Spin glasses, where do we stand?

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Many progresses have recently done in spin glasses: theory, experiments, simulations and theorems!

In this talk I will present:

- A very brief introduction to spin glasses.
- The topics on which progresses have been done or we hope to reach in the near future
- A selection of recent developments in which I have been involved.

In the study of spin glasses two themes cross:

- We have only a partial knowledge of the Hamiltonian. We know the probability distribution of the Hamiltonian. In other words the Hamiltonian is generic, with some constraints.

  Examples: disordered systems, optimization (Mézard's and Monasson's talks), coding theory, structural glasses, evolution (Derrida's talk)...

  (Spin glasses are also interesting because they are the prototype of a *complex* physical, Hamiltonian system.)

- What is glassiness? Often simple Hamiltonians have simple ground states.

  In some cases the computation of the ground states has high algorithmic complexity.

  Here there are many local minima at low energy in phase space and the approach to equilibrium is very slow.

  In these cases some well know results may be not valid, e.g. the Gibbs rule for the number of equilibrium states or the fluctuation dissipation relation when the system is slightly out of equilibrium.

The simplest Ising spin glasses has the following Hamiltonian:

$$H_J = \sum_{i,k=1,N} J_{i,k}\sigma_i\sigma_k$$

$\sigma = \pm 1$, i and k are *neighbours*. The J are random (e.g. $\pm C$).

There are many variations:

- All points are connected: Sherrington Kirkpatrick model: Mean field theory and infinite dimensions limit.
- Points lives on a Bethe lattice: infinite range, but finite connectivity.
- Nearest neighbour on a regular lattice: Edwards Anderson model in finite dimensions.
- Points lives on a one dimensional lattice with long range connections (Levy Lattice): Poor man versions of the Edwards Anderson model.
- Points lives on a hierarchical lattice: poor man version of the Levy Lattice.

Random and non-random systems may share some properties.

The Hamiltonian

$$H = D_{i,k} \sigma_i \sigma_k$$

with

$$D_{i,k} = \sin\left(\frac{ik\pi}{N}\right)$$

may have similar properties to those of a random Hamiltonian with the constraint that the spectrum of J is equal to the spectrum of D.

There are connection to structural glasses (Binder's talk).

Are there applications to string theory (in the landscape framework)?

There are many theoretical problems, some are well understood, some are at their infancy.

- Mean field $\rightarrow$ Qualitative understanding of the low temperature phase.
- Perturbative expansion $\rightarrow$ Range of validity of the mean field theory.
- Non-perturbative corrections $\rightarrow$ Range of validity of the perturbative expansion.
- Renormalization group + Perturbative expansion $\rightarrow$ Upper critical dimension.
- Renormalization group + Goldstone Bosons and/or non-perturbative phenomena $\rightarrow$ Lower critical dimension.

In this period we are doing many progresses for different reasons:

- We are hopefully near to prove that **all** the features of the the analytical solution using the replica method are correct (Aizenman's and Guerra's talks).
- New observables have been measured in numerical simulations.
- A new model (Levy Lattice) has been introduced, that helps us to understand the behaviour near the upper critical dimension.
- New hardware has been constructed (Janus) that allows the simulations of large lattices for $O(10^{12})$ Montecarlo sweeps, i.e about one second in physical scale.
- There are new non-perturbative computations (Brézin's talk).
- There are very interesting new experiments (Vincent's talk).

Deep studies of spin glass theory started with the Edwards Anderson paper and the Sherrington Kirkpatrick model.

The SK model is conceptually simple, it defines the mean field theory of spin glasses. However the theory of spin glasses is difficult. Already in mean field theory it took a long time to clarify the theory (Sherrington's talk).

- 5 years to solve a soluble model. This has been done using replica theory. It was first stated in an abstract algebraic approach (analytic continuation to $n \to 0$ of a problem involving $n \times n$ matrices. It has been later put in a probabilistic setting, i.e. cavity (Mézard's talk), that inspired the rigorous proofs.
- 5 years to find the full physical meaning of the physical solution after it was found.
- Other 10 years in order to found out the origin of some properties of the solution (i.e. stochastic stability).
- 25 years to prove that the free energy computation in mean field is correct.
- ?? years to prove that the computation of **all** observables in mean field theory is correct (Aizenman's and Guerra's talks).

In the study of spin glasses we found new phenomena that nobody was forecasting at the beginning.

In mean field theory there are many equilibrium states (an infinite number in the infinite volume limit). The states are locally different: they are not mosaic states done using two different colors with large tiles.

Each point is a critical point (first or second order according to the model). We have found SOCE (Self Organized Criticality at Equilibrium).

This violates the Gibbs rule. Apparently this is not possible. The system would be unstable under generic perturbation unless something special happens.

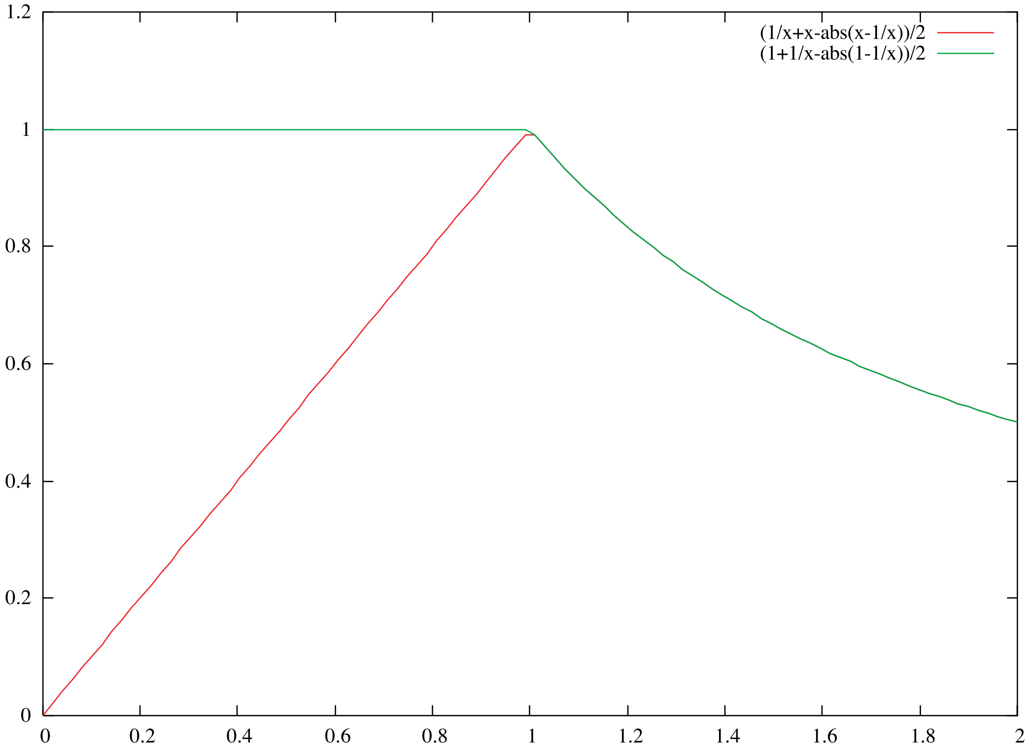
We can turn around the argument: *something special happens.* This argument leads to the stochastic stability identities of Guerra.

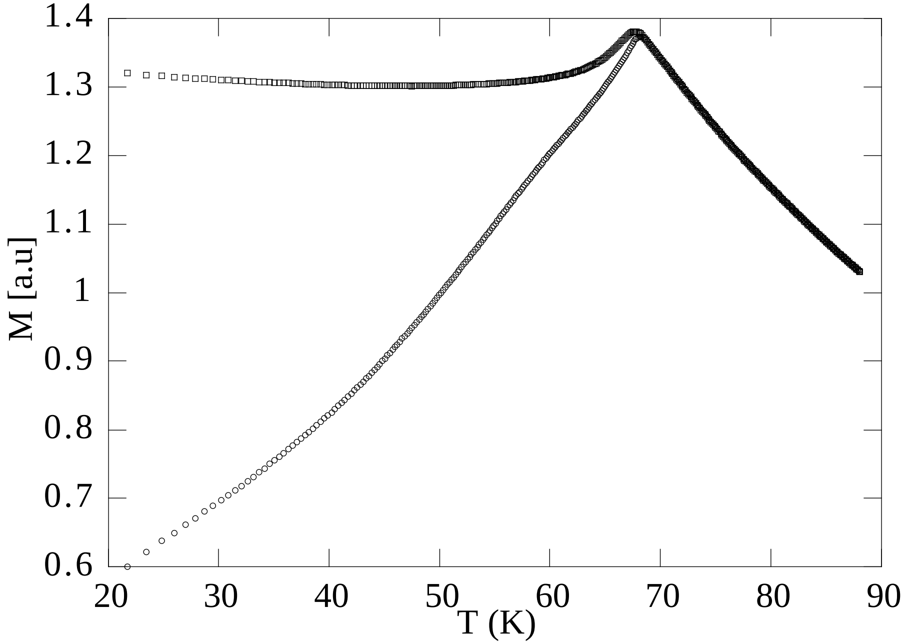
We can define two physically relevant susceptibilities.

χ_{LR}, i.e. the response within a state, that is observable when one changes the magnetic field at fixed temperature and one does not wait too much.

χ_{eq}, the true equilibrium susceptibility, that is very near to χ_{FC}, the field cooled susceptibility, where one cools the system in presence of a field.

The difference of the two susceptibilities is the hallmark of replica symmetry breaking.





At the left we show the results for the SK model, at the right we have experimental data on metallic spin glasses. The similarities among the two are striking.

How to define the order parameter? We do not know the ground state.

Great idea of Edwards and Anderson: Take two systems σ and τ:

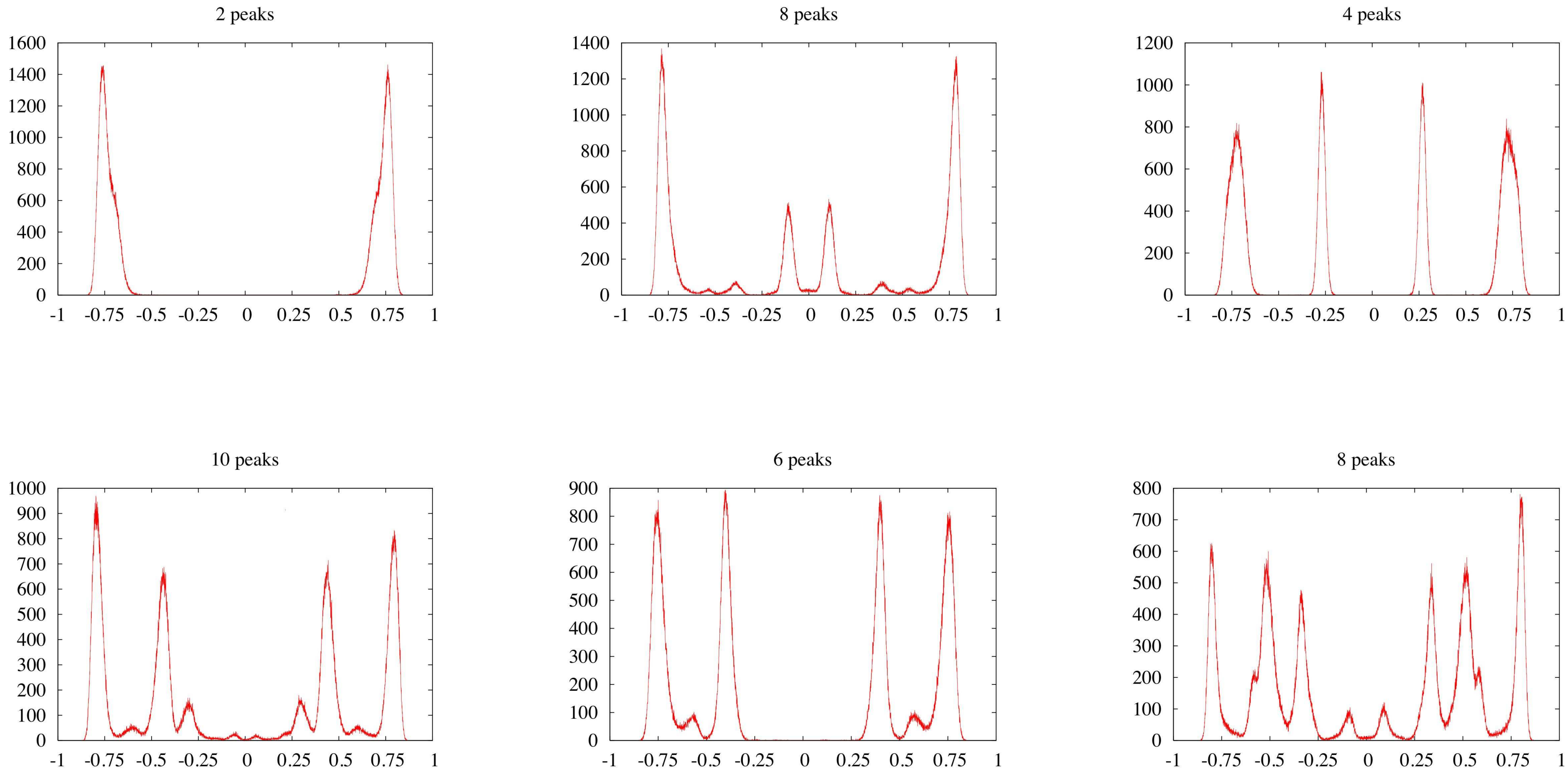
The overlap is

$$q = \frac{\sum_i \sigma_i \tau_i}{N} \qquad \langle q \rangle = \frac{\sum_i m_i^2}{N}$$

If for a given system m points in a random direction, q is non-zero.

If there is only one ground state $P(q)$ has one peak (if we restrict ourselves to positive q).

Multiple states appear as multiple peaks of $P(q)$.



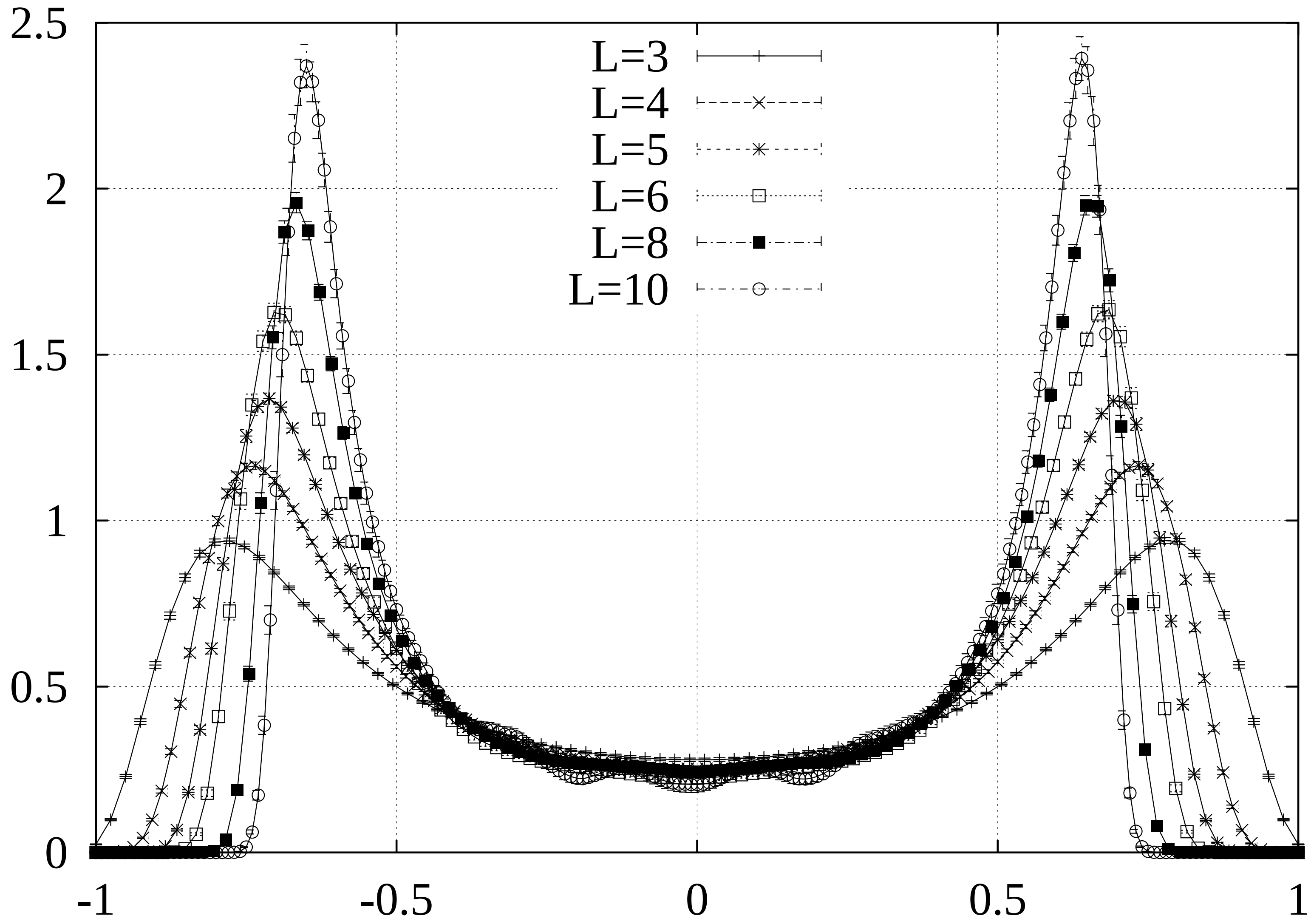
$P_J(q)$ for six different disorder realizations in the Sherrington-Kirkpatrick model: here $N = 4096$ and $T = 0.4T_c$. From *Finite size corrections in the Sherrington-Kirkpatrick model* by T. Aspelmeier, A. Billoire, E. Marinari, M.A. Moore.

For each system we define a function $P_J(q)$, q being the overlap. Its average is a function $P(q)$ that is not trivial also in finite dimensions.

The equilibrium function $P(q)$ is non trivial and it change from system to system. All $P_J(q)$ have a peak at $q = q_{EA} = \overline{m_i^2}$.

No doubts from simulations, also in three dimensions, at least at zero magnetic field.

If the function $P(q)$ is non trivial (also at infinitesimal h different from zero), we have two different susceptibilities.



The function $P(q) = \overline{P_J(q)}$ after average over many samples (L=3…10) in $D = 4$.

$P(q)$ being non-trivial does imply something very special?

Two scenarios:

- Case (a) Ising ferromagnetic with antiperiodic boundary conditions: many states and non trivial $P(q)$. These states are locally identical (apart from a spin flip) apart form a region whose relative volume goes to zero as L^{-a} with $a = 1$. An infinitesimal magnetic field $(1/V)$ destroys everything.

- Case (b) —The states are not locally congruent. The picture remains the same if we consider window overlaps, i.e. overlaps in a region R when L goes to infinity first.

In order to see the difference:

Take two replicas: define the correlation among the two local overlaps at distance x between to replicas with overlap q: $G(x|q)$.

We introduce the conditioned correlation:

$$G(x|q) = \frac{\overline{\langle q_x q_0 \delta(q[\sigma,\tau] - q)\rangle_J}}{\overline{\langle \delta(q[\sigma,\tau] - q)\rangle_J}} \neq \overline{\left(\frac{\langle q_x q_0 \delta(q[\sigma,\tau] - q)\rangle_J}{\langle \delta(q[\sigma,\tau] - q)\rangle_J} \right)}$$

where

$$q_x = \sigma_x \tau_x \quad q[\sigma,\tau] = \frac{\sum_x q\sigma_x \tau_x}{V} \equiv \frac{\sum_x q_x}{V}$$

Case (a)

For small x, $G(x|q)$ does not depend on q for not too large q.

For large x, $G(x|q)$ is a function of x/L.

Case (b)

For all x, $G(x|q)$ depends on q.

For large x, $G(x|q) \to q^2$.

The correlation at distance 1, $G(1|q)$, is also called the link overlap $Q_L(q)$.

Simulations shows that $q_L(q)$ depends on q. This shows that case (b) is correct.

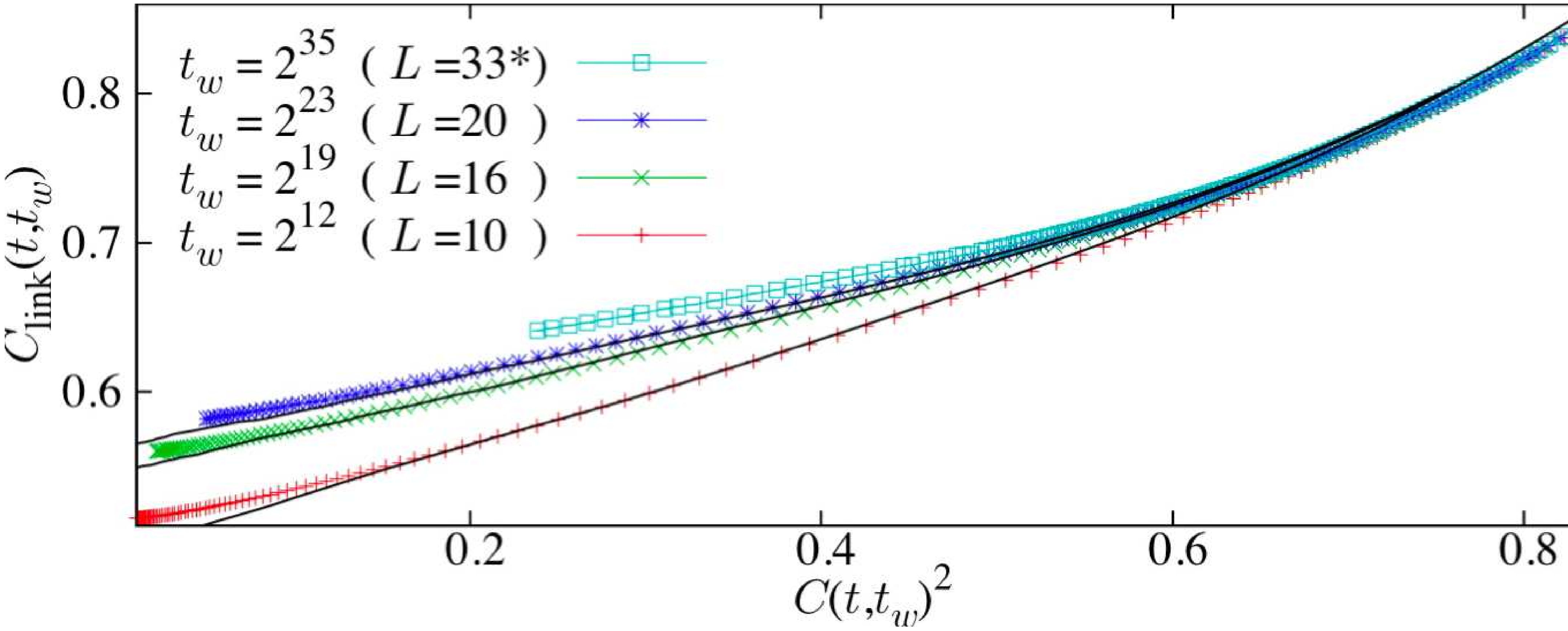
This was done by Marinari Parisi at $T = 0$ and at $T \neq 0$ using and external forcing for going to different states.

Contucci, Giardinà, Gilbert, Parisi and Vierna at finite temperature.

Also data form the Janus collaboration, using the dynamics to be discussed later (they depend on t_w and the equilibrium is reached at $t_w = \infty$).

The equilibrium data are at $L = 10, 16, 20$.

The curves from the dynamics (Janus) are labeled by t_w.

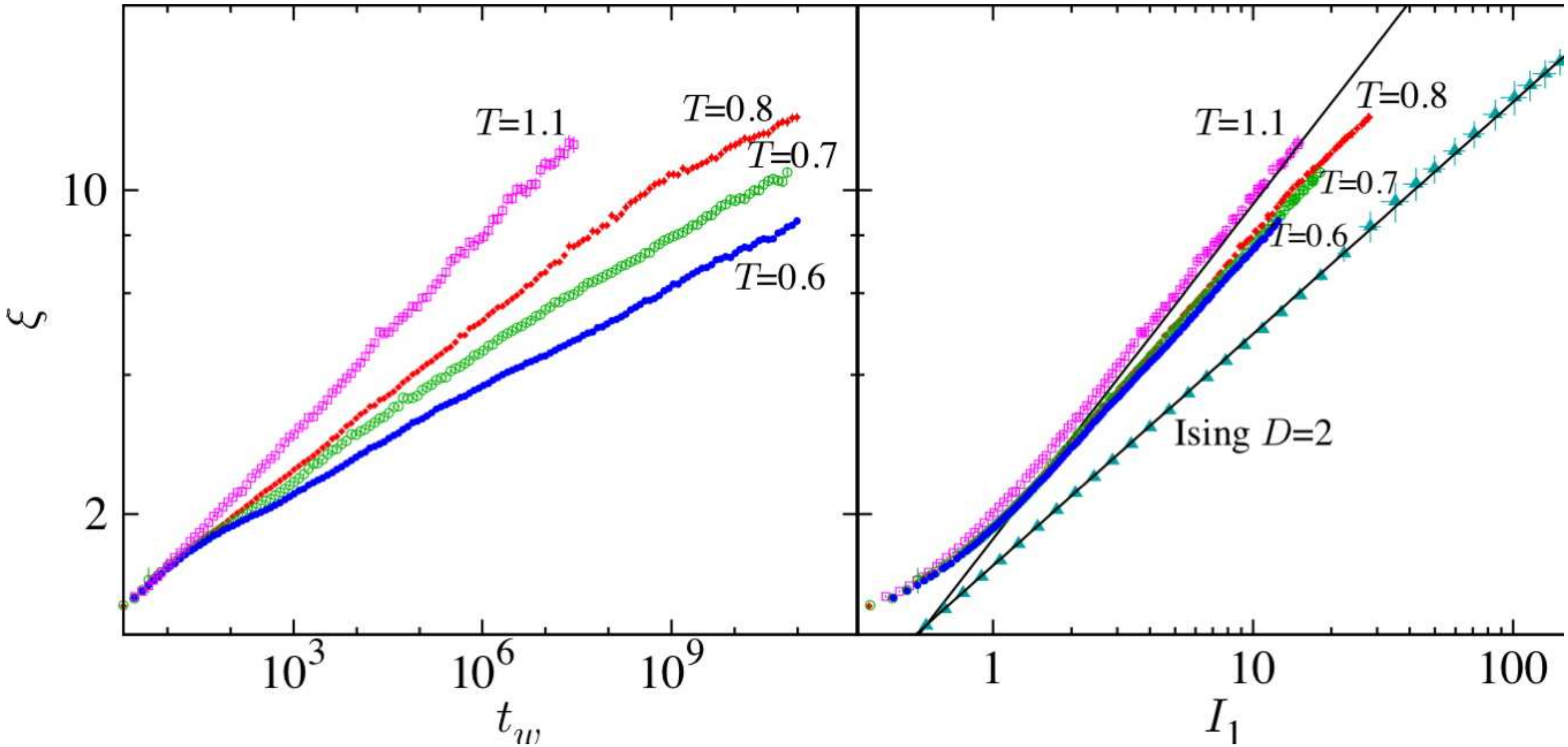


One consider an configuration that evolves in time, starting from a random one. The two configurations at two different times (i.e. $\sigma(t_w)$ and $\sigma(t)$) play the role of σ and τ in the equilibrium formulae. The system has size ($L = 80$).

Correlations at $q = 0$. They can be computed in the dynamics: for large systems, if $q = 0$ at time zero, $q = 0$ at all times.

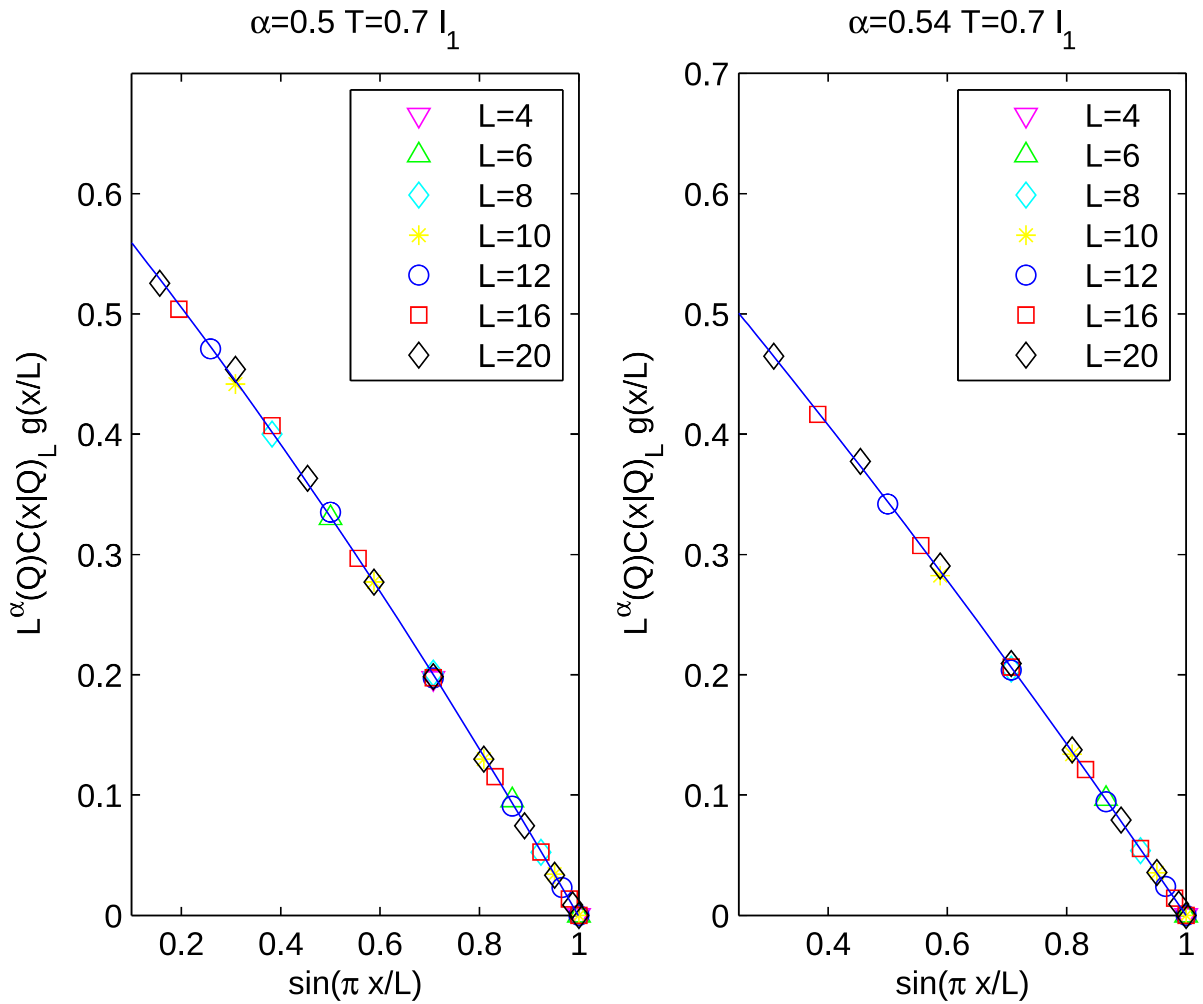
We start from two random configurations and we look at $x >> 1$.

We should have $G(x,t) = x^{-\alpha}\ f(x/\xi(t))$

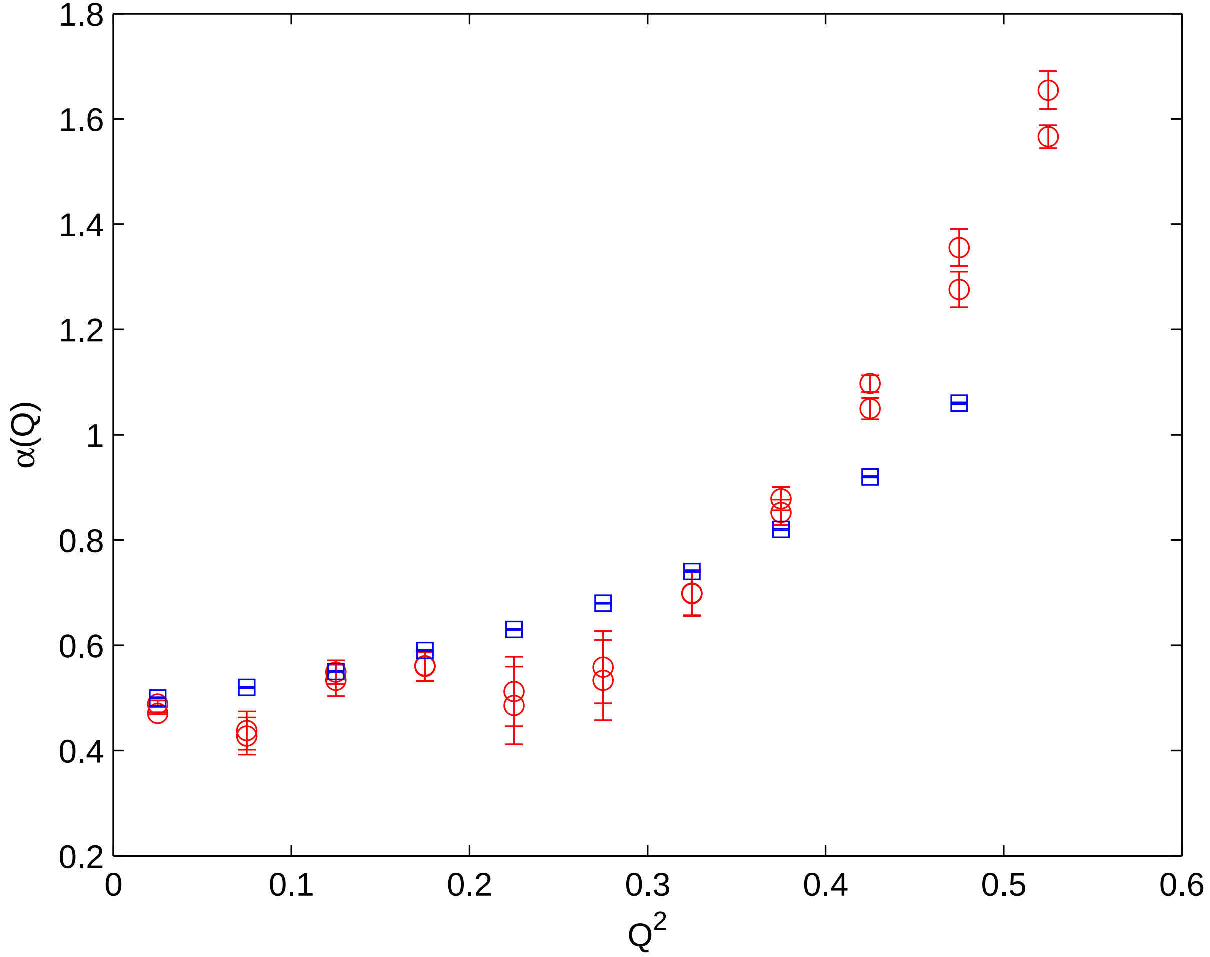


$\alpha \approx .4$ in the low temperature phase ($T < T_c \equiv 1.1$).

We consider the correlation function $G(x|q)$ at equilibrium from $L = 4$ to $L = 20$. We do a global fit of the data at fixed q, assuming that the behaviour of the correlation can be approximated by $G(x|q)_L = 1/x^{\alpha} F(x/L)$ where the unknown function F takes care of fine volume effects.



The values of the exponent α as function of q^2 (Q^2 (from Contucci, Giardinà, Gilbert, Parisi and Vierna in preparation).

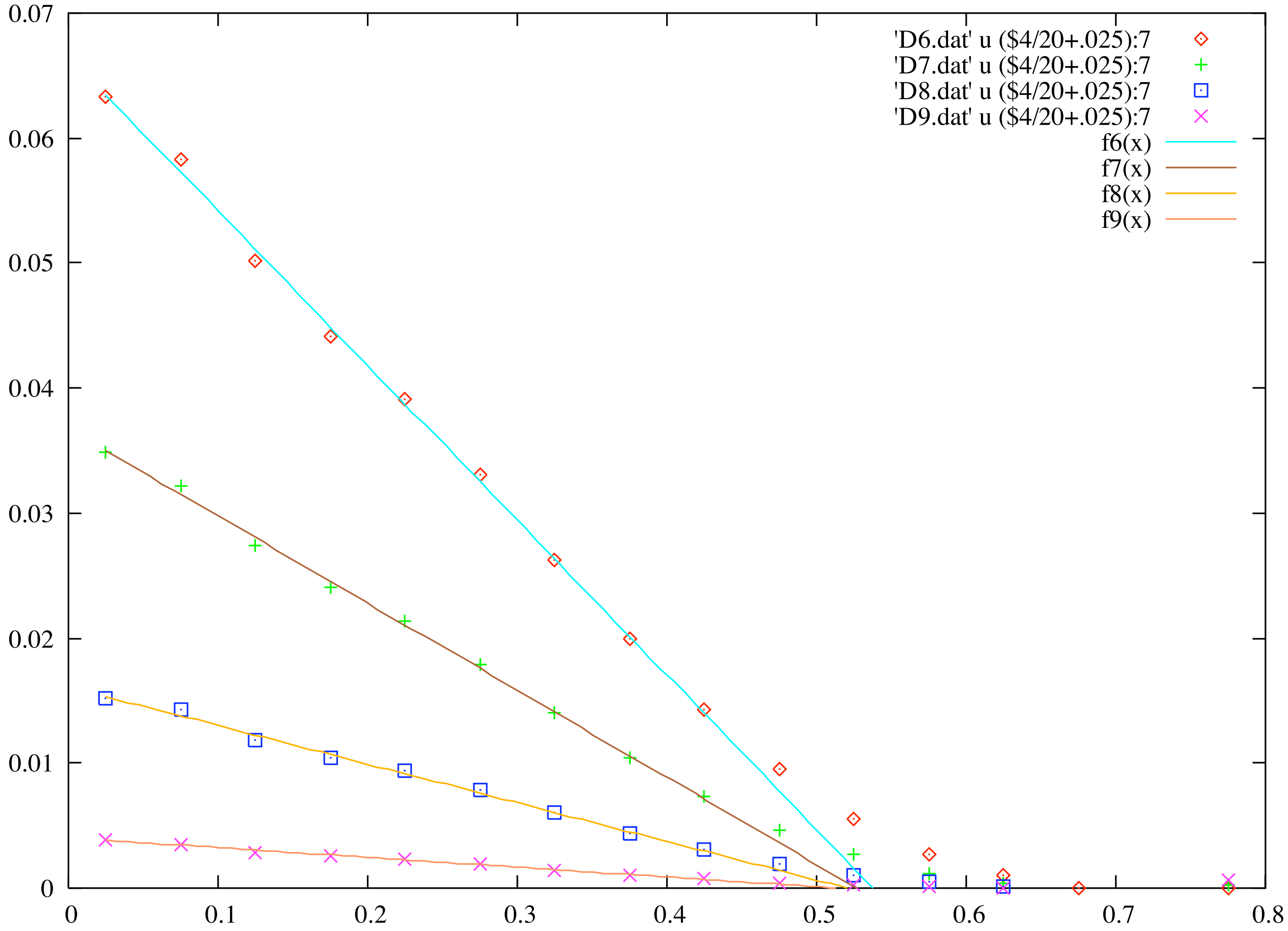


Notice that q_{EA}^2 is about 0.45-0.5.

Does the exponent α depends on q^2? The answer is **NO**

We can fit the correlations at fixed L (e.g. $L = 20$) for large x as:

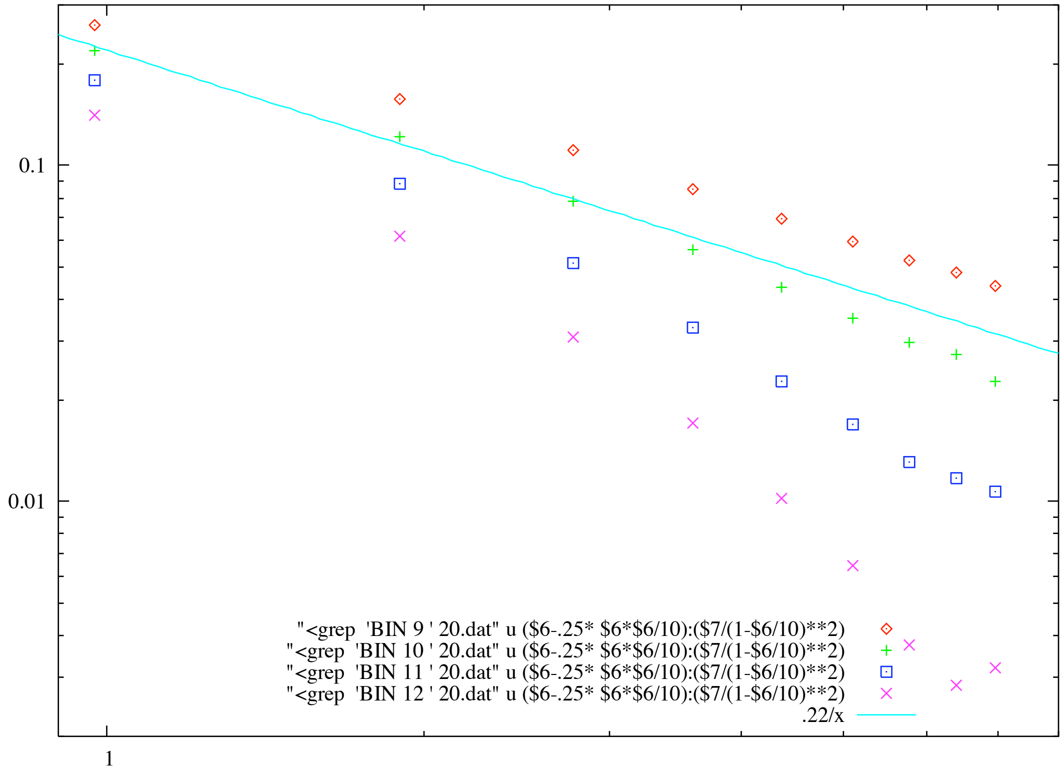
$$G(x|q)_L - G(L/2|q)_L = A(x, L)(q^2 - q_{EA}^2)$$



$L = 6, 7, 8, 9.$

At $q = q_{EA}$ the data are consistent with $G(x|q) \propto x^{-1}$

At $q > q_{EA}$ $G(x|q)$ goes to zero faster than a power at large distances.



Correlations functions at $L = 20$ for $q^2 = 0.475, 0.525, 0.575, 0.625$.

It is difficult to extract precise quantitative conclusions.

Issue: ultrametricity (Contucci, Giardinà, Gilbert, Parisi, Vierna)

Ultrametricity states a very striking property for a physical system: essentially it says that the equilibrium configurations of a large system can be classified in a taxonomic (hierarchical) way (as animal in different taxa): configurations are grouped in states, states are grouped in families, families are grouped in superfamilies.

Ultrametricity implies that sampling three configurations independently with respect to their common Boltzmann-Gibbs state and averaging over the disorder, the distribution of the distances among them is supported, in the limit of very large systems, only on equilateral and isosceles triangles with no scalene triangles contribution.

We define the three replicas overlap distribution $\mathcal{P}_3$.

$$\mathcal{P}_3(q_{1,2}, q_{2,3}, q_{3,1}) = \langle \delta(q_{1,2} - q(\sigma, \tau))\delta(q_{2,3} - c(\tau, \gamma))\delta(q_{3,1} - q(\gamma, \sigma))\rangle \ ,$$

where σ, τ, γ denote three different equilibrium configurations.

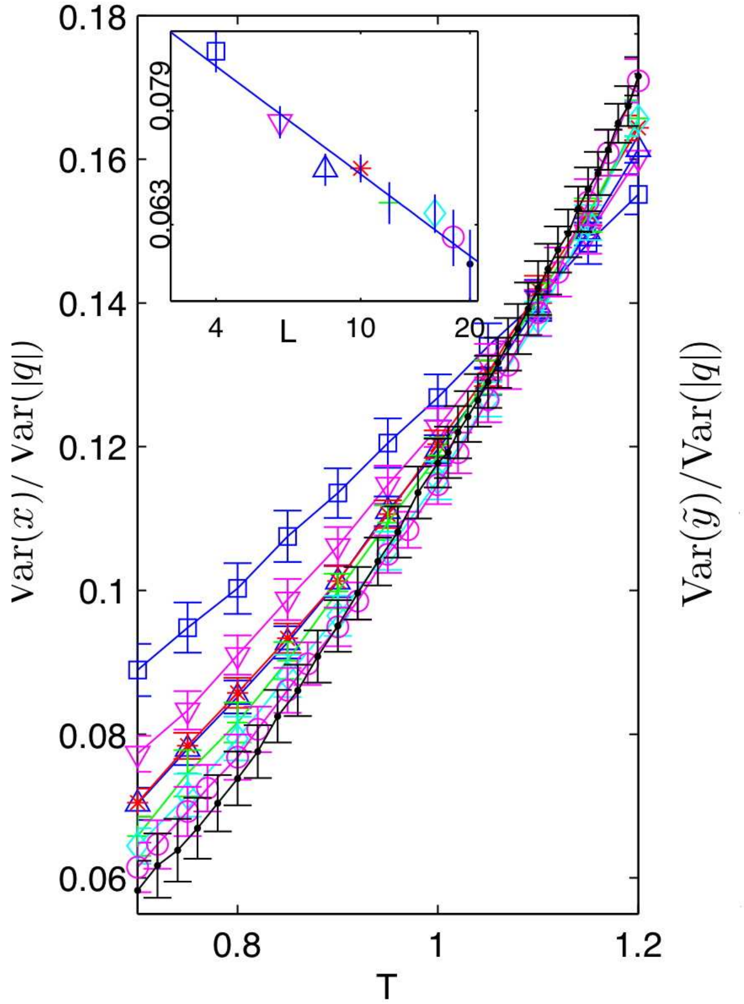
Ultrametricity implies that

$$\mathcal{P}_3(q_{1,2}, q_{2,3}, q_{3,1}) = 0$$

if the following inequality is violated

$$q_{1,3} \geq \max(q_{1,2}, q_{2,3})$$

Is ultrametricity present in D=3? At T_C is not ultrametric but it nearly ultrametric. Ultrametricity seems to be present below T_c, but it sets in slowly $L^{-.3}$.



One dimensional model with N sites with long range couplings.

$\overline{J_{i,k}^2} \propto |i-k|^{-2\sigma} = |i-k|^{-\rho}$

Poor man version of the D dimensional model.

$\rho = 1 + 2/D$

Same upper critical dimension, different lower critical dimension.

This system can be studied N large, $O(10^3)$, by Young et al. Finite volume corrections are proportional to $L^{-2/D}$ (naive estimate; x is the proxy of $r^{1/D}$). We need to go to much larger systems.

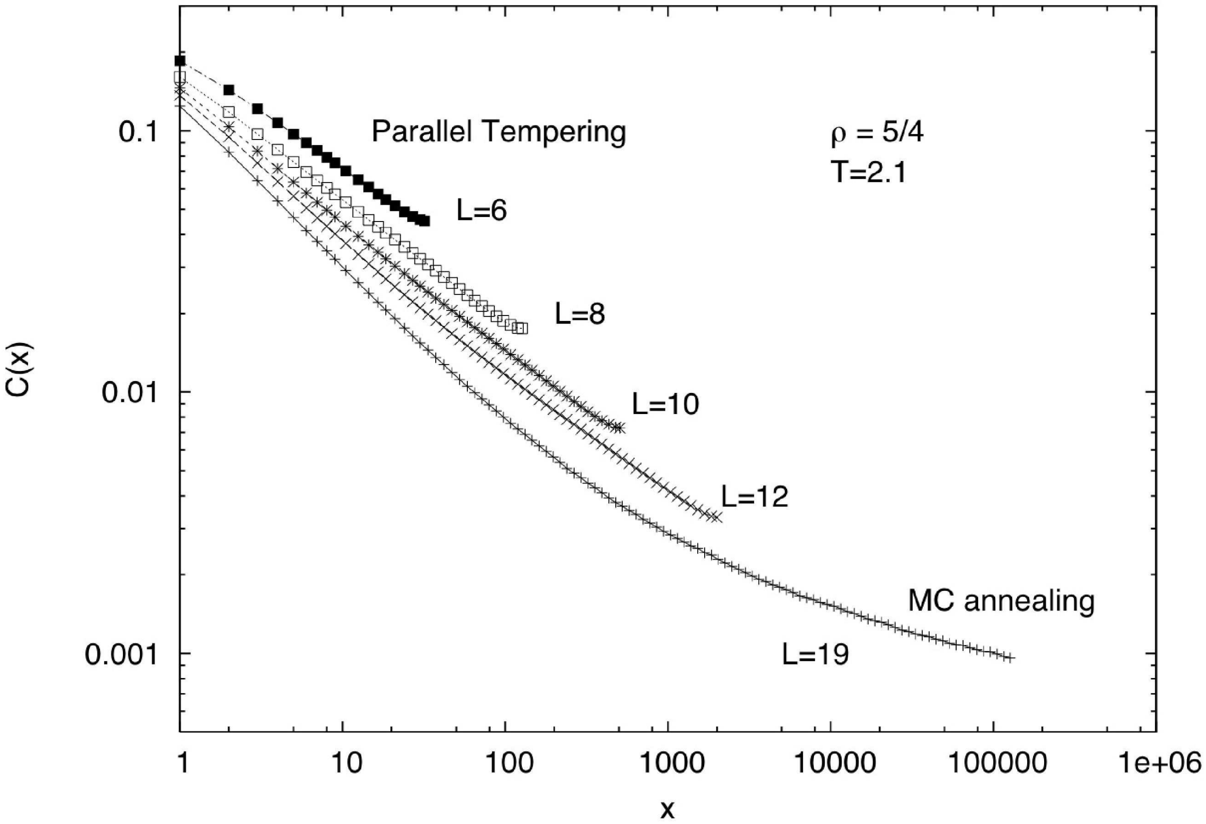
$J_{i,k} = \pm|i-k|^{-\sigma}$

$J_{i,k} = \pm 1$ with probability $|i-k|^{-2\sigma}$, otherwise 0.

The second model has been studied by Leuzzi, Parisi, Ricci-Tersenghi and Ruiz-Lorenzo. One gains a factor L in CPU and memory with respect to the first model.

Static simulations up to $N = 2^L = 2^{14}$. Everything is OK, critical temperatures and exponents.

Strong finite volume effects for the correlations (slightly below T_c, $D = 8$, i.e $\rho = 1.25$)



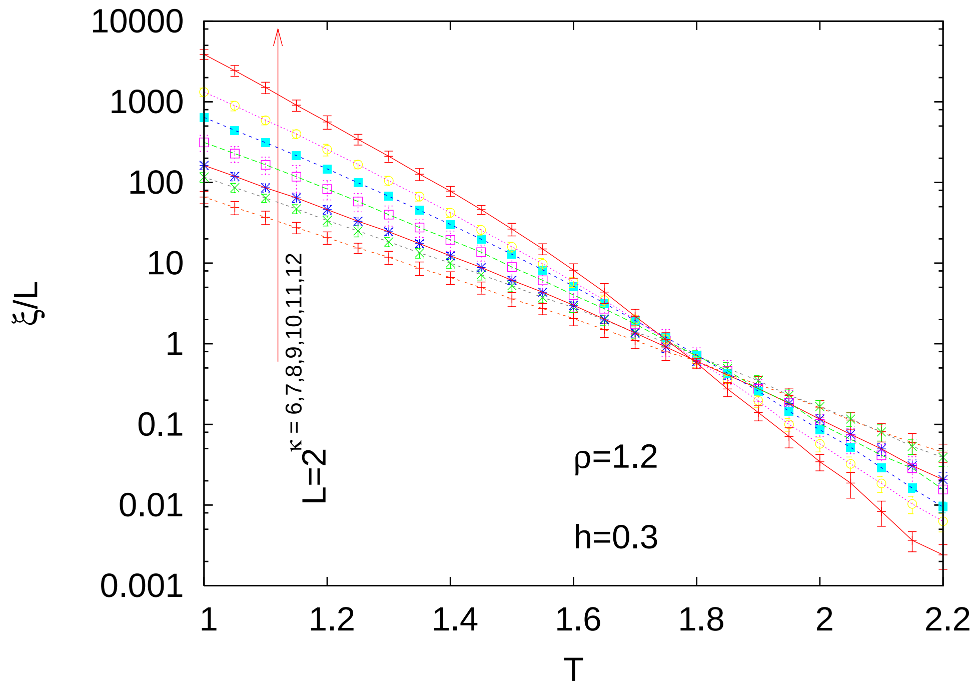
We have done dynamic simulations mostly at $L = 17$, with some data at $L = 21$ ($N = 2\ 10^6$) at $D = 4$, i.e. $\rho = 1.5$.

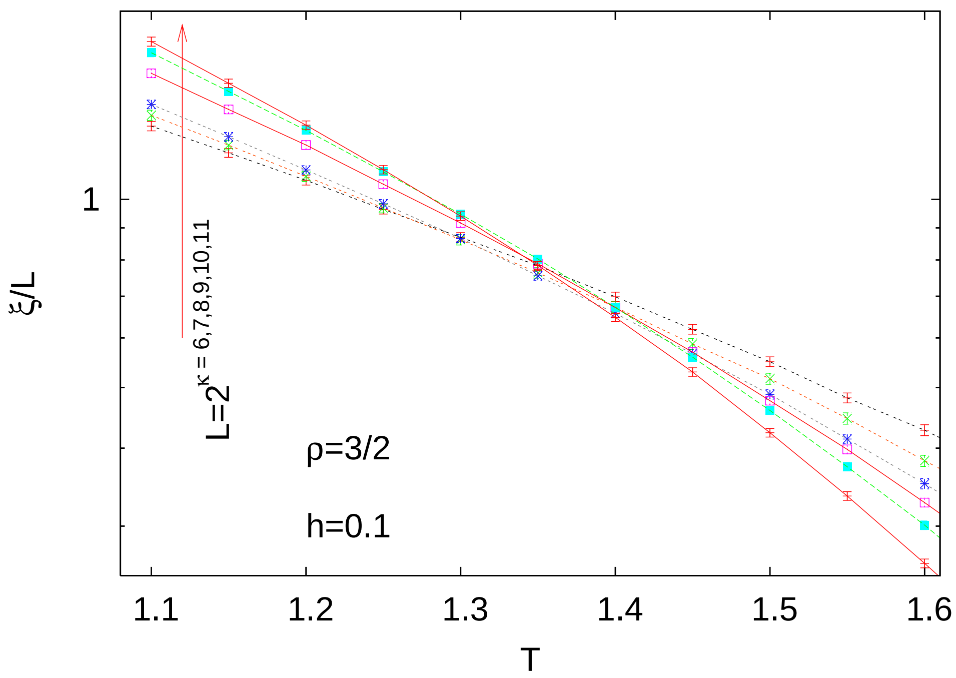
Very nice fits with power on three decades on x.

$$C(x) = \frac{a}{x^\alpha} \frac{1}{\left[1 + (x/\ell)^{\delta(\rho-\alpha)}\right]^{1/\delta}}$$

$\xi(t) \equiv l \propto B \exp(A \log(t)^{1/2}\ )$ at low temperature

There is a phase transition in magnetic field (De Almeida Thouless transition) both in the mean field that in the non mean field case. (Leuzzi, Parisi, Ricci-Tersenghi and Ruiz-Lorenzo)





The result is consistent with previous studies in the long range case that has looked at larger value $h/T_c(h=0)$, where we do not find a transition.

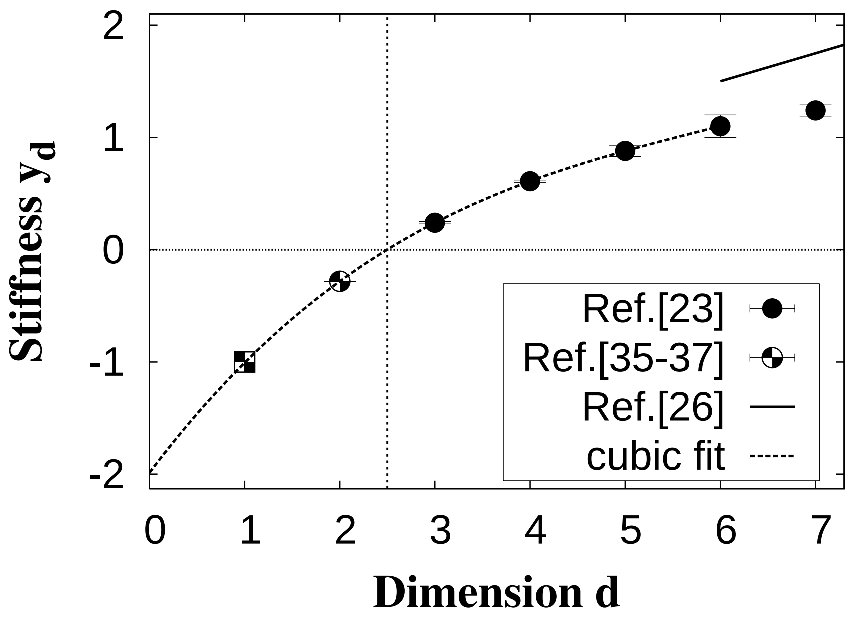
Non-perturbative corrections.

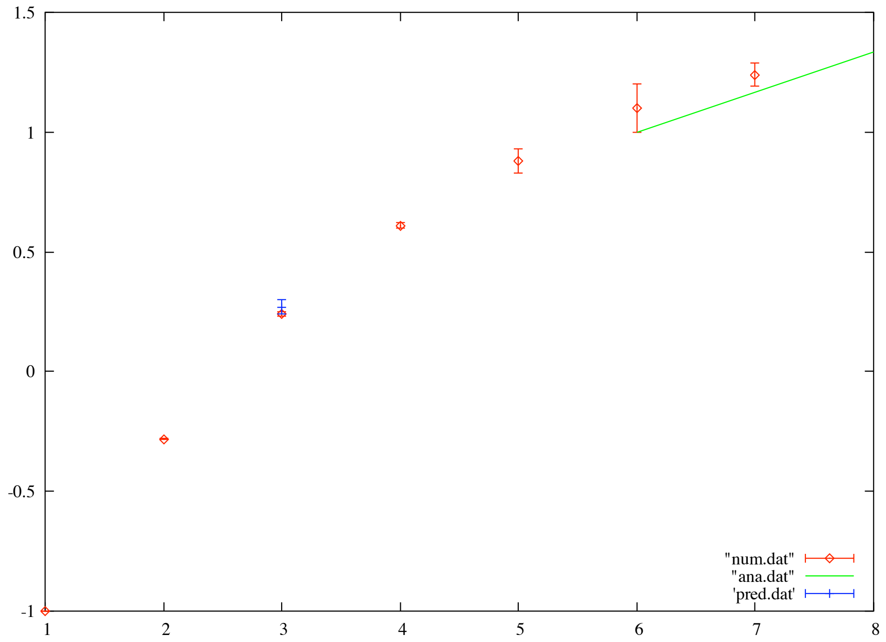
Parisi, Rizzo (following Aspelmeier, Moore and Young): periodic antiperiodic conditions.

$\overline{|\Delta E|} \propto L^{\theta}$, $D > 6$, $\theta = D/6$

$D < 6$, $\theta = (D - 2 - \alpha)/2$; for $D = 6$ α, should be equal to 2 and $\theta = 1$.

$D = 3$, $\alpha = 0.4 - 0.5$, $\theta = 0.3 - 0.25$. Direct estimates $\theta = 0.24$.





Lower critical dimension=2.5 as predicted by an interface computation.

Non-perturbative phenomena in Hierarchical Spin-Glass models.

Overlap Interfaces in Hierarchical Spin-Glass models by Franz, Jorg, Parisi.

Real space renormalization, ongoing research with Franz, Mézard, Ricci-Tersenghi.

Conclusions and questions to be answered

Replica symmetry breaking picture is confirmed.

Ultrametricity seems to present in the statics (measurements are difficult, strong finite volume corrections).

We start to have numeric information on the correlations.

We clearly see the transition in a external field outside mean field

We need non-perturbative computations of barriers of various type.

We need better understanding of why the lower critical dimension is 2.5 (as predicted by a detailed analytic computation by Franz, Parisi and Virasoro long time ago).

I would like to thank all the many friends I worked with on spin glasses.

I would also use the occasion to that thank those friends, I have never worked with on spin glasses, but whose repeated discussions, suggestions and remarks have been very useful to me.

Moreover I would like to thank all the many people that have contributed in one way or in the other to a better understanding of spin glasses, especially in the framework of broken replica symmetry.