A note on weakly discontinuous dynamical transitions

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We analyze mode coupling discontinuous transition in the limit of vanishing discontinuity, approaching the so called "A_3" point. In these conditions structural relaxation and fluctuations appear to have universal form independent from the details of the system. The analysis of this limiting case suggests new ways for looking at the mode coupling equations in the general case. © *2013 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4790517]

I. INTRODUCTION

The dynamics of supercooled liquids is characterized by a two step relaxation. After a rapid decay, the dynamical correlation function displays a plateau where relaxation is arrested before decaying on a much larger time scale. Mode coupling theory (MCT) describes the formation of the plateau in terms of a discontinuous dynamical transition where the length of the plateau diverges as the temperature becomes close to the dynamical transition point.[1]

The approach and the departure from the plateau are described by power laws, respectively, t^{-a} and t^b, where the powers a and b are system dependent but obey the universal relation:

$$\lambda = \frac{\Gamma^2(1+b)}{\Gamma(1+2b)} = \frac{\Gamma^2(1-a)}{\Gamma(1-2a)}. \tag{1}$$

The exponent parameter λ also appears in replica theory, where it has been related to the ratio between six point static correlation functions that can in principle be measured or computed directly using the Boltzmann measure.[2] Explicit analytic computations have been performed in mean-field schematic models[3–6] and in liquids.[7,8]

However, a discontinuous glass transition is not the only possibility. A different transition mechanism is found, for example, in spin glasses with full replica symmetry breaking, where the long time limit of the dynamical correlation function passes continuously from zero to a non zero value when the transition is crossed. Within MCT, Götze and Sjögren[9] have proposed a schematic model whose dynamical transition can be tuned smoothly from a discontinuous one to a continuous one through the variation of a parameter. The resulting critical point has been recognized as "A_3" singularity in Arnol'd terminology.

The study of discontinuous/continuous crossover is not a mere academic exercise. Realistic systems where this is found include disordered spin models in presence of a magnetic field, kinetic models on random graphs,[10,11] liquids in porous media both in the MCT[12,13] and in the hypernetted chain (HNC) approximations and liquid models with pinned particles.[14]

In Ref. 9, Götze and Sjögren initiated the study of the problem presenting various remarkable results, including the characterization of the dynamics at the critical point and the identification of two diverging time-scales relevant for the β-relaxation dynamics near the critical point. They also pointed out that in some sectors of the parameter space the final decay happens on even larger scales but a full characterization of the correlation function of these α processes is still to be obtained. More recently a series of result concerning the α-regime have been obtained in some sector of the parameter space by Götze and Sperl.[15,16] Here we will consider a different sector and show that in this case α-relaxation takes a universal form.

In Sec. II we set up the problem and discuss the dynamical correlation function for weakly discontinuous transitions both in equilibrium and in the aging regime. The first section is organized in Subsections II A–II D, and in Subsection II B we will briefly recall the theoretical results of Götze, Sjögren, and Sperl on A_3 in order to make contact with ours. In Sec. III we extend our analysis to the study of fluctuations and compute the four point susceptibility. Finally we draw our conclusions.

II. MCT EQUATIONS NEAR A CONTINUOUS TRANSITION

Mode-coupling theory provides a description of the dynamical correlation and response functions in terms of a system of integro-differential equations.[1] In the general theory of liquids these are equations for the dynamical structure factor and they contain information about the spatial structure of this quantity. However, close to the transition the spatial structure can be neglected in a first approximation by looking at the peak of the static structure factor.[17] Using this fact one can produce a dynamical equation describing the evolution of a single mode, that is called the schematic MCT equation.

It is well known that this equation is exactly the one that describes the Langevin dynamics of fully connected spherical p-spin model with a 1 step replica symmetry breaking (1RSB) dynamical transition.[18–20] The schematic MCT equation for the correlation $C(t)$ reads:

$$\frac{dC(t)}{dt} = -TC(t) + \frac{1}{T}(1 - C(t))\hat{M}[C(t)] - \frac{1}{T}\int_0^t du\, \frac{dC(u)}{du}(\hat{M}[C(t-u)] - \hat{M}[C(t)]), \tag{2}$$

where $\hat{M}[C(t)]$ is the memory kernel that depends on the parameters of the problem, temperature, and/or density (for example, in the p-spin spherical model we have $\hat{M}[q] = pq^{p-1}/2$); the initial condition is $C(0) = 1$. Depending on the nature of the memory kernel, different kinds of dynamical transitions are possible. Well known instances are the F_{12} and F_{13} models that are specified, respectively, by a memory kernel of the form $\hat{M}(q) = \hat{v}_1 q + \hat{v}_2 q^2$ and $\hat{M}(q) = \hat{v}_1 q + \hat{v}_3 q^3$.

It can be shown that the long-time limit of the correlation (the non-ergodicity parameter) satisfies the following equation:

$$q = (1 - q)M[q], \tag{3}$$

where we have defined $M[q] = \hat{M}[q]/T^2$. Glassy states are characterized by a solution with $q > 0$. When multiple solutions exist a maximum theorem of MCT[1] states that the one has to chose the solution with the higher value of q provided $0 < q < 1$. Glass transitions singularities are identified with the bifurcation singularities of the above equation. The simple singularity is called A_2 while the next type of singularity is called A_3, both will be discussed in Subsections II A–II D.

A. Glassy dynamics near an A_2 singularity

The A_2 singularity is the simplest scenario and it is relevant for supercooled liquids. In the typical case at high temperature Eq. (3) has a single solution $q_0 = 0$ corresponding to the liquid phase while lowering the temperature another solution $q_1 > 0$ appears abruptly at some temperature (called T_c (for critical) in MCT literature and T_d (for dynamical) in spin-glass literature). Since the dynamics decays from 1 for $T < T_d$ the solution remains stuck in the glassy solution with the highest value of $q > 0$. Depending on the structure of $M(q)$ we can also have transition called glass-glass if $q_0 > 0$, in this case at the critical point it appears abruptly a solution $q_1 > q_0$ and again the dynamics remains stuck in the glassy state with highest value of q, this can be observed for instance in the F_{13} model. It follows that at a generic A_2 singularity the novel solution q_1 satisfies both (3) and the following equation:

$$1 = \frac{d}{dq}(1 - q)M[q]|_{q=q_1}, \tag{4}$$

which expresses a marginal stability condition of the dynamics at criticality.[1] A possible way to prove this is by using the physical condition that $dC/dt \le 0$ we can argue from (2) that

$$-TC(t) + \frac{1}{T}(1 - C(t))\hat{M}[C(t)] \le 0. \tag{5}$$

To have a glassy behavior, the function on the right should have a maximum between 0 and 1 and the relation above is satisfied at high enough temperature. At the dynamical transition the maximum touches zero and the system remains stuck in the metastable minimum.[21] It follows that for this value of the correlation one also has the relation (4). This means that q_1 is a double root for Eq. (3).

Dynamics near an A_2 singularity displays the well-known two-step relaxation scenario. Approaching the transition from the liquid or low-q phase (if $q_0 > 0$) the correlation $C(t)$ remains near the value q_1 for an increasing time (the so-called β regime), while if finally decays to q_0 in the so-called α regime. The β regime displays universal properties[22] that are obtained expanding the dynamical equation for $C(t) = q_1 + G(t)$ as the relevant control parameter is the so-called separation parameter σ which is negative in the low-q phase and positive in the high-q phase. In the β regime we have

$$G(t) = |\sigma|^{1/2} f_{\pm}(t/\tau_\beta) \quad t \gg 1, \quad \tau_\beta \propto \frac{1}{|\sigma|^{\frac{1}{2a}}}, \tag{6}$$

where the function f_- has to be chosen in the low-q phase ($\sigma < 0$) while the function f_+ has to be chosen in the high-q phase ($\sigma > 0$). The two universal scaling functions f_+ and f_- obey universal equations.[22] They both diverge as $1/x^a$ for $x \to 0$ while their behavior at large value of x is completely different. In the low-q region we have to choose f_- that goes to $-\infty$ as x^b for large x where b is given by the well-known equation:

$$\lambda = \frac{\Gamma^2(1-a)}{\Gamma(1-2a)} = \frac{\Gamma^2(1+b)}{\Gamma(1+2b)}. \tag{7}$$

Below the dynamical temperature instead we have to choose f_+ that decays exponentially to the constant $(1 - \lambda)^{-1/2}$ for $x \to \infty$.

In the low-q region and close to the transition the decay of the correlation from $C(t) \sim q_1$ to $C(\infty) = q_0$, verifies the "time-temperature superposition principle," i.e., it has a scaling form:

$$C(t) \approx \mathbf{C}(t/\tau_\alpha), \tag{8}$$

where the $\tau_\alpha(T) \sim |\sigma|^{-\gamma}$ is the relaxation time as a function of the temperature and $\mathbf{C}(u)$ is a scaling function independent of the temperature. The matching with the β regime implies the following expression for the exponent γ:

$$\gamma = \frac{1}{2a} + \frac{1}{2b}. \tag{9}$$

The scaling $\mathbf{C}(u)$ function can be computed solving the following equation obtained from (2) setting to zero the time derivative:

$$0 = -TC(t) + \frac{1}{T}(1 - C(t))\hat{M}[C(t)] - \frac{1}{T}\int_0^t du\, \frac{dC(u)}{du}(\hat{M}[C(t-u)] - \hat{M}[C(t)]), \tag{10}$$

If we are at the critical point the corresponding equation admits a solution $\mathbf{C}(u)$ such that $\mathbf{C}(0) = q_1$ and $\mathbf{C}(\infty) = q_0$. Much as the functions $f_+(x)$ and $f_-(x)$ of the β regime the solution is scale invariant meaning that $\mathbf{C}(su)$ is also a solution for any $s > 0$. However, at variance with $f_+(x)$ and $f_-(x)$ that depend only on the exponent parameter λ the α-regime scaling function depends on the whole memory kernel $M(q)$ for $q_0 < q < q_1$ and therefore it is not universal.

B. Glassy dynamics near an A_3 singularity

The A_3 singularity is defined as the endpoint of a line of A_2 transitions. In the general case this means that near a critical point A_3 Eq. (3) can be expanded in powers of q around some finite value q_c and the coefficients of the constant, linear, and quadratic terms are small as a function of the external parameters. In order to discuss the structure of the solutions it is convenient following[9] to apply a shift to q_c in order to have a vanishing quadratic coefficient. The resulting equation has the following structure:

$$0 = \xi + \eta\delta q - \mu\delta q^3, \tag{11}$$

where the two coefficients ξ and η can be expressed in terms of derivatives of $M(q)$ at q_c and vanish at the A_3 singularity while μ remains finite. The structure of the solutions of the above equation in the (η, ξ) plane is the following: (i) on the line $(\xi = 0, \eta < 0)$ we have only the solution $\delta q = 0$; (ii) proceeding counter-clockwise the solution becomes negative $\delta q < 0$; (iii) on the critical line $\xi = -2\eta(\eta/3\mu)^{1/2}$ and $\eta > 0$ a couple of new solutions with a higher value of δq appear discontinuously, it is a line of standard A_2 singularities and near this line the dynamics is that of Subsection II A; (iv) proceeding counter-clockwise we reach the line $\xi = 2\eta(\eta/3\mu)^{1/2}$ and $\eta > 0$ where the intermediate and smaller solutions merge and disappear abruptly and Eq. (11) has a again only one real solution. This second transition plays no role because for the aforementioned maximum theorem the dynamics remains blocked in the glassy state described by the solution with higher value of δq.

We note that near the A_3 singularity the value of δq is always small, meaning that for generic transitions where $q_c > 0$ the critical line describes a glass-glass transition. In the special case $q_c = 0$ (e.g., in the F_{12} model) we have instead a transition from a liquid $q = 0$ to a glass with a small value of q.

As shown in Ref. 9 at the critical point $\xi = \eta = 0$ the correlator displays a logarithmic decay at leading order:

$$C(t) = \frac{\rho^2}{\ln^2 t}, \tag{12}$$

where $\rho^2 = 4\pi^2/(6\mu)$. In order to see deviations from this behavior for small non-zero ξ and ν we need to reach extremely large times and correspondingly very small values of the correlator. Indeed combining the expression for the critical decay with Eq. (11) we obtain the two-time scales over which non-zero ξ and ν can be detected:

$$t_\xi \propto \exp[\rho(\mu/|\xi|)^{1/6}], \tag{13}$$

$$t_\eta \propto \exp[\rho(\mu/|\eta|)^{1/4}]. \tag{14}$$

The dynamics on this times scales is described by the general form:

$$C(t) = \rho^2 p(\ln(t/t_1)), \tag{15}$$

where t_1 is an unknown constant that cannot be fixed because of the scale invariant nature of the equations considered and $p(y)$ is a solution of the following equation:[23]

$$p' = -(4p^3 - g_2 p - g_3)^{1/2}, \tag{16}$$

where $g_2 \equiv 4\eta/(\mu\rho^4)$ and $g_3 \equiv 4\xi/\mu(\rho^6)$. In some regions of the (ξ, η) plane the above solution describes the cross-over from the critical behavior (12) at small values of y to the long-time limit of the dynamic given by the solution of Eq. (11) at large values of y. In some other regions however the above solution describes solely a change in the decay rate and the final decay must occur on even larger time scales. For instance on the line $(\eta = 0, \xi < 0)$ Eq. (16) yields a cross-over to a simple logarithmic decay $C(t) \propto -\ln t$ on the time scale (13), similarly on the line $(\xi = 0, \eta < 0)$ we have $C(t) \propto -\ln^2 t$ on the time-scale (14).[9]

A complete characterization of the α-regime near the A_3 point is still an open problem. In Ref. 15 this problem was studied considering corrections of the form $\ln^k t$ to the leading term $\ln t$ on the line $(\eta = 0, \xi < 0)$. In the following instead we will focus on the behavior on the critical line $\xi = -2\eta(\eta/3\mu)^{1/2}$. According to Subsection II A the α regime near an A_2 singularity is obtained by taking the limit of $\sigma \to 0^-$ while considering the dynamics on time-scales that diverge as $|\sigma|^{-\gamma}$. Given a point on the critical line (η_0, ξ_0) the separation parameter is a linear function of the distance between a generic point (η, ξ) and (η_0, ξ_0). We will *take the* $\sigma \to 0^-$ *limit first and then take the* $(\eta_0, \xi_0) \to (0, 0)$ *limit*. As we said before the α-regime scaling function $\mathbf{C}(u)$ is not universal, but we will show that near an A_3 point it has instead a universal shape.

C. α-relaxation near weakly discontinuous transitions

We are interested in the case of weakly discontinuous A_2 transitions close to a A_3 critical point where q_1 and q_0 are almost degenerate. Thanks to the vicinity to criticality we can characterize these transitions in a universal way. For small $q_1 - q_0$, the exponent parameter λ, which is in general determined by the relation:

$$\lambda = \frac{\hat{M}''(q_1)}{2(\hat{M}'(q_1))^{3/2}}, \tag{17}$$

is near to 1 and both the exponents a and b are close to zero. At the leading order $a = b = \sqrt{\frac{6}{\pi^2}(1-\lambda)} \sim \sqrt{q_1 - q_0}$. We choose to parameterize the distance from the A_3 critical point by the value of b itself (so that $q_1 - q_0$ is a vanishing function of b in the limit $b \to 0$).

As discussed already in Ref. 23, at small argument the function $\mathbf{C}(t)$ admits a regular short time series expansion in terms of the parameter $y = t^b$, whose coefficients can be computed solving recursively Eq. (10). Unfortunately this expansion is not convergent in the general case, but for $b \to 0$ we can compute the solution directly from the equation. More precisely, we suppose the existence of the limit:

$$\lim_{\substack{b\to 0;\, t\to\infty \\ y=(t/\tau_\alpha)^b}} (\mathbf{C}(t, b) - q_0)/(q_1 - q_0) = G(y), \tag{18}$$

with $G(y)$ a well defined function of its argument.

Let us now rewrite Eq. (10). We get:

$$C(t) = M[C(t)](1 - C(t)) - \int_0^t \mathrm{d}u\, \frac{\mathrm{d}C(u)}{\mathrm{d}u}(M[C(t-u)] - M[C(t)]). \quad (19)$$

We now consider the various terms in Eq. (19). We first consider the memory term in the integral; in the $b \to 0$ limit:

$$M[C(t-u)] - M[C(t)] \simeq M'[C(t)](C(t-u) - C(t)) \simeq byM'(q_1)(q_1 - q_0)\frac{\mathrm{d}G(y)}{\mathrm{d}y}\ln\left(1 - \frac{u}{t}\right). \quad (20)$$

In an analogous way we have

$$C'(u) = \simeq \frac{b}{u} y(q_1 - q_0)\frac{\mathrm{d}G(y)}{\mathrm{d}y}. \quad (21)$$

Next we observe that generically at the transition point the function $N(C) = -C + (1 - C)M(C)$ has a single root in q_0 and a double root in q_1. For small $q_1 - q_0$ its form should read $N(C) = -A(C - q_0)(q_1 - C)^2$, where by using the relations (3), (4), and (17) we have $A = \frac{M'(q_1)(1-\lambda)}{q_1 - q_0}$. It follows that to the leading order the mode coupling equation can be rewritten as

$$0 = M'(q_1)(q_1 - q_0)^2[(1 - \lambda)G(1 - G)^2 - (b\,y)^2[G'(y)]^2 \int_0^1 \frac{\mathrm{d}u}{u}\ln(1 - u). \quad (22)$$

Now, taking into account that $1 - \lambda = b^2 \int_0^1 \frac{\mathrm{d}u}{u}\ln(1 - u) = b^2\frac{\pi^2}{6}$, we obtain the following equation for G:

$$G(1 - G)^2 = y^2[G'(y)]^2. \quad (23)$$

Note the similarity between this equation and Eq. (16) that describes the β regime. Recasting it under the form:

$$\frac{dG}{\sqrt{G}(1 - G)} = -\frac{dy}{y}, \quad (24)$$

we find that it admits the solutions

$$G(y) = \left(\frac{1 - y/y_0}{1 + y/y_0}\right)^2. \quad (25)$$

The value of y_0 cannot be computed, as a consequence of scaling invariance of the MCT equation (10) and we choose $y_0 = 1$. We notice that $G(y)$ decreases from 1 to 0, vanishing at finite $y = y_0$. This is not in contradiction with the fact that the correlation is positive for all times at finite b, but is a consequence of the fact that we have taken the limit $b \to 0$. A detailed computation for small but finite b tells us that for $y > y_0$ $C(y) \sim e^{-A(y/y_0)^{1/b}}$. This expression is exponentially small for $b \to 0$ and corresponds to the simple exponential $C(t) \sim e^{-At/t_0}$ in terms of t, where $t_0 = y_0^{1/b}$.

We can compare this asymptotic solution with the Padé approximants of the series expansion of Eq. (10) for small values of b. This is done in Figure 1 for the schematic F_{12} model[1] where $M(C) = \frac{(2\lambda-1)C+C^2}{\lambda^2}$. The curves show that the Padé approximants give an accurate description of the function at time smaller than 1, and that the limit $\lambda \to 1$ is achieved smoothly. We recall that although we tested our result on the F_{12} model characterized by $q_c = 0$, it is completely general and it holds also for non-zero q_c, notably the F_{13} model. The key point is that in Eq. (20) we can replace the general kernel $M[C(t-u)] - M[C(t)]$ with $M'[q_1](C(t-u) - C(t))$, i.e., the one of the F_{12} model.

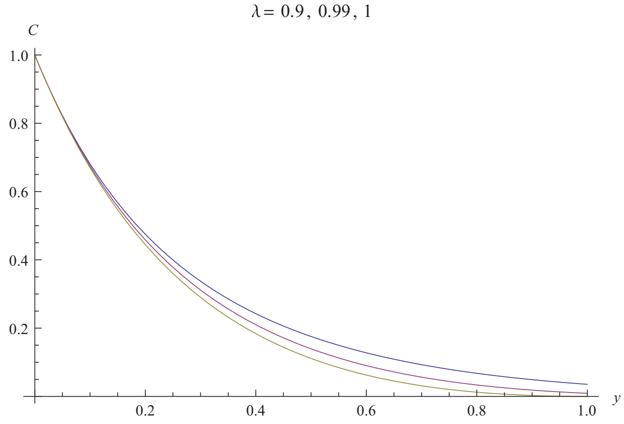

FIG. 1. Scaling function $\mathbf{C}(y)$. From top to bottom $\lambda = 0.9, 0.99, 1$. The first two curves are obtained from the (20, 20) Padé approximants of the small time expansion in t^b. The last curve is the function $(\frac{1-y}{1+y})^2$.

D. Aging

The previous analysis can be generalized to the aging dynamics. We specialize to the case of the generalized spherical p-spin model where the temperature appears explicitly into the equation. The structure of the equation in the aging alpha regime is similar to the equilibrium case and one has[24]

$$\begin{aligned}0 = &-TC(t,t') + \beta[q_1 f'(q_1)(1 - x) - q_0 f'(q_0)x]C(t,t') \\ &+ \beta f'(C(t,t'))(1 - q_1) - \beta f'(q_1)(1 - x)C(t,t') \\ &- \beta x q_0 f'(q_0) + \beta x f'(C(t,t'))(q_1 - C(t,t')) \\ &- \beta x \int_{t'}^{t} ds\, \frac{\partial C(t',s)}{\partial s}[f'(C(t,s)) - f'(C(t,t'))]. \quad (26)\end{aligned}$$

Here $f'(C)$ generalizes the memory kernel M of the equilibrium case.

The quantity x is the so called fluctuation-dissipation ratio, fixed by the condition that the function:

$$\begin{aligned}K(C) = &-TC + \beta[q_1 f'(q_1)(1 - x) - q_0 f'(q_0)x]C \\ &+ \beta f'(C)(1 - q_1) - \beta f'(q_1)(1 - x)C - \beta x q_0 f'(q_0) \\ &- \beta x f'(C)(q_1 - C), \quad (27)\end{aligned}$$

has a double root in $C = q_1$.

It is well known that Eq. (26) is reparametrization invariant and admits scaling solutions of the form $C(t, t') = \mathbf{C}(g(t) - g(t'))$ where the reparametrization function $g(t)$ is left undetermined. The short time expansion of the equation predicts a behavior of the kind:

$$\mathbf{C}(u) = q_1 + (u)^b, \quad (28)$$

where b is determined by the condition:[20]

$$\lambda = \frac{T}{2}\frac{f'''(q_1)}{f''(q_1)^{\frac{3}{2}}} = x\frac{\Gamma(1+b)^2}{\Gamma(1+2b)}. \tag{29}$$

As in the equilibrium case, for q_1 close to q_0 the function $K(C)$ behaves as $K(C) = A(C - q_0)(q_1 - C)^2)$. We can suppose that $\mathbf{C}$ becomes an analytic function of $y = (g(t) - g(t'))^b$. Notice that if the function $g(t)$ is such that $g''(t)/g'(t)^2 \ll 1$ for large t, then one can equivalently write $y = (\frac{t-t'}{\tau_{t'}})^b$. We can then define the scaling function:

$$G(y) = \lim_{\substack{b\to 0,\, t,t'\to\infty \\ y=(g(t)-g(t'))^b}} \frac{C(t,t') - q_0}{q_1 - q_0} \tag{30}$$

and repeat verbatim the analysis of the equilibrium case. It turns out that the equation verified by G coincides with the one found at the critical point. A fortiori, the same is true for the function $G(y)$.

III. FLUCTUATIONS

In this section we would like exploit our analysis to investigate fluctuations in the alpha regime. In the last years, research has concentrated in the study of fluctuations of the time dependent correlation functions in terms of 4-point functions. As it is usual in disordered systems one can define different kinds of correlation functions with *a priori* different scaling properties. It has been recently proposed that it is useful to disentangle the fluctuations of correlations with respect to thermal noise for fixed initial condition from the fluctuations with respect to initial conditions.[25]

Denoting by $\langle\cdot\rangle$ the thermal average for fixed initial condition (iso-configurational average) and by $[\cdot]$ the average initial condition, we define[25,26]

$$\begin{aligned} \chi_{th}(t) &= [\langle C(t)^2\rangle] - [\langle C(t)\rangle^2], \\ \chi_{het}(t) &= [\langle C(t)\rangle^2] - [\langle C(t)\rangle]^2. \end{aligned} \tag{31}$$

A theory for this kind of fluctuations in the beta regime has been proposed in Ref. 25, using a "reparametrization invariant" formulation where time is eliminated in favour of the average correlation function. Within a Gaussian fluctuation theory it is found that the singularity of χ_{het} doubles the one of χ_{th}.

The basic observation allowing to study now the functions in the α regime is the fact that, as proposed in Ref. 27, the leading behavior of $\chi_{th}(t)$ can be obtained as

$$\chi_{th}(t) \propto \frac{\partial C(t)}{\partial T}. \tag{32}$$

Before exploiting this relation we would like to note that it appears naturally in the theory put forward in Ref. 25. In that context that fluctuations can be described through a field theory where the correlation function, which plays the role of fundamental field, couples linearly to the temperature. Moreover, the dependence with respect to the initial configuration turns out to be parameterized by a random variation of the temperature. This has the consequence that the susceptibility χ_{het} is the square of the thermal one multiplied by the variance of the random temperature.

While these considerations strictly hold for the beta and early alpha regime, the time-temperature superposition principle shows how the correlation is very sensitive to any temperature change which can induce large changes in the relaxation time. This is a sort of "beta imprinting" indicating that large fluctuations of the correlation function in the alpha regime could be just consequence of fluctuations in the initial time of relaxation. In last instance this is a consequence of the emerging scale invariance of the MCT equation when the critical temperature is approached. We see here a link with the theory of fluctuations during aging dynamics below T_d developed by Cugliandolo, Chamon, and co-workers[28–31] where fluctuations are ascribed to the large time emergence of reparametrization invariance

With all this in mind, we can write:

$$\begin{aligned} N\chi_{th}(t) &= \frac{\partial \mathbf{C}(t/\tau(T))}{\partial T}, \\ N\chi_{het}(t) &= [\delta T^2]\chi_{th}(t)^2. \end{aligned} \tag{33}$$

The first one of these equations has been proved in Ref. 27. Even if they are valid in the liquid phase, both of them can be derived in a Gaussian approximation coming from the glassy phase.[25] Using the relation $\tau(T) \sim (T - T_d)^{-\frac{1}{2a}+\frac{1}{2b}}$ with $a \approx b$ for $b \to 0$ and $\mathbf{C}(u) = G(u^b)$, one gets

$$\begin{aligned} \chi_{th}(t) &= 2\frac{1}{T - T_d}(q_1 - q_0)zG'(z) \\ &= 2\frac{1}{T - T_d}(q_1 - q_0)\sqrt{G}(1 - G), \\ \chi_{het}(t) &= 4[\delta T^2]\frac{1}{(T - T_d)^2}(q_1 - q_0)^2 G(1 - G)^2. \end{aligned} \tag{34}$$

The divergence as a function of $T - T_d$ which just depends on the power law behavior of the relaxation time, confirms the direct dynamical analysis of Ref. 27.

Notice that for a finite system the divergence should be cut-off by a function of the volume. It was found in Refs. 25 and 32 that the scaling variable describing the cross-over is $x = (T - T_d)N^{1/2}$. This predicts an alpha relaxation scaling at T_d where $\chi_{th} \sim \frac{1}{\sqrt{N}}$ and a finite χ_{het}.

In Ref. 25 it was shown that if $C(t)$ follows a bimodal distribution as it would be implied by a simple jump process, one should expect the dependence $\chi_{het} \sim G(1 - G)$. Notice the form we find differs from this expectation.

We would like to remark that while the square root behavior of χ_{th} at small G is only valid in the limit of small $q_1 - q_0$ that we are considering, the linear behavior for $G \approx 1$ is more general: it is a consequence of the initial power law relaxation of the correlation function $C(t) = q_1 - at^b$, that holds whenever there is a discontinuous transition. As far as the small C behavior for finite b is concerned, the final exponential relaxation suggests a behavior $\chi_{th} \sim -C\log C$.

IV. CONCLUSIONS

The point where the discontinuous transition becomes continuous can be seen as a critical point for mode coupling theory. As such universal properties emerge which do not depend of the details of the model.[1] In this note we have computed the scaling functions for the correlation function both at the MCT transition and in the aging regime, finding that they take the same universal form. We have also analyzed the behavior of fluctuations, finding general expressions of the four point functions as a function of the correlations.

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