Mean field spin glasses and models of evolution

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OUTLINE

Mean field spin glasses

Simple models of evolution without or with selection

Effect of noise on traveling waves

Directed Polymers in a random medium

Ratios of coalescence times and phase transitions

Mean field spin glasses (SK model)

N Ising spins $S_i = \pm 1$ with random interactions J_{ij}

$$H_J(\{S\}) = -\sum_{i,j} J_{i,j} S_i S_j$$

$\Rightarrow$ random energy landscape with lots of valleys

Take two spin configurations $\{S^{(1)}\}$ and $\{S^{(2)}\}$

in the **same energy landscape** and

define their overlap $Q^{(1,2)}$

$$Q^{(1,2)} = \frac{1}{N} \sum_i S_i^{(1)} S_i^{(2)}$$

At thermal equilibrium try to measure or to calculate

$$P_J(q) = \sum_{\{S^{(1)}\}} \sum_{\{S^{(2)}\}} \delta(q - Q^{(1,2)}) \frac{\exp\left[-\beta H_J(\{S^{(1)}\}) - \beta H_J(\{S^{(1)}\})\right]}{Z^2}$$

Consequences of Parisi's theory of spin glasses

Parisi 79-80-83

Mézard-Parisi-Sourlas-Toulouse-Virasoro 84

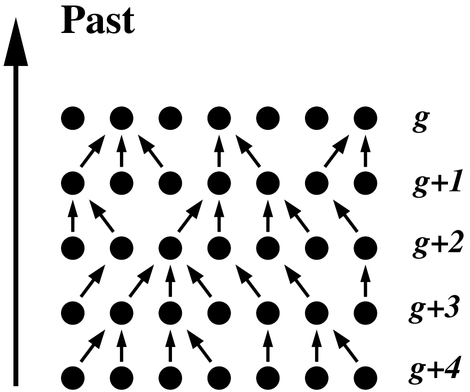
For large N:

- $P_J(q)$ is **not self-averaging**
- The average $\langle P_J(q) \rangle$ over J remains **broad**
- The moments of $P_J(q)$ can be expressed in terms of $\langle P_J(q) \rangle$
- Ultrametricity $P_J\left(q^{(1,2)}, q^{(2,3)}, q^{(1,3)}\right)$
- Statistics of the trees
- ...

tree	**probability**
	$\frac{3}{4}$
	$\frac{1}{4}$

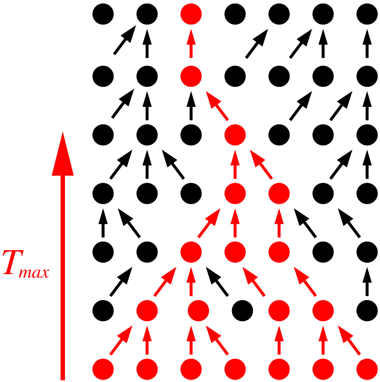
Wright-Fisher model (1930-1931)

- One parent model
- Population of fixed size N
- Each individual has its parent chosen at random in the previous generation (neutrality)



Coalescence times:

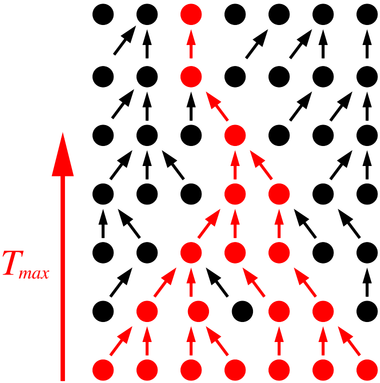
Age of the most recent common ancestor T_{max}



T_{max} depends on g

Coalescence times:

Age of the most recent common ancestor T_{max}



T_{max} depends on g

$$\langle T_{max} \rangle \simeq 2N$$

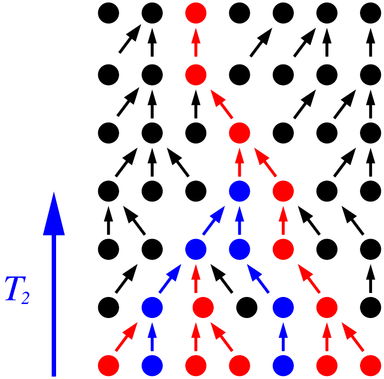
$$\langle {T_{max}}^2 \rangle - \langle T_{max} \rangle^2 \simeq C \ N^2$$

where $C = \frac{4\pi^2}{3} - 12 \simeq 1.6$

T_{max} is a non self-averaging quantity

Coalescence times:

Age T_2



The time T_2 depends on g and on the pair of individuals

Its average $\overline{T}_2$ over the population still depends on g

$$\langle T_2 \rangle \simeq N$$

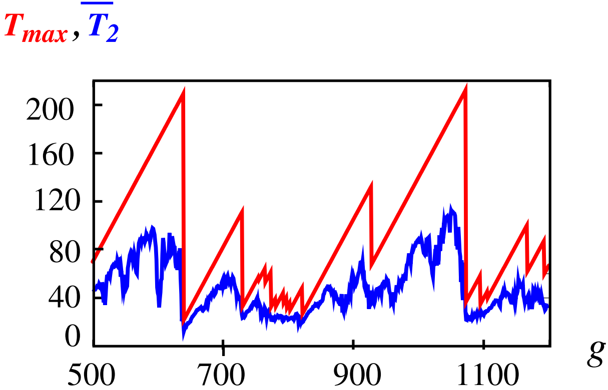
$$\langle {T_2}^2 \rangle - \langle T_2 \rangle^2 \simeq N^2$$

Two sorts of averages: { over the population; over g }

Evolution of T_{max} and $\overline{T}_2$

T_{max} = age of the most recent common ancestor

$\overline{T}_2$ = average of T_2 over the whole population

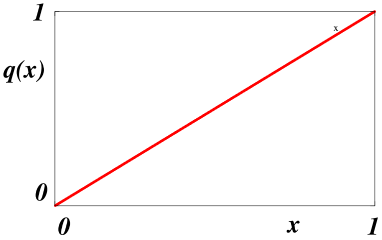


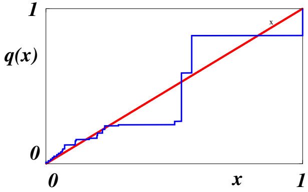
Serva 2005
Simon D. 2006

Giorgio's $q(x)$

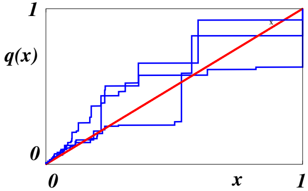
e^{-T_2} plays a role very similar to $Q^{(1,2)}$ in spin-glasses. Then

$$q(x) = x$$





Full RSB



DISORDERD SYSTEMS - MODELS OF EVOLUTION

D. Peliti 1991

DISORDERED SYSTEMS SK model	NEUTRAL MODELS of EVOLUTION
overlaps	coalescence times
thermal average	average over the population
average over the quenched disorder	average over generations

Non-self-averaging ultrametric structure

possibility of multiple coalescences	only pair coalescences

Statistics of the trees

	spin–glass	neutral
	$\frac{3}{4}$	1
	$\frac{1}{4}$	0

spin-glass $\equiv$ mean-field spin glasses

Parisi 79-80

Mézard-Parisi-Sourlas-Toulouse-Virasoro 84

... Bolthausen-Sznitman 98

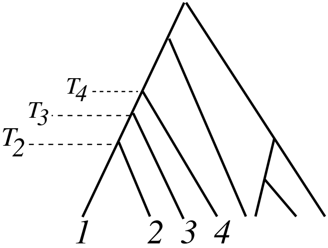
neutral $\equiv$ Wright-Fisher model

	spin–glass	neutral
	$\frac{1}{3}$	$\frac{2}{3}$
	$\frac{1}{6}$	$\frac{1}{3}$
	$\frac{1}{6}$	0
	$\frac{2}{9}$	0
	$\frac{1}{9}$	0

Coalescence times:

Age T_p

T_p = age of the most recent common ancestor of p individuals chosen at random



$$\langle T_p \rangle \simeq \frac{2(p-1)}{p} N$$

Kingman's coalescent

Kingman 1982

$$p \underset{t_p}{\longrightarrow} p-1 \underset{t_{p-1}}{\longrightarrow} \cdots 2 \underset{t_2}{\longrightarrow} 1$$

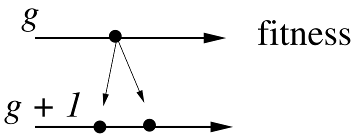
and

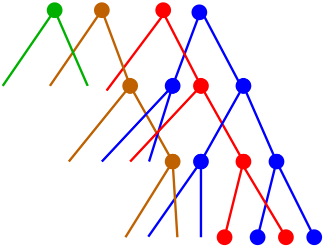
$$\boxed{T_p = t_p + t_{p-1} + \ldots t_2}$$

The times t_p are independent and have exponential distibutions with $\langle t_p \rangle = 2N/p(p-1)$

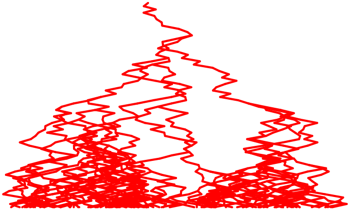
Models of evolution with selection

- Population of size N
- Each individual has 2 offspring at the next generation
- The fitness is transmitted up to some small change due to mutations
- The N right-most individuals are selected

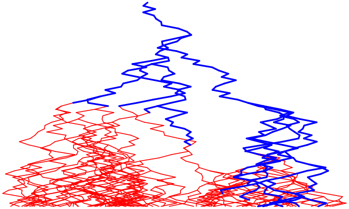




Branching random walk

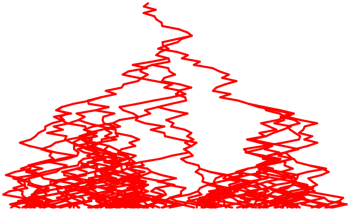


Branching random walk + selection

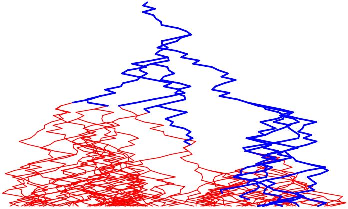


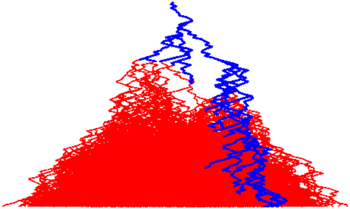
$N \leq 5$

Branching random walk



Branching random walk + selection

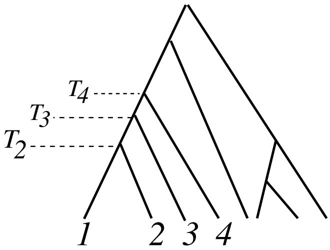




Questions

For a population of fixed size N

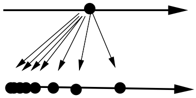
- v_N velocity of the population
- D_N diffusion constant
- Genealogy
  - Ages of the most recent common ancestors



  - Shape of the genealogical trees

Exponential model: exact solution

- Population of size N
- Each individual has infinitely many offspring at the next generation
- An individual at position x has an offspring in $(x+y, x+y+dy)$ with probability $e^{-y}dy$ (Poisson process).



- The N right-most individuals are selected

Brunet D. Mueller Munier 2006-2007

$$v_N \simeq \log\log N$$

$$D_N \sim \frac{1}{\log N}$$

$$\langle T_2 \rangle \simeq \log N$$

$$\frac{\langle T_3 \rangle}{\langle T_2 \rangle} \to \frac{5}{4} \neq \frac{4}{3}$$ Kingman

$$\frac{\langle T_4 \rangle}{\langle T_2 \rangle} \to \frac{25}{18} \neq \frac{3}{2}$$

Trees in the exponential model

	Selection	Neutral
	$\frac{3}{4}$	1
	$\frac{1}{4}$	0

Selection $\equiv$ mean-field spin glasses

Parisi 79-80

Mézard-Parisi-Sourlas-Toulouse-Virasoro 84

... Bolthausen-Sznitman 98

Neutral Kingman 82

	Selection	Neutral
	$\frac{1}{3}$	$\frac{2}{3}$
	$\frac{1}{6}$	$\frac{1}{3}$
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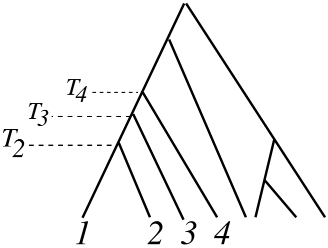
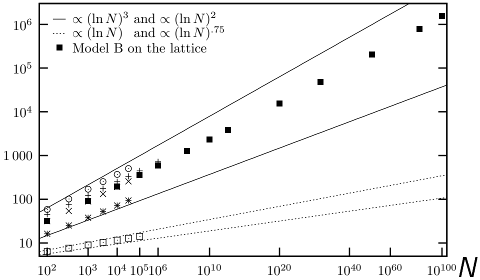
Generic case: simulations + phenomenoligical theory



T_p = age of the most common ancestor of p individuals chosen at random

Time $\langle T_2 \rangle$

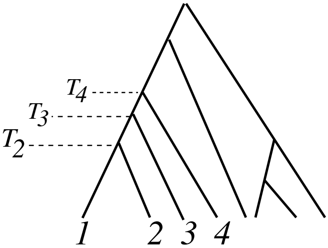


$$\langle T_2 \rangle \sim \log^3 N$$

exponential model:
$\langle T_2 \rangle \sim \log N$

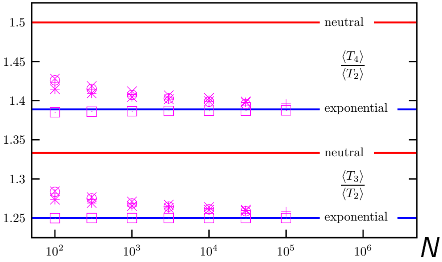
neutral model:
$\langle T_2 \rangle \sim N$

Generic case: simulations + phenomenoligical theory



T_p = age of the most common ancestor of p individuals chosen at random

Ratios $\frac{\langle T_p \rangle}{\langle T_2 \rangle}$



$$\frac{\langle T_3 \rangle}{\langle T_2 \rangle} \rightarrow \frac{5}{4} \neq \frac{4}{3}$$

$$\frac{\langle T_4 \rangle}{\langle T_2 \rangle} \rightarrow \frac{25}{18} \neq \frac{3}{2}$$

selection $\neq$ neutral

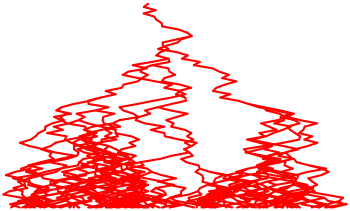
Fisher equation and branching random walk

The Fisher-KPP equation

$$\frac{dc}{dt} = \frac{d^2c}{dx^2} + c - c^2$$

Fisher 1937
Kolmogorov Petrovsky Piscounov 1937

McKean 1975



$Q(x,t)$ probability that the right-most walker is at the right of x

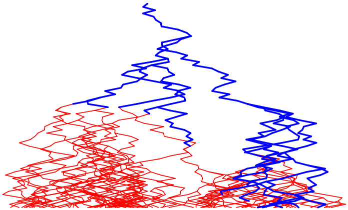
$$\frac{dQ}{dt} = \frac{d^2Q}{dx^2} + Q - Q^2$$

Fisher equation and branching random walk

The Fisher-KPP equation

Fisher 1937
Kolmogorov Petrovsky Piscounov 1937

$$\frac{dc}{dt} = \frac{d^2c}{dx^2} + c - c^2$$



selection

$Q(x, t)$ probability that the right-most walker is at the right of x

$$\frac{dQ}{dt} = \frac{d^2Q}{dx^2} + Q - Q^2 + \text{Noise}$$

Traveling wave equation + noise

Branching random walk + saturation

$$\frac{dc}{dt} = \frac{d^2c}{dx^2} + c - c^2 + \epsilon\, \eta(x,t)\sqrt{c(1-c)}$$

Branching random walk + selection

N size of the population

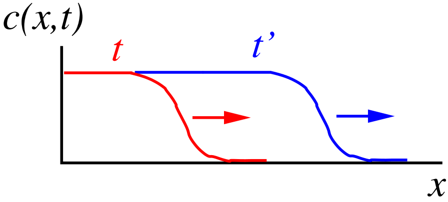
$c(x,t) =$ fraction of the population at the right of x

$$\text{Traveling wave equation} + \frac{1}{\sqrt{N}}\ \text{noise}$$

Traveling wave equation + noise

$$\frac{dc}{dt} = \frac{d^2c}{dx^2} + c - c^2 + \frac{1}{\sqrt{N}} \ \eta(x,t)\sqrt{c(1-c)}$$

Brunet D. 1997



Brunet D. Mueller Munier 2006

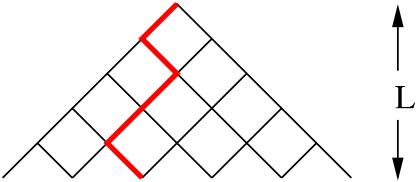
Mueller Mytnik Quastel 2008

$$v_N \simeq 2 - \frac{\pi^2}{\log^2 N} + \frac{6\pi^2 \log\log N}{\log^3 N}$$

$$D_N \simeq \frac{2\pi^4}{3\log^3 N}$$

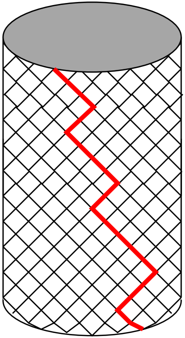
Directed Polymers in a random medium

1+1 dimension



Random energy ϵ_b on each bond b

Torus

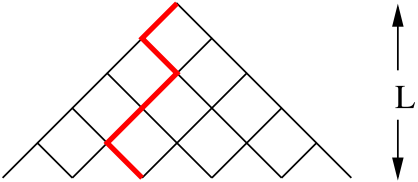


$$E_W = \sum_{b \in W} \epsilon_b$$

Energy of a path = position of a random walker in the branching radom walk problem

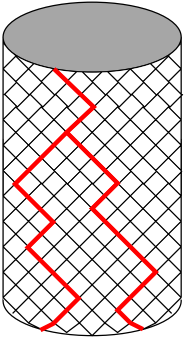
Directed Polymers in a random medium

1+1 dimension



Random energy ϵ_b
on each bond b

Torus



$$E_W = \sum_{b \in W} \epsilon_b$$

Energy of a path =
position of a random
walker in the branching
radom walk problem

Universal ratios

	T_3/T_2	T_4/T_2
Neutral $d \geq 2$= Kingman	$4/3$	$3/2$
Neutral $d = 1$	$7/5$	$8/5$
Directed Polymers $1 + \infty$ Selection = Parisi	$5/4 = 1.25$	$25/18 \simeq 1.39$
Directed Polymers $1 + 2$	1.29	1.42
Directed Polymers $1 + 1$	1.36	1.55

Brunet D. Simon 2008

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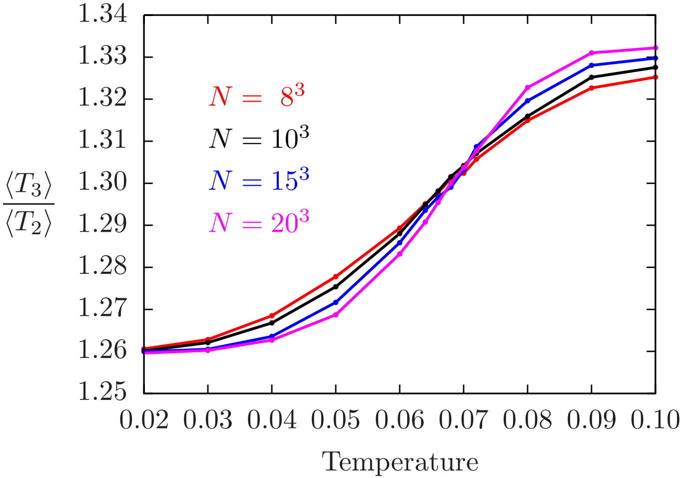
Brunet D. Simon 2008

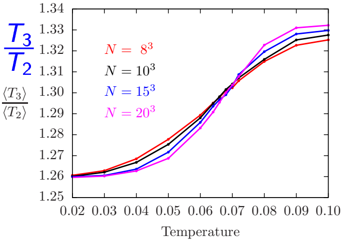
Phase transitions in dimension $3+1$

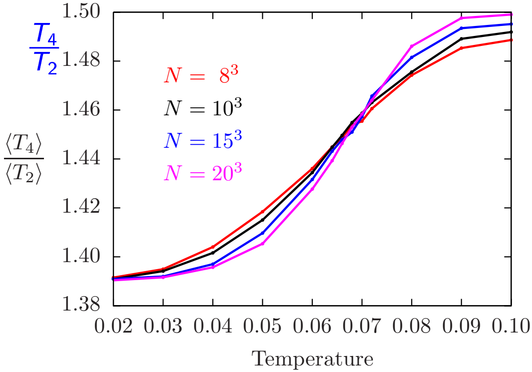
Brunet D. 2009?

Lattice infinite in the special direction and finite of N sites in the transverse direction



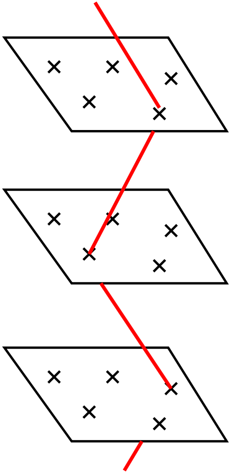
Phase transitions in dimension $3+1$





Directed Polymers in $1 + \infty$ dimension

N sites in each section



Ground state energy$/L = v_N$

Variance of G.S. energy$/L = D_N$

Replicas for the generic case?

$$v_N \simeq 2 - \frac{\pi^2}{\log^2 N} + \frac{6\pi^2 \log\log N}{\log^3 N}$$

$$D_N \simeq \frac{2\pi^4}{3\log^3 N}$$

(For $\rho(\epsilon) = \exp[\epsilon - \exp[\epsilon]]$, exact expressions of v_N and D_N

Brunet D. 2004)