31**, 784 (1970).
[2]   S. Fishman and A. Aharony, J. Phys. C **12**, L729 (1979).
[3]   P. G. de Gennes, J. Phys. Chem. **88**, 6469 (1984).
[4]   A. L. Efros *et al.*, J. Phys. C **8**, L49 (1975).
[5]   T. R. Kirkpatrick and D. Belitz, Phys. Rev. Lett. **73**, 862 (1994).
[6]   J. P. Sethna *et al.*, Phys. Rev. Lett. **70**, 3347 (1993).
[7]   S. Galam, Physica (Amsterdam) **238A**, 66 (1997); Q. Michard and J.-P. Bouchaud, Eur. Phys. J. B **47**, 151 (2005).
[8]   Y. Imry and S.-K. Ma, Phys. Rev. Lett. **35**, 1399 (1975).
[9]   J. Bricmont and A. Kupiainen, Phys. Rev. Lett. **59**, 1829 (1987).
[10]   G. Parisi and N. Sourlas, Phys. Rev. Lett. **43**, 744 (1979).
[11]   M. Mézard and G. Parisi, J. Phys. A **23**, L1229 (1990); M. Mézard and A. P. Young, Europhys. Lett. **18**, 653 (1992); M. Mézard and R. Monasson, Phys. Rev. B **50**, 7199 (1994).
[12]   M. Mézard, G. Parisi, and M. A. Virasoro, *Spin Glass Theory and Beyond* (World Scientific, Singapore, 1987).
[13]   C. De Dominicis, H. Orland, and T. Temesvari, J. Phys. I **5**, 987 (1995); E. Brezin and C. De Dominicis, Europhys. Lett. **44**, 13 (1998); Eur. Phys. J. B **19**, 467 (2001); C. De Dominicis and I. Giardina, *Random Fields and Spin Glasses: A Field Theory Approach* (Cambridge University Press, Cambridge, England, 2006).
[14]   G. Parisi and V. S. Dotsenko, J. Phys. A **25**, 3143 (1992).
[15]   G. Tarjus and M. Tissier, Phys. Rev. B **78**, 024203 (2008).
[16]   T. Schneider and E. Pytte, Phys. Rev. B **15**, 1519 (1977).
[17]   Y. Shapir, Phys. Rev. Lett. **57**, 271 (1986); R. Bruinsma, Phys. Rev. Lett. **57**, 272 (1986); J. R. L. de Almeida and R. Bruinsma, Phys. Rev. B **35**, 7267 (1987); A. A. Pastor, V. Dobrosavljevic, and M. L. Horbach, Phys. Rev. B **66**, 014413 (2002).
[18]   D. J. Thouless, Phys. Rev. Lett. **57**, 273 (1986); M. O. Hase, J. R. L. de Almeida, and S. Salinas, Eur. Phys. J. B **47**, 245 (2005).
[19]   M. Bleher, J. Ruiz, and V. A. Zagrebnov, J. Stat. Phys. **93**, 33 (1998); T. Nowotny, H. Patzlaff, and U. Behn, Phys. Rev. E **65**, 016127 (2001).
[20]   D. Lancaster, E. Marinari, and G. Parisi, J. Phys. A **28**, 3959 (1995).
[21]   M. E. J. Newman and G. T. Barkema, Phys. Rev. E **53**, 393 (1996); A. Middleton and D. S. Fisher, Phys. Rev. B **65**, 134411 (2002).
[22]   G. Parisi, F. Ricci-Tersenghi, and J. J. Ruiz-Lorenzo, Eur. Phys. J. B **11**, 317 (1999).
[23]   K. H. Fischer and J. A. Hertz, *Spin Glasses* (Cambridge University Press, Cambridge, England, 1991).
[24]   J. R. L. de Almeida and D. J. Thouless, J. Phys. A **11**, 983 (1978).
[25]   In the dynamical replica symmetry breaking scenario [see, e.g., A. Montanari and G. Semerjian, J. Stat. Phys. **124**, 103 (2006)], the spin-glass susceptibility does not diverge. This scenario is, however, impossible in finite dimensional systems and we checked that it does not occur for the RFIM on the Bethe lattice. In addition, this transition is only a topological one, with no associated singularity in the free energy.
[26]   D. S. Fisher and D. A. Huse, J. Phys. A **20**, L1005 (1987).
[27]   C. M. Fortuin, P. W. Kasteleyn, and J. Ginibre, Commun. Math. Phys. **22**, 89 (1971).
[28]   M. Mézard and G. Parisi, Eur. Phys. J. B **20**, 217 (2001).
[29]   M. Mézard and A. Montanari, *Information, Physics, and Computation* (Oxford University Press, Oxford, 2009).
[30]   L. Zdeborová, Acta Phys. Slovaca **59**, 169 (2009).
[31]   A. Pagnani, G. Parisi, and M. Ratieville, Phys. Rev. E **68**, 046706 (2003).