Critical properties of the disordered XY model on random graphs

Federico Ricci-Tersenghi
(Univ. Roma La Sapienza)

in collaboration with Cosimo Lupo and Giorgio Parisi

RG and Disordered Systems, Paris 26-07-2016

Why studying the XY model

- Simplest vector model, have different critical properties than Ising models
- Experiments on spin glass materials show some glassy features (e.g. rejuvenation and memory) are reduced by the spin anisotropy (i.e. in Ising models)
- Vector models allow for small perturbation to study low-energy excitations via the spectrum of the Hessian

Why on a sparse random graph

- Analytically solvable (although more complex solution than on fully-connected topology, i.e. SK limit)
- Distances can be defined, correlations can be measured
- More similar to finite dimensional lattices:
    - local fluctuations
    - finite critical field at T=0
- The SK limit can be recovered for diverging degree

XY model

$|\vec{\sigma}_i| = 1$, unit vectors in $\mathbb{R}^2$

$$\mathcal{H}[\{\vec{\sigma}_i\}] = - \sum_{(i,j)\in E} \vec{\sigma}_i U_{ij} \vec{\sigma}_j - \sum_i \vec{H}_i \vec{\sigma}_i$$

rotates by ω_{ij}
multiplies by J_{ij}

E defines the interaction graph

- a regular lattice
- a random graph

we study random regular graphs with fixed degree equal to 3

XY model

change of variables $\vec{\sigma}_i = e^{i\,\theta_i}$

$$\mathcal{H}[\{\theta_i\}] = -\sum_{(i,j)\in E} J_{ij}\cos(\theta_i - \theta_j + \omega_{ij}) - \sum_i H_i \cos(\theta_i - \phi_i)$$

XY model

change of variables $\vec{\sigma}_i = e^{i\,\theta_i}$

$$\mathcal{H}[\{\theta_i\}] = -\sum_{(i,j)\in E} J_{ij}\cos(\theta_i - \theta_j + \omega_{ij}) - \sum_i H_i \cos(\theta_i - \phi_i)$$

Ferro	Spin glass			Gauge glass
$J_{ij} = J$	$J_{ij} = \pm J$	$\Leftrightarrow$	$J_{ij} = J$	$J_{ij} = J$
$\omega_{ij} = 0$	$\omega_{ij} = 0$		$\omega_{ij} \in \{0, \pi\}$	$\omega_{ij} \in [0, 2\pi)$

XY model

change of variables $\vec{\sigma}_i = e^{i\,\theta_i}$

$$\mathcal{H}[\{\theta_i\}] = -\sum_{(i,j)\in E} J_{ij}\cos(\theta_i - \theta_j + \omega_{ij}) - \sum_i H_i \cos(\theta_i - \phi_i)$$

Ferro	Spin glass		Gauge glass
$J_{ij} = J$	$J_{ij} = \pm J$	$\Leftrightarrow$ $J_{ij} = J$	$J_{ij} = J$
$\omega_{ij} = 0$	$\omega_{ij} = 0$	$\omega_{ij} \in \{0, \pi\}$	$\omega_{ij} \in [0, 2\pi)$

XY model

change of variables $\vec{\sigma}_i = e^{i\,\theta_i}$

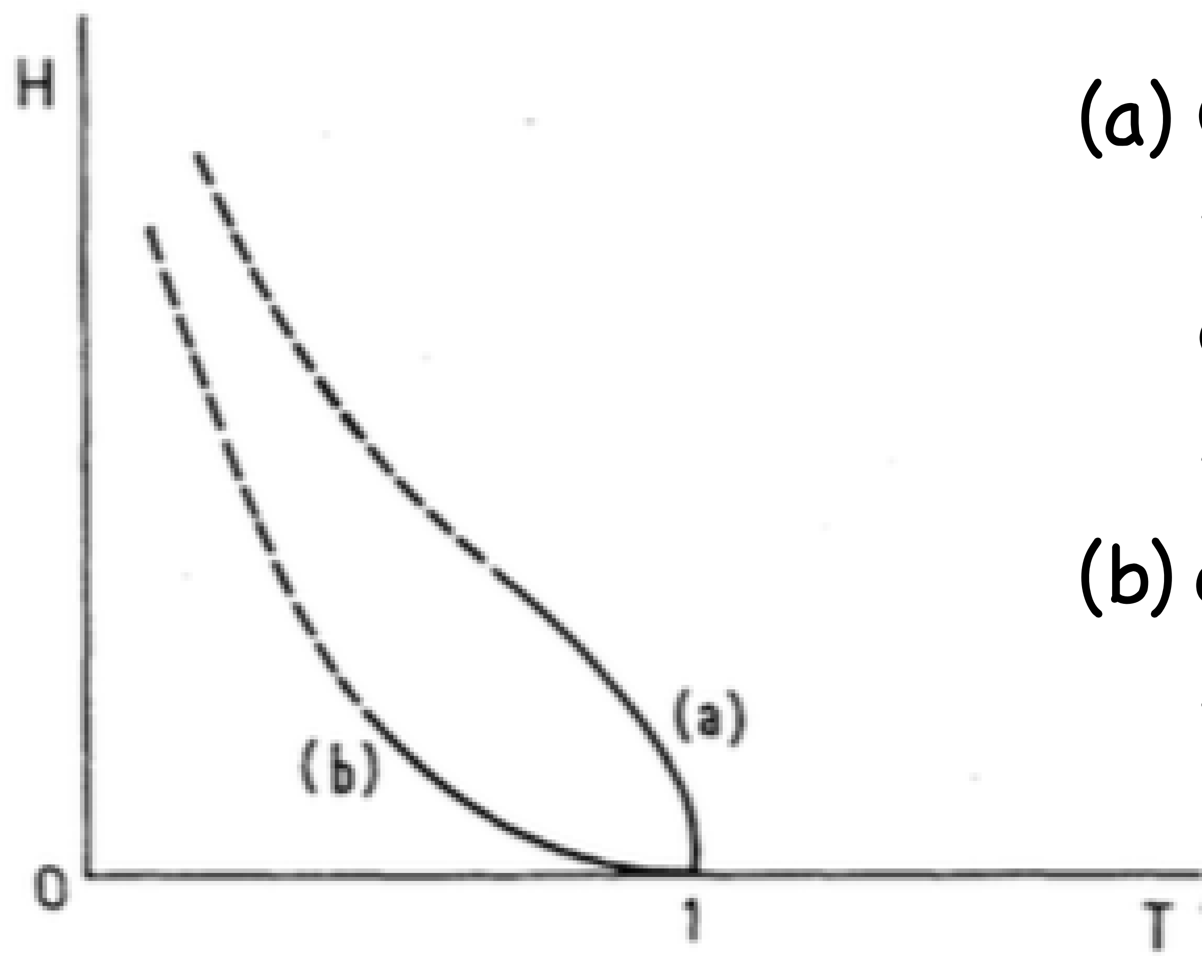
$$\mathcal{H}[\{\theta_i\}] = -\sum_{(i,j)\in E} J_{ij}\cos(\theta_i - \theta_j + \omega_{ij}) - \sum_i H_i \cos(\theta_i - \phi_i)$$

Ferro	Spin glass		Gauge glass
$J_{ij} = J$	$J_{ij} = \pm J$ $\Leftrightarrow$	$J_{ij} = J$	$J_{ij} = J$
$\omega_{ij} = 0$	$\omega_{ij} = 0$	$\omega_{ij} \in \{0, \pi\}$	$\omega_{ij} \in [0, 2\pi)$

Field:	uniform	random
$H_i = H$	$\phi_i = 0$	$\phi_i \in [0, 2\pi)$

Known phase diagrams

Fully-connected spin glass (SK-like) model

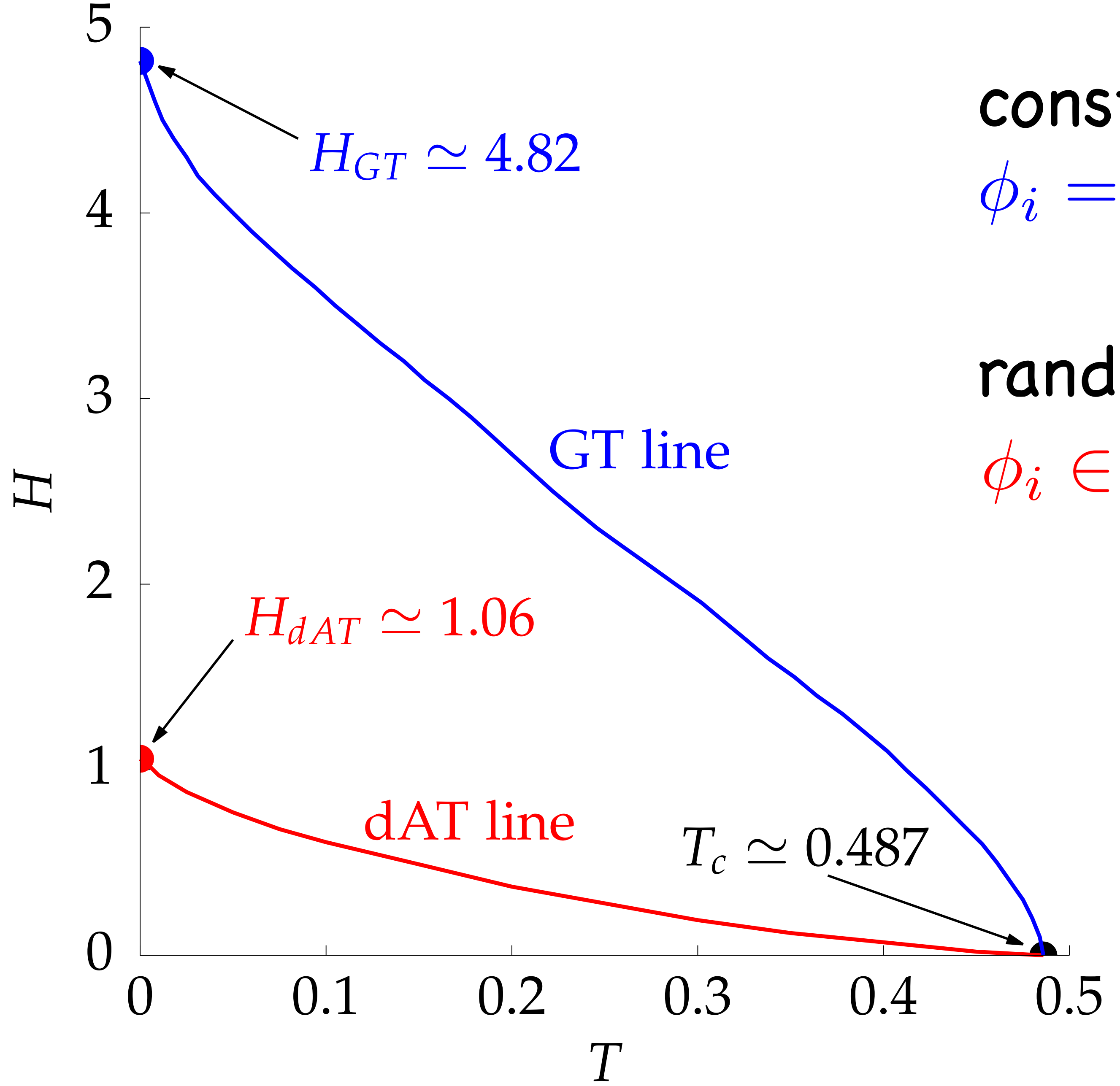


Gabay-Toulouse (1981)

(a) GT line, only if H=const.
freezing of transverse d.o.f.
effectively in zero field
breaking of spin symmetry

(b) dAT line, for any H
breaking of replica symmetry

GT and dAT lines for SG XY model on c=3 RRG



constant field

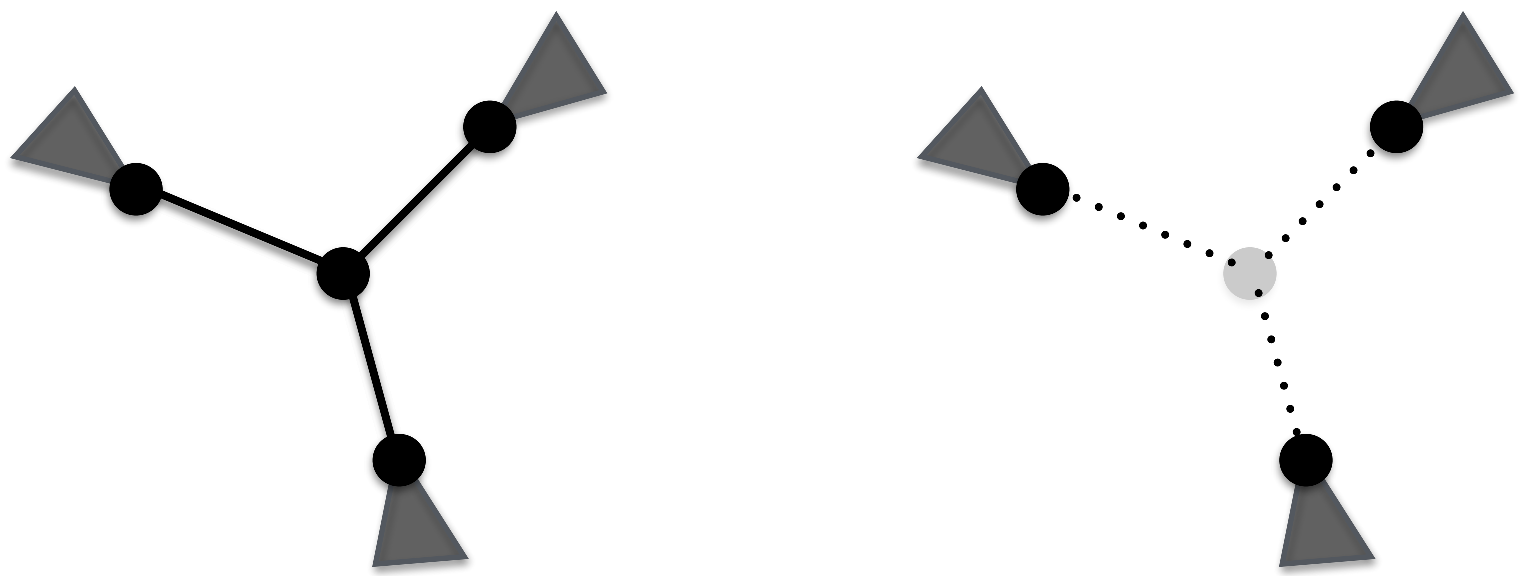
$\phi_i = 0$

random field

$\phi_i \in [0, 2\pi)$

How to solve models on sparse random graphs

Use cavity method (i.e. Bethe approximation)

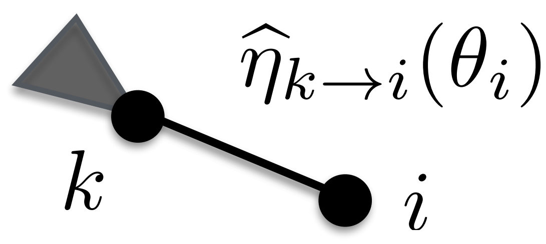


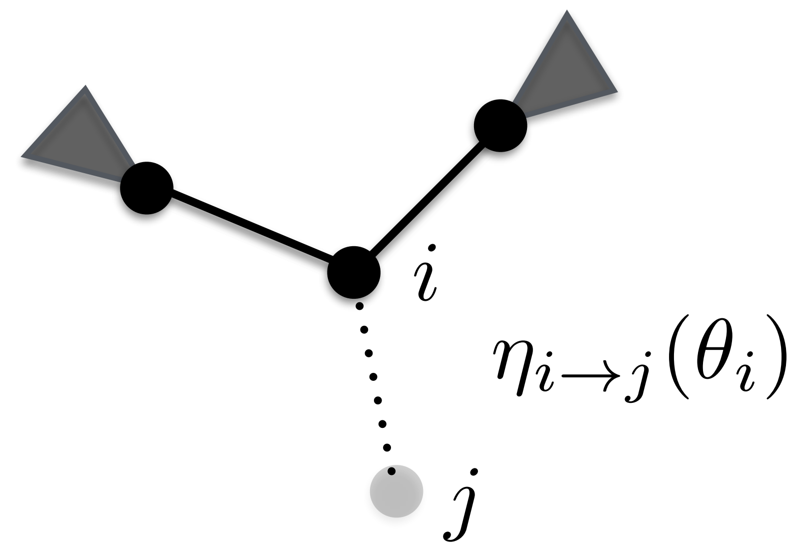
that is assume neighbors of the cavity vertex to be uncorrelated in the cavity graph

It is correct for trees and for locally tree-like graphs if correlations are not too strong

How to solve models on sparse random graphs

Cavity marginals





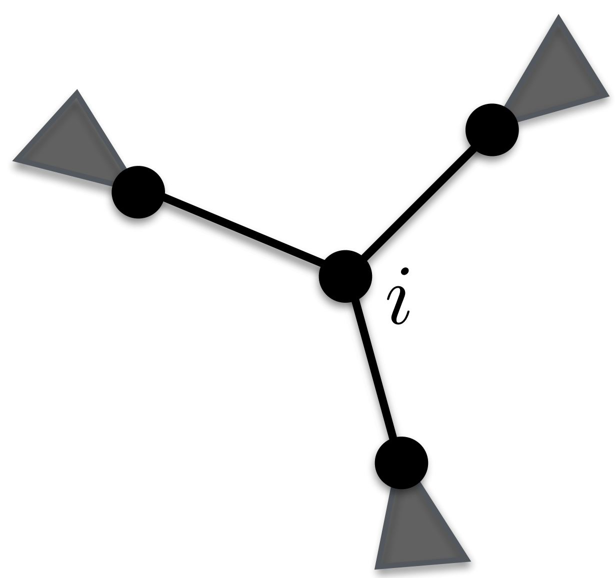
Self-consistency equations

$$\eta_{i\to j}(\theta_i) = \frac{1}{Z_{i\to j}} e^{\beta H \cos(\theta_i - \phi_i)} \prod_{k\in\partial i\setminus j} \widehat{\eta}_{k\to i}(\theta_i)$$

$$\widehat{\eta}_{i\to j}(\theta_j) = \frac{1}{\widehat{Z}_{i\to j}} \int d\theta_i \, e^{\beta J_{ij} \cos(\theta_i - \theta_j - \omega_{ij})} \, \eta_{i\to j}(\theta_i)$$

How to solve models on sparse random graphs

From cavity marginals to (full) marginals



$$\eta_i(\theta_i) = \frac{1}{Z_i} e^{\beta H \cos(\theta_i - \phi_i)} \prod_{k \in \partial i} \widehat{\eta}_{k \to i}(\theta_i)$$

Range of validity

- The above cavity method is exact:
  - on trees (no loops)
  - on random graph (long loops) in replica symmetric (RS) phases (paramagnetic, ferromagnetic)
- May give a very good approximation in graph with loops (regular lattices) in RS phases (e.g. RFIM)
- 1RSB cavity method (Mézard & Parisi, 2001 & 2003) more complex, requires populations of cavity marginals

Complexity of the cavity solution

- Ising (m=1)

  just 1 scalar per marginal -> mean local magnetization

$$\eta(s_i) = \frac{1 + m_i s_i}{2}$$

- XY (m=2)

  the mean local magnetization $\langle \vec{\sigma}_i \rangle$ is not enough!

  a function in 1 variable $\eta_i(\theta_i)$ is required (∞ d.o.f.)

- m>2 (e.g. Heisenberg, m=3)

  $\eta_i(\vec{\sigma}_i)$ is a measure on the m-dimensional unit sphere

How to solve the XY model cavity eqns.

- A discretization is needed for the numerical solution

- Fourier series

$$\eta_{k\to i}(\theta_k) = \frac{1}{2\pi}\left[1 + \sum_{l=1}^{\infty}\left(a_l^{(k\to i)}\cos(l\theta_k) + b_l^{(k\to i)}\sin(l\theta_k)\right)\right]$$

- Approximate via the clock model with q states

$$\theta_i \in \left\{0, \frac{2\pi}{q}, \frac{4\pi}{q}, \ldots, \frac{q-1}{q}2\pi\right\}$$

Hard for m>2 !!

Expanding in Fourier series (H=0)

- High temperature -> all coefficients null $\eta(\theta_i) = \frac{1}{2\pi}$
- For continuous transitions -> coefficients small close to the critical temperature -> expand to linear order

$$a_l^{(i \to j)} = \sum_{k \in \partial i \setminus j} \frac{I_l(\beta J_{ik})}{I_0(\beta J_{ik})} a_l^{(k \to i)}$$

- Ferromagnetic order -> mean of a_1 grows
  Spin glass order -> variance of a_1 grows
- Good for critical lines at H=0, bad for the low T and H≠0

q states clock model

- Marginals $\eta_i(\theta_i)$ described by q-1 numbers (q=2 is Ising) Same XY model eqns. with integrals -> sums

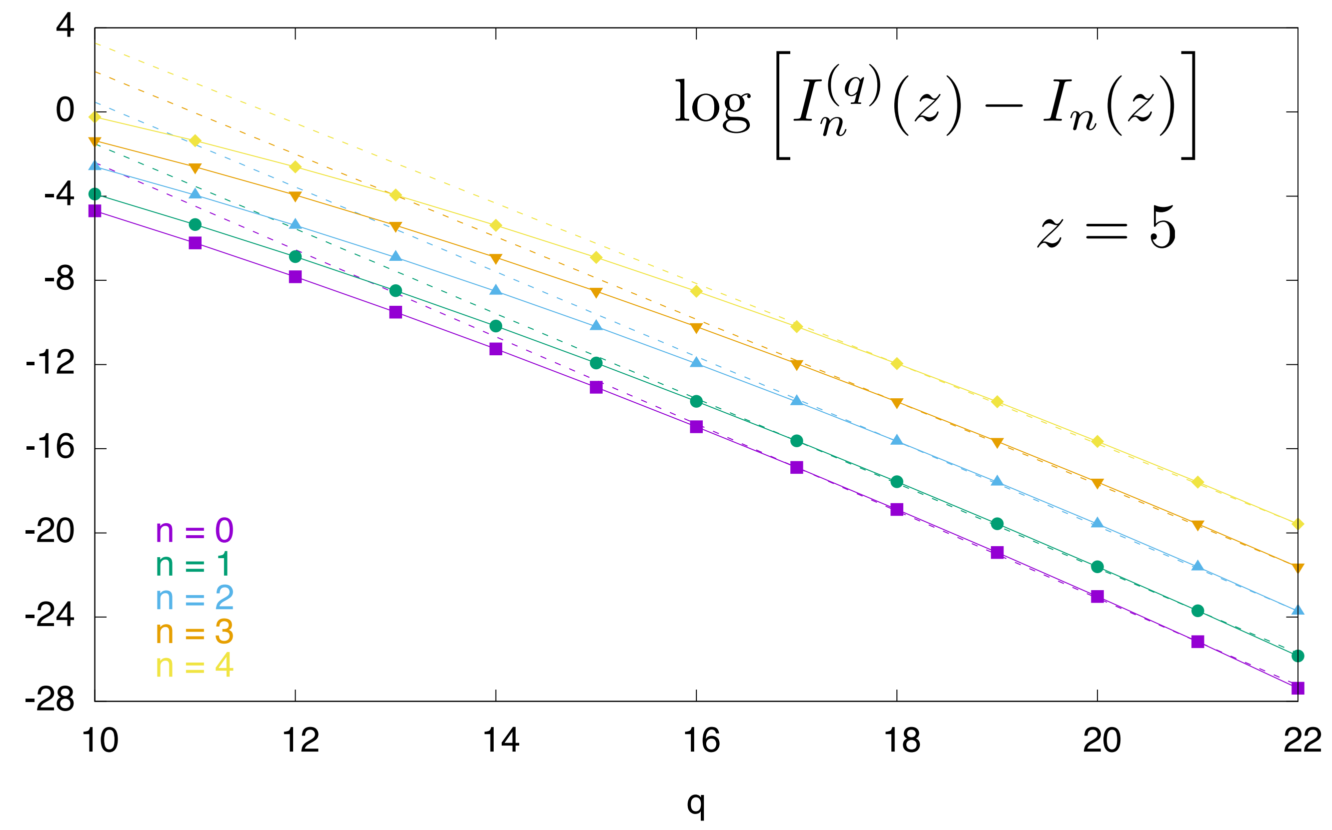
- We expect fast convergence in q to the XY model

E.g., Bessel function

$$I_n(z) = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, e^{z\cos\theta}\cos(n\theta)$$

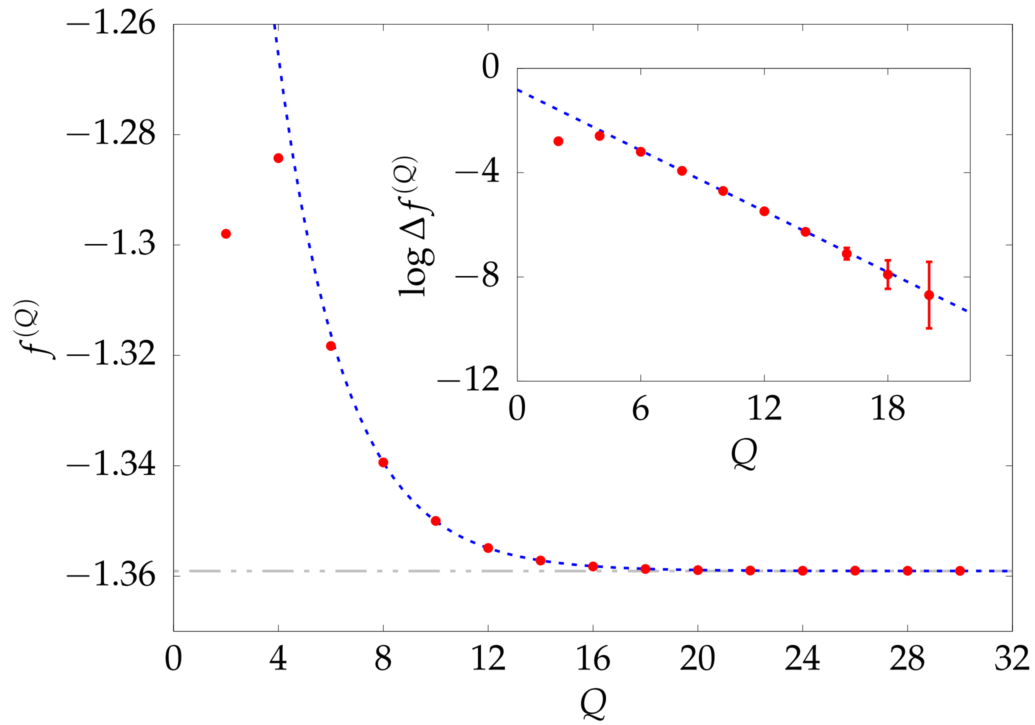
discretized Bessel fn.

$$I_n^{(q)}(z) = \frac{1}{q}\sum_{a=0}^{q-1} e^{\,z\cos\theta_a}\cos(n\theta_a)$$

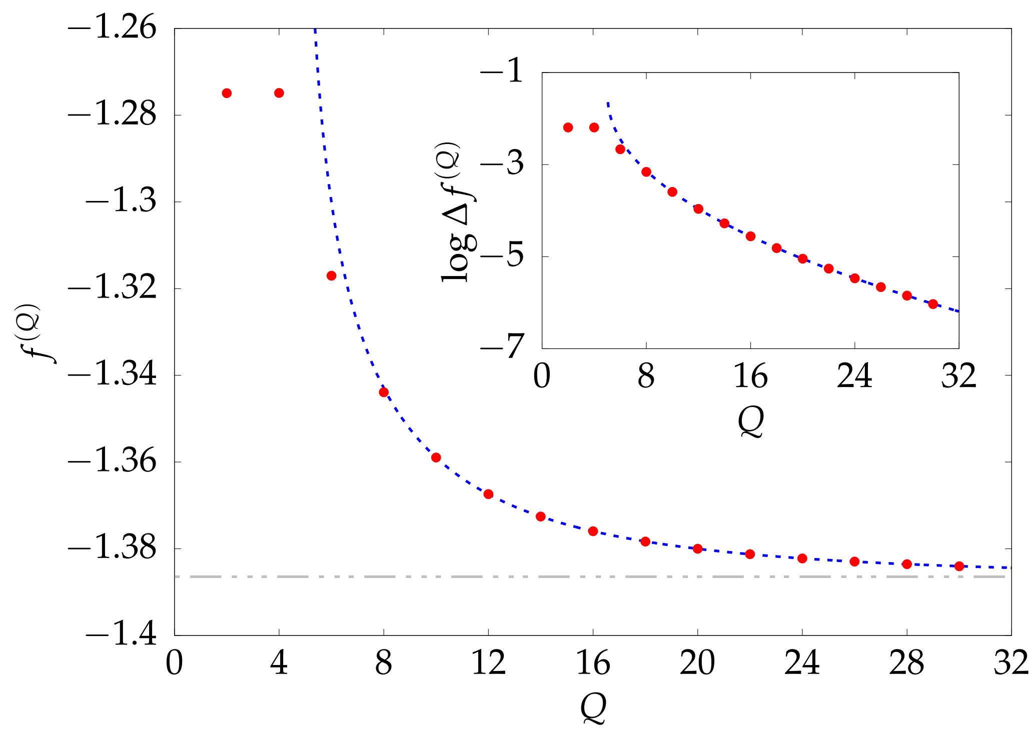


Clock model converges fast to XY model

- Exponential convergence
  - critical lines
  - observables at T>0



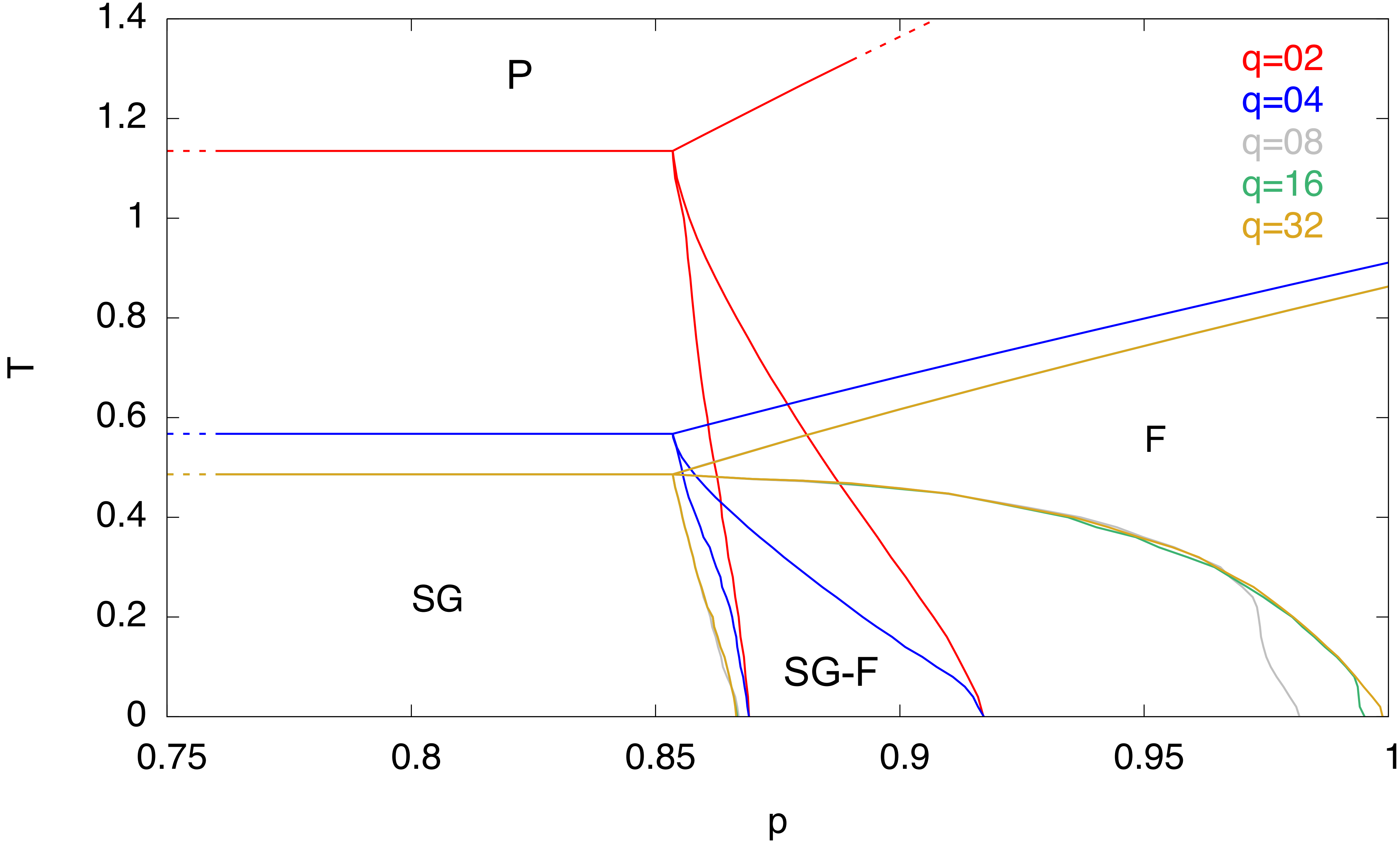
- Stretched exponential convergence with $\beta \simeq 0.5$ at T≈0



q state clock model (H=0)

±J spin glass

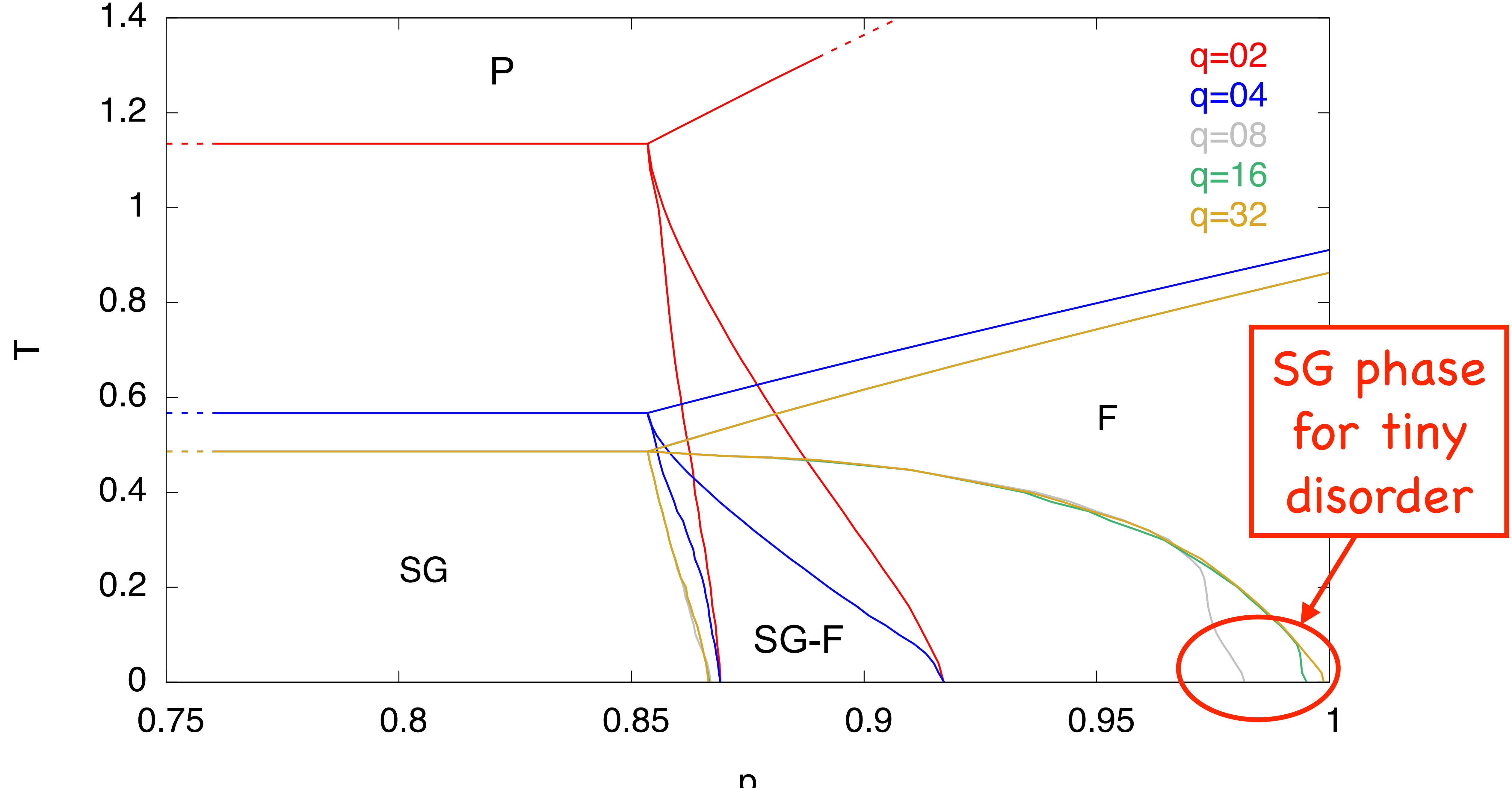
$$J_{ij} = +1 \quad \text{prob. } p$$
$$J_{ij} = -1 \quad \text{prob. } 1-p$$



q state clock model (H=0)

±J spin glass

$$J_{ij} = +1 \quad \text{prob. } p$$
$$J_{ij} = -1 \quad \text{prob. } 1-p$$



q=64 clock model ≈ XY model

- Hereafter q=64, s.t. clock model ≈ XY model
- Random field $P(\omega_{ij}) = \frac{1}{q}\sum_{a=0}^{q-1}\delta\left(\omega_{ij} - \frac{2\pi a}{q}\right)$

  of intensity H
- Couplings
  - ferro $J_{ij} = 1$
  - spin glass $J_{ij} = \pm 1$

Computing critical lines for H≠0

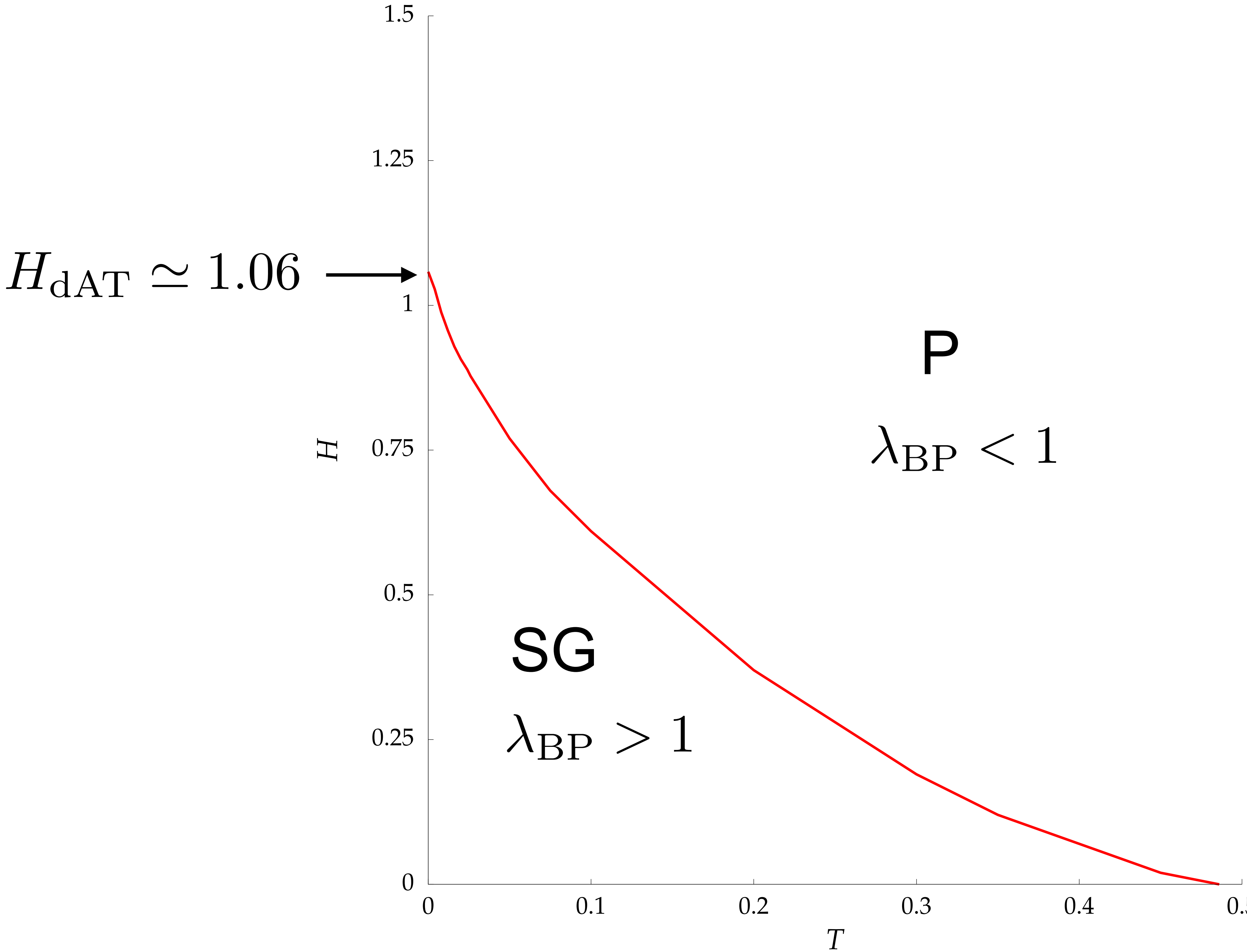
- Stability of BP fixed point
- Perturb marginals $\eta(\theta) + \delta\eta(\theta)\,, \quad \widehat{\eta}(\theta) + \delta\widehat{\eta}(\theta)$

$$\int \delta\eta(\theta)\, d\theta = 0\,, \quad \int \widehat{\delta}\eta(\theta)\, d\theta = 0\,, \quad ||\delta\eta(\theta)||, \, ||\delta\widehat{\eta}(\theta)|| \ll 1$$

- Write linear equations for perturbations
- Check growing rate of their norm

$$||\delta\eta^{(t+1)}|| = \lambda_{\text{BP}}\, ||\delta\eta^{(t)}||$$

XY spin glass model (J=±1)



In a SG phase

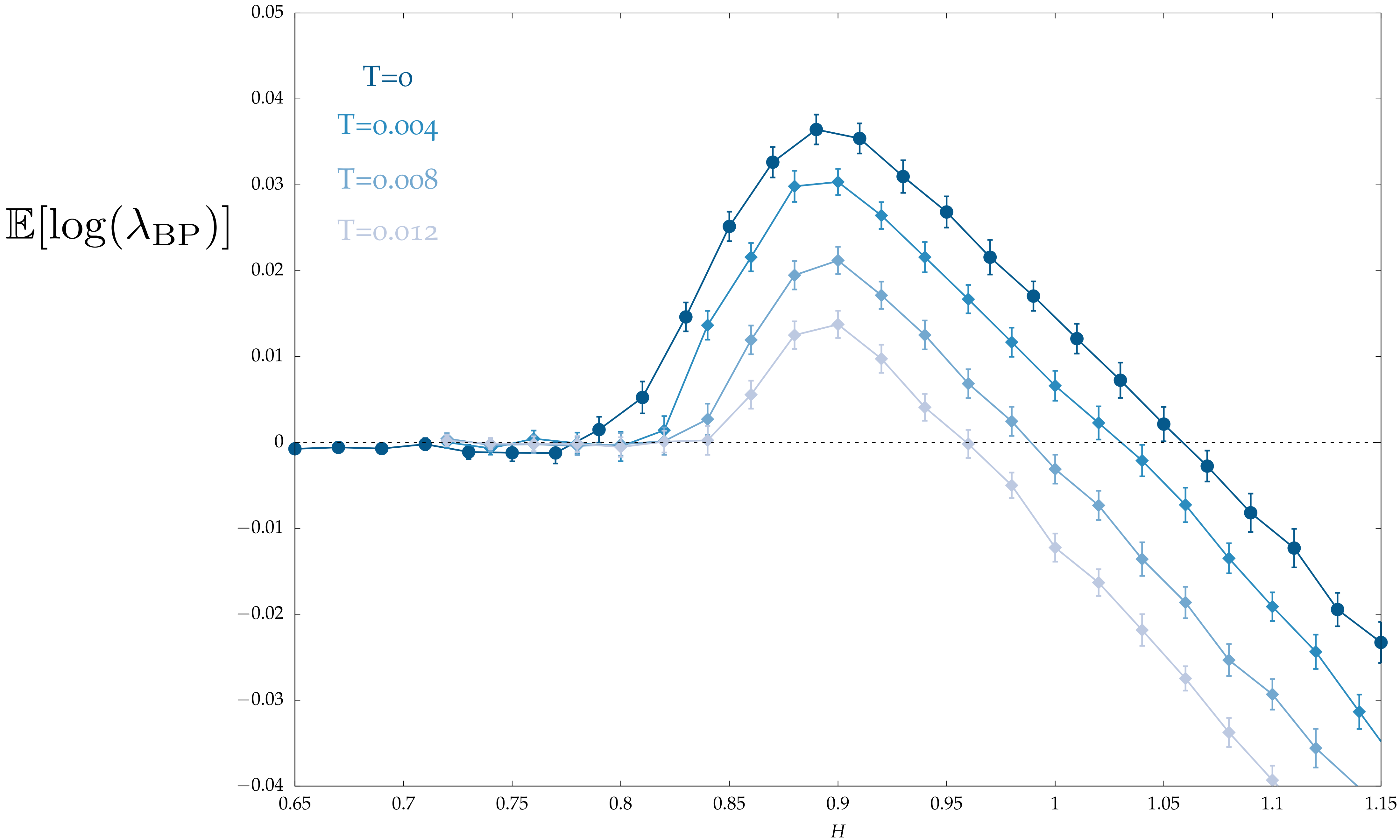
- BP run on a given graph does not converge

- Population dynamics solves

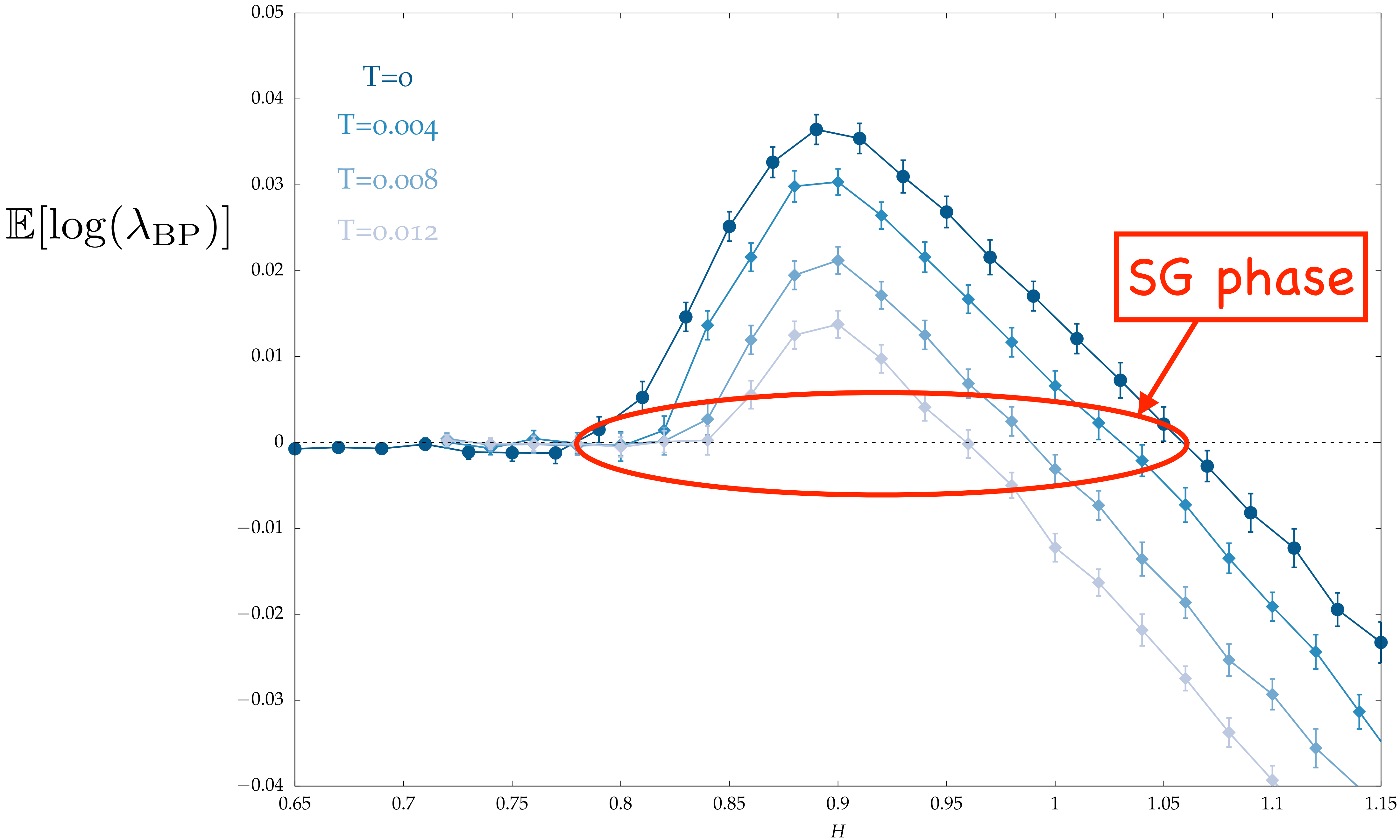
$$\eta^{(t+1)}(\theta) \stackrel{d}{=} \frac{1}{Z} e^{\beta H \cos(\theta-\phi)} \prod_{k}^{c-1} \int d\theta_k \, e^{\beta J_k \cos(\theta-\theta_k)} \, \eta_k^{(t)}(\theta_k)$$

- The linear stability of BP is measured on the stationary distribution of marginals $\{\eta^{(\infty)}(\theta)\}$

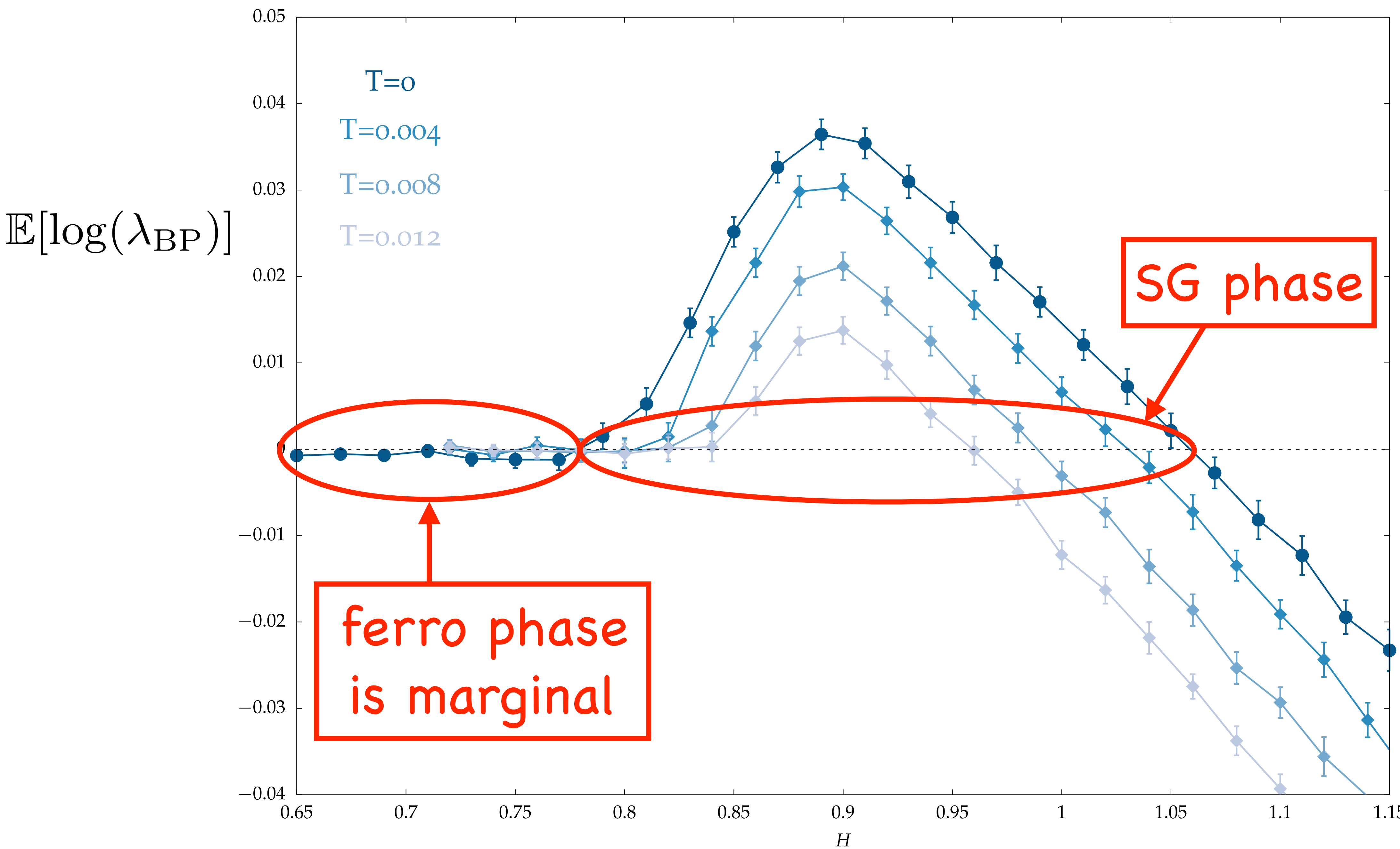
Random field XY model (J=1)



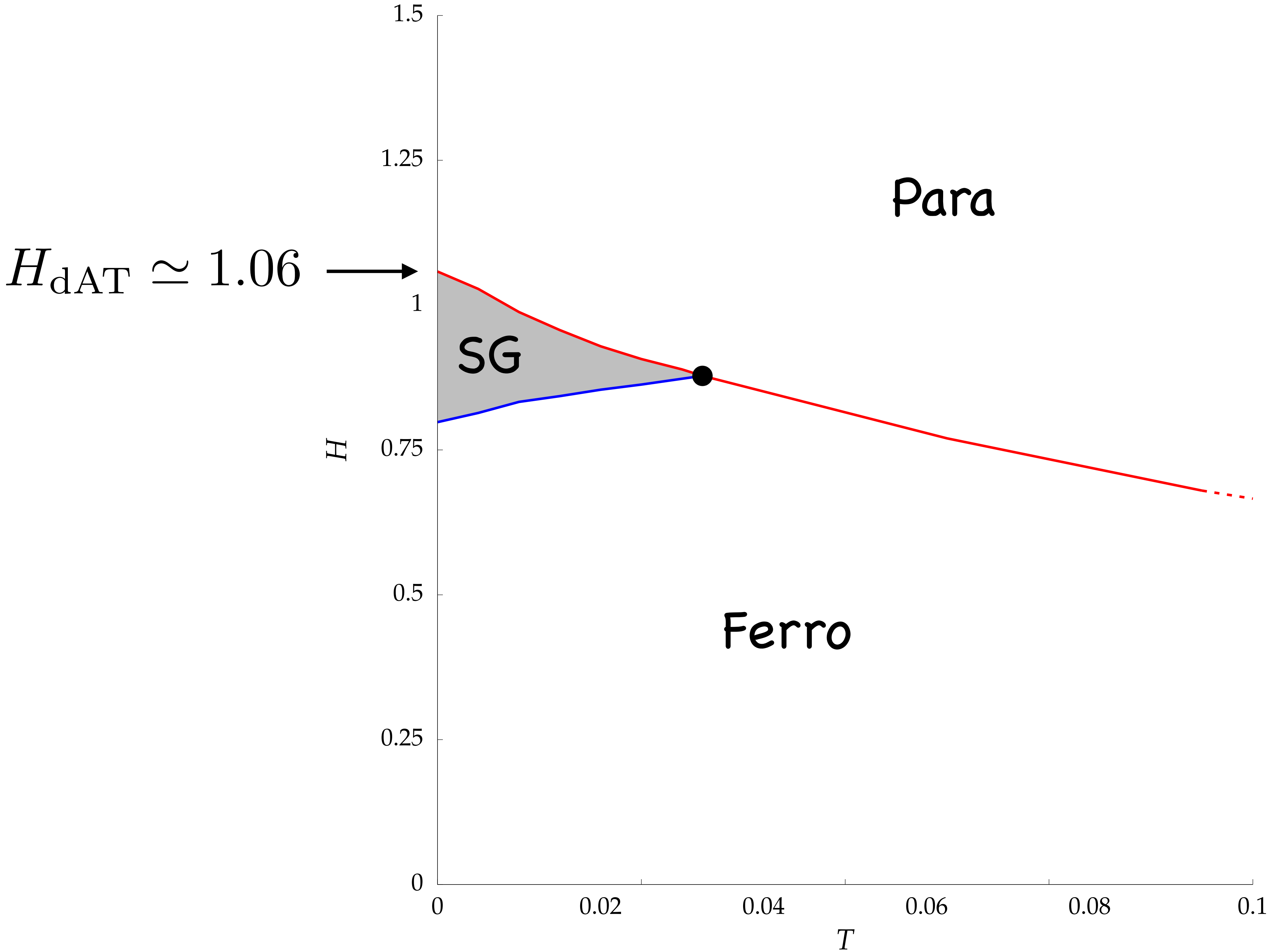
Random field XY model (J=1)



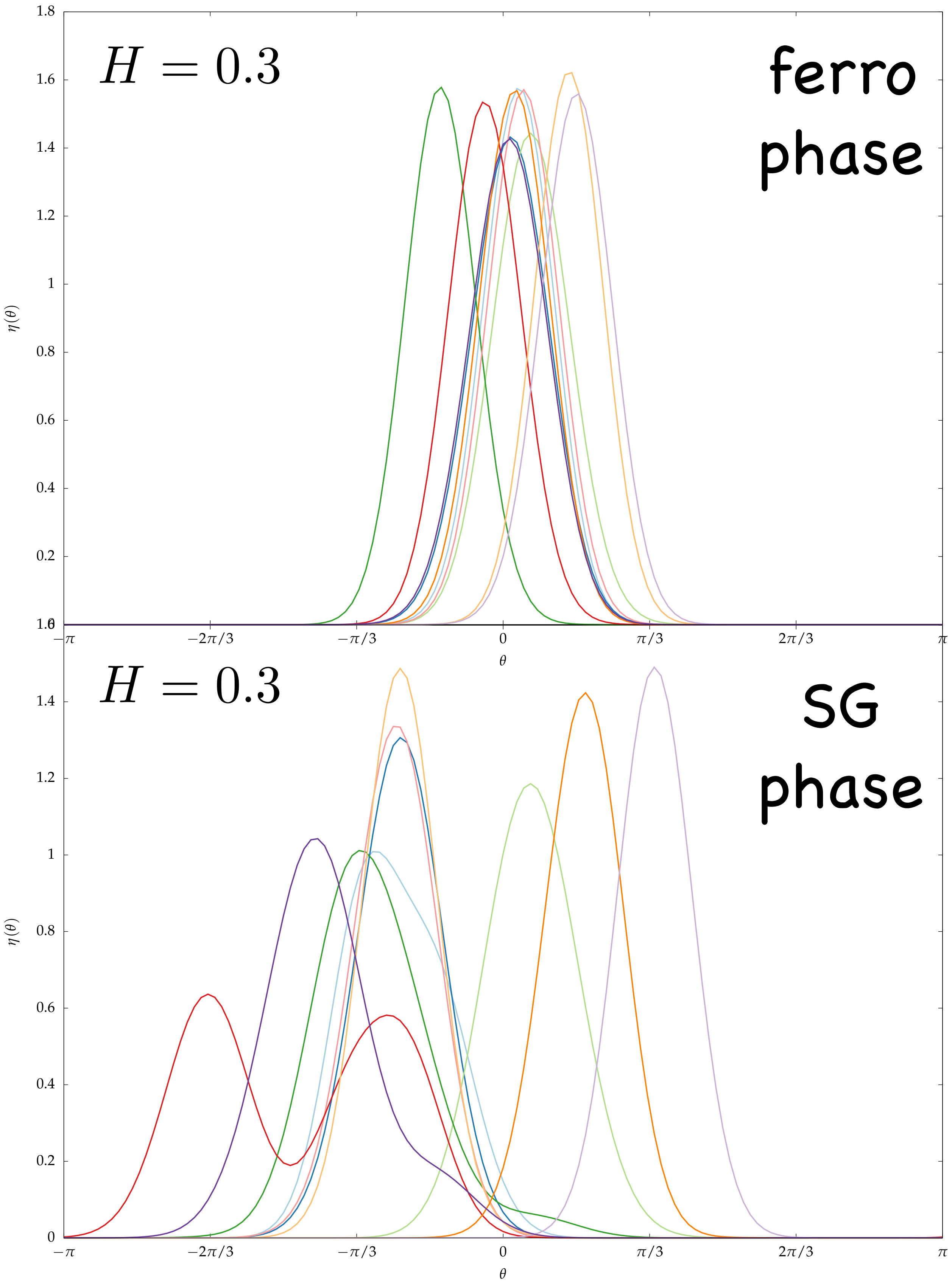
Random field XY model (J=1)



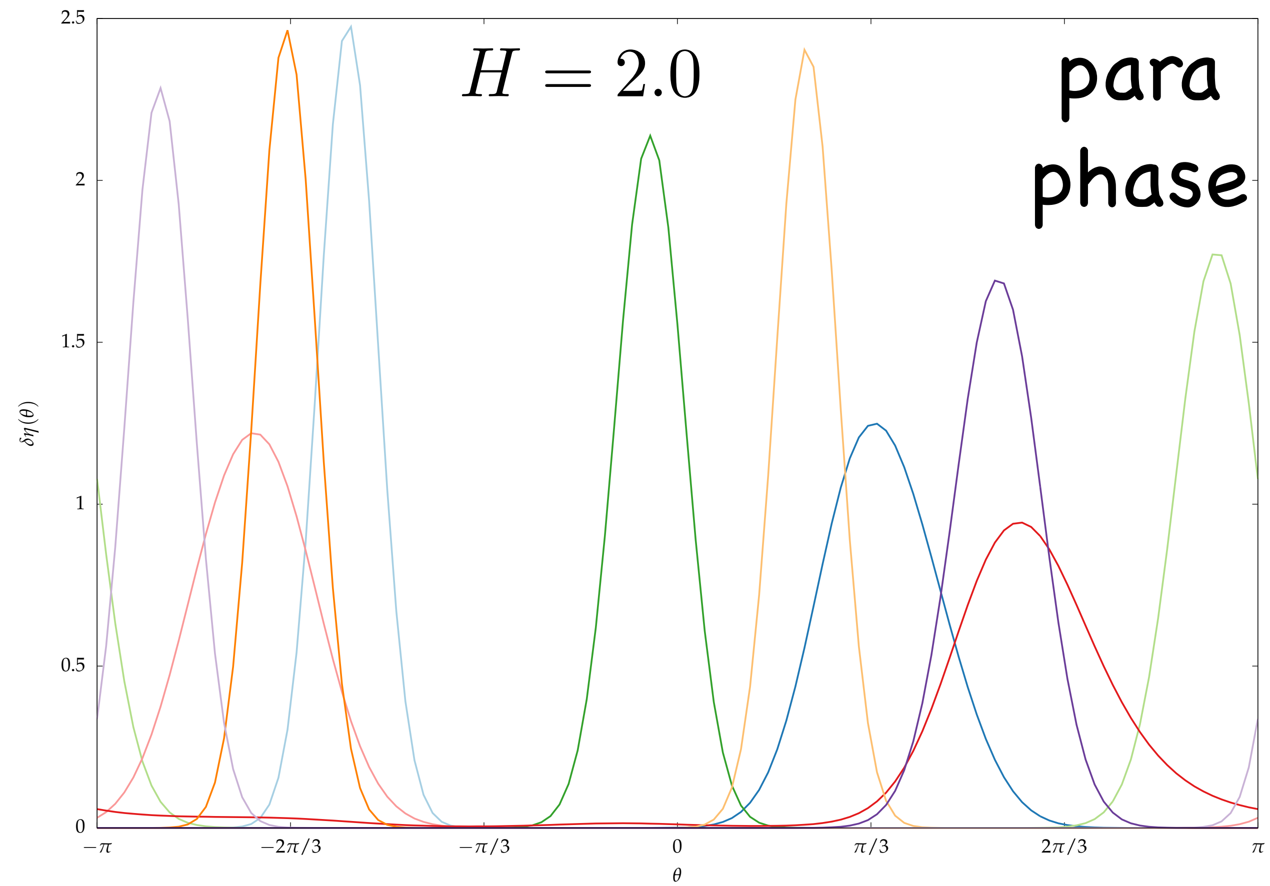
Random field XY model (J=1)



Some marginals



Low temperature T=0.1



Simpler BP equations at T=0

- Rewrite marginals as large deviation functions

$$\eta(\theta) \propto e^{\beta h(\theta)} \qquad \widehat{\eta}(\theta) \propto e^{\beta u(\theta)}$$

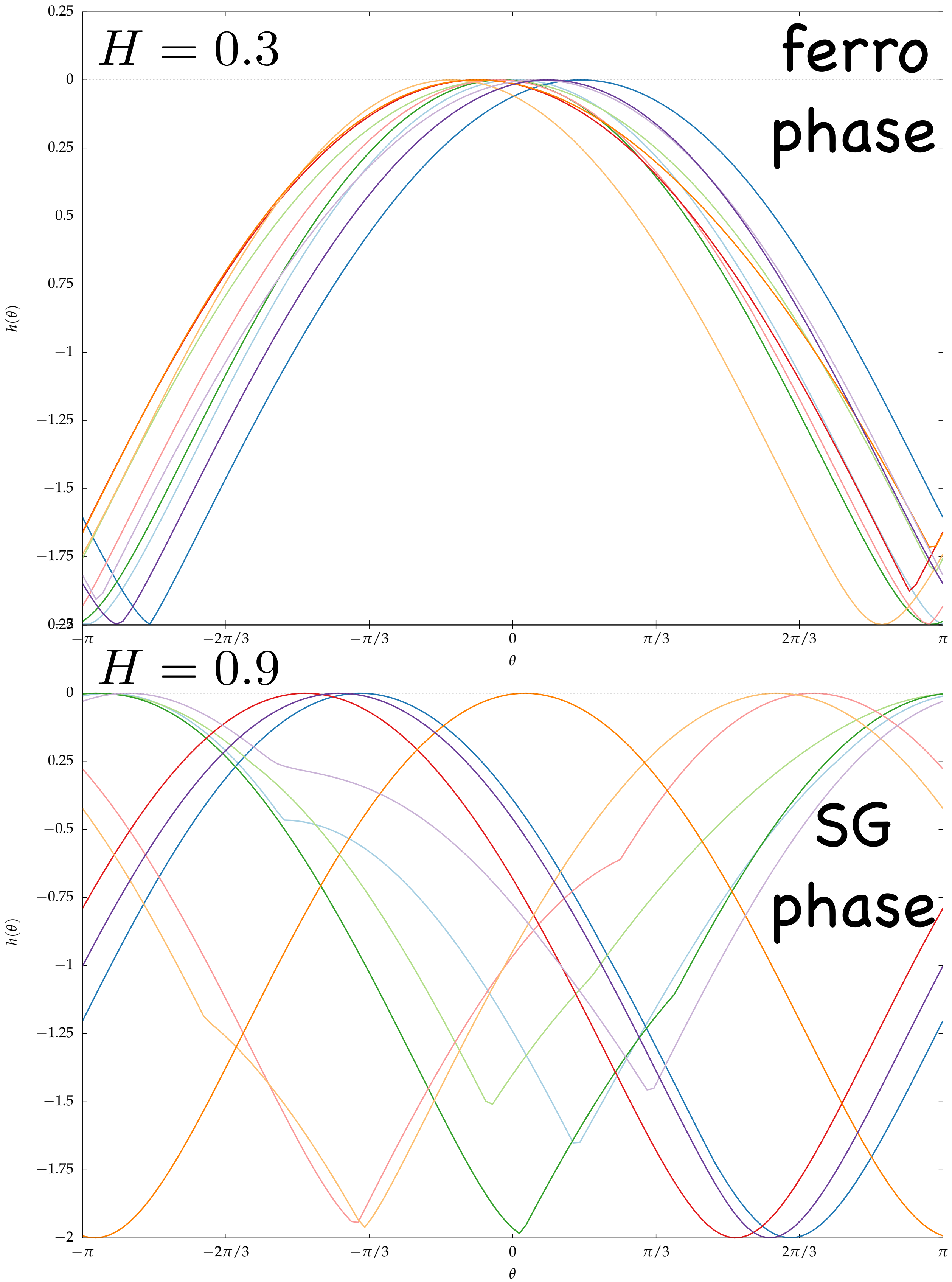
- Take the limit $\beta \to \infty$

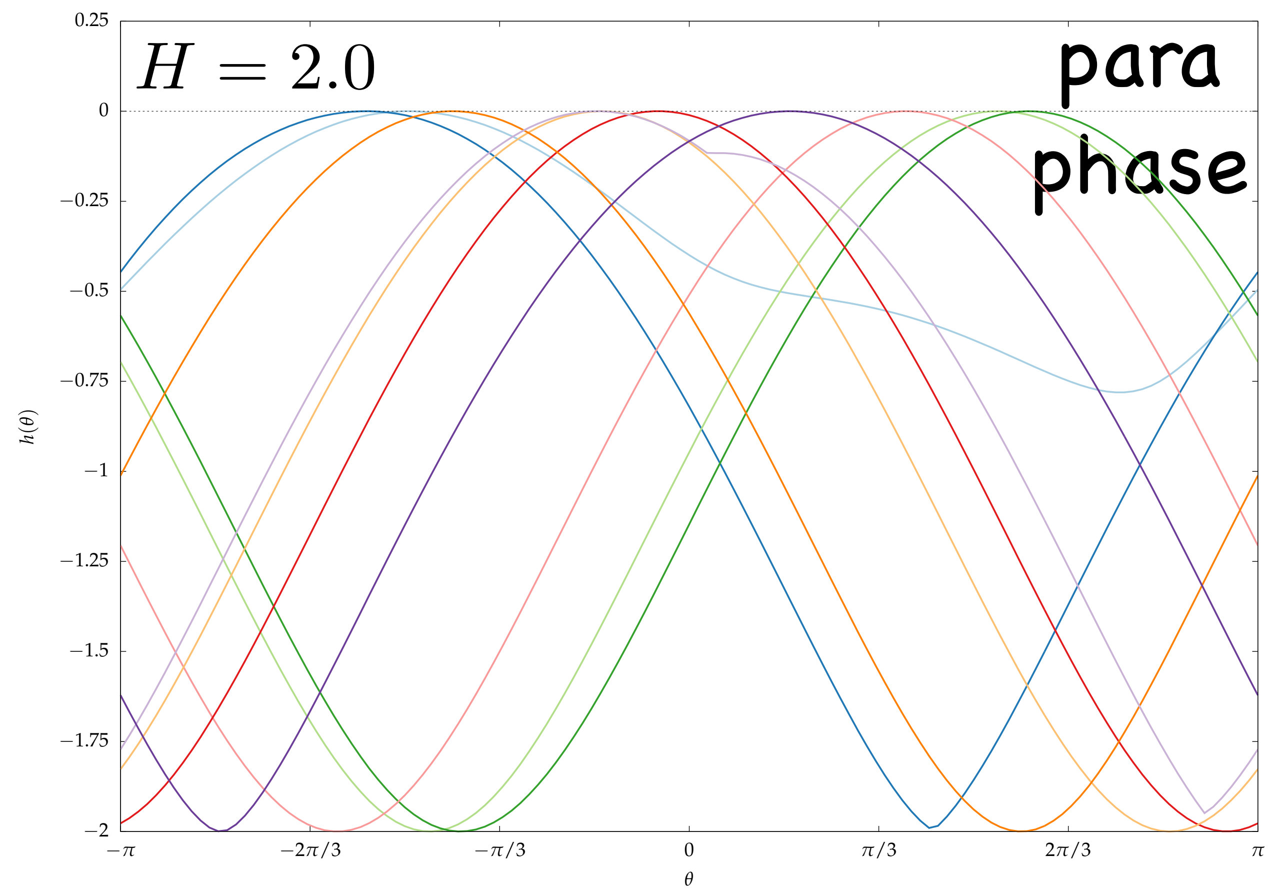
$$h_{i\to j}(\theta_i) = H \cos(\theta_i - \phi_i) + \sum_{k \in \partial i \setminus j} u_{k\to i}(\theta_i) + \text{cost.}$$

$$u_{i\to j}(\theta_j) = \max_{\theta_i} \left[h_{i\to j}(\theta_i) + J_{ij} \cos(\theta_i - \theta_j) \right] + \text{cost.}$$

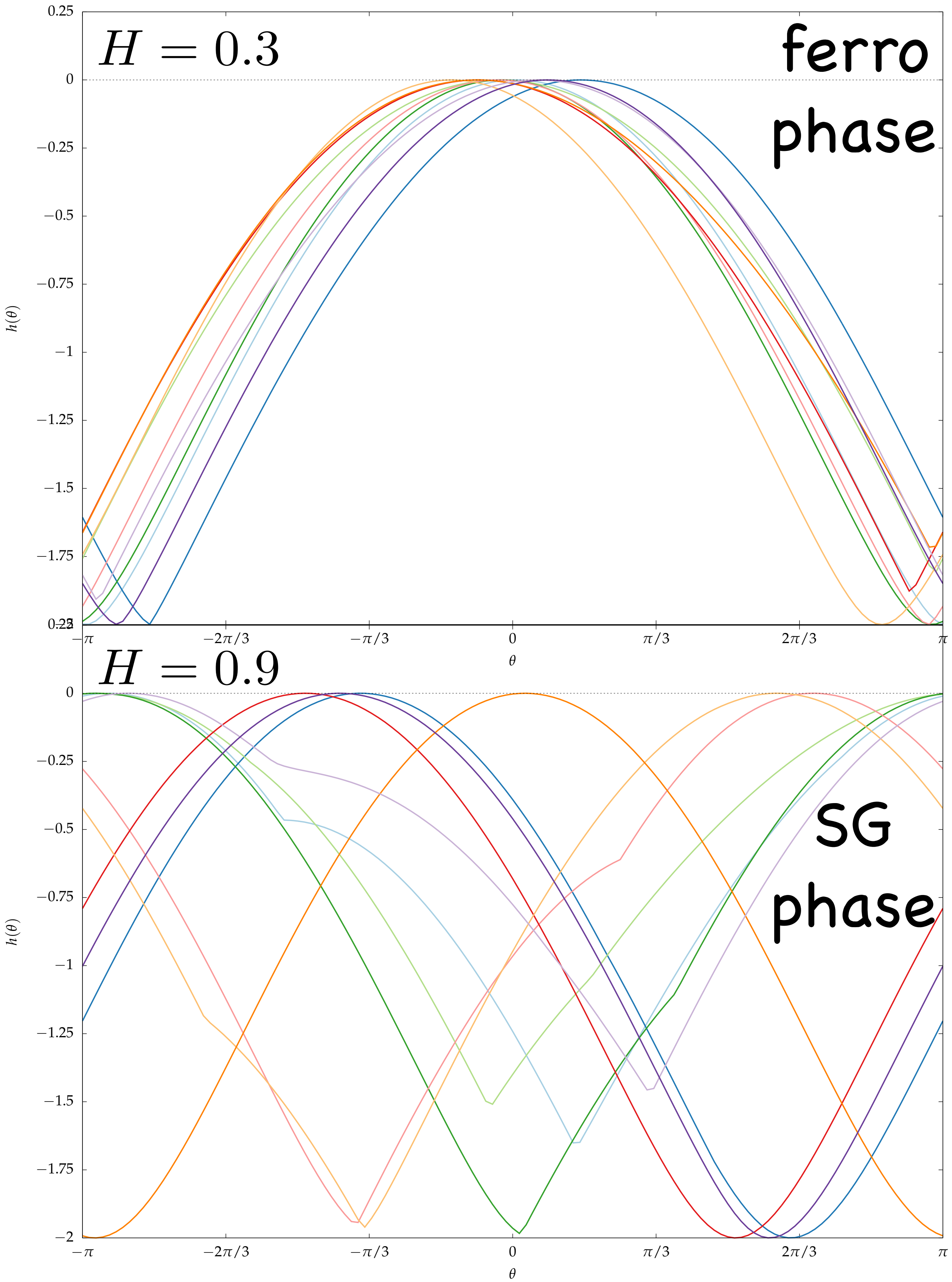
- Taking the max on the reals or on the discrete set?

Some marginals at T=0

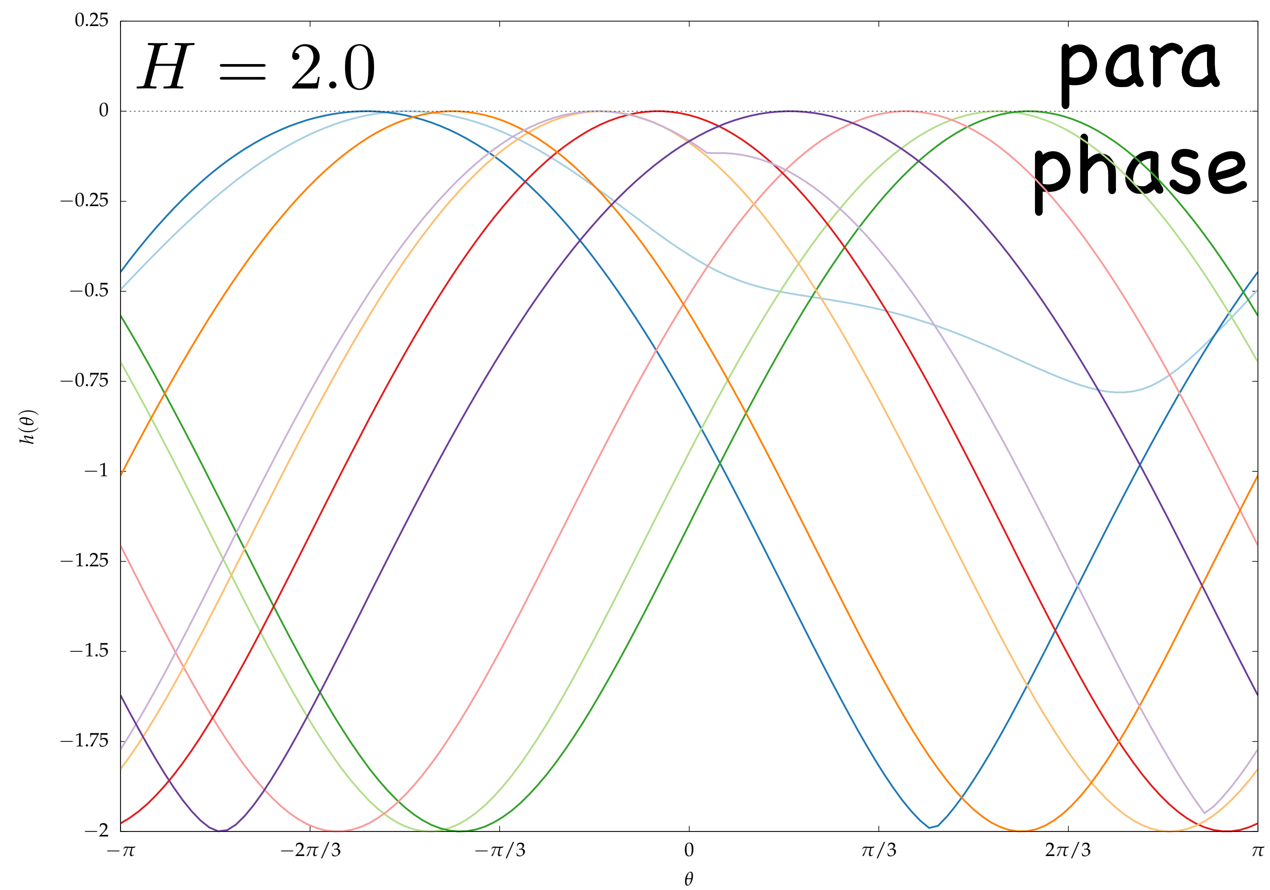




Some marginals at T=0



All the large deviation function is required for the exact solution!



BP perturbations at T=0

$$\delta h_{i\to j}(\theta_i) = \sum_{k\in\partial i\setminus j} \delta u_{k\to i}(\theta_i) + \text{cost.}$$

$$\delta u_{i\to j}(\theta_j) = \delta h_{i\to j}(\theta_i^*(\theta_j)) + \text{cost.}$$

$$\theta_i^*(\theta_j) = \text{argmax}_{\theta_i}\left[h_{i\to j}(\theta_i) + J_{ij}\cos(\theta_i - \theta_j)\right]$$

Perturbations can vanish in case

$$\theta_i^*(\theta_j) = \bar{\theta}_i \;\forall i \quad \Longrightarrow \quad \delta u_{i\to j}(\theta_j) = 0$$

e.g. in presence of a very strong field

How perturbations evolve at T=0

- Models with a finite alphabet (Ising, clock, ...) have a finite probability p_0 of generating the same perturbation on all the q values -> null perturbation
  Non-zero perturbations remain O(1), do not shrink/grow

  $$\lambda_{\mathrm{BP}} = (1 - p_0)(c - 1)$$

  At criticality ($\lambda_{\mathrm{BP}} = 1$) the fraction of spins correlated at distance r decays as $\left(\frac{1}{c-1}\right)^r$

- Vector models (XY, Heisenberg, ...), infinite alphabet, $p_0 = 0$ all perturbations are non-null, study their shrinking rate

Clock model + interpolations

- To avoid problems related to finite alphabet

$$\theta_i^*(\theta_j) = \mathrm{argmax}_{\theta_i} \left[h_{i\to j}(\theta_i) + J_{ij}\cos(\theta_i - \theta_j) \right]$$

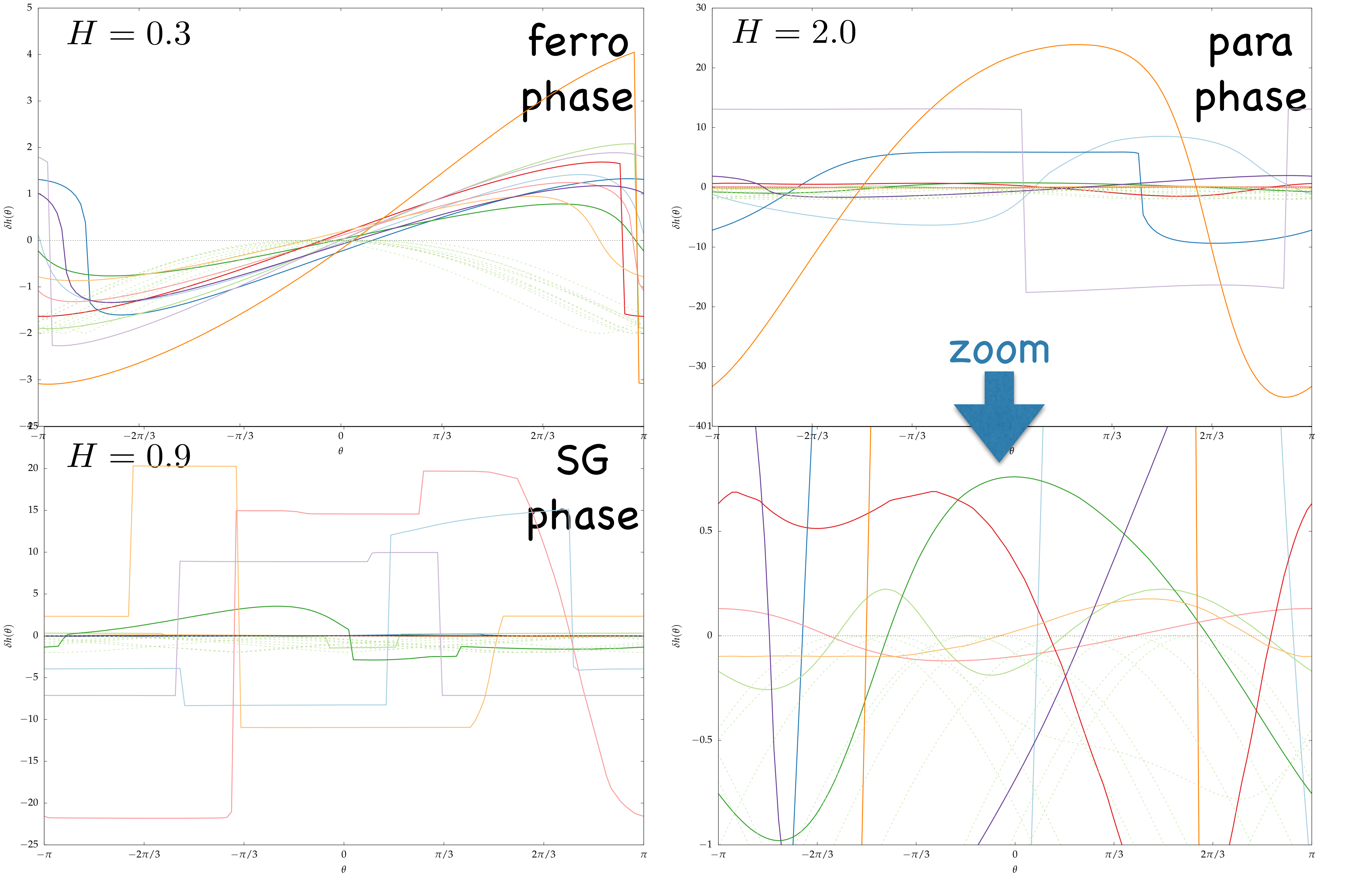
takes q values

interpolate values around maximum with a parabola

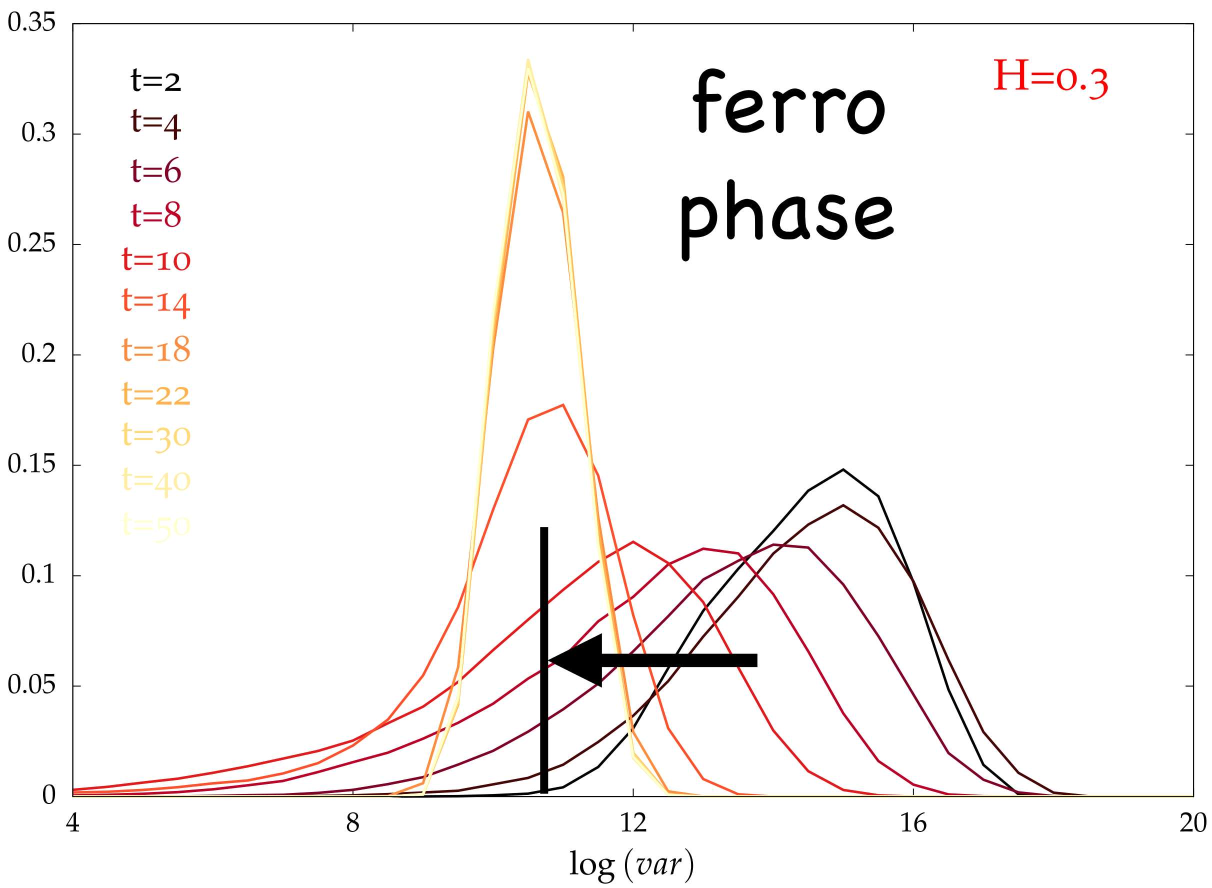
- Even in case of strong fields is unlikely to have the maximum always on exactly the same **real** value

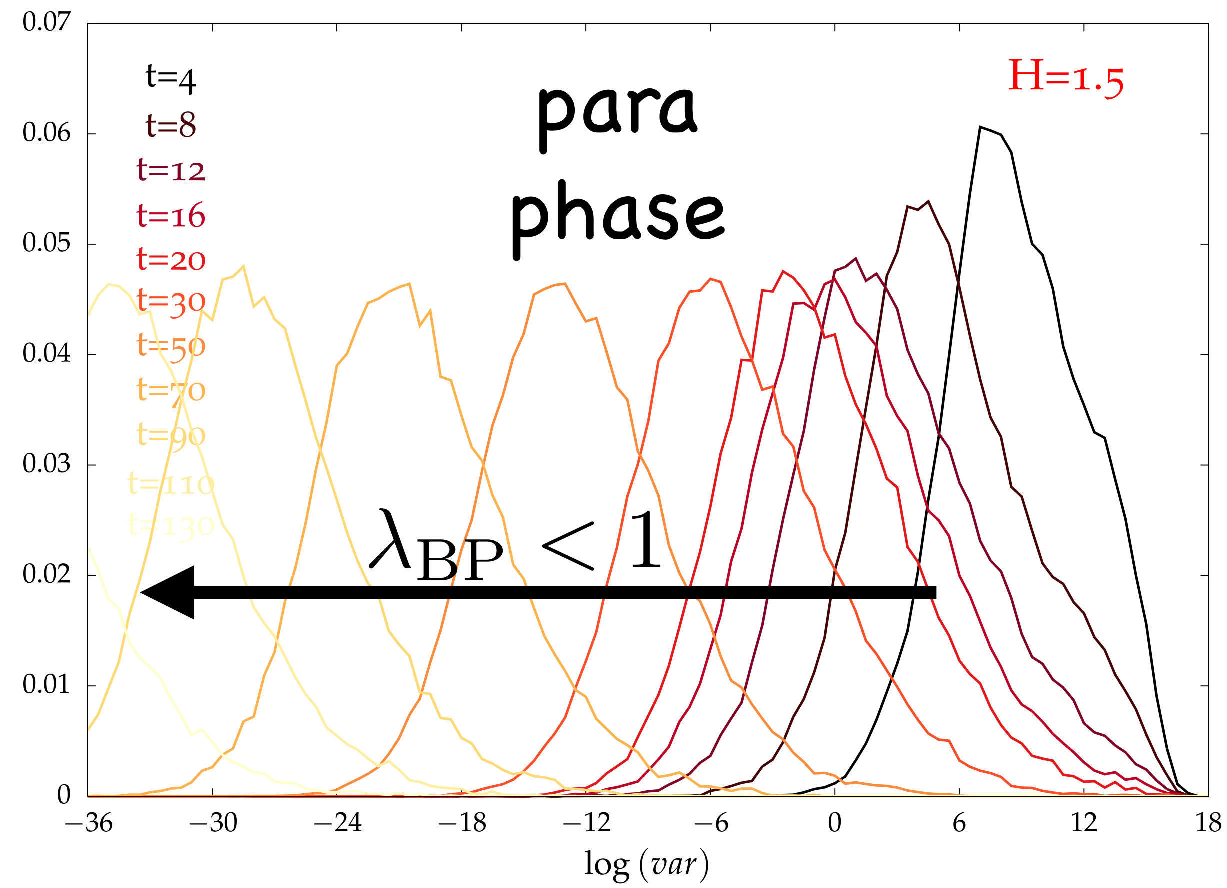
- Perturbations may become small, but never vanish (very small fluctuations are always allowed)

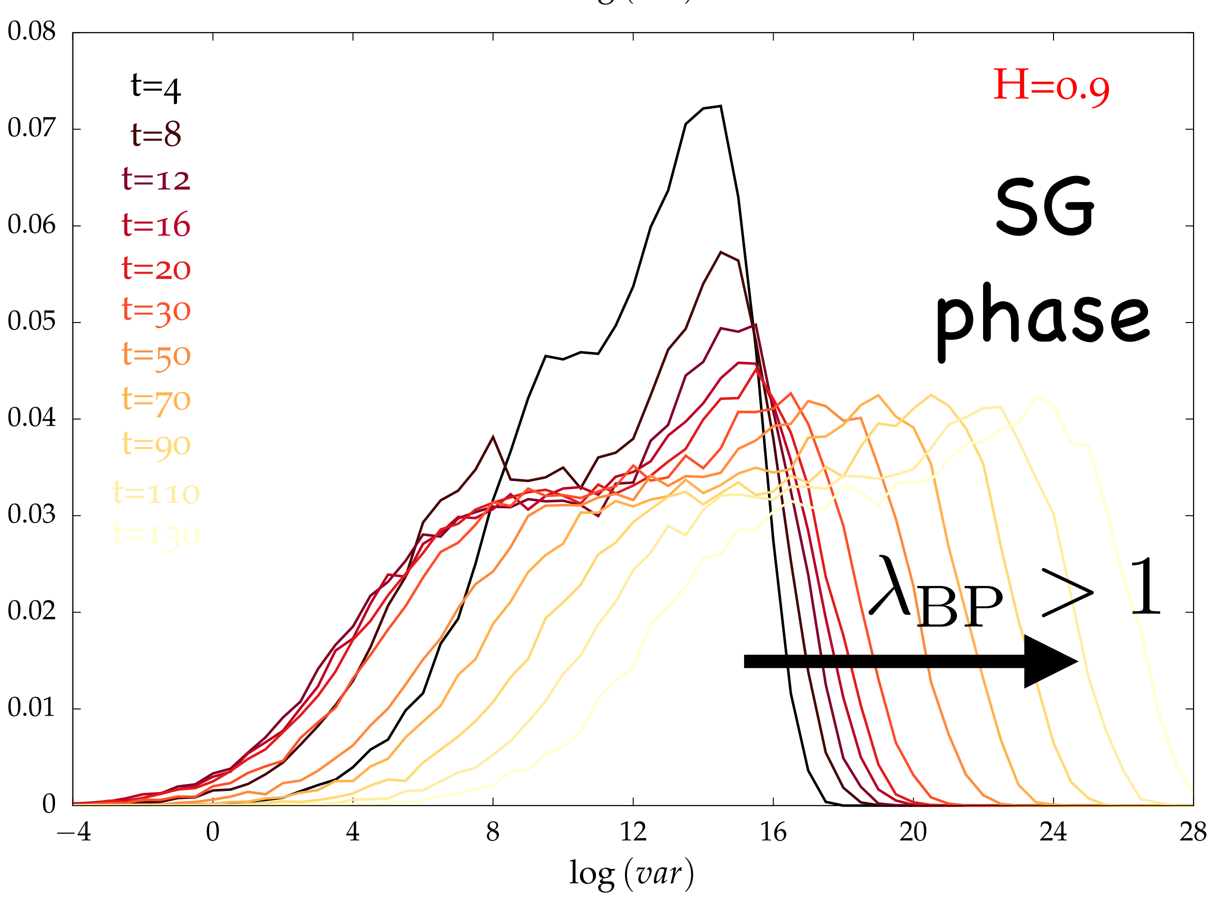
Some perturbations at T=0



Evolution of perturbations at T=0



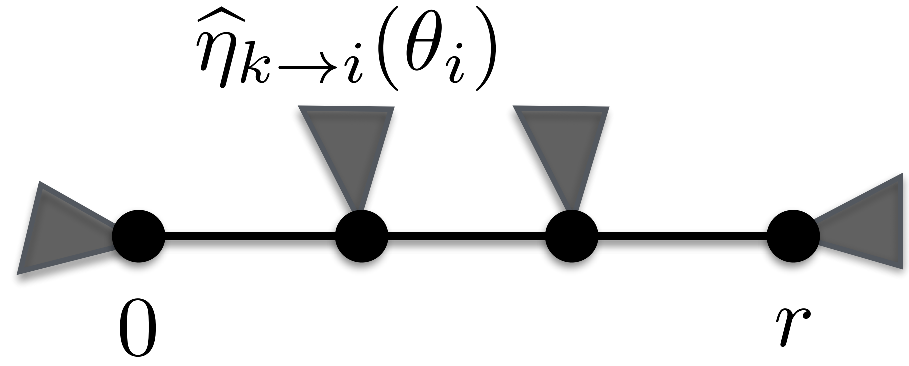




Perturbations extend over several order of magnitude

Connected correlations

Compute exactly the joint marginal of spins at distance r



$$\eta^{(r+1)}(\theta_0, \theta_{r+1}) \stackrel{d}{=} \frac{1}{Z} \int d\theta_r \eta^{(r)}(\theta_0, \theta_r) \times$$

$$e^{\beta H \cos(\theta_r - \phi)} \prod_{k}^{c-2} \widehat{\eta}_k(\theta_r) \times e^{\beta J \cos(\theta_r - \theta_{r+1})}$$

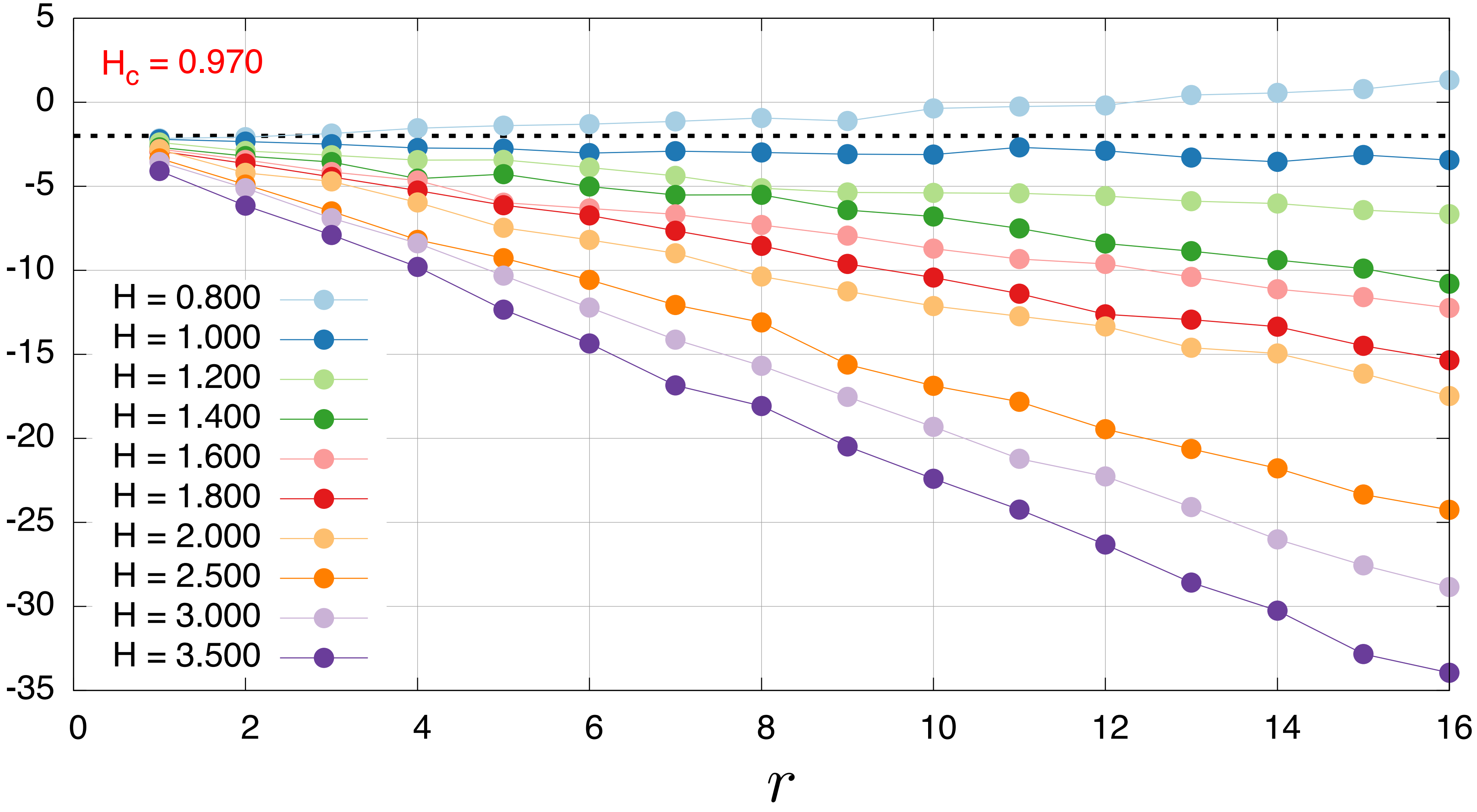
Outcome: <u>population</u> of joint marginals

$$\mu(\theta_0, \theta_r) \stackrel{d}{=} \eta(\theta_0)\eta^{(r)}(\theta_0, \theta_r)\eta(\theta_r)$$

Connected correlations

$$\mu_0(\theta_0) = \int d\theta_r \mu(\theta_0, \theta_r) \quad \mu_r(\theta_r) = \int d\theta_0 \mu(\theta_0, \theta_r)$$

$$C(\theta_0, \theta_r) \approx ||\mu(\theta_0, \theta_r) - \mu_0(\theta_0)\mu_r(\theta_r)||$$



T=0.01

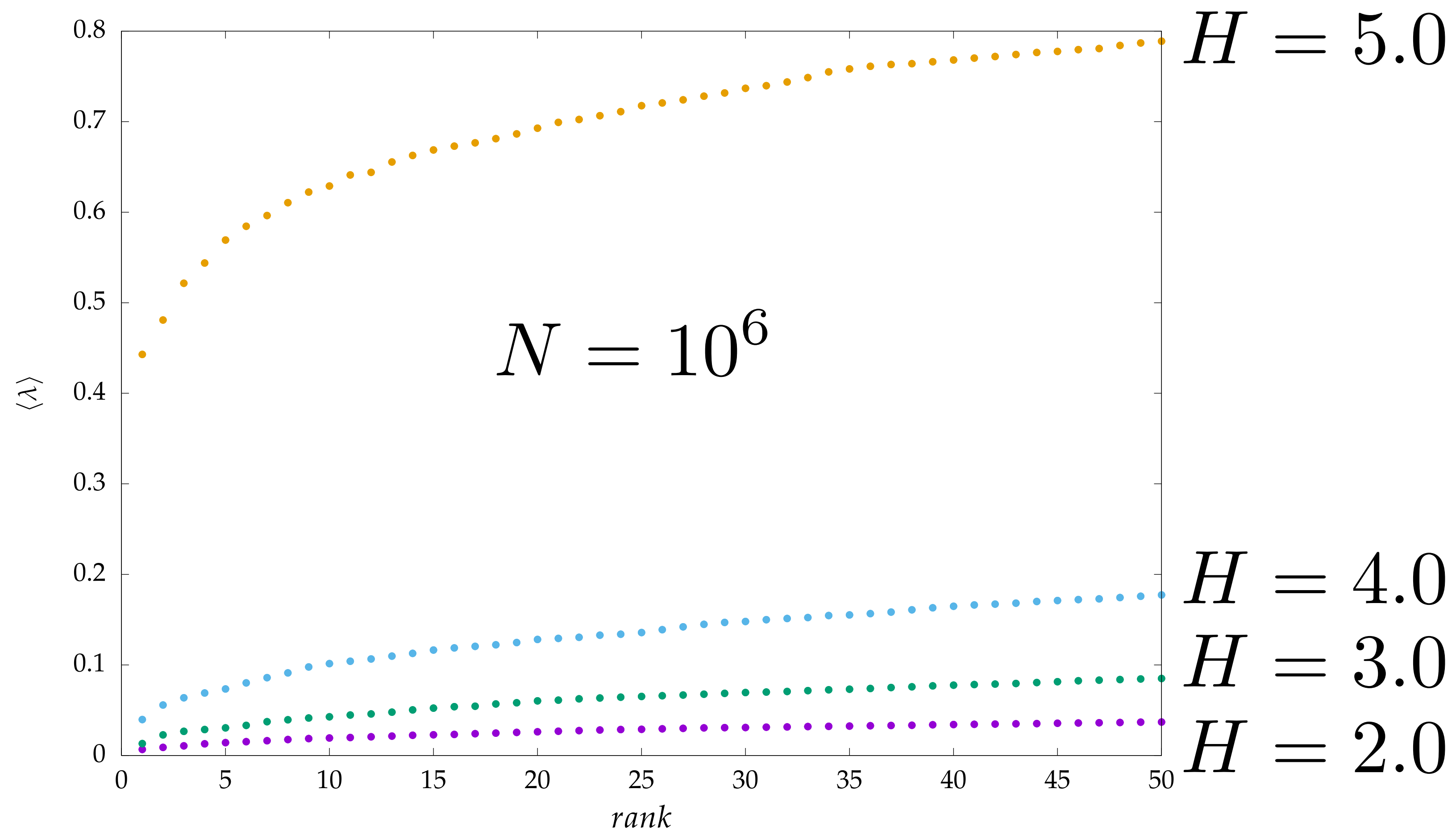
Hessian and low energy excitations

$$\mathcal{H} = -H \sum_i \cos(\theta_i - \phi_i) - \sum_{(i,j)\in E} J_{ij} \cos(\theta_i - \theta_j)$$

- Ground state configuration $\{\theta_i^*\}$
- Hessian matrix $\left.\frac{\partial^2 \mathcal{H}}{\partial\theta_i \partial\theta_j}\right|_{\theta^*}$
- Above the critical line ($H > H_{\mathrm{dAT}} \simeq 1.06$ at T=0) we compute true ground states, not just metastable states
  - run BP until convergence
  - few steps of T=0 Monte Carlo
- Below the critical line, we reach low energy states

Hessian becomes gapless at large fields

- The gap closes at very large fields H≈4

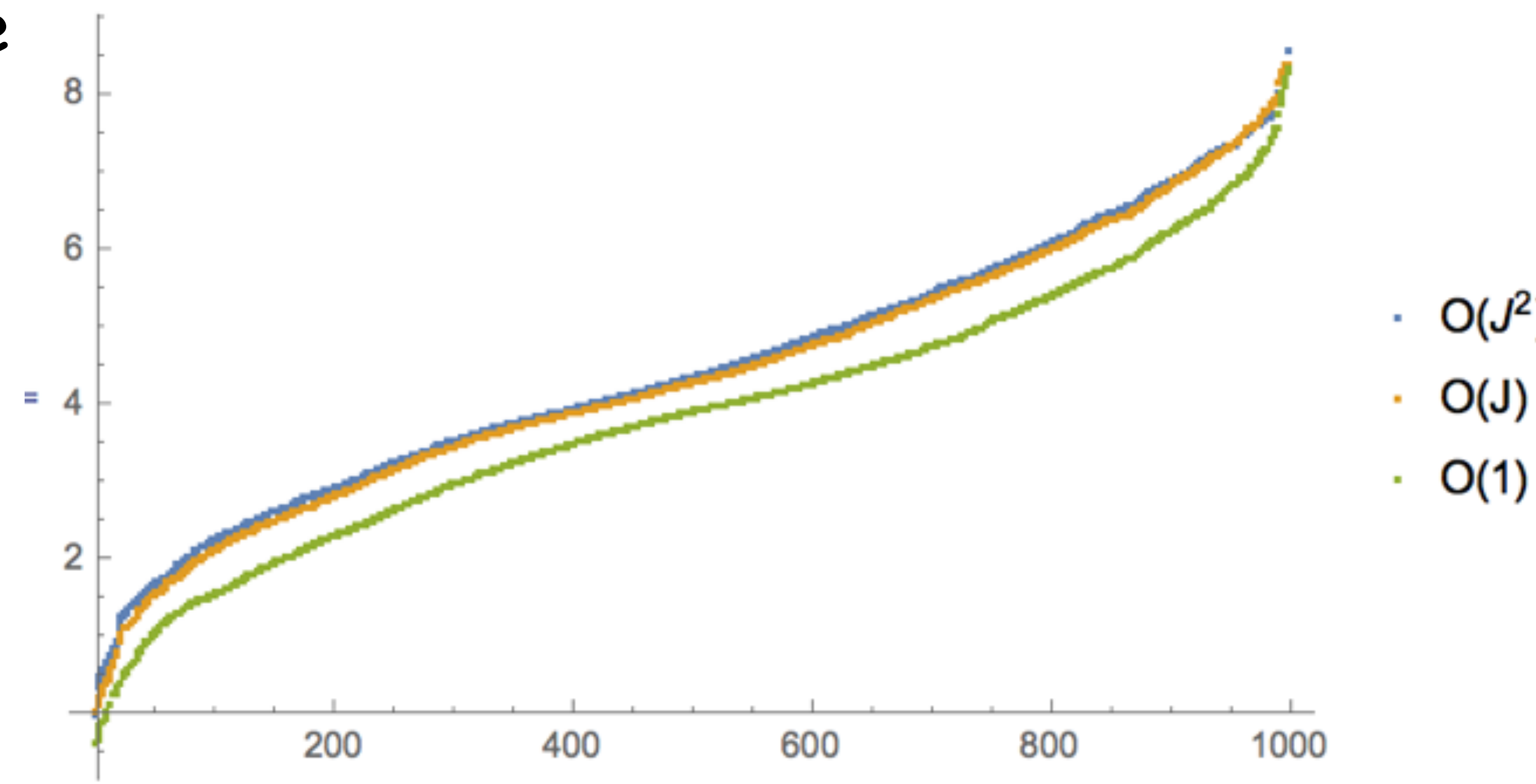


- The system has many localized low-energy excitations, that have nothing to do with long range correlations

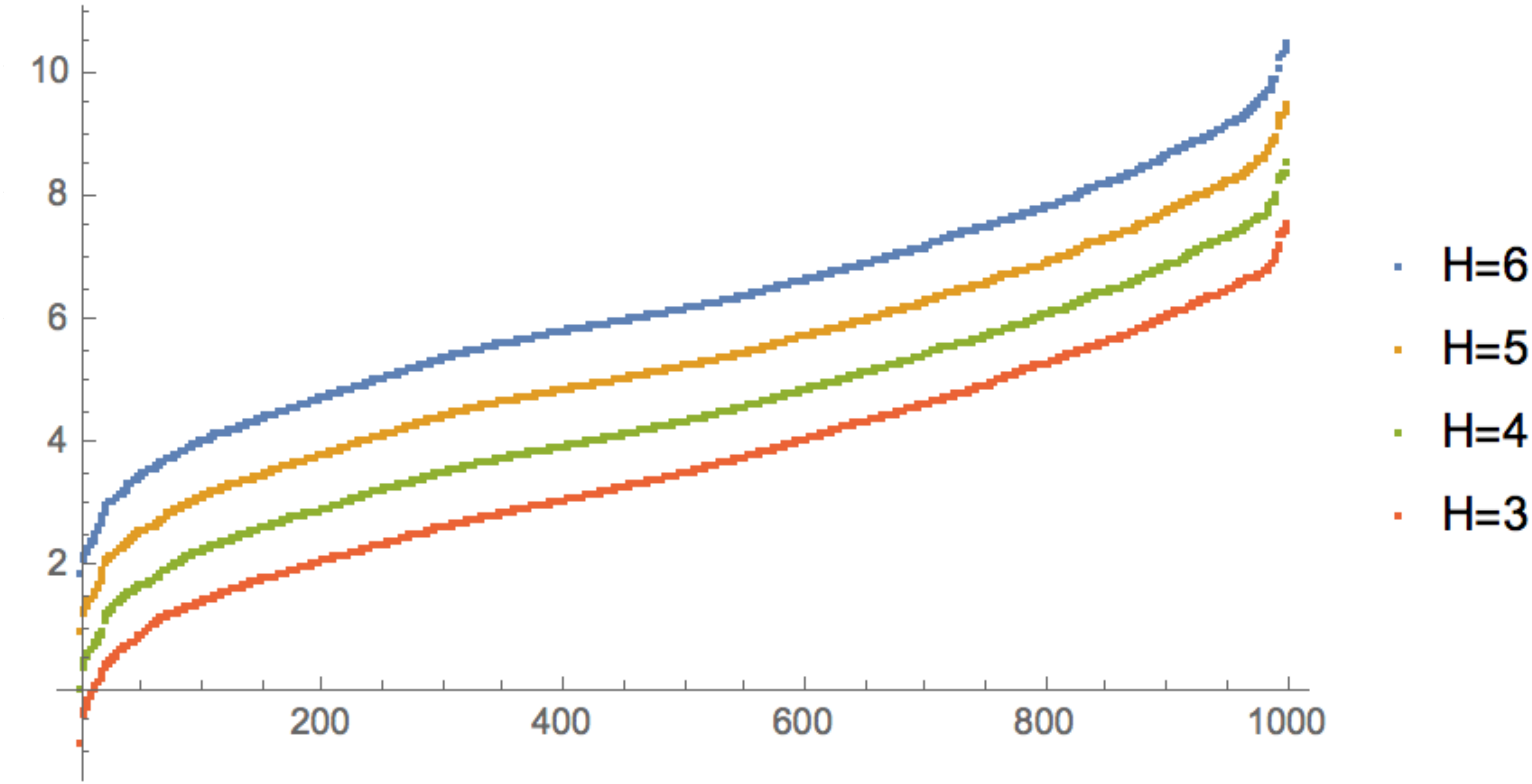
Why Hessian is gapless at large fields

- Very simple explanation in terms of random matrices by expanding in J/H
- Ground state $\theta_i^* = \phi_i + \sum_{j \in \partial i} \frac{J_{ij}}{H} \sin(\phi_j - \phi_i)$
- Computing the Hessian up to second order in J/H

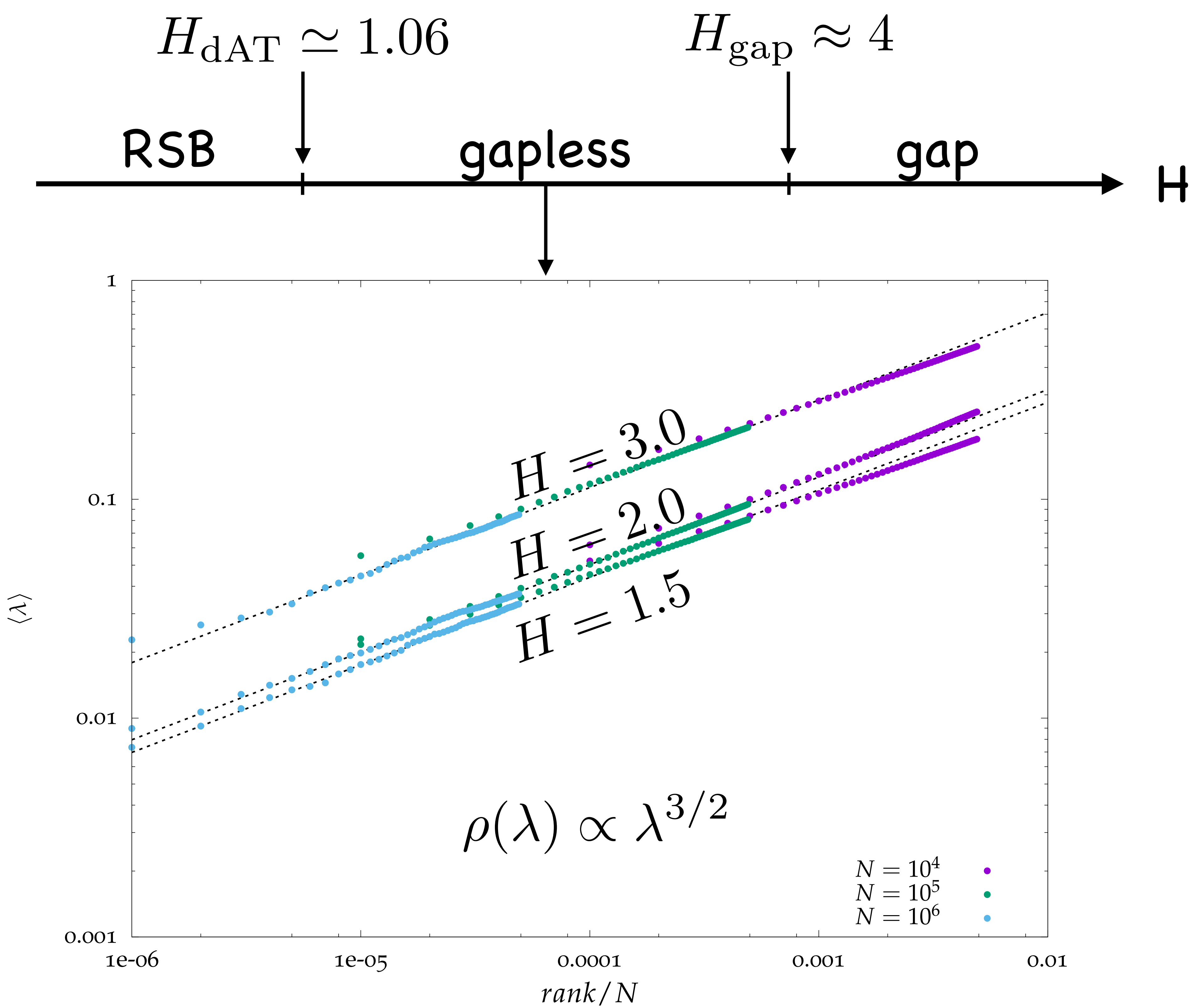
$$J = \pm 1$$
$$H = 4.0$$



Why Hessian is gapless at large fields

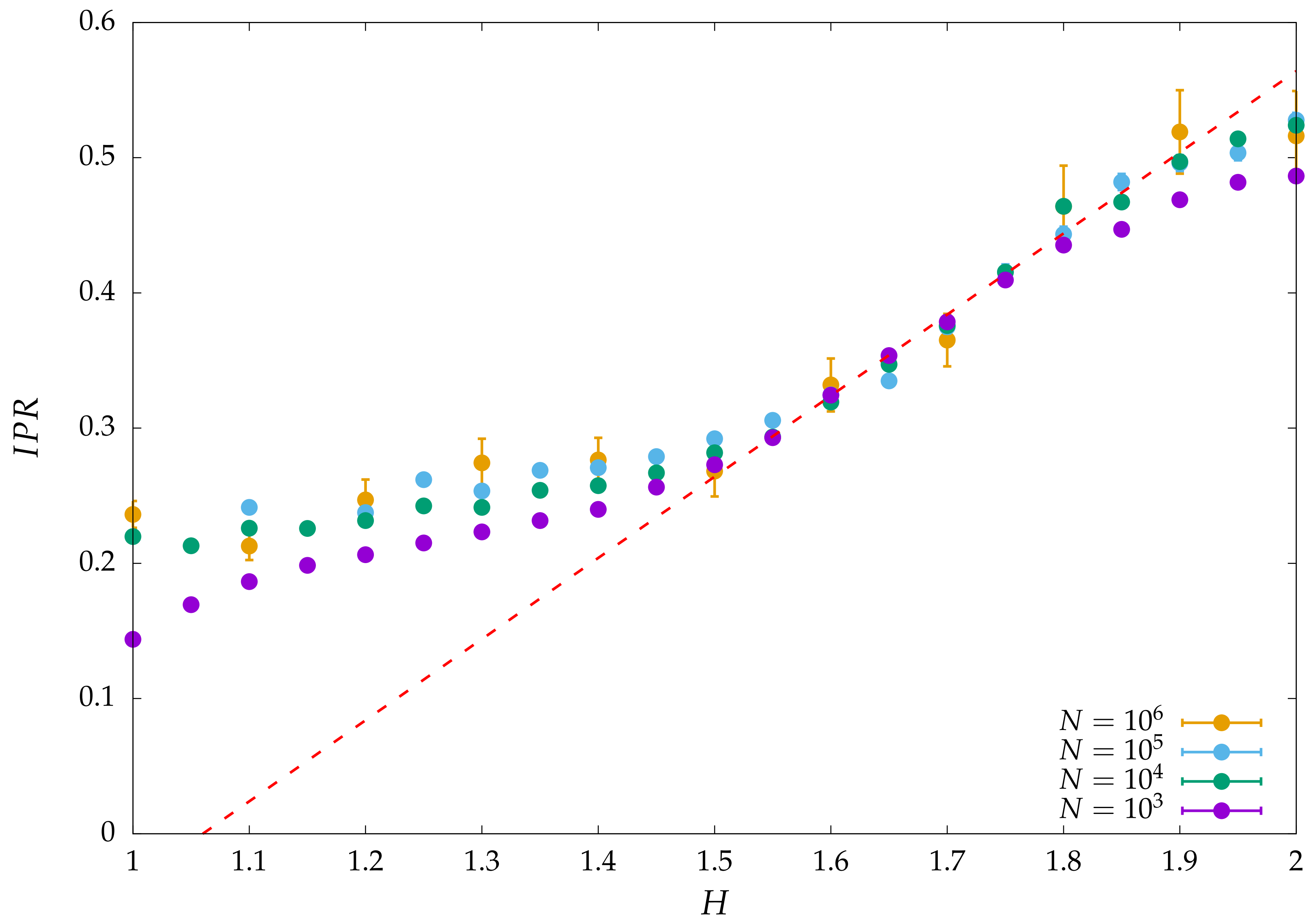


Hessian low-energy spectrum



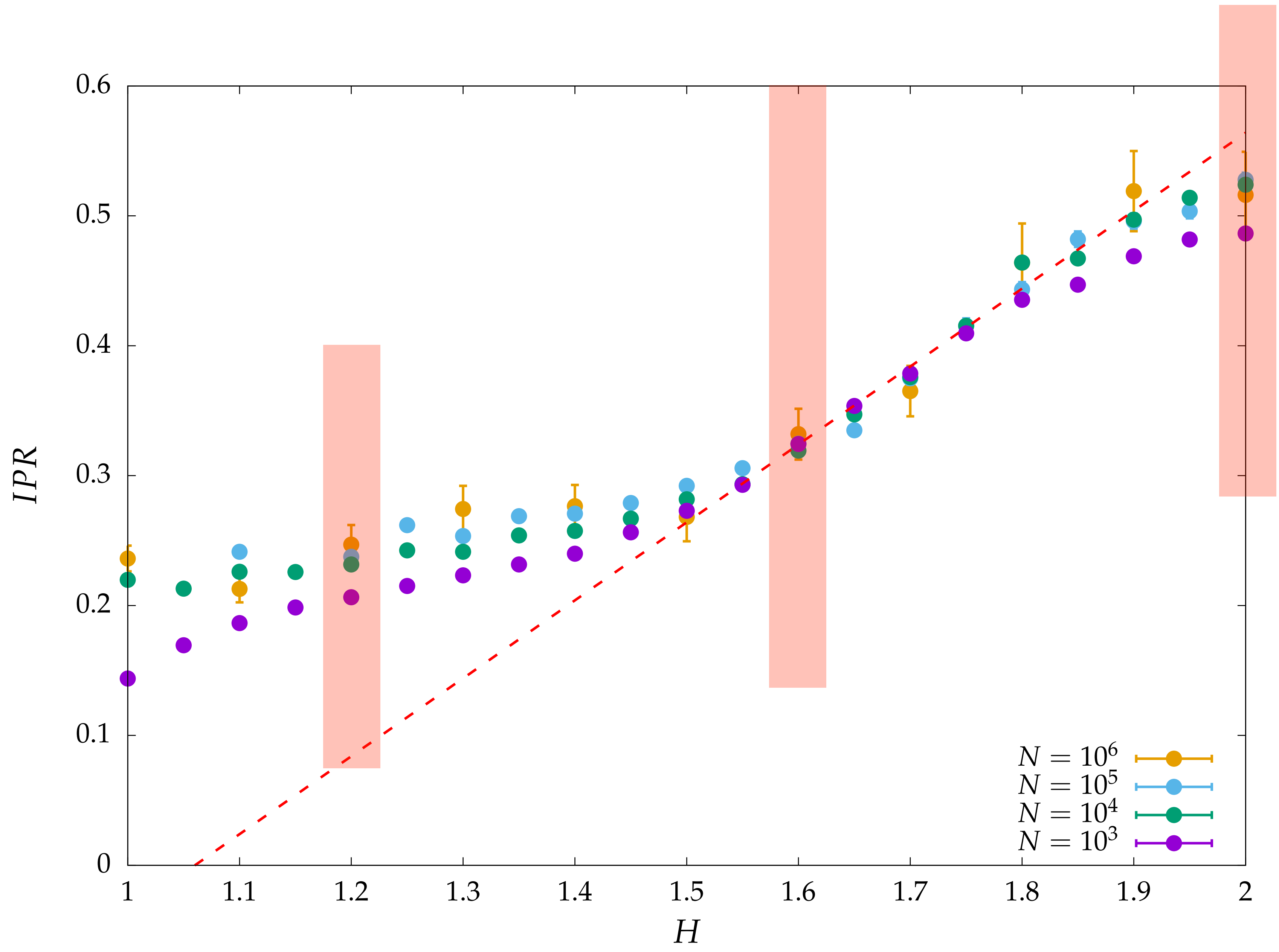
Hessian lowest eigenvectors

Standard analysis of Inverse Participation Ratio (IPR) is very sensitive to finite size effects and not conclusive



Hessian lowest eigenvectors

Standard analysis of Inverse Participation Ratio (IPR) is very sensitive to finite size effects and not conclusive



Conclusions

- Clock model approximate well XY model (useful for numerical simulations)
- XY more glassy than Ising: may reproduce experimental results?
- Random field XY model has RSB phase and marginal ferro phase
- Different nature of long range correlations:
    - Ising -> few spins highly correlated
    - XY -> all spins correlated, but correlations are spread over many different orders of magnitude
- Low-energy excitations arise very far from the critical point
- Very hard to infer the critical point from Hessian spectrum