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No spin glass phase in the ferromagnetic random-field random-temperature scalar Ginzburg–Landau model

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Abstract
Krzakala, Ricci-Tersenghi and Zdeborová have recently shown that the random field Ising model with non-negative interactions and an arbitrary external magnetic field on an arbitrary lattice does not have a static spin-glass phase. In this communication we generalize the proof to a soft scalar spin version of the Ising model: the Ginzburg–Landau model with a random magnetic field and a random temperature parameter. We do so by proving that the spin glass susceptibility cannot diverge unless the ferromagnetic susceptibility does.

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1. Introduction

A widely studied class of disordered systems in statistical physics consists in adding local randomness in fields coupled to the order parameter. A textbook example of such a system is the random field Ising model (RFIM), introduced in [1], that has been a very useful playground for theoretical ideas. The Hamiltonian of the standard RFIM reads

$$\mathcal{H} = -\sum_{i<j} J_{ij} S_i S_j + \sum_i h_i S_i, \tag{1.1}$$

where all the non-zero interactions are ferromagnetic, i.e. $J_{ij} \geqslant 0$. The N Ising spins $S_i = \pm 1$, $i = 1, \ldots, N$, are placed at the vertices of a graph (lattice), and the h_i are quenched random magnetic fields. The fact that all the interactions J_{ij} are non-negative is fundamental; it means that in the absence of the fields there is no explicit frustration in the problem.

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The case where the graph of interactions is a finite-dimensional lattice and where the fields are taken from a Gaussian distribution with zero mean and a variance H_R has received a lot of attention. Of particular interest is the phase diagram in the T–H_R plane, where T is the temperature. Several authors have suggested, based on non-rigorous field theoretic arguments, that there exists an equilibrium spin glass phase in the three-dimensional RFIM that is a phase with a random frozen ordering [2–6]. These suggestions were disproved rigorously in [7] for the RFIM defined by the Hamiltonian (1.1). In particular reference [7] showed that for the RFIM (1.1) a special case of the Fortuin–Kasteleyn–Ginibre (FKG) inequality [8] implies that the spin glass susceptibility is upper-bounded by the ferromagnetic susceptibility. Since the spin glass susceptibility diverges in the whole spin glass phase, a spin glass phase cannot exist away from the ferromagnetic critical point/line in the RFIM.

The field theoretic approach of [2–6], however, was not formulated with the Ising spin Hamiltonian (1.1) but instead with the soft-spin description of the random field model. This is the well-known Ginzburg–Landau model (or the so-called ϕ^4-theory) which is defined by the following Hamiltonian:

$$\mathcal{H}_N = -\sum_{ij} J_{ij}\phi_i\phi_j - \sum_i h_i\phi_i + \sum_i r_i\phi_i^2 + \sum_i u_i\phi_i^4, \tag{1.2}$$

where ϕ_i are now real numbers, $\phi_i \in \mathbb{R}$ and the interactions are ferromagnetic, $J_{ij} \geqslant 0$ (this will be the case in the whole communication).

The generalized model (1.2) includes several special cases. The Ising model is recovered in the limit where $r_i = -2u_i$ and $u_i \to \infty$. The most common *random field model* is obtained when the h_i are independent random variables while $r_i = r$ and $u_i = u$ are fixed and $J_{ij} = 0$ or $J_{ij} = 1$ depending upon whether the spins ij interact or not. Another version that was considered in the literature, the *random temperature model*, is where the r_i are random while $h_i = 0$, $u_i = u$ and $J_{ij} \in \{0, 1\}$. The existence of a spin glass phase was also predicted in the random temperature model [9, 10], based again on non-rigorous arguments using perturbation theory; this result was, however, questioned in [11].

Our results work even for the slightly more general Hamiltonian

$$\mathcal{H}_N = -\sum_{ij} J_{ij}\phi_i\phi_j + \sum_i f_i(\phi_i), \tag{1.3}$$

where $J_{ij} \geqslant 0$ and the local constraining potentials $f_i()$ are arbitrary analytic functions except for the requirement that the partition function

$$Z_N = \int_{-\infty}^{\infty} \prod_{i=1}^{N} \mathrm{d}\phi_i \ \mathrm{e}^{-\beta\mathcal{H}_N(\{\phi_i\})} \tag{1.4}$$

must exist for any $N \in \mathbb{N}$. This is the most minimalist requirement, since the non-convergence of the integral in (1.4) would make the Gibbs–Boltzmann measure ill defined and the model would not be a physical one.

The Gibbs–Boltzmann average at temperature $T = \beta^{-1}$ is defined by

$$\langle A \rangle^{(N)} = \frac{1}{Z_N} \int_{-\infty}^{\infty} \prod_{i=1}^{N} \mathrm{d}\phi_i \, A \, \mathrm{e}^{-\beta\mathcal{H}_N(\{\phi_i\})}. \tag{1.5}$$

The superscript (N) on the angular brackets will be written explicitly only when the size dependence is crucial, while the temperature dependence is always made implicit. Connected correlation functions are defined as

$$\langle AB \rangle_c = \langle AB \rangle - \langle A \rangle\langle B \rangle. \tag{1.6}$$

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It is worth noticing that the convergence of the integral in (1.4) ensures that the partition function Z_N is analytic if regarded as a function of all the J_{ij}, including self-interactions J_{ii}, irrespective of whether the corresponding interaction terms are non-zero in the original Hamiltonian. Consequently, the derivatives with respect to J_{ij}

$$\beta^{-1} \frac{\partial \ln Z_N}{\partial J_{ij}} = \langle \phi_i \phi_j \rangle \qquad (1.7)$$

show that the $\langle \phi_i \phi_j \rangle$ exist (i.e. are bounded) for any pair of indices i, j. In particular, by setting $i = j$ it is easy to prove the finiteness of any $\langle \phi_i^2 \rangle$ and, as this bounds $\langle \phi_i \rangle^2$, also that the $\langle \phi_i \rangle$ exist: so single-variable marginal probability distributions have the first and the second moments, $\langle \phi_i \rangle$ and $\langle \phi_i^2 \rangle$. Actually in soft-spin models used in the literature, such as the spherical model and the ϕ^4 model, single-variable marginal probabilities decay exponentially fast for large values of ϕ_i, and so all the moments $\langle \phi_i^k \rangle$ exist. However, our proof only requires the first two moments to exist.

Note also that any type of lattice can be encoded in the framework of model (1.3) by setting $J_{ij} = 0$ if spins i and j do not interact.

The main contribution of this communication is a rigorous proof that the soft-spin random-field random-temperature model defined by (1.3) does not have a spin glass phase as long as the interactions are ferromagnetic (non-negative). This generalizes the result of [7].

2. Definitions of susceptibilities

We define the ferromagnetic and the spin glass phases using the properties of the ferromagnetic and spin glass susceptibilities.

The order parameter that characterizes a ferromagnetic transition is the magnetization $m = \sum_i \langle \phi_i \rangle / N$. However, a non-zero magnetization does not imply a ferromagnetic phase. Indeed, $m > 0$ even at large temperatures when a uniform positive external magnetic field is applied. A convenient way to characterize the ferromagnetic phase is to define the ferromagnetic susceptibility as

$$\chi_F^0(N) = \frac{1}{N} \sum_{ij} \langle \delta \phi_i \delta \phi_j \rangle, \qquad (2.1)$$

where

$$\delta \phi_i = \frac{\phi_i - \langle \phi_i \rangle}{\sqrt{\langle \phi_i^2 \rangle - \langle \phi_i \rangle^2}} \qquad (2.2)$$

are the fluctuations with respect to the average values, normalized by the variances.

In the thermodynamic limit ($N \to \infty$), $\chi_F^0(\infty)$ is finite in the high temperature ($T > T_c$) paramagnetic phase and it diverges approaching the ferromagnetic critical point from above ($T \searrow T_c$). Right at the critical point ($T = T_c$), $\chi_F^0(N)$ diverges with $N \to \infty$ signaling that the system is critical, i.e. has long-range correlations between fluctuations of its variables. Unfortunately, the ferromagnetic susceptibility $\chi_F^0(N)$ diverges with N also in the whole low temperature ($T < T_c$) ferromagnetic phase: however, this divergence is not due to criticality (i.e. long-range correlation of fluctuations), but only because below T_c two ferromagnetic states coexist[6].

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[6] In the presence of two or more equivalent states, an appropriately chosen perturbation, although of infinitesimal strength, may induce a macroscopic change of state, thus leading to an infinite susceptibility.

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Given that we are interested in finding critical points and critical lines where a phase transition takes place, we would like to measure an observable that diverges only at criticality, and so we consider the following *ferromagnetic susceptibility*:

$$\chi_F = \lim_{h \searrow 0} \lim_{N \to \infty} \chi_F(h, N) = \lim_{h \searrow 0} \lim_{N \to \infty} \frac{1}{N} \sum_{ij} \langle \delta\phi_i \delta\phi_j \rangle, \tag{2.3}$$

where h is an auxiliary uniform magnetic field (in practice one needs to add a term $-h \sum_i \phi_i$ in the Hamiltonian). Due to the order of the limits in (2.3), below T_c, the infinitesimal external field h makes the two ferromagnetic states no longer equivalent, and consequently χ_F is finite everywhere except at the critical point T_c (which is indeed defined as the point where χ_F diverges).

In general to define a susceptibility that diverges only when a critical state is present one should explicitly break (by adding infinitesimal perturbations) all the symmetries of the Hamiltonian. In our case, the Hamiltonian (1.3) is very general, but the first term is invariant under the transformation $\phi_i \to -\phi_i \ \forall i$. In case the potentials too are invariant under such a transformation, $f_i(\phi) = f_i(-\phi)$, then the infinitesimal auxiliary uniform field in (2.3) is strictly required.

The spin glass phase is characterized by a freezing of spins in random directions [12]. Hence the *spin glass susceptibility* is defined as

$$\chi_{SG} = \lim_{h \searrow 0} \lim_{N \to \infty} \chi_{SG}(h, N) = \lim_{h \searrow 0} \lim_{N \to \infty} \frac{1}{N} \sum_{ij} \langle \delta\phi_i \delta\phi_j \rangle^2. \tag{2.4}$$

Again we use the infinitesimal auxiliary external field to break the $\phi \to -\phi$ symmetry, if present. The susceptibility χ_{SG} is related closely to what is measured in simulations and experiments [13] and is predicted to diverge at the critical point in spin glass theories, such as replica symmetry breaking [14] or the droplet description [15]. In a spin glass phase χ_{SG} is infinite, because of the presence of at least two states[7] related by symmetries, which are not broken by the auxiliary field. For this reason we can define that a system is in a spin glass phase *if and only if* the ferromagnetic susceptibility (2.3) is finite, while the spin glass susceptibility (2.4) is infinite.

More precisely the computation of these two susceptibilities must proceed by first taking the thermodynamic limit in the presence of the external field, $\chi_F(h, \infty)$ and $\chi_{SG}(h, \infty)$, and then studying the limit $h \searrow 0$ of these two functions. If such a limit exists, then we say that the susceptibility is finite and we are away from the critical point, while if a divergence is found while decreasing h, then we say that the susceptibility is infinite.

In the next section we prove that $\chi_{SG}(h, N) \leqslant \chi_F(h, N)$, for any value of h and N, thus excluding the possibility of a spin glass phase (defined by $\chi_F < \infty$ and $\chi_{SG} \to \infty$) in the model (1.3) in the absence of explicit frustration in the couplings.

3. Results

We start by proving a generalization of the second Griffith's inequality [16]. The following lemma is a consequence of much more general FKG inequalities [8, 17], but we believe it useful to present an independent and more elementary proof.

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[7] The number of states depends on the model and for some models, like the 3D Edwards–Anderson model, it is still a matter of debate.

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Lemma 3.1. *In the model defined by the Hamiltonian (1.3) with non-negative couplings, $J_{ij} \geqslant 0 \ \forall i, j$, under the Gibbs–Boltzmann measure $\mathrm{e}^{-\beta \mathcal{H}_N}/Z_N$, the correlation between fluctuations of any two variables is non-negative and bounded by 1:*

$$0 \leqslant \langle \delta\phi_i \delta\phi_j \rangle \leqslant 1 \qquad \forall i, j. \tag{3.1}$$

Proof. Let us prove first the second inequality in (3.1). From the definition (2.2) of the relative fluctuations we have that $\langle \delta\phi_i^2 \rangle = 1$ for any i. Moreover, for any pair of indices i, j we have that

$$0 \leqslant \langle (\delta\phi_i - \delta\phi_j)^2 \rangle = \langle \delta\phi_i^2 \rangle + \langle \delta\phi_j^2 \rangle - 2\langle \delta\phi_i \delta\phi_j \rangle = 2(1 - \langle \delta\phi_i \delta\phi_j \rangle) \tag{3.2}$$

from which $\langle \delta\phi_i \, \delta\phi_j \rangle \leqslant 1$ follows.

In order to prove the first inequality in (3.1) we note that it is equivalent to the inequality

$$\langle \phi_i \phi_j \rangle_c \geqslant 0, \tag{3.3}$$

thanks to the fact that all denominators in the definition (2.2) of $\delta\phi_i$ are positive and can be canceled without changing the sign of the correlation.

Then we prove (3.3), by induction in the system size. In a system with a single spin $(N = 1)$

$$\langle \phi_1^2 \rangle_c^{(1)} \geqslant 0, \tag{3.4}$$

since the variance is always non-negative. Then we assume the property to hold in a system of N spins, consider a system with $(N + 1)$ spins and demonstrate that the corresponding property holds also for that system.

The Hamiltonian of the $N + 1$ spin system is related to that of the N-spin system by

$$\mathcal{H}_{N+1} = \mathcal{H}_N - \sum_{i=1}^{N} J_{N+1,i} \phi_{N+1} \phi_i + f_{N+1}(\phi_{N+1}). \tag{3.5}$$

We denote[8]

$$P(x) = \frac{1}{Z_{N+1}} \int_{-\infty}^{\infty} \prod_{i=1}^{N} \mathrm{d}\phi_i \exp\left[-\beta \mathcal{H}_N(\{\phi_i\}) + \beta \sum_{i=1}^{N} J_{N+1,i} x \phi_i - \beta f_{N+1}(x) \right] \tag{3.6}$$

and the thermodynamic average of an N-spin system in a correspondingly modified external magnetic field by

$$\langle A \rangle_x^{(N)} = \frac{\int \prod_{i=1}^{N} \mathrm{d}\phi_i \, A \, \mathrm{e}^{-\beta \mathcal{H}_N(\{\phi_i\}) + \beta \sum_{i=1}^{N} J_{N+1,i} x \phi_i}}{\int \prod_{i=1}^{N} \mathrm{d}\phi_i \, \mathrm{e}^{-\beta \mathcal{H}_N(\{\phi_i\}) + \beta \sum_{i=1}^{N} J_{N+1,i} x \phi_i}}. \tag{3.7}$$

The connected correlation between spins ϕ_{N+1} and ϕ_i in the $(N + 1)$-spin system can then be rewritten as

$$\begin{aligned}
\langle \phi_{N+1} \phi_i \rangle_c^{(N+1)} &= \int_{-\infty}^{\infty} \mathrm{d}x \, x \, P(x) \langle \phi_i \rangle_x^{(N)} - \int_{-\infty}^{\infty} \mathrm{d}y \, y \, P(y) \int_{-\infty}^{\infty} \mathrm{d}x \, P(x) \langle \phi_i \rangle_x^{(N)} \\
&= \int_{-\infty}^{\infty} \mathrm{d}x \left[x - \int_{-\infty}^{\infty} \mathrm{d}y \, y \, P(y) \right] P(x) \langle \phi_i \rangle_x^{(N)}.
\end{aligned} \tag{3.8}$$

We can then use the following inequality. For any real non-decreasing function $g(x)$ such that

$$\int_{-\infty}^{\infty} \mathrm{d}x \, g(x) = 0, \tag{3.9}$$

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[8] We suppress explicit indication of the dependence on the parameters J and f.

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and any non-decreasing function $f(x)$ one has

$$\int_{-\infty}^{\infty} \mathrm{d}x g(x) f(x) \geqslant 0. \tag{3.10}$$

Proof of this statement is elementary. The function $g(x)$ has to be non-positive for some $x \leqslant x_0$ and non-negative for $x \geqslant x_0$. Since $f(x)$ is non-decreasing one has

$$\int_{-\infty}^{x_0} \mathrm{d}x |g(x)| f(x) \leqslant \int_{x_0}^{\infty} \mathrm{d}x |g(x)| f(x), \tag{3.11}$$

from which (3.10) follows.

We observe that $\langle \phi_i \rangle_x^{(N)}$ is a non-decreasing function of x. Indeed we have

$$\frac{\mathrm{d}\langle \phi_i \rangle_x^{(N)}}{\mathrm{d}x} = \beta \sum_{j=1}^{N} J_{N+1,j} \langle \phi_j \phi_i \rangle_c^{(N)} \geqslant 0, \tag{3.12}$$

where the last inequality follows from the inductive assumption and since $J_{N+1,j} \geqslant 0$. We also observe that

$$\int_{-\infty}^{\infty} \mathrm{d}x \left[x - \int_{-\infty}^{\infty} \mathrm{d}y y P(y) \right] P(x) = 0. \tag{3.13}$$

Hence equation (3.10) implies that

$$\langle \phi_{N+1} \phi_i \rangle_c^{(N+1)} \geqslant 0. \tag{3.14}$$

We proceed similarly for the correlation function in the $(N+1)$-spin system between two spins that were already present in the N-spin system

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\begin{aligned}
\langle \phi_i \phi_j \rangle_c^{(N+1)} &= \int_{-\infty}^{\infty} \mathrm{d}x P(x) \langle \phi_i \phi_j \rangle_x^{(N)} - \int_{-\infty}^{\infty} \mathrm{d}x P(x) \langle \phi_i \rangle_x^{(N)} \int_{-\infty}^{\infty} \mathrm{d}y P(y) \langle \phi_j \rangle_y^{(N)} \\
&= \int_{-\infty}^{\infty} \mathrm{d}x P(x) \langle \phi_i \phi_j \rangle_{x,c}^{(N)} \\
&\quad + \int_{-\infty}^{\infty} \mathrm{d}x P(x) \langle \phi_i \rangle_x^{(N)} \left[\langle \phi_j \rangle_x^{(N)} - \int_{-\infty}^{\infty} \mathrm{d}y P(y) \langle \phi_j \rangle_y^{(N)} \right].
\end{aligned} \tag{3.15}
$$

The first term is non-negative by the inductive assumption, and the second term is non-negative according to (3.10), because $\langle \phi_i \rangle_x^{(N)}$ and $\langle \phi_j \rangle_x^{(N)}$ are non-decreasing functions of x and

$$\int_{-\infty}^{\infty} \mathrm{d}x P(x) \left[\langle \phi_j \rangle_x^{(N)} - \int_{-\infty}^{\infty} \mathrm{d}y P(y) \langle \phi_j \rangle_y^{(N)} \right] = 0. \tag{3.16}$$

Hence

$$\langle \phi_i \phi_j \rangle_c^{(N+1)} \geqslant 0. \tag{3.17}$$

This concludes the proof of lemma 3.1. □

Based on this lemma we can now easily state the main result of this communication.

Theorem 3.1. *In the model defined by the Hamiltonian (1.3) with non-negative couplings, $J_{ij} \geqslant 0 \ \forall i, j$, under the Gibbs–Boltzmann measure $\mathrm{e}^{-\beta \mathcal{H}_N}/Z_N$, the spin glass susceptibility $\chi_{SG}(h, N)$ is always upper-bounded by the ferromagnetic susceptibility $\chi_F(h, N)$. Consequently, the model does not posses a thermodynamic spin glass phase.*

Proof. The assumptions of this theorem are the same as those of lemma 3.1, with the only difference in that to properly define the susceptibilities we need to add the external auxiliary field term to the original model Hamiltonian. Then lemma 3.1 can be used provided that

$$Z_N(h) = \int_{-\infty}^{\infty} \prod_{i=1}^{N} \mathrm{d}\phi_i \ \mathrm{e}^{-\beta \mathcal{H}_N(\{\phi_i\}) + \beta h \sum_i \phi_i} \tag{3.18}$$

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exists also for $h > 0$, and this is easy to prove. Indeed $Z_N(0)$ exists (otherwise the Gibbs–Boltzmann measure would be ill-defined) and also the first two derivatives of $Z_N(h)$ with respect to h exist (because $\langle \phi_i \rangle$ and $\langle \phi_i^2 \rangle$ exist): so $Z_N(h)$ can be continued in a neighborhood around $h = 0$, that we call S_0, and this is enough to take the limit $h \searrow 0$ that is required to define properly the susceptibility. Note that the region S_0 coincides with $\mathbb{R}$ for all the models used previously in the literature, such as the spherical model and the ϕ^4 model.

Given that the assumptions of lemma 3.1 are satisfied in S_0, we can make use of the inequalities in (3.1) and find that

$$\langle \delta \phi_i \delta \phi_j \rangle^2 \leqslant \langle \delta \phi_i \delta \phi_j \rangle \quad \implies \quad \chi_{SG}(h, N) \leqslant \chi_F(h, N) \tag{3.19}$$

for any value of N and $h \in S_0$. The inequality holds even in the thermodynamic limit:

$$\chi_{SG}(h, \infty) \leqslant \chi_F(h, \infty), \tag{3.20}$$

so the spin glass susceptibility cannot diverge if the ferromagnetic susceptibility stays finite.

In other words, from the definitions given in the previous section it is clear that if a thermodynamic spin glass phase exists, then for a sufficiently large value of N and a sufficiently small value of h the spin glass susceptibility must be larger than the ferromagnetic one and this would violate the inequality in (3.19). Hence we conclude that a thermodynamic spin glass phase does not exist in the model defined in the assumptions. □

4. Discussion

We have shown rigorously that there is no spin glass phase in the scalar soft-spin random-field random-temperature Ginzburg–Landau model with ferromagnetic interactions defined by (1.3). This shows that with two-body interactions and a scalar order parameter one cannot obtain a genuine spin glass phase at equilibrium without *explicit frustration* in the couplings in the absence of other constraints[9].

Our proof contradicts the conclusions of a number of works that used field theoretic arguments [2–6, 9, 10]. It is yet to be discovered where the problem lies in those approaches. One possibility is that the spin glass instability could be an artifact of some truncation in the perturbative expansion. For some of these works the discrepancy may stem from the use of vectorial soft-spin models instead of scalar ones. Another possibility, that is related to a recent suggestion [18], is that the observed 'replica symmetry breaking' instabilities arise only in disorder averaged quantities and never in the thermodynamic limit of single instance quantities. These instabilities would then not be equivalent to the divergence of the spin glass susceptibility (which we prove impossible away from the ferromagnetic critical point), but they could instead be connected to some subtle non-self-averaging effects between different realizations of the system. Indeed all the above-mentioned works, that predicted a spin glass phase contrary to our demonstration, considered 'replicated' field theories, that is, field theories averaged over many realizations of the disorder. The divergences that they observed could hence be coming from strong sample-to-sample fluctuation. The fact that some non-self-averaging is present in the RFIM has been suggested by Parisi and Sourlas [19], who argued that the correlation function, or equivalently the ferromagnetic susceptibility, of the RFIM is non-self-averaging in the critical region and suggested that this is the source of problems with perturbative expansions. Note, however, that such simple non-self-averaging effects can *only* take place *at the ferromagnetic critical point* in any finite-dimensional system. This is a straightforward consequence of a theorem by Wehr and Aizenman [20] who proved that any extensive quantity (such as the ferromagnetic susceptibility away from the critical point) is

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[9] Another possibility of frustrating the system is to impose a non-equilibrium value of magnetization, see [7].

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self-averaging in finite-dimensional systems. In other words, if this effect is the one observed in the field theoretic approaches, it has to be limited to the ferromagnetic critical point itself.

Finally, it would be very interesting to see if our proof can be generalized further. There are two interesting counterexamples that seem to put strong limits to such generalizations. Matsuda and Nishimori [21] showed that a random field Ising model with 3-spin interactions on the Bethe lattice has a thermodynamic spin glass phase. And so moving beyond pairwise interacting models seems impossible in full generality. Moreover, Parisi [22] provided an interesting example of a pairwise interacting ($n = 2$)-component vector spin system where the two-point connected correlation can be negative even if all couplings are positive, namely a chain of spins with an external field that smoothly rotates by $180°$ along the chain, such that the field on the last spin is opposite to the field on the first spin. If the field strength is strong enough, each spin will be mostly aligned along the local field and will thermally fluctuate around this position. However, given that the extremal spins are in opposite directions, their thermal fluctuations will be negatively correlated. This is a very specific configuration which may not occur in typical samples, but its existence implies that the proof strategy presented in this communication cannot be straightforwardly generalized to vector spin models.

Acknowledgments

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