Critical Slowing Down Exponents of Mode Coupling Theory

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(Received 5 December 2011; published 24 February 2012)

A method is provided to compute the exponent parameter λ yielding the dynamic exponents of critical slowing down in mode coupling theory. It is independent from the dynamic approach and based on the formulation of an effective static field theory. Expressions of λ in terms of third order coefficients of the action expansion or, equivalently, in terms of six point cumulants are provided. Applications are reported to a number of mean-field models: with hard and soft variables and both fully connected and dilute interactions. Comparisons with existing results for the Potts glass model, the random orthogonal model, hard and soft-spin Sherrington-Kirkpatrick, and p-spin models are presented.

DOI: 10.1103/PhysRevLett.108.085702 PACS numbers: 64.70.Q−, 64.60.Ht, 75.10.Nr

In the framework of glassy mean-field (MF) models with a quenched or built-in disorder, the dynamics of models whose glassy phase is consistently described by a replica symmetry breaking (RSB) solution with a finite number of breakings displays critical slowing down and a dynamic transition [1–4]. The equations governing the relaxation dynamics down to the dynamic critical temperature T_d are those pertaining to the schematic mode-coupling theory (MCT) developed in the context of supercooled liquids [5–9]. The "guide observable" is the correlation function $C(t)$, i.e., the overlap between a given initial equilibrium configuration of the dynamics and the configuration at time t. In the discontinuous one step RSB case near T_d the relaxation of $C(t)$ is a two step process in which the system spends a large amount of time, algebraically diverging as $T \to T_d$, around a plateau value q_{EA}. Models with a discontinuous dynamic transition include, e.g., the spin-glass (SG) p-spin model, the Potts glass model, and the random orthogonal model. MCT predicts that two exponents control the whole dynamics. In the β regime $C(t)$ approaches the plateau value with a power law $C(t) \approx q_{EA} + c_a/t^a$, while in the early α regime $C(t) = q_{EA} - c_b t^b$ [8]. A major prediction of MCT is the relationship between a, b, and the *exponent parameter* λ:

$$\frac{\Gamma^2(1-a)}{\Gamma(1-2a)} = \frac{\Gamma^2(1+b)}{\Gamma(1+2b)} = \lambda. \tag{1}$$

In case of a continuous transition, there is no dynamic arrest and no b exponent is defined. Well known instances are, e.g., the paramagnet to full-RSB SG transition along the de Almeida Thouless (dAT) line in mean-field SG models, either fully connected [10,11] or on random graphs [12], as well as the SG transition in Potts models with $p \leq 4$ [13,14] and in the p-spin spherical model with a large external magnetic field [3]. Equation (1) is usually assumed to be the correct relationship between exponents a and b and it has been proved to be robust against higher order corrections to standard MCT [8]. This robustness has been recently confirmed by formulating an equivalent Landau theory [15]. Apart from schematic MCT cases, λ is, instead, simply considered a tunable parameter, generically connected to the static structure function at T_d through an often explicitly unknown functional [16]. Here we identify the connection with physical observables working within a "static-driven" Landau theory of dynamics: we will put forward an independent formulation of λ and apply it to some paradigmatic SG models. In general, the analytic treatment of dynamics is more complicated than the statics and only a few models have been studied so far: the soft-spin Sherrington-Kirkpatrick model [10,11], schematic MCT's [6], soft-spin p-spin, and Potts glass models, for which the tie with MCT was first identified [2]. This prompted us to consider the spherical p-spin SG [3,17] in all details as a MF structural glass, cf., e.g., Ref. [18], even in the off-equilibrium regime below T_d [19]. In these cases, dynamics is explicitly solved and λ exactly computed. In particular, one finds that *it is not universal* and depends on model and external parameters. On the other hand, its computation becomes difficult when we consider more complicated MF systems and finite-dimensional ones.

Static definition of the exponent parameter λ.—Similar to the static transition, the dynamic one can also be located as the critical point of an appropriate replicated Gibbs free energy Γ [20–22]. This is a function of the replicated dynamic variables (e.g., spins σ), Legendre transform of the replicated free energy $\Phi[\epsilon_{ab}]$

$$\Gamma[\delta Q_{ab}] = \Phi[\epsilon_{ab}] + \sum_{ab} \epsilon_{ab}\delta Q_{ab}, \quad \frac{\partial \Phi}{\partial \epsilon} = \delta Q_{ab}, \tag{2}$$

with respect to a field ϵ, conjugated to the elements of the overlap matrix Q_{ab}. We used $\delta Q_{ab} = \sigma_a\sigma_b - q$, with $q = \langle \sigma_a\sigma_b \rangle$ being the Edwards Anderson parameter: the value of Q_{ab} elements for the replica symmetric (RS) solution. Average $\langle\ldots\rangle$ is performed over the proper replicated ensemble [23,24]. One then expands Γ around the RS critical point, this being the paramagnetic solution for continuous transitions (number of replicas $n\to 0$) [28–30] or the dynamic threshold state solution for discontinuous transitions ($n\to 1$) [20–22,31]

$$\Gamma(\delta Q)=\frac{1}{2}\sum_{(ab),(cd)}\delta Q_{ab}M_{ab,cd}\delta Q_{cd}-\frac{w_1}{6}\,\mathrm{Tr}\delta Q^3-\frac{w_2}{6}\sum_{ab}\delta Q_{ab}^3, \tag{3}$$

with $a,b,c,d=1,\ldots,n$. We have retained only two of all the cubic coefficients for they are the relevant ones at criticality [32]. Our main result is:

$$\lambda=\frac{w_2}{w_1}. \tag{4}$$

The starting point for deriving the above result is to reproduce static replica solutions as a long time limit within a dynamic formulation (details elsewhere [33]). This can be obtained, e.g., using a supersymmetric formulation [34]. Then one computes the corrections around the long time limit solution assuming large but not infinitely long times, obtaining equations formally equivalent to the scaling mode-coupling equations [from which Eq. (1) is derived] [8]. On the other hand, one sees that the coefficients of the dynamic equations are equal to those of the replicated Landau expansion for the statics, thus, leading to Eq. (4). When $\lambda\to 1$ a crossover from a discontinuous dynamic to a continuous transition occurs. Technically, the static replica solution for $n=0$ coincides with the one at $n=1$, $w_1=w_2$ and the critical slowing down is no more power-law but logarithmic. In MCT language, we are in the presence of glass transition singularities [7]. We, further, note that for continuous transitions Eq. (4) yields the breaking point x of the replica symmetry at the transition [13].

Equation (4) holds in full generality above the upper critical dimension. In general, we do not have an analytic expression of the Gibbs free energy, e.g., for MF models defined on finite-connectivity random graphs. However, Γ is defined as the Legendre transform of Φ, cf. Eq. (2) and its proper vertices can be associated to cumulants of the replicated order parameter. We, thus, face the problem of computing Γ from Φ in the presence of fields ϵ_{ab} coupled to $\sum_i s_i^a s_i^b/N$. The free energy Φ needs to be computed at the third order, i.e., dealing with eight RS cumulants $\omega_{1,\ldots,8}$ [32]. The coefficients w_1 and w_2 of the Gibbs potential, cf. Eq. (3), however, are expressed as

$$w_1=r^3\omega_1,\qquad w_2=r^3\omega_2, \tag{5}$$

where r is the inverse of the SG susceptibility, $r=(\sum_{ij}\overline{\langle s_is_j\rangle_c^2}/N)^{-1}$ also called the replicon, and ω_1 and ω_2 can be written in terms of six spin correlations [33]:

$$\omega_1=\frac{1}{N}\sum_{ijk}\overline{\langle s_is_j\rangle_c\langle s_js_k\rangle_c\langle s_ks_i\rangle_c}, \tag{6}$$

$$\omega_2=\frac{1}{2N}\sum_{ijk}\overline{\langle s_is_js_k\rangle_c^2}, \tag{7}$$

where $\overline{\cdots}$ means the average over the disorder and $\langle\ldots\rangle_c$ denotes the connected thermal average. In the case of discontinuous transition, it is implicit that different thermal averages in the above expression are all computed within the same state, selected by the initial condition. We underline that while vertices w_1 and w_2 remain finite at T_d, the corresponding ω cumulants diverge as r^{-3}.

Method validation and λ computation.—Applying Eq. (4) and computing w_1 and w_2 within the above mentioned static approach, we have tested our prediction on various models. In those few cases where λ is exactly known from the dynamics [3,11,35], we verify that its formal expressions are identical. In more complicated systems the dynamic MCT phenomenology has been studied numerically and estimates of the exponents are available in the literature. These are instances in which Eq. (4) yields actual analytic predictions for λ.

Fully connected models.—In the following, we consider a family of models with a Hamiltonian of the kind:

$$\mathcal{H}=-\sum_{i<j}J_{ij}\sigma_i\sigma_j-\sum_{p=1}^{\infty}\sqrt{\frac{R^{(p)}}{p!}}\sum_{i_1<\ldots<i_p}K^p_{i_1,\ldots,i_p}\sigma_{i_1}\cdots\sigma_{i_p},$$

where σ_i are N Ising spins, or soft or spherical ones. The 2-body interaction matrix is constructed as $J=\mathcal{O}^T\mathcal{M}\mathcal{O}$, where $\mathcal{O}$ is a random $O(N)$ matrix chosen with the rotational invariant Haar measure and $\mathcal{M}$ is a diagonal matrix with elements independently chosen from a distribution $p(\mu)$ [36]. In order to ensure the existence of the thermodynamic limit, the support of $p(\mu)$ must be finite and independent of N. The p-body interactions $K^{(p)}$ are independent identically distributed Gaussian variables with zero mean and variance $p!/N^{p-1}$ and $R^{(p)}=d^pR(x)/dx^p(x)|_{x=0}$ for some real valued function $R(x)$. For the MF Ising SG [1,4,29], as well as for spherical SG's [17,37,38], the general form of the replicated Gibbs free energy is:

$$-n\beta\Gamma=\mathrm{extr}_{Q,\Lambda}\mathcal{S}[Q,\Lambda], \tag{8}$$

$$\mathcal{S}[Q,\Lambda]=\frac{1}{2}\,\mathrm{Tr}G(\beta Q)+\frac{\beta^2}{2}\sum_{ab}R(Q_{ab})-\frac{1}{2}\,\mathrm{Tr}Q\Lambda+\ln\mathrm{Tr}_{\{\sigma\}}\mathcal{W}[\Lambda;\{\sigma\}], \tag{9}$$

$$\mathcal{W}[\Lambda;\{\sigma\}] = \exp\left\{\frac{1}{2}\sum_{a,b}\Lambda_{ab}\sigma_a\sigma_b\right\}, \quad (10)$$

where $G:M_{n\times n}\to M_{n\times n}$ is a function in the space of $n\times n$ matrices, formally defined through its power series around zero. Its form depends on the choice of the eigenvalue distribution $p(\mu)$ of the $\mathcal{M}$ matrix.

Given this effective action $\mathcal{S}[Q,\Lambda]$, the saddle point equations in Λ and Q respectively read

$$Q_{ab} = \langle\sigma_a\sigma_b\rangle_{\mathcal{W}}, \quad (11)$$

$$\Lambda_{ab} = \beta[G'(\beta Q)]_{ab} + \beta^2 R'(Q_{ab}),$$

that, in the RS ansatz, become

$$q = \langle m^2(z)\rangle_z; \qquad m(z) = \langle\sigma\rangle_\sigma$$
$$\Lambda = \frac{\beta}{n}[G'\{\beta[1+(n-1)q]\} - G'\{\beta(1-q)\}] + \beta^2 R'(q), \quad (12)$$

where the weights over which $\langle\ldots\rangle_z$ and $\langle\ldots\rangle_\sigma$ are proportional to the following distributions:

	$\langle\ldots\rangle_\sigma$	$\langle\ldots\rangle_z$
Ising	$e^{z\sigma}[\delta(\sigma+1)+(\sigma-1)]$	$e^{-z^2/(2\Lambda)}\cosh^n(z)$,
Spher.	$e^{z\sigma}\exp\{-\sigma^2/[2(1-q)]\}$	$e^{-z^2/(2\Lambda)}e^{n(1-q)z^2/2}$

implying $m(z)=\tanh(z)$ for Ising and $(1-q)z$ for spherical spins. To compute $w_{1,2}$, cf. Eq. (3), one has to expand Eq. (9) to third order around the saddle point value. Considering the second order term and imposing that the Hessian determinant vanishes (criticality) we obtain

$$\langle\langle(\sigma-m)^2\rangle_\sigma^2\rangle_z = [\beta^2 G''\{\beta(1-q)\} + \beta^2 R''(q)]^{-1}. \quad (13)$$

Using Eq. (13) and expanding Eq. (11) to second order, for fully connected systems we derive:

$$\frac{3!}{\beta^2}w_1 = \frac{\beta}{2}G'''[\beta(1-q)] + A(q)\langle\langle(\sigma-m)^2\rangle_\sigma^3\rangle_z, \quad (14)$$

$$\frac{3!}{\beta^2}w_2 = \frac{R'''(q)}{2} + 2A(q)\langle\langle(\sigma-m)^3\rangle_\sigma^2\rangle_z,$$
$$A(q) \equiv \beta^4\{G''[\beta(1-q)] + R''(q)\}^3, \quad (15)$$

holding both for $n=0$ and $n=1$. They can be used to compute λ in different cases. We now exemplify a few.

For the Sherrington-Kirkpatrick model, $R(x)=x$, $G(x)=x^2/2$, $\rho(\mu)=\sqrt{4-\mu^2}/(2\pi)$, Sompolinsky's result for Ising spins along the dAT line is recovered, cf. Eq. 6.21 of Ref. [11].

For the random orthogonal model model [4,36,39], one has $R(x)=0$,

$$2G(x) = v_\alpha(x) - 1 + 2(\alpha-1)\ln\frac{v_\alpha(x)+2x+2\alpha-1}{2\alpha} - \ln\frac{v_\alpha(x)+1+2x(2\alpha-1)}{2}$$
$$v_\alpha(x) \equiv \sqrt{1+4x(2\alpha-1+x)}$$

$\rho(\mu)=\alpha\delta(\mu-1)+(1-\alpha)\delta(\mu+1)$, and the transition is dynamic. MCT dynamics in the ergodic phase has been numerically studied in Ref. [40] for $\alpha=13/32$, where strong finite-size effects are observed and two different estimates for the exponent provided: $b=0.62$, from the fit of the von Schweidler law, while $b=0.75$, from the fit of the equilibrium α relaxation time vs temperature. Our "static-driven" computation yields $b=0.628$ ($\lambda=0.7077$), allowing for a validation of the first numerical estimate.

For the Ising p spin, $G=0$, and $R(x)=x^p/2$, λ values are in Table I. For $p\to 2$ we retrieve the result of Ref. [2].

For spherical models, $G=0$, $R(x)=\sum_r x^r/2$, the exact analytic form $\lambda=\Lambda''(q_d)(1-q_d)^3/2$ is retrieved [3,35,41], equivalent to the schematic MCT prediction for very long times at criticality, i.e., for $C(t)=q_d$ [8,9], $\Lambda(C(t))$ being the MCT equations memory kernel. In the simpler case $R(x)=x^p$, $\lambda=1/2$ for any p, confirming the dynamic results for the spherical model [3] and the soft-spin model studied in the approximation of Ref. [2].

Yet another example is the Potts glass:

$$\mathcal{H} = -\sum_{ij}J_{ij}(p\delta_{\sigma_i\sigma_j}-1). \quad (16)$$

The model has a discontinuous glass transition for $p>4$ [13]. Brangian *et al.* studied the Potts glass with $p=10$ by means of Monte Carlo numerical simulations [42]. Approaching the dynamic transition, finite-size effects turn out to be large, implying that the *plateau* is almost invisible also for very large sizes: this makes the numerical estimation of exponents very difficult. Their interpolation yields $a=0.33\pm 0.04$. For $p=10$, from the expansion of $\Gamma[\delta Q]$ around $q=q_d$, we obtain the exact values $\lambda=0.8053$ and $a=0.2759$ [43], compatible with, though not extremely near to, the numerical estimate.

Models on diluted random graphs.—To study glassy models on random graphs we set up an apart technical method to analytically compute dynamic exponents. To

TABLE I. Dynamic exponents in the Ising p spin model.

p	$\to 2$	2.05	2.2	2.5	3	4	5	6	7	8	9
λ	0.5	0.556	0.652	0.719	0.743	0.746	0.743	0.739	0.736	0.733	0.731
a	0.395	0.379	0.346	0.32	0.308	0.306	0.308	0.31	0.311	0.313	0.314

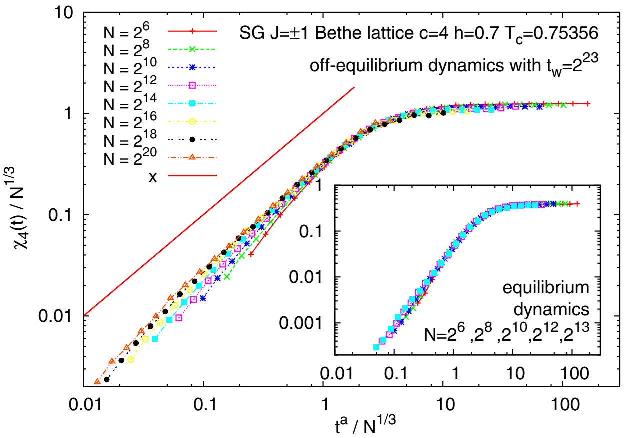

FIG. 1 (color online). Rescaled $\chi_4(t)$ for the Viana-Bray model with $c = 4$, $h = .7$, and $T = .735\,36(1)$, along the dAT line with $a_{\rm th} = 0.406$. No fitting interpolation is performed. The line $y = x$ is parallel to the expected behavior for $t < t^*(N)$ (see text).

frame the results, we first recall that there is no single instance of these classes of models whose dynamics have been solved explicitly. Equation (4) would allow us to bypass the problem provided one had the replicated action Γ. Unfortunately, the action is hard to compute in diluted models, thus preventing w's derivation. We can, however, explicitly compute the related cumulants $\omega_{1,2}$ at criticality, cf. Eqs. (6) and (7). Details of the derivation will be given elsewhere [43], while here we present a validation of the results for the Viana-Bray model [44] in a field, displaying a continuous transition along the dAT line.

Let us consider a Bethe lattice with connectivity $c = 4$, in a field $h = 0.7$ at the corresponding critical temperature $T_c(h) = 0.735\,36(1)$ [45]. The cumulants ratio yields $\lambda = 0.461$. Our analytic prediction for the β decay exponent is, thus, $a_{\rm th} = 0.406$. We, then, numerically study the system by means of both equilibrium and off-equilibrium Monte Carlo simulations. At equilibrium sizes $2^8 \leq N \leq 2^{13}$ have been probed. Starting from an equilibrated configuration we measure the decay of $C(t) \equiv \sum_i s_i(0)s_i(t)/N$ with time. For each sample 6 replicas are simulated and 16 uncorrelated measurements taken for each replica. Number of samples: from 1280 ($N = 2^{13}$) to 5120 ($N = 2^6$). Off-equilibrium, the two time $C(t, t_w)$ is measured. For very long t_w, $C(t, t_w)$ tends to the equilibrium $C(t)$. For each sample we take one measure (at the longest simulated t_w). The number of samples goes from 1280 ($L = 2^{20}$) to 51 200 ($L = 2^6$). Because of finite-size effects, $C(t)$ (at and off- equilibrium) displays a power-law behavior only for times smaller than a time scale $t^*(N)$ that diverges with N. This makes the estimation of the exponent hard. Therefore, rather than $C(t)$, we probe its thermal and disorder fluctuations,

$$\chi_4(t) = N[\overline{\langle C(t)^2\rangle} - \overline{\langle C(t)\rangle^2}], \qquad (17)$$

remaining finite for large N at 'finite t. Using finite-time-scaling arguments it can be argued that (i) $\chi_4(t)$ diverges as t^a at large times on a scale smaller than $t^*(N)$, (ii) the critical region diverges as $t^*(N) = N^{1/(3a)}$, (iii) on times scales larger than the critical region the fluctuations scale as $N^{1/3}$ [46,47]. As a consequence, if a has the correct value, the rescaled dynamic critical $\chi_4(t)/N^{1/3}$ vs $t/N^{1/(3a)}$, should be size independent. We plot it in Fig. 1 with $a = a_{\rm th} = 0.406$. Collapse appears excellent both at and off-equilibrium.

Concluding, we have introduced a "static-driven" method to obtain, by means of a replica field theory, the dynamic exponents of the critical slowing down. The method allows us to determine the MCT parameter exponent λ as the ratio of coefficients of third-order terms of the Gibbs free energy action expanded around the critical point, cf., e.g., Eqs. (14) and (15). Equivalently, λ is shown to be equal to the ratio between six point cumulants of a theory whose action is the Legendre transformed of the Gibbs free energy, cf. Eqs. (6) and (7). Indeed, the dynamical exponents can be associated to the ratio between two physical observables computed within a static framework. We verified the method's prediction in various MF models, both on fully connected and diluted graphs, successfully comparing with previous analytical and numerical results. The method can be exported to any glass models whose Gibbs action is computable or whose six point cumulants can be estimated.

We thank A. Crisanti, S. Franz, and E. Zaccarelli for useful discussions.

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