The cavity method

"Vingt ans après"

Early days with Giorgio

Les Houches lectures 1982

COURSE 6

AN INTRODUCTION TO THE STATISTICAL MECHANICS OF AMORPHOUS SYSTEMS

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to f_0, f_1, f_2 being ...

It is clear that there is no bound to the number and to the form of the solutions we can consider: when the replica symmetry group P_n is broken, (instantons have a definite orientation in the replica space), Pandora's box is opened, and we gain nothing by closing it again.

We shall not try to classify the solutions and to interpret them from the physical point of view: we limit ourselves to a few comments.

Instantons in the replica space seem to be connected to Griffith singularities [18]; a careful analysis of the dependence of their contribution as a function of the various parameters g, λ, m^2 and D would be welcome.

... fixed the mean field

SK model

$$E = -\sum_{i<j} J_{ij} s_i s_j$$

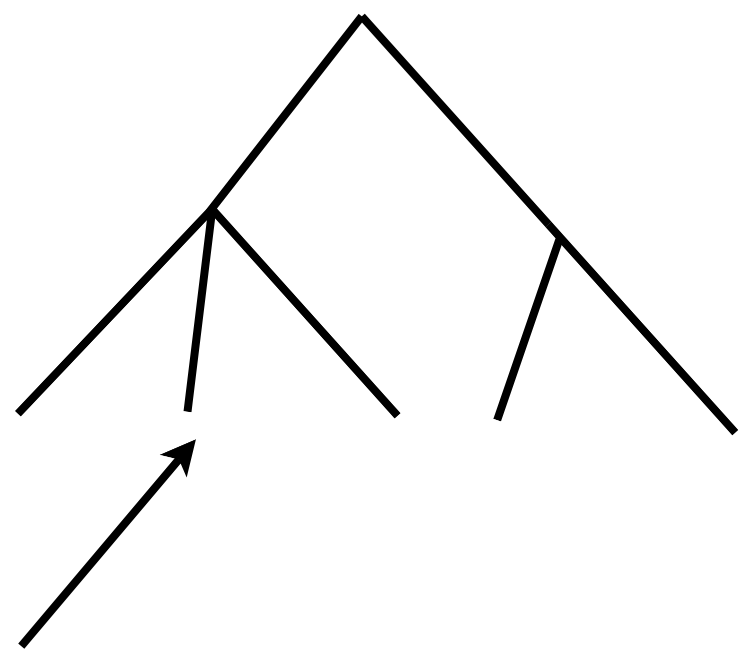
$$P(\underline{s}) = \sum_{\alpha} W_\alpha P_\alpha(\underline{s})$$

1983 Ultrametricity

Non self-averageness

(M, Parisi, Sourlas, Toulouse, Virasoro)

Many pure states, organized in a hierarchical structure: phase transition without a clear symmetry breaking



State α, free energy F_α

SK model

1983
Ultrametricity

Non self-averaging

(M, Parisi, Sourlas, Toulouse, Virasoro)

Referee's Report on "On the Nature of the Spin Glass Phase"
by M. Mézard et al (LM2400)

I believe this paper should not be published in P.R.L.

It falls into two parts. The first part is essentially an elaboration of an earlier letter of one of the authors (Ref. 11). The "ultrametric topology" is a trivial consequence of their eq. 1 once the acceptance of "overlap" of states with the $q^{\alpha\beta}$ of the replica approval is accepted. (This, however, was the message of Ref. 11.) That this is the case can be seen by taking their Fig. 1 and regarding it as a diagram for describing the Parisi replica symmetry breaking scheme. The $q^{\alpha\beta}$ between states α and β is the "q" at the common mode. Once this is realised, their result

$$q_3 \geq q_2 = q_1$$

is obvious. In itself this result does not seem of much immediate physical interest.

The second part of the paper is the discussion that $P_J\{q\}$ has a probability distribution, which they attempt to calculate. The more relevant question of the physical significance of such a result

The cavity method for the SK model

M, Parisi, Virasoro, 1985

$$E = -\sum_{i<j} J_{ij} s_i s_j$$

Motivation: understanding replica symmetry breaking

David Sherrington: "*Replica Symmetry Breaking and the conceptual, mathematical and physical challenges it raised have been a rich and fruitful source from which new knowledge and application have flowed profusely since it was invented 30 years ago by Giorgio Parisi and show no sign of abating.*"

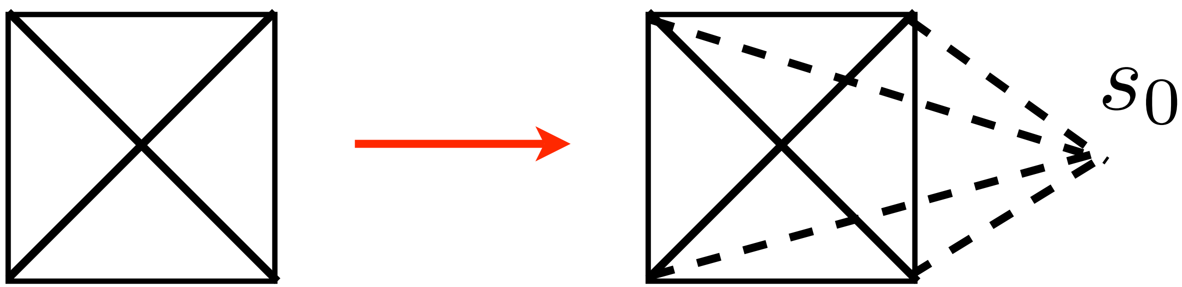
abate |əˈbāt|

verb [intrans.]
(of something perceived as hostile, threatening, or negative) become less intense or widespread : *the storm suddenly abated.*

The cavity method for the SK model

$$E = -\sum_{i<j} J_{ij} s_i s_j$$

$N \to N+1$



Hyp. $F_N \sim N f$

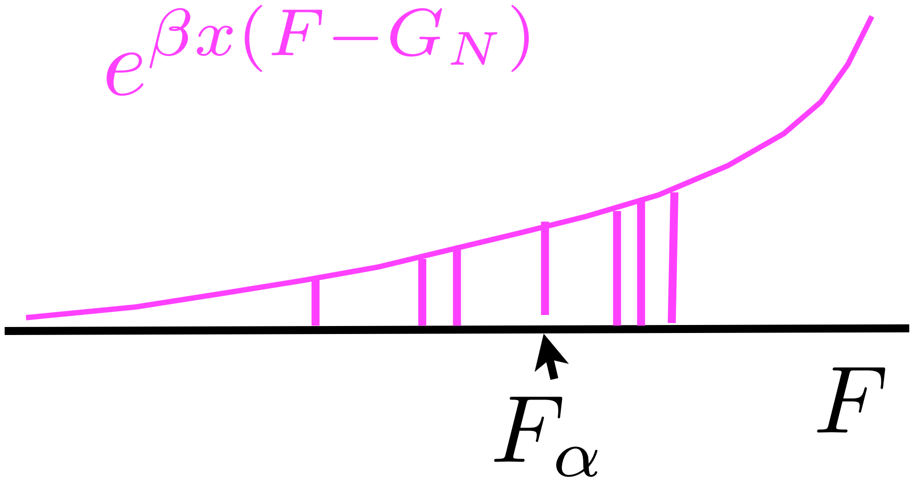
$$f = \overline{F_{N+1} - F_N}$$

New spin s_0 sees a local magnetic field $\sum_i J_{0i} s_i$

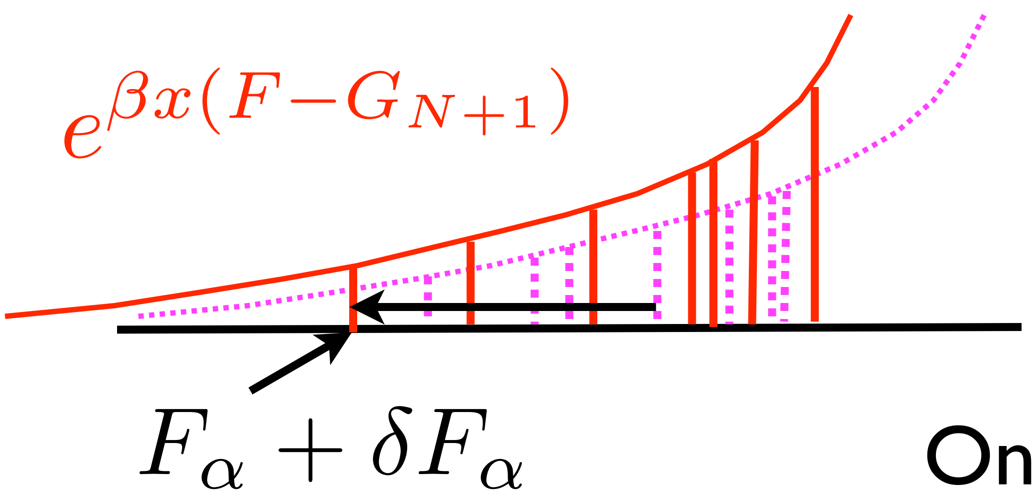
which has a Gaussian distribution ... within one pure state

SK model $N \to N+1$

Distribution of free energies $W_\alpha = C e^{-\beta F_\alpha}$



<u>Two main ingredients:</u>
<u>Ultrametricity and</u>
<u>exponential distribution of</u>
<u>low-lying free energies</u>



$$f = \overline{F_{N+1} - F_N}$$
$$= G_{N+1} - G_N$$

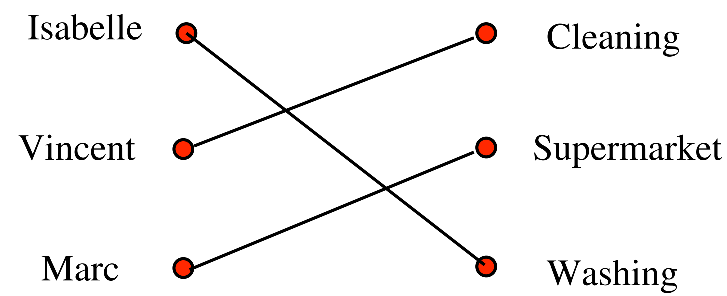
Only exp. distribution is stable

The cavity method beyond SK

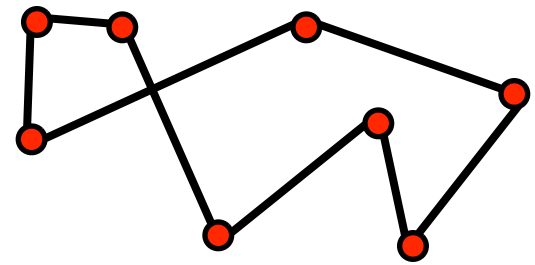
- Diluted systems (where the number of variables interacting with a given one is finite) at the RS level: 'simple' optimization problems (assignment) and Bethe lattice spin glasses (1986-1987)
- Diluted systems at the 1RSB level (one level of hierarchy in the ultrametric tree): RSB effects in Bethe lattice spin glasses, phase diagram and new algorithms in hard optimization problems (satisfiability, coloring) (2001-2008)

Optimisation problems

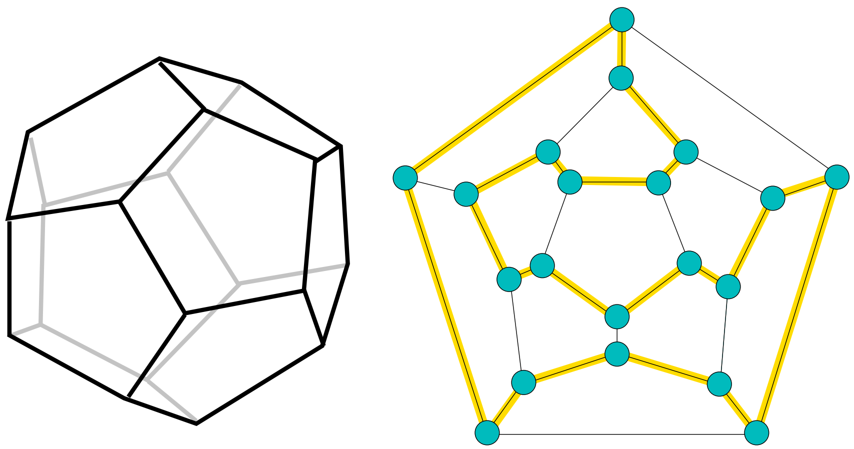
Assignment ("easy", in P)



Travelling salesman ("hard", NPC)



Hamiltonian path ("hard", NPC)



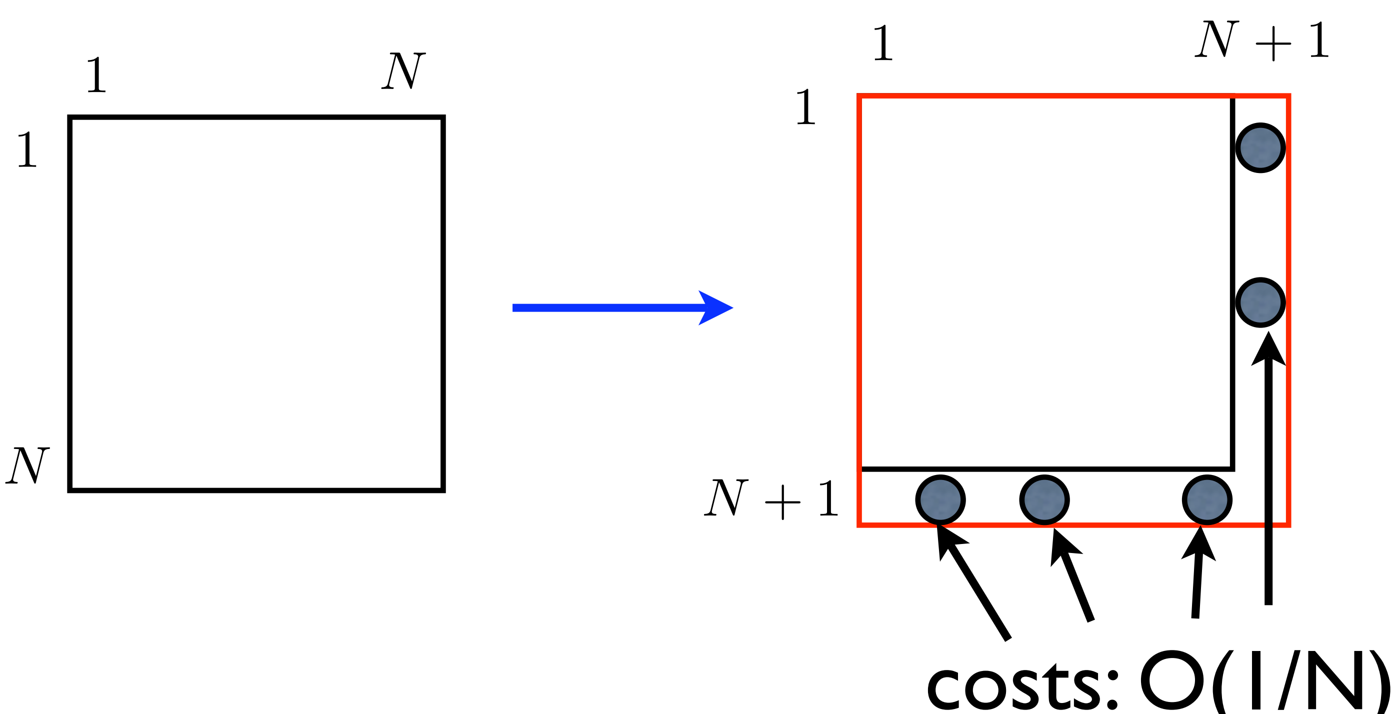
Simple (RS) optimization pbs

Random cost assignment and TSP (M, Parisi, 86)

Assign job a to person j: cost E_{aj} iid on $[0,1]$

Pb: find the lowest cost assignment (permutation)

Selected costs: O(1/N) ⟶ “Diluted system”



Finite number of possibilities for the newly connected job (or person)

The cavity method for simple (RS) optimization pbs

e.g.: Random cost assignment (M,Parisi 86)

Field theoretic representation with spin variables.
Local field: sum of a finite number of fluctuating terms, non-gaussian.
Cavity $\longrightarrow$ integral equations for local field

Distribution of rescaled edges d/N in the ground state

$$P(d) = \frac{d - e^{-d} \sinh d}{\sinh^2 d}$$

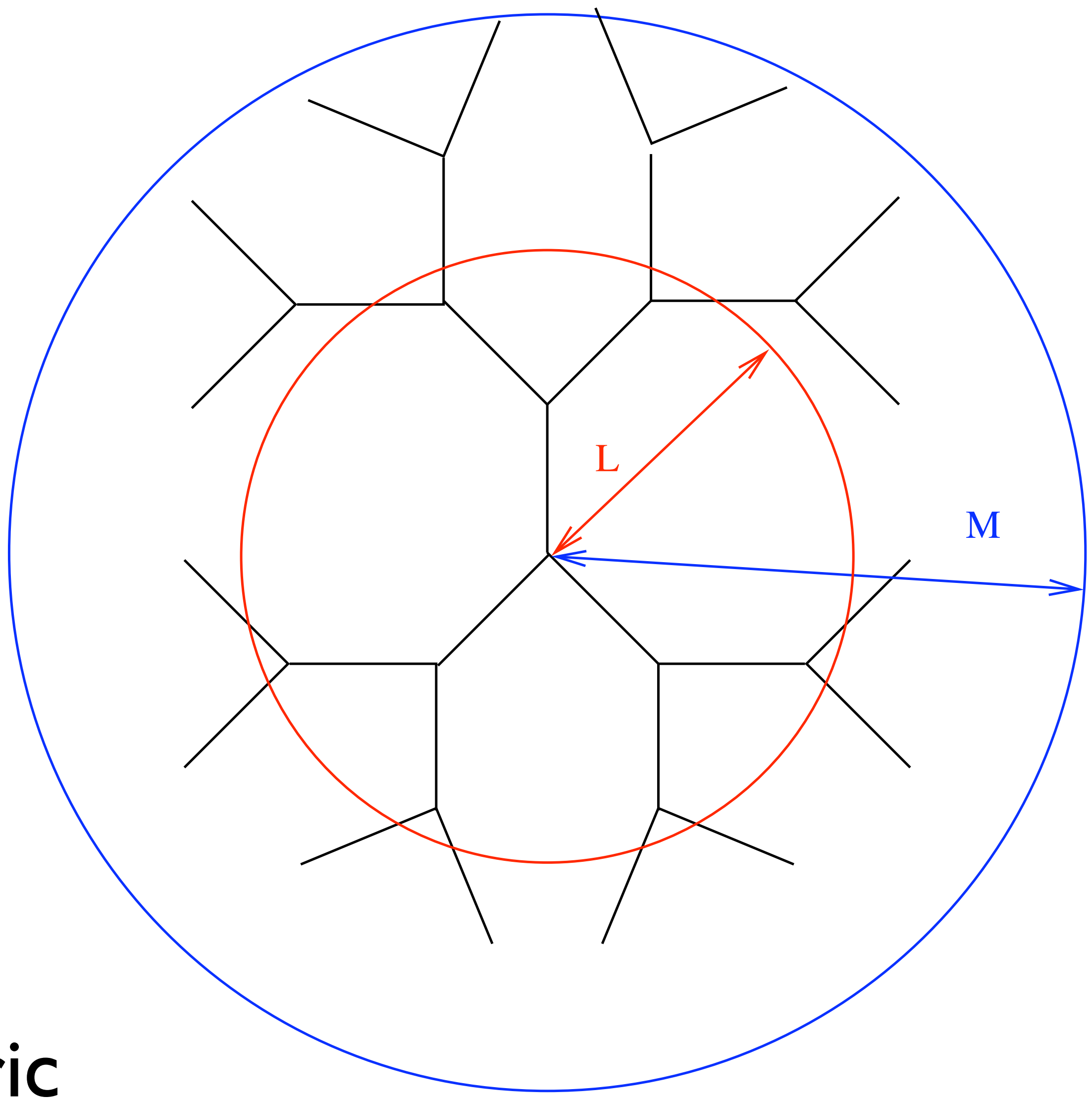
Optimal cost: $\zeta(2) = \pi^2/6$

(Rigorous proof with cavity method: Aldous 2001)

Finite connectivity spin glasses

Bethe lattice, usually:

$$\lim_{L\to\infty} \lim_{M\to\infty}$$



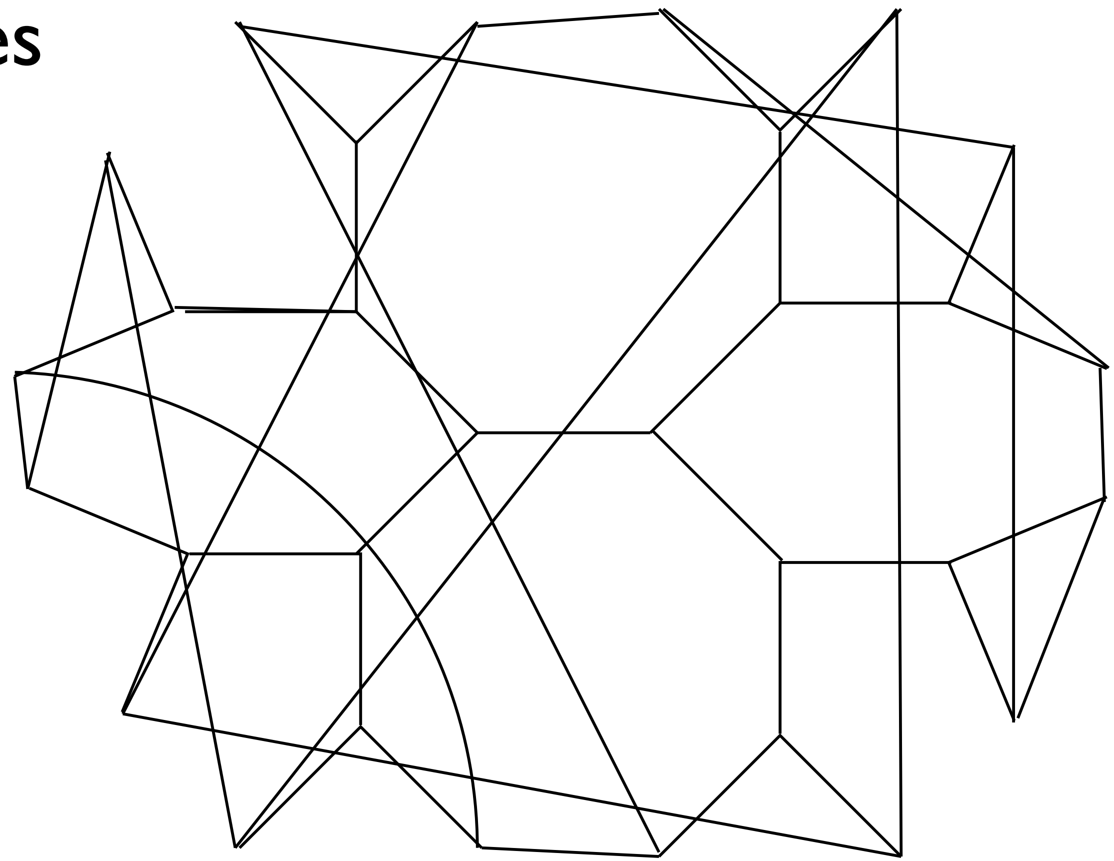
Spin glass: boundary conditions??

i.i.d: only replica symmetric

Finite connectivity spin glasses

Bethe lattice for spin glasses

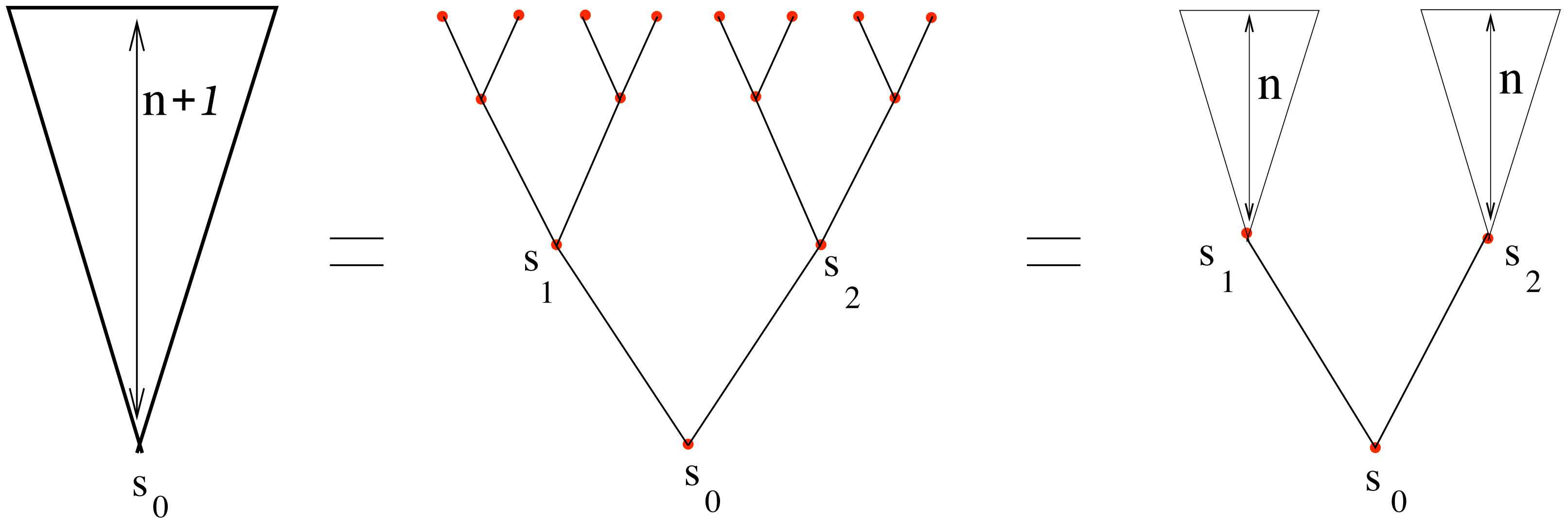
Random regular graph
(fixed degree $k+1$)



- Locally tree-like
- Loops of length $O(\log N)$

Finite connectivity spin glasses

Cavity recursion: local field on s_0 : h_0



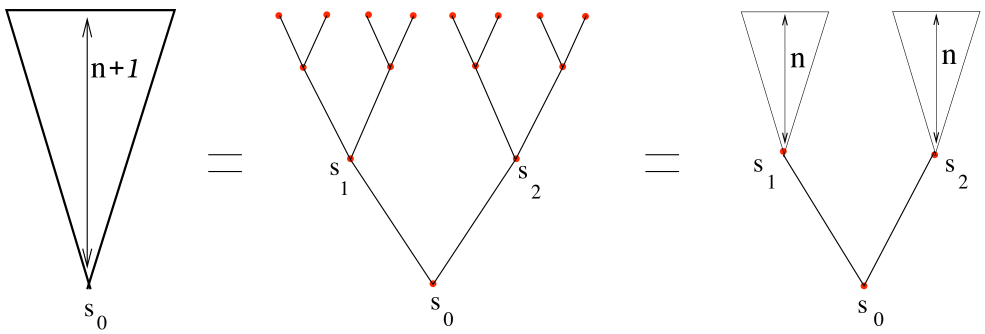
$$h_0 = f(h_1, h_2)$$

Large n : h_0, h_1, h_2 i.i.d. from $P(h)$ → Self consistent

Only true if one pure state

→ Replica symmetric approximation

Finite connectivity spin glasses



$$h_0 = f(h_1, h_2)$$

Connectivity $k+1$	3	4	5	∞
Ground state energy $E_0/\sqrt{k+1}$	$-.738$	$-.744$	$-.756$	$-.798$

= RS result for SK.

Correct RSB result: $-.763$

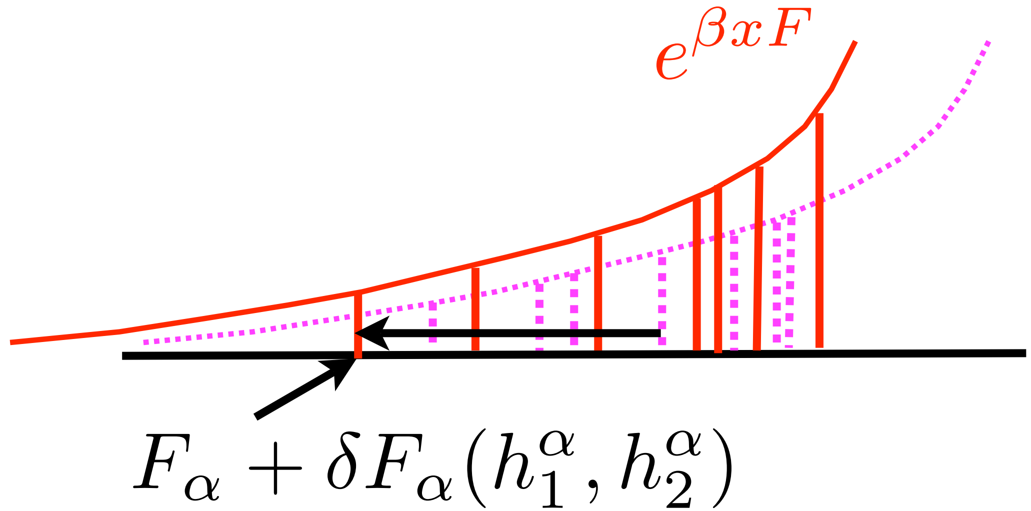
⟶ Replica symmetric approximation (M, Parisi 87):

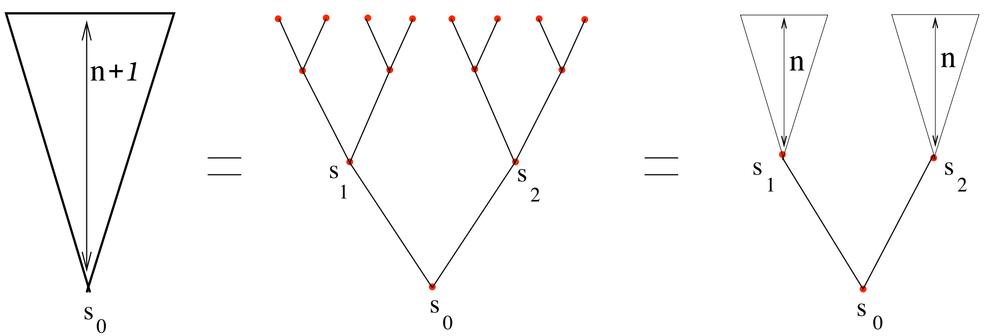
"*Maybe the most interesting perspectives are linked to the problem of replica symmetry breaking effects [] on the ground state energy of optimization problems*"

... 14 years later:

1RSB in finite connectivity spin glasses (M, Parisi, 2001):

- State crossing
- Non gaussian local fields





$$h_0^\alpha = f(h_1^\alpha, h_2^\alpha)$$

$$\longrightarrow \quad P(h, \delta F)$$

On a given site s_0 : Distribution of fields h for states at a given free energy F: integral equation with reweighting

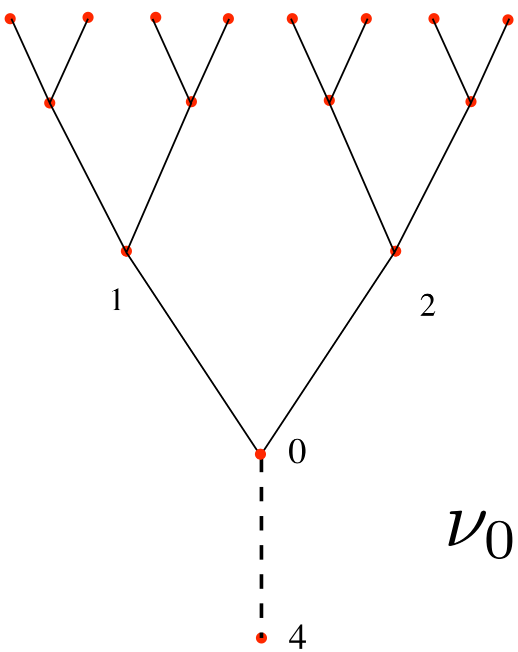
$$P_0(h) = \int dP_1(h_1) dP_2(h_2) e^{-\beta x \delta F(h_1,h_2)} \delta(h - f(h_1, h_2))$$

1RSB in finite connectivity spin glasses

Integral equation with reweighting

$$P_0(h) = \int dP_1(h_1)dP_2(h_2)e^{-\beta x \delta F(h_1,h_2)}\delta(h - f(h_1,h_2))$$

Instead of the RS recursion: $h_0 = f(h_1, h_2)$



GS energy (k=4): -0.749 instead of -0.756

NB: one probability distribution on each edge $P_0(h) = \nu_{0\to 4}(h)$

$$\nu_{0\to 4} = \mathcal{G}\left(\nu_{1\to 0}, \nu_{2\to 0}\right)$$

A broad class of problems

- Spin glasses
- Structural glasses (lattice models)
- Information theory decoding/compression
- Coloring
- Satisfiability of logical propositions
- ...

Constraint satisfaction problems: Many simple variables, related by local constraints... finite connectivity

$$P(x_1, ..., x_N) = C \prod_{a=1}^{M} \psi_a(X_a)$$

$$X_a = \{x_{i_1(a)}, ..., x_{i_K(a)}\}$$

A broad class of problems

$$P(x_1, ..., x_N) = C \prod_{a=1}^{M} \psi_a(X_a)$$

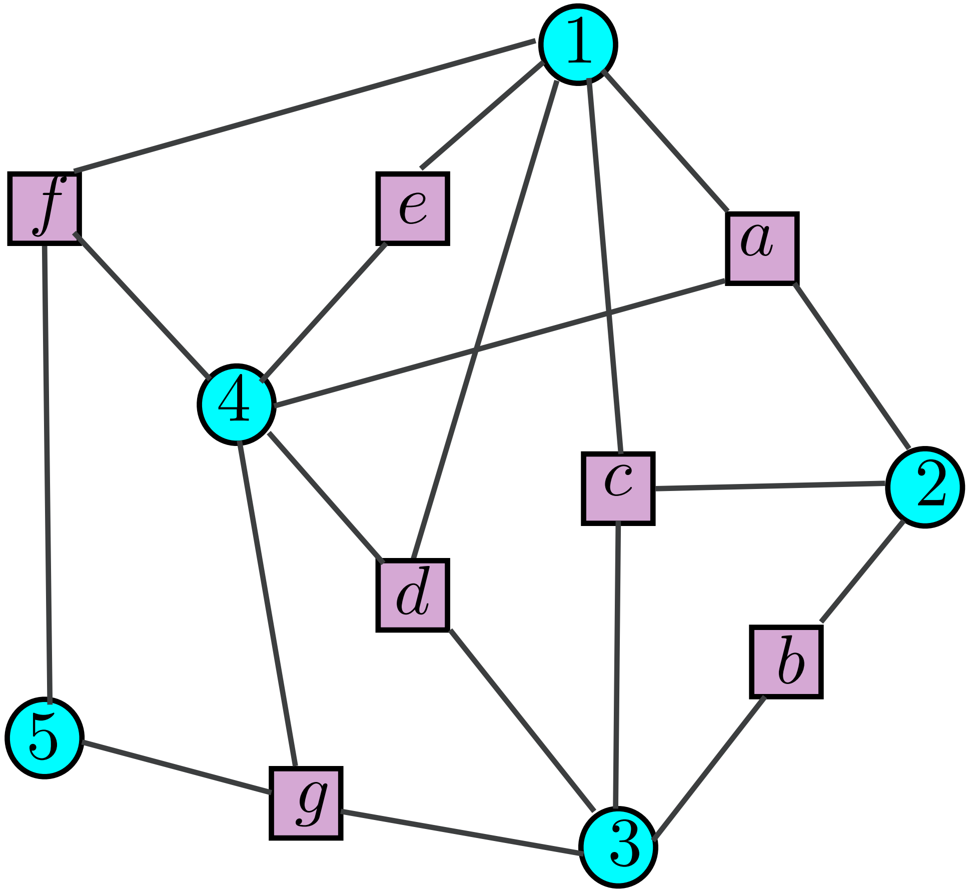
$$X_a = \{x_{i_1(a)}, ..., x_{i_K(a)}\}$$

- Compute marginals?
- Sample from P?
- Compute free-energy 1/C?

... efficiently: not in $O\left(e^{aN}\right)$ operations, but $O(N^c)$

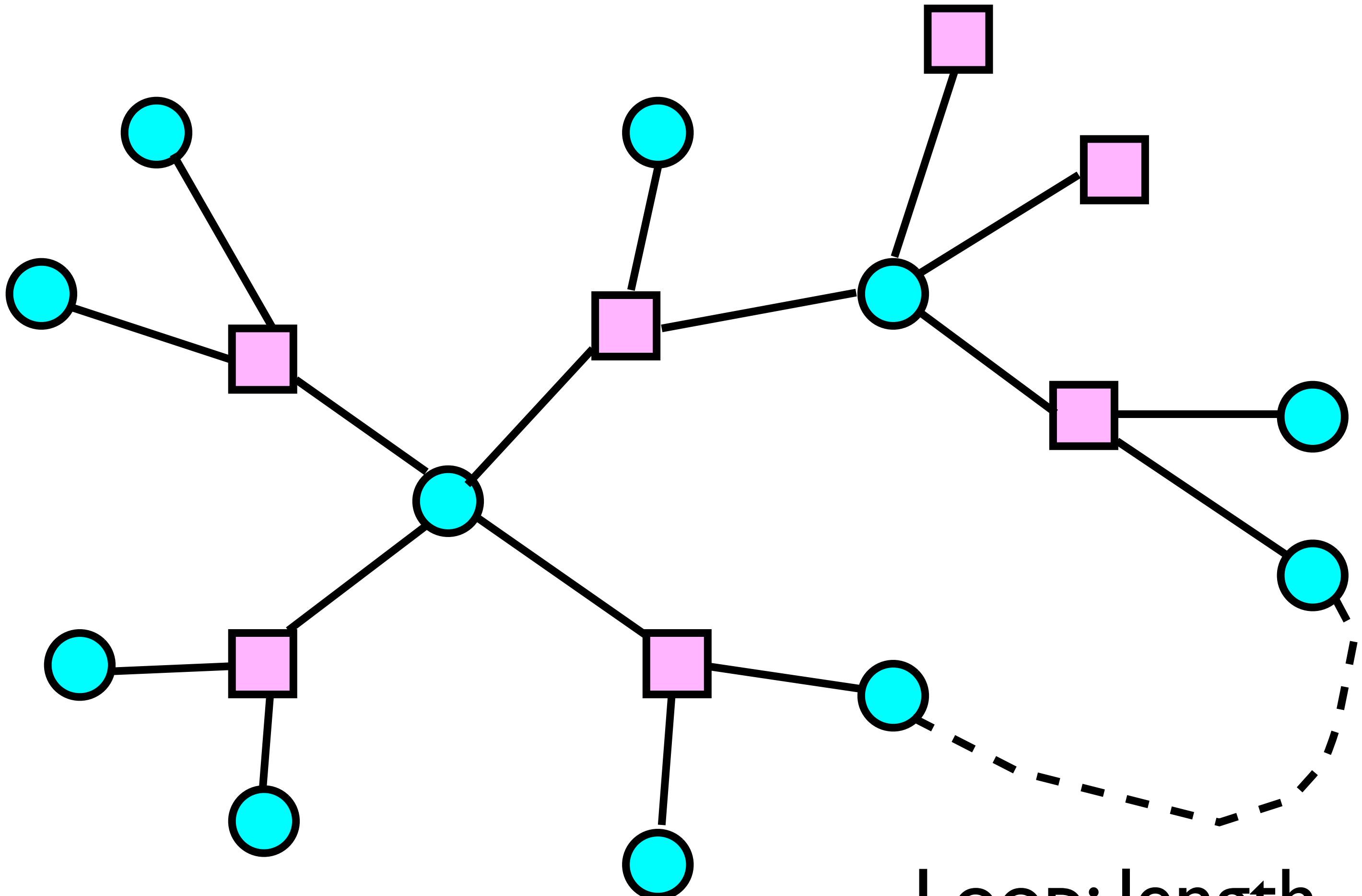
Cavity method

Factor graph representation



$$P(x_1, \ldots, x_5) = \psi_a(x_1, x_2, x_4)\psi_b(x_2, x_3)\psi_c(x_1, x_2, x_3) \ldots$$

Locally tree-like: OK for large random factor graphs



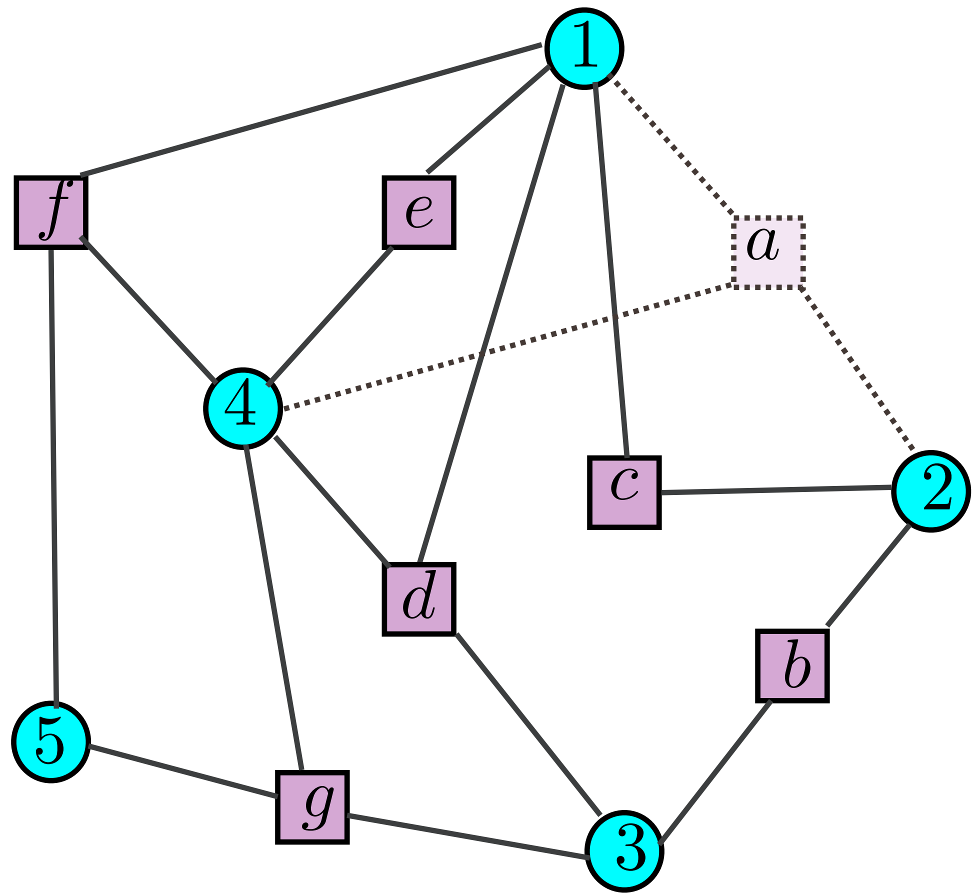
Loop: length $O(\log N)$

RS cavity method

Dig a cavity

Compute probability of x_1 in absence of a

$m_{1\to a}(x_1)$



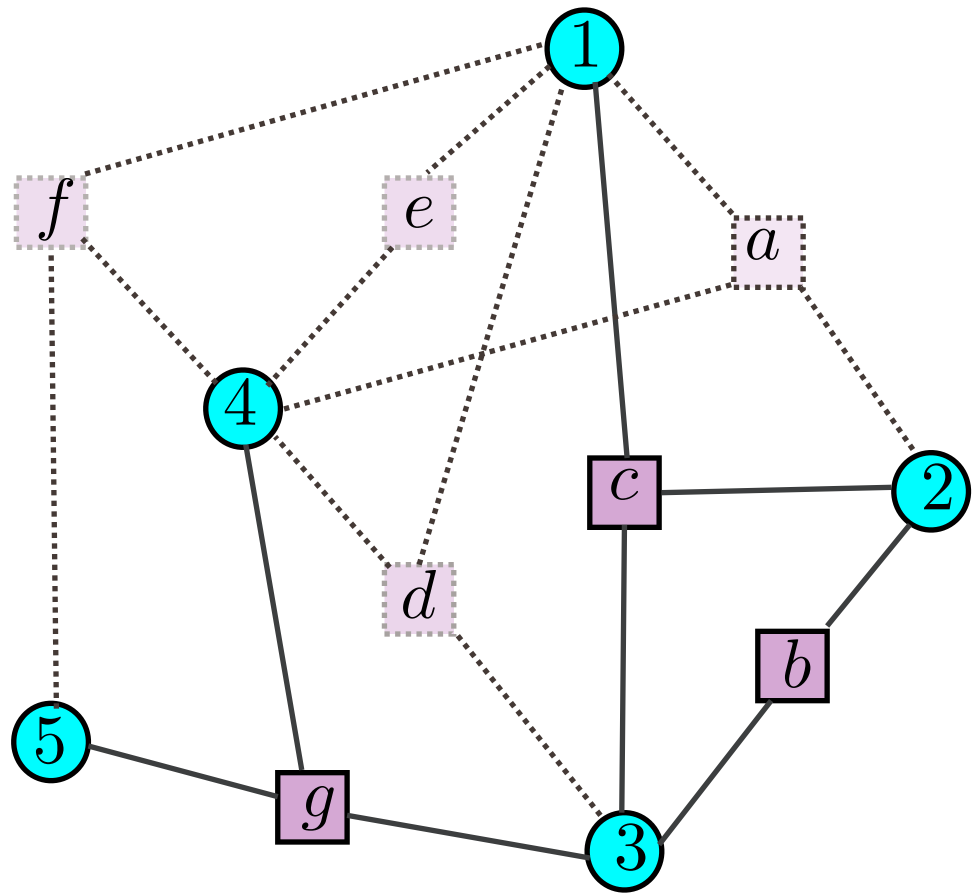
⟶ "Message= local field" $h_{1\to a}$

$$m_{1\to a}(x_1) = \frac{\exp\left(h_{1\to a} x_1\right)}{2\cosh\left(h_{1\to a}\right)}$$

RS cavity method

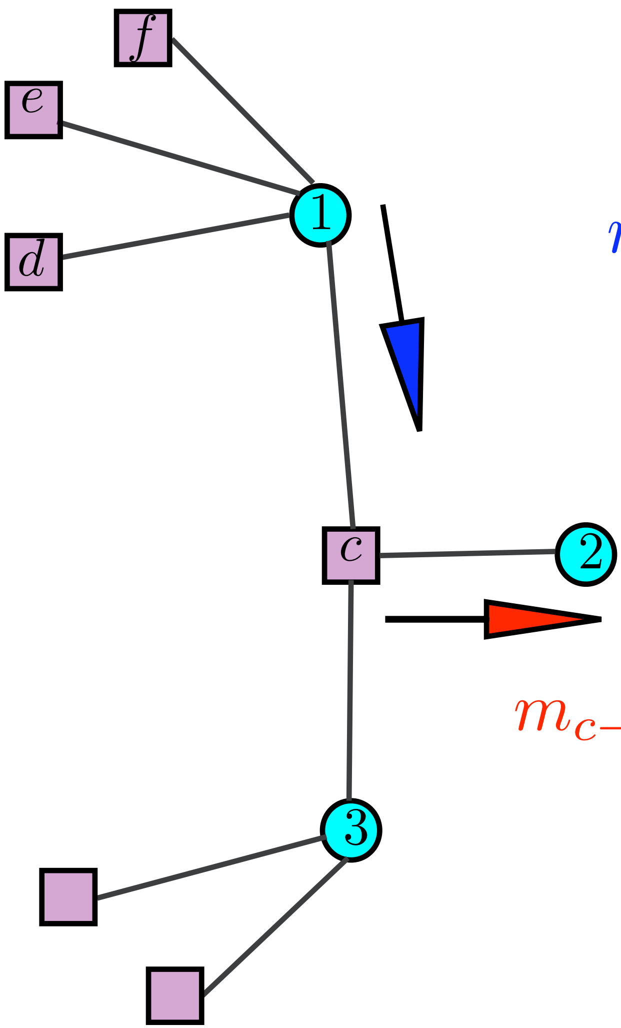
Dig a cavity

Compute probability of x_1 when it is connected only to c



⟶ "Message" $h_{c \to 1}$

Replica symmetric cavity equations



$$m_{1\to c}(x_1) = C m_{d\to 1}(x_1) m_{e\to 1}(x_1) m_{f\to 1}(x_1)$$

$$m_{c\to 2}(x_2) = \sum_{x_1,x_3} \psi_c(x_1,x_2,x_3) m_{1\to c}(x_1) m_{3\to c}(x_3)$$

Closed set of equations: two messages propagate on each edge of the factor graph

Replica symmetric cavity equations
= Belief Propagation
= Bethe Peierls with disorder

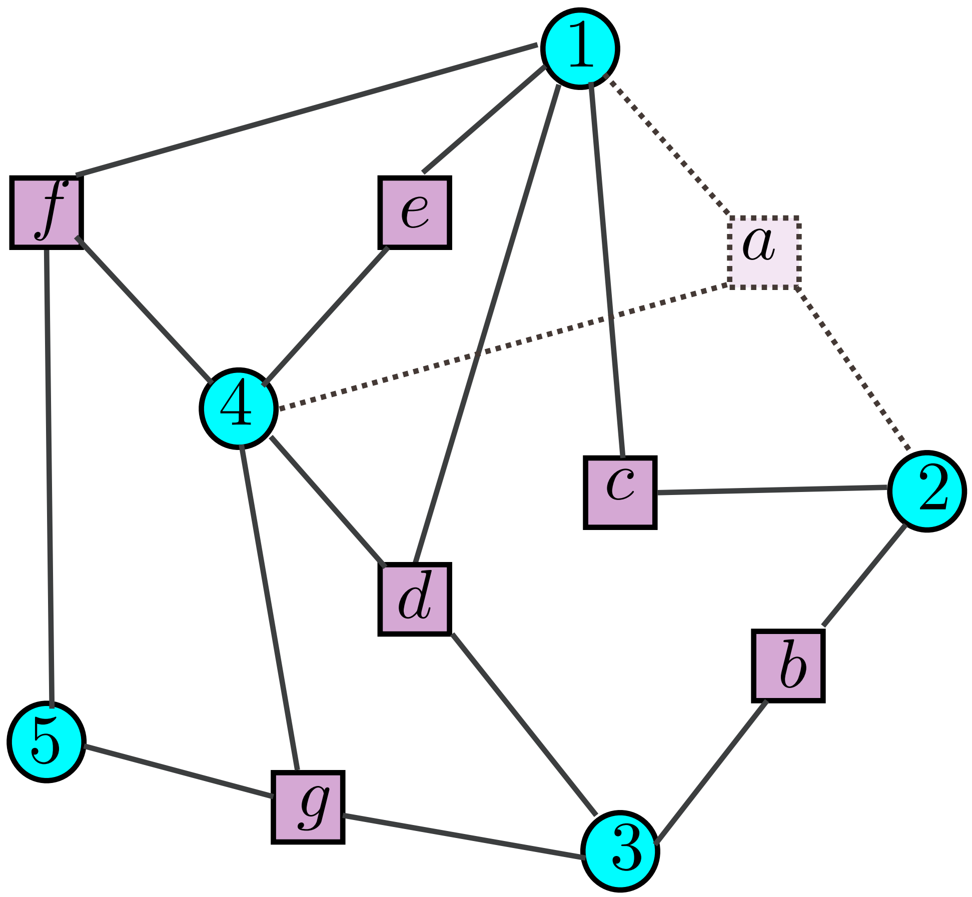
Successful in some problems (fast decoding of LDPC codes)... Only RS phases

Modification in presence of glassy phase:
1RSB cavity:

- Statistical analysis: 1RSB phase diagram (M, Parisi)
- Algorithm on a single instance (M, Zecchina)

1RSB cavity method

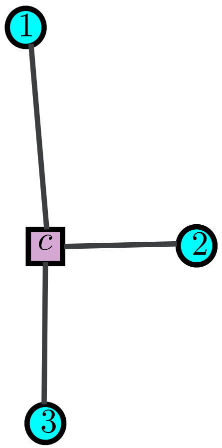
Message= survey of the local fields in the various states



$P_{1\to a}(h)$: Probability that $h^{\alpha}_{1\to a}$ is equal to h , when α is picked up at random

$$\longrightarrow \qquad P_{1\to a} = \mathcal{G}\left(P_{c\to 1}, P_{d\to 1}, P_{e\to 1}, P_{f\to 1}\right)$$

Application: satisfiability



Constraint = clause like $x_1 \vee \overline{x}_2 \vee x_3$

The grand father of hard problems (Cook 71)

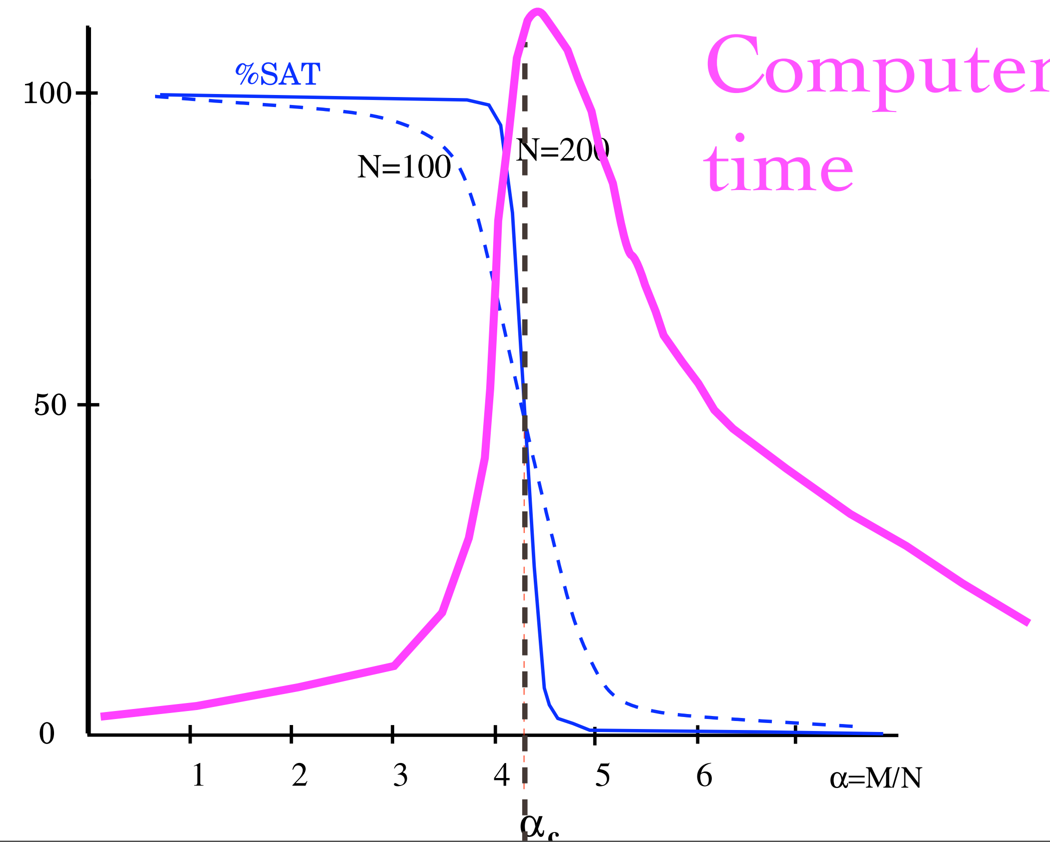
N variables, M clauses,
density of constraints $\alpha = M/N$

Large N limit:

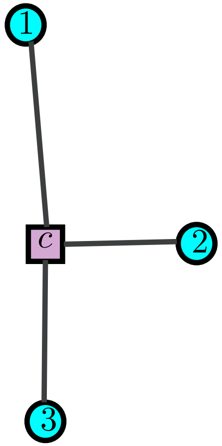
SAT for $\alpha < \alpha_c$

UNSAT for $\alpha > \alpha_c$

"Phase transition"



Application: satisfiability



Constraint = clause like $x_1 \vee \overline{x}_2 \vee x_3$

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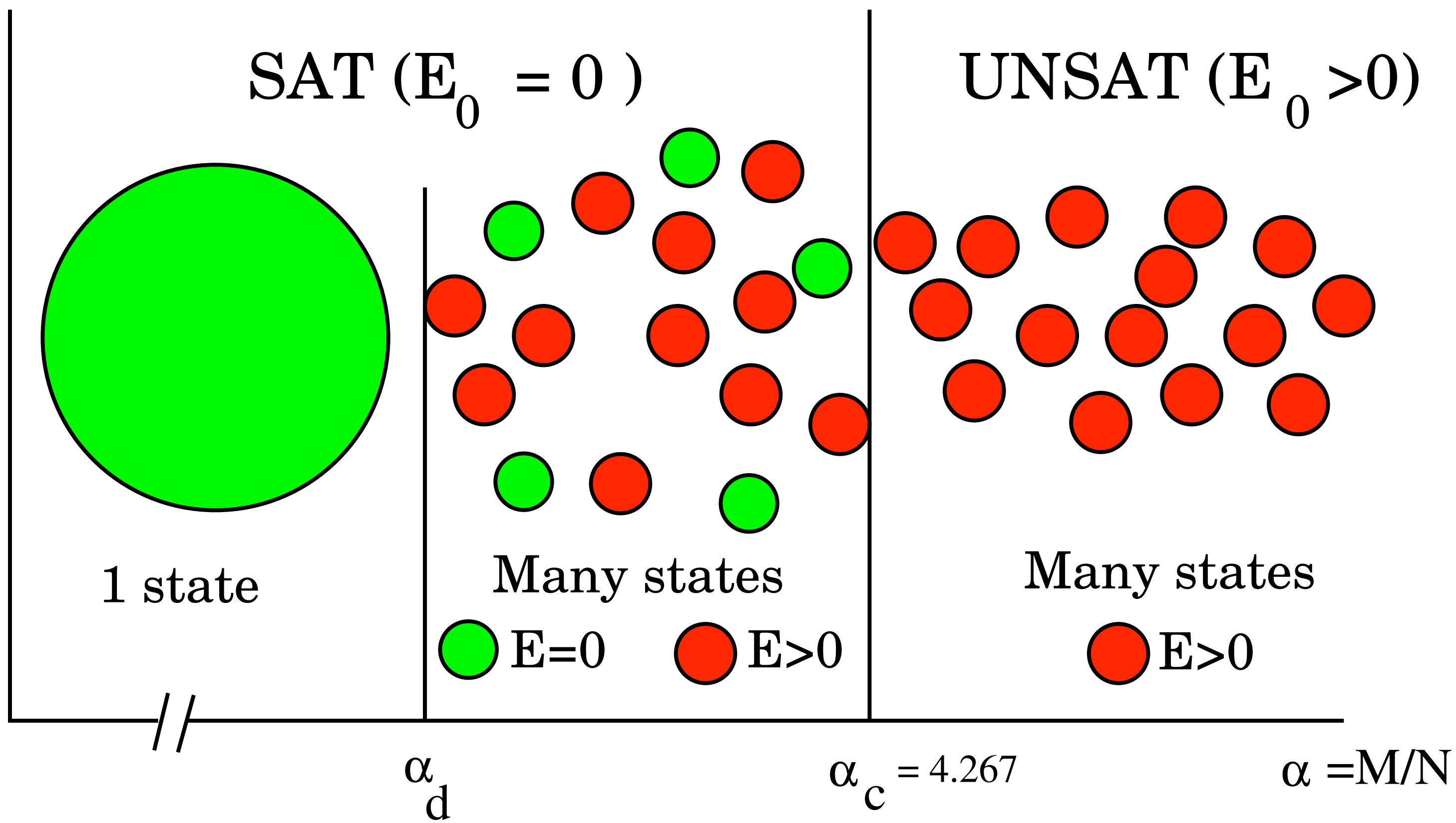
Phase diagram:



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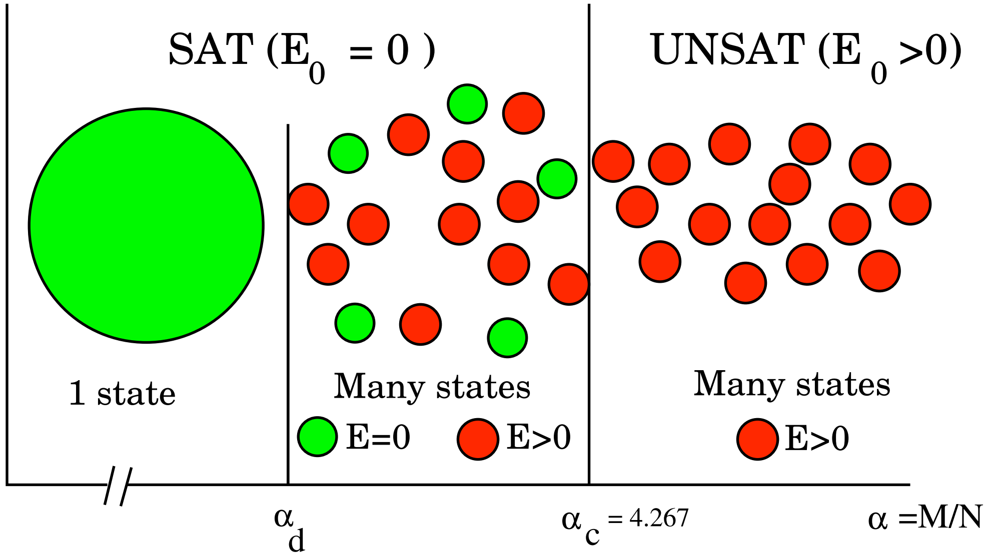


Phase diagram= properties of almost all samples

On a single instance/sample: use the 1RSB cavity equations $P_{1\to a} = \mathcal{G}\left(P_{c\to 1}, P_{d\to 1}, P_{e\to 1}, P_{f\to 1}\right)$ as a message passing algorithm: survey propagation (M,Zecchina) solves systems of 10^7 variables at $\alpha = 4.25$

The cavity method as a powerful message passing algorithm

- Local exchange of messages along a factor graph
- Simple computations at each node
- Solves very complicated global constraint satisfaction /optimization problems (in spite of "anarchy" !) in a distributed way

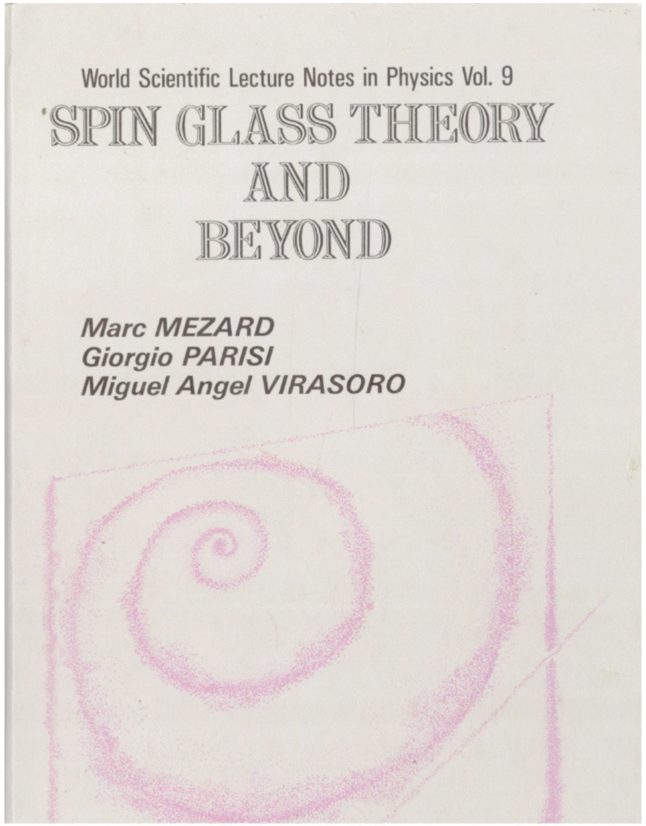
What are the problems it does not solve?

- Local structures... Need more work
- UNSAT phase (does solve, but no signature)
- "Point-like" clusters from locked constraint satisfaction problems

Spin glasses: Totally useless (few grams) of boring material...

Intellectual interest. Tens of thousands of papers over the last 30 years. Some of the most fascinating developments in statistical physics

A beautiful conceptual scheme, for many topics ranging from distributed computing (neural networks) to portfolio optimization in finance, to basic computer science problems like coloring, satisfiability,... Unexpected offsprings and applications



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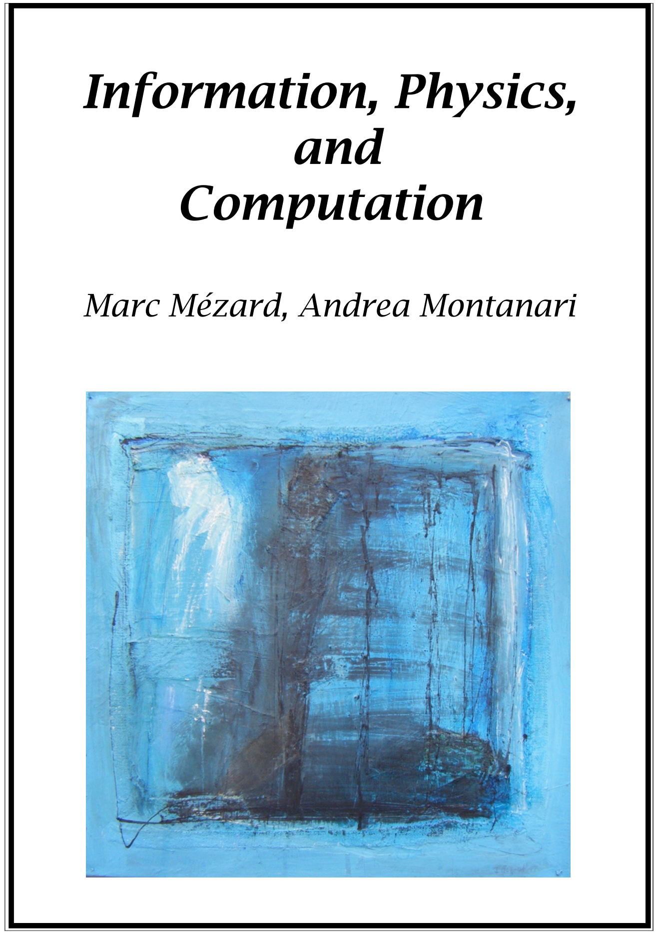

With Giorgio:

- 34 papers, on spin glasses, structural glasses, polymers, manifolds, interfaces, vortices, random matrices, Burgers turbulence, optimization, computer science...
- one book,
- a lot of disorder
- many cavities
- ~~many~~ replicas...

zero

...and an infinite amount of thanks!
Let’s keep wandering with curiosity in complex landscapes...

Many thanks to

Enzo

and all the local organizers, for this wonderful conference

And all the staff for the perfect organization:

- Manuela Marchetti
- Adriana Vescera
- Angelo Campus

And the whole team working with them