The full replica symmetry breaking in the Ising spin glass on random regular graph

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Abstract

In this paper, we extend the full replica symmetry breaking scheme to the Ising spin glass on a random regular graph. We propose a new martingale approach, that overcomes the limits of the Parisi-Mézard cavity method, providing a well-defined formulation of the full replica symmetry breaking problem in random regular graphs. Finally, we define the order parameters of the system and get a set of self-consistency equations for the order parameters and the free energy. We face up the problem only from a technical point of view: the physical meaning of this approach and the quantitative evaluation of the solution of the self-consistency equations will be discussed in next works.

1 Introduction

The mean field theory of spin glasses has attracted a considerable interest in the last forty years, as a promising theory to describe the statistical mechanics of glassiness and disorder systems.

The Sherrington-Kirkpatrick [1] model has represented a fruitful source of insights in this field. The complete solution was derived by Parisi via the replica method [2], with the introduction of a clever Replica-Symmetry-Breaking ansatz (RSB) [3]. The Parisi RSB ansatz was further clarified by Mézard, Parisi and Virasoro [4] as related to the decomposition of the Gibbs state in a mixture of a large number of pure equilibrium states, recognizing the Parisi RSB order parameters [5], i.e. the overlaps between replicas, as the distribution of the mutual overlap between two equilibrium states on the phase space.

The replica method was successfully used in many problems on fully-connected networks [4, 6]. Various models show different patterns of RSB, depending on the way the states are "distant" to each other.

- The overlaps between different states can take (almost surely) only two different values. In this case, we speak about "one-step replica symmetry breaking"(1-RSB) solution. The states are scattered randomly in the phase space and correspond to stable well-defined minima (genuine minima) of the free energy landscape [7, 8].

- The overlaps can take a discrete number $r + 1 \in \mathbb{N}$ of values in the interval $[q_m : q_M]$. In this case, we speak about "r-step replica symmetry breaking"(r-RSB) solution. The equilibrium states exhibit a hierarchal structure, where clusters of states with a given mutual overlap are grouped in a progressively wider level of clusters with a progressively lower overlap, for r levels [4, 9]. Each state enters in the Gibbs decomposition with a random weight, which is generated according to the Derrida's REM and GREM calculations [10, 11]. Such solutions can be considered as an iterative composition of 1-RSB solutions. As far I know this situation is very rare.

- The overlaps among states can take all possible values in the interval $[q_m : q_M]$. In this case, we speak about "full replica symmetry breaking"($full$-RSB) and it can be considered as a $r \to \infty$ of the

preceding case. The equilibrium states exhibit a continuous fractal clustering and the random weights are configurations of a Ruelle random probability cascade [12–14], that provides a continuous extension of GREM. It is worth stressing that, unlike the preceding case, the equilibrium states can be arbitrarily close and the barriers between the states may be very small. The minima are marginal, with many flat directions (infinite in the thermodynamic limit) [15].

Most of the results of these results have been reproduced using a probabilistic iterative approach, the cavity method, which avoids the mathematical weirdnesses of the replica method [4, 13]. The replica jargon, however, is used in spin glass theory, regardless the approach considered: actually, we speak about RSB if the system exhibits many pure states, organized according to one of the schemes described before.

The rigorous proof of the Parisi os the SK model solution was derived by Talagrand [16], with the Guerra's interpolation scheme, that allows a rigorous handling of the replica symmetry breaking [17]. The ultrametricity of the states was proved soon after by Panchenko [18], in relation to the notion of stochastic stability [19].

If and how the RSB scheme applies also in non-fully-connected systems is still debated, in spite of recent results.

We concentrate our attention on spin glasses defined on a random regular graph (RRG), with finite connectivity [20].

The interest in sparse graph models is motivated by the fact that they represent a more realistic class of mean field models, including the notion of neighborhood which is absent in the infinite range case. They attract a big interest also in computer science, since many random optimization problems turn out to have a finite connectivity structure.

The 1-RSB scheme was successfully extended to the Ising spin glass on sparse graphs by Parisi and Mézard (PM) with the cavity method [21], improving the Bethe–Peierls method in order to deal with many equilibrium states. The approach can be easily generalized to the case with r steps of replica symmetry breaking, imposing the GREM scheme for the states distribution and the ultrametricity as in the fully-connected case [24].

It was proved, via interpolation arguments, that this approach should provide a rigorous lower bound for the free energy [25].

The 1-RSB PM cavity method has been very successful even now, since, in certain regimes, this approach provides an algorithmic solution of random satisfiability problems with finite connectivity [22, 23].

Unfortunately, in the PM cavity method approach, the r-RSB order parameter is a distribution of $(r-1)$-RSB order parameters. Since the replica symmetric solution already involves an order parameter which is the local fields distribution, going to a 1- RSB solution the replica order parameter becomes now the probability distribution over the space of local field distributions and the 2-RSB free energy order parameter is a distribution of distributions of distributions [21]. As a consequence, the cavity method turns to be inadequate to achieve a $full$-RSB theory for RRG spin glasses, indeed the $r \to \infty$ limit of the order parameter has no mathematical meaning. Also, the high levels of replica symmetry breaking are actually numerically intractable.

Actually, the 1-RSB cavity method is commonly used also to describes models where the equilibrium states are not independent and well-defined minima (1-RSB approximation). This approximation, however, cannot grab the marginality of the states and then completely misses the right evaluation of such quantities that have very different properties in the marginally stable phase, as the spectrum of small oscillations, nonlinear susceptibilities and so on [27]. This limitedness entails the impossibility to describe, with the cavity method, a glassy phase with many marginal equilibrium states.

In this paper, we propose an alternative approach, that overcomes the difficulties presented by the cavity method and allows to get a mathematical formulation of the $full$-RSB free energy for Ising spin glasses on RRG.

We manage the progressive branching of the clusters of states with a martingale approach [26], improving the idea suggested in [27]. At the end, we reduce the computation to a series of variational problems, where variational parameters are martingales. The order parameter, then, is not a deterministic distribution, as in the cavity method, but it is a stochastic process.

The paper is organized as follows.

In the next section, we present the model. In the third section, we present the cavity variational free energy with the r-RSB ansatz that extends to RRG the hierarchical ROSt formalism proposed by Azienman, Sims and Starr for the SK model [13]. Using the properties of GREM and standard technics of statistical mechanics, we rewrite the r-RSB as the rth iteration of a discrete time recursive low, starting from the RS free-energy and going down along the various levels of clustering of the equilibrium states. The previous recursion low is then reconsidered using a discrete time martingale approach.

In the last section, the formalism can be extended to continuous time, obtaining an auxiliary variational representation à la Bouè and Dupuis [28] for the $full$-RSB variational free energy functional. Suitable $full$-RSB order parameters are defined and the variational free energy functional is finally derived as a generalization of the Chen and Auffinger representation of the Parisi free energy for SK model [29]. The self-consistency equations for the order parameters are then derived from the first variation of the free energy functional.

Whilst a rigorous proof is still lacking, it is generally believed that the $full$-RSB ansatz is correct for Ising spin glasses on sparse graphs. If it is the case, the equations presented in this work should provide the right solution for this model. In any case, the mathematical formulation of the $full$-RSB theory provides a basic groundwork for complete mathematical proof about RSB on non-fully-connected models, extending the recent results on discrete RSB [24].

The aim of the paper is to provide a clear formal definition and a non-ambiguous mathematical setting for the $full$-RSB problem for spin glass models in random regular graphs. We will tackle this problem only from a technical point of view. The physical interpretation of our approach will be discussed further in next works.

2 The model

We consider a system of N Isisng spins $\boldsymbol{\sigma} := (\sigma_1, \sigma_2, \cdots, \sigma_N) \in \{-1, 1\}^N$ with the Hamiltonian:

$$H[J, \mathcal{G}_c, \boldsymbol{\sigma}] = \sum_{\langle ij \rangle_{\mathcal{G}_c}} J_{i,j}\sigma_i\sigma_j \tag{1}$$

The sum runs over the edges of a random regular graph $\mathcal{G}_c$, with connectivity c (RRG_c). The couplings $\{J_{i,j}\}$ are independent $\{-1, 1\}-$valued random variables. Our aim is to compute the free energy density, defined as:

$$F_{J,\mathcal{G}_c} = \lim_{N\to\infty} -\frac{\log Z_{N,J,\mathcal{G}_c}}{\beta N} = \lim_{N\to\infty} -\frac{1}{\beta N} \log \sum_{\boldsymbol{\sigma}\in\{-1,1\}^N} e^{-\beta H[J,\mathcal{G}_c,\boldsymbol{\sigma}]}\,, \tag{2}$$

where $Z_{N,J,\mathcal{G}_c}$ is the partition function on an N spins system. The couplings and the graph constitute the quenched disorder of the system.

It can be shown that, in the thermodynamic limit, the free energy does not depend on the realization of the quenched disorder with probability one. As a consequence, we will concentrate on the computation of the average free energy:

$$F = \lim_{N\to\infty} -\frac{1}{\beta N}\overline{\log\, Z_{N,J,\mathcal{G}_c}} \tag{3}$$

where $\overline{\cdot}$ denotes the average over the random couplings and the realizations of the random graph.

Because of the randomness of the edges, for large N and $c > 2$, the typical size of a loop is of order $\sim \log N$, so the probability to have finite loops vanishes in the $N \to \infty$ limit [20]. As consequence, random regular graphs are locally isomorphic to a tree: such models are exactly solvable in mean field theory. Large loops, however, can induce frustration, so the system exhibits a glassy behavior at low temperature.

At high temperature, the system exhibits a single pure state (the paramagnetic one), and the free energy can be exactly solved by the Bethe-Peierls (BP) approximation. However the BP approach turns to be wrong at low temperature, where the Gibbs state is no more a clustering state, i.e. the correlation function does not vanish exponentially with the distance, invalidating the BP assumptions [30]. A more sophisticated cavity approach must be considered, dealing with the presence of many clustering states. The BP approximation

still holds within each state separately, whilst the average over the states generate non-vanishing correlation functions for the Gibbs state [21, 27].

In the $c \to \infty$ limit, keeping $c \langle J^2 \rangle = 1$, the free energy becomes independent of the probability distribution of the J and we obtain the same free energy of the SK model.

3 The finite replica symmetry breaking problem

In this section, we describe the discrete RSB scheme for this model.

In the first subsection, we define the r-RSB cavity free energy functional for sparse graphs. We recall the notion of discrete Ruelle random probability cascade, or GREM [10, 12–14], and generalize the Parisi RSB ansatz for diluted models [32–34].

In the second subsection, we recast the progressive steps of replica symmetry breaking in a discrete time recursive map, that generalizes the Parisi replica computation for the SK models [3, 4].

In the last subsection, we derive a new variational representation of the r-RSB free energy, using a progressive iteration of the Gibbs variational principle.

3.1 The finite RSB free energy functional

Let us assume that the system has many equilibrium states, that are labeled by an index $\boldsymbol{\alpha}$. The cavity spins are uncorrelated within a given state $\boldsymbol{\alpha}$, leading to a factorized cavity spins distribution, that depends on the label $\boldsymbol{\alpha}$. Since each spin σ_i, with $1 \leq i \leq N$, can take only two values, the cavity probability distribution, for a given state $\boldsymbol{\alpha}$, depends only on the cavity magnetization $m_{i|\boldsymbol{\alpha}}$ or, equivalently, on the cavity field $h_{i|\boldsymbol{\alpha}} = 1/\beta \ \mathrm{atanh}\, m_{i|\boldsymbol{\alpha}}$. The cavity fields depend on the random couplings, so they are also random quantities and their distribution is not known a priori. The equilibrium free energy is finally given by the Gibbs state, that is a statistical mixture of the states $\boldsymbol{\alpha}$.

The cavity free energy functional is given by

$$\Phi = \overline{\mathbb{E} \log \left(\frac{\sum_{\boldsymbol{\alpha}} \xi_{\boldsymbol{\alpha}} \Delta^{(v)}(\vec{J}, \vec{h}_{\boldsymbol{\alpha}})}{\sum_{\boldsymbol{\alpha}} \xi_{\boldsymbol{\alpha}}} \right)} - \frac{c}{2} \overline{\mathbb{E} \log \left(\frac{\sum_{\boldsymbol{\alpha}} \xi_{\boldsymbol{\alpha}} \Delta^{(e)}(J_{1,2}, h_{1|\boldsymbol{\alpha}}, h_{2|\boldsymbol{\alpha}})}{\sum_{\boldsymbol{\alpha}} \xi_{\boldsymbol{\alpha}}} \right)} . \tag{4}$$

Here $\vec{J} = (J_{0,1}, J_{0,2}, \cdots, J_{0,c})$ and $\vec{h}_{\boldsymbol{\alpha}} = (h_{1|\boldsymbol{\alpha}}, h_{2|\boldsymbol{\alpha}}, \cdots, h_{c|\boldsymbol{\alpha}})$ and the variables $\{\xi_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha}}$ are the (non-normalized) statistical weights of the states. All the $c+2$ couplings in the functional (4) are independent.

The functions $\Delta^{(v)}$ and $\Delta^{(e)}$ are defined as:

$$\Delta^{(v)}(\vec{J}, \vec{h}_{\boldsymbol{\alpha}}) = \cosh\big(\beta U_c(\vec{J}, \vec{h}_{\boldsymbol{\alpha}})\big) \prod_{i=1}^{c} \frac{\cosh(\beta J_{0,i})}{\cosh\big(\beta u(J_{0,i}, h_{i|\boldsymbol{\alpha}})\big)} , \tag{5}$$

$$\Delta^{(e)}(J_{1,2}, h_{1|\boldsymbol{\alpha}}, h_{2|\boldsymbol{\alpha}}) = \cosh(\beta J_{1,2}) \big(1 + \tanh(\beta J_{1,2}) \tanh(\beta h_{1|\boldsymbol{\alpha}}) \tanh(\beta h_{2|\boldsymbol{\alpha}})\big) , \tag{6}$$

with

$$u(J_{1,2}, h_{2|\boldsymbol{\alpha}}) = \frac{1}{\beta} \mathrm{atanh}\big(\tanh(\beta J_{1,2}) \tanh(\beta h_{2|\boldsymbol{\alpha}})\big) , \tag{7}$$

$$U_c(\vec{J}, \vec{h}_{\boldsymbol{\alpha}}) = \sum_{i=1}^{c} u(J_{0,i}, h_{i|\boldsymbol{\alpha}}) . \tag{8}$$

The overline $\overline{\cdot}$ stands for the average over the quenched disorder and the expectation value $\mathbb{E}$ is over all the cavity fields and the random weights $\{\xi_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha}}$.

The contribution to the free energy (4) that depends on the function $\Delta^{(v)}$ is usually called *vertex contribution*, whilst the contribution depending on $\Delta^{(e)}$ is the *edge contribution*.

The equilibrium free energy is, formally given by:

$$-\beta F = \sup_{\mathcal{P}(\{\xi_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha}},\{h_{i|\boldsymbol{\alpha}}\}_{i,\boldsymbol{\alpha}})} \Phi\,, \tag{9}$$

where the supremum must be take over the set of all the possible probability distributions of the cavity fields $\{h_{i|\boldsymbol{\alpha}}\}_{i,\boldsymbol{\alpha}}$ and the random weight of the state $\{\xi_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha}}$. This set is huge and too general, then further assumptions are needed to face up the problem.

In the RSB assumption, the sum $\sum_{\boldsymbol{\alpha}} \cdot$ runs over the leaves of an infinitary rooted taxonomic tree and $\boldsymbol{\xi} := \{\xi_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha}}$ is a collection of positive random variables generated by a Ruelle random probability cascade defined along the tree; for each vertex i, the set $\{h_{i|\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha}}$ is a random hierarchical population of fields generated along the same tree. Such hierarchical populations are independent for different i.

More specifically, the r-RSB ansatz, for a finite integer r, is defined as follow.

Let X be a non-decreasing sequence of $r+2$ numbers, for some $r \in \mathbb{N}$:

$$0 = x_0 \leq x_1 \leq \cdots \leq x_r \leq x_{r+1} = 1\,. \tag{10}$$

We first define a Poisson point process $\boldsymbol{\xi}^{(1)} := \{\xi^{(1)}_{\alpha_1};\, \alpha_1 \in \mathbb{N}\}$ on $[0,\infty)$, with density given by $\rho(d\xi) = x_1 \xi^{-x_1-1} d\xi$; such a process is usually referred to as REM_{x_1}.

Next, for each α_1, a REM_{x_2} process $\boldsymbol{\xi}^{(2)}_{\alpha_1} := \{\xi^{(2)}_{(\alpha_1,\alpha_2)};\, \alpha_2 \in \mathbb{N}\}$ is generated, independently for different values of α_1. We then iterate the procedure: at the $n-$th step, up to $n = r$, independent realizations of the REM_{x_n} process $\boldsymbol{\xi}^{(n)}_{(\alpha_1,\cdots,\alpha_{n-1})} := \{\xi^{(n)}_{(\alpha_1,\cdots,\alpha_{n-1},\alpha_n)};\, \alpha_n \in \mathbb{N}\}$ are generated for each of the distinct values of the multi-index $(\alpha_1,\alpha_2,\cdots,\alpha_{n-1})$ of the previous iteration.

Such structure defines an infinitary rooted taxonomic tree of depth r, with the vertex set given by

$$\mathcal{A} = \mathbb{N}^1 \cup \mathbb{N}^2 \cup \cdots \cup \mathbb{N}^r \tag{11}$$

with each vertex $(\alpha_1,\cdots,\alpha_{n-1})$ branching to the vertexes $(\alpha_1,\cdots,\alpha_{n-1},\alpha_n)$, for all $\alpha_n \in \mathbb{N}$.

Each $\boldsymbol{\alpha} = (\alpha_1,\alpha_2,\cdots,\alpha_r) \in \mathbb{N}^r$ at the boundary identifies a path along the tree, defined as:

$$\boldsymbol{\alpha} \mapsto p(\boldsymbol{\alpha}) = (\,(\alpha_1),(\alpha_1,\alpha_2),\cdots(\alpha_1,\alpha_2,\cdots,\alpha_r)\,) \tag{12}$$

The $r-$step Ruelle random probability cascade, for the sequence X, or GREM_X, is then defined as the point process $\{\xi_{\boldsymbol{\alpha},r};\boldsymbol{\alpha} \in \mathbb{N}^r\}$ such that:

$$\xi_{\boldsymbol{\alpha},r} = \prod_{\boldsymbol{\beta} \in p(\boldsymbol{\alpha})} \xi^{(n)}_{\boldsymbol{\beta}}. \tag{13}$$

Note that a rigorous definition of the Ruelle probability cascade point process requires the reordering, for each level $0 \leq k \leq r+1$, of the random variables, generated in REM_{x_k}, in a decreasing order [10, 12–14, 24].

The cavity fields $h_{i|\boldsymbol{\alpha},r}$, for a given vertex i and label $\boldsymbol{\alpha} = (\alpha_1,\cdots,\alpha_r)$, are written as the sum over random variables (*random steps*) defined along the path $p(\boldsymbol{\alpha})$:

$$h_{i|\boldsymbol{\alpha},r} = h^{(0)}_i + h^{(1)}_{i|(\alpha_1)} + h^{(2)}_{i|(\alpha_1,\alpha_2)} + \cdots + h^{(r)}_{i|(\alpha_1,\alpha_2,\cdots,\alpha_r)}\,, \tag{14}$$

where, for fixed $(\alpha_1,\cdots,\alpha_{n-1})$ and vertex i, with $1 \leq n \leq r$, all the random steps $h^{(n)}_{i|(\alpha_1,\alpha_2,\cdots,\alpha_n)}$, with $\alpha_n \in \mathbb{N}$, are identical distributed, with distribution depending on n and on the *preceding* steps $h^{(0)}_i$,$h^{(1)}_{i|(\alpha_1)}$, $h^{(2)}_{i|(\alpha_1,\alpha_2)}$, $\cdots$, $h^{(n-1)}_{i|(\alpha_1,\alpha_2,\cdots,\alpha_{n-1})}$, and conditionally independent given the preceding steps. The variable $h^{(0)}_i$ is the root field of the vertex i and it is shared amongst all the $\boldsymbol{\alpha}$s.

In the Parisi solution of the SK model, the analog random quantities are Gaussian and fully independent, not only conditionally.

By general argument, we can argue that all the random steps must have symmetric distribution, with mean zero, and the cavity fields must be almost surely bounded:

$$\mathbb{E}[|h_{i|\boldsymbol{\alpha},r}|] < \infty \quad \text{for all allowed } i \text{ and } \boldsymbol{\alpha} \tag{15}$$

The random steps are generated independently for different cavity labels.

Taking the expectation value over all the random quantities, the functional Φ will depends only on the sequence X and on the distributions of the random steps.

It will be useful, in the following, defining the truncated fields:

$$h_{i|\boldsymbol{\alpha},n} = h_i^{(0)} + h_{i|(\alpha_1)}^{(1)} + h_{i|(\alpha_1,\alpha_2)}^{(2)} + \cdots + h_{i|(\alpha_1,\alpha_2,\cdots,\alpha_n)}^{(n)} \tag{16}$$

$$\text{with} \quad \text{and} \quad h_{i|\boldsymbol{\alpha}}^{(0)} = h_i^{(0)} \tag{17}$$

For fixed $\boldsymbol{\alpha}$ and i, the collection $h_{i|\boldsymbol{\alpha}} := \{h_{i|\boldsymbol{\alpha},n}, 1 \leq n \leq r\}$ is a discrete time stochastic process. In the following, we will refer to the label n as *level* of the process.

The probability law of $h_{i|\boldsymbol{\alpha},n}$, for any allowed n, depends on the history of the process $h_{i|\boldsymbol{\alpha}}$ it self-up to the level $n-1$.

The distribution of the whole process $h_{i|\boldsymbol{\alpha}}$ does not depend explicitly on the label $\boldsymbol{\alpha}$, i.e. the system is invariant over permutation of the multi-indexes $\boldsymbol{\alpha}$ and we can drop it away without ambiguities. We then consider an $\boldsymbol{\alpha}-$independent stochastic process, that we will refer to as $r-$steps field process, or, simply, field process.

In the following, we will indicate with h_i the whole $r-$step field process, associated to a given vertex i. The shorthand notation $\{h_i\}$ will denote a particular realization of the process h_i (i.e. the realization of all the variables $h_i^{(n)}$, $0 \leq n \leq r$), and $\{h_i\}_n$ will denote the realization up to the level $n < r$.

$$\begin{aligned} \{h_i\} &\longrightarrow \text{(realization of) } \{h_i^{(1)}, h_i^{(2)}, \cdots, h^{(r)}\}, \\ \{h_i\}_n &\longrightarrow \text{(realization of) } \{h_i^{(1)}, h_i^{(2)}, \cdots, h_i^{(n)}\}. \end{aligned} \tag{18}$$

Because of the properties of the random steps, at any level n the $r-$steps field process satisfies the martingale properties [26]:

$$\begin{aligned} \mathbb{E}[|h_{i|n}|] < \infty \quad \text{and} \\ \mathbb{E}[h_{i|n}|\{h_i\}_{n-1}] = h_{i|n-1} + \mathbb{E}[h_i^{(n)}|\{h_i\}_{n-1}] = h_{i|n-1}. \end{aligned} \tag{19}$$

This observation will be crucial for the approach presented in this paper.

We have not defined the probability distribution of the $r-$steps field process. Let us consider the space $\Omega_r = \mathbb{R}^r$ as the $r-$steps sample space, endowed with the Borel $\sigma-$algebra $\mathcal{B}_r$ [31], and let $\mathcal{P}_r(h)$ be the set of all the probability measures on $(\Omega_r, \mathcal{B}_r)$ such as to the $r-$steps field process h is a martingale (with respect its natural filtration[1]) [26]. In the following, we will consider that each field process h_i, has a generic distribution $\mathbb{P}_i \in \mathcal{P}_r(h)$. We also define the probability spaces $(\Omega_r, \mathcal{B}_r)^{\otimes 2}$ and $(\Omega_r, \mathcal{B}_r)^{\otimes c}$, given, respectively, by the $2-$fold and $c-$fold Cartesian product of the probability space $(\Omega_r, \mathcal{B}_r)$.

3.2 The free energy as a composition of non-linear expectation values

One of the most remarkable properties of the Ruelle random probability cascade point processes is the quasi-stationarity under a class of time evolutions, which includes the cavity dynamics. In particular, it implies that the average of a population of hierarchical random variables, weighted by the Ruelle random probability cascade configurations $\{\xi_{\boldsymbol{\alpha},r}; \boldsymbol{\alpha} \in \mathbb{N}^r\}$, can be represented with a recursive composition of non-linear expectation values that run only over such variables, getting rid of the cumbersome random weights; this is actually the case of vertex and the edge contribution of the free energy (4).

Note that the edge and the vertex contributions to the free energy have a quite similar form, then, for simplicity, we will use a unique notation representing both the cases.

In the rest of this paper, the edge/vertex superscript $\cdot^{(e/v)}$ will denote that a given result must be considered both for the two contributions. The symbols $\vec{J}$ and $\vec{h}_{\boldsymbol{\alpha},r}$ represents the set of (2 or c) couplings

[1]The natural filtration associated to a stochastic process $\{X_n; n \in I\}$, is an incresing family of σ-algebras $\{\mathcal{F}_n; n \in I, \mathcal{F}_n \subseteq \mathcal{F}_m \forall n \leq m\}$, such as, $\forall n \in I$, $\mathcal{F}_n$ is the smallest σ-algebra that contains all the possible realizations of the process X up to the time n.

and cavity fields in which the considered function depends, according to the definitions (5). The quantities without the edge/vertex superscript have the same probability law in both the contributions.

The vertex and the edge contributions verify:

$$\mathbb{E}\log\left(\frac{\sum_{\boldsymbol{\alpha}}\xi_{\boldsymbol{\alpha}}\Delta^{(e/v)}(\vec{J},\vec{h}_{\boldsymbol{\alpha},r})}{\sum_{\boldsymbol{\alpha}}\xi_{\boldsymbol{\alpha}}}\right)=\int\left(\prod_{i}^{(2/c)}dP_0(h_i^{(0)})\right)\,\phi_0^{(e/v)}(\vec{J},\vec{h}^{(0)})\,, \tag{20}$$

where dP_0 is the root field probability distribution and $\phi_0^{(e/v)}(\vec{J},\vec{h}^{(0)})$ is the end point of the following backtracking recursive map:

$$\begin{aligned}&\phi_r^{(e/v)}(\vec{J},\{\vec{h}\}\,)=\log\left(\,\Delta^{(e/v)}(\vec{J},\vec{h}_r)\right)\,,\\&\phi_n^{(e/v)}(\vec{J},\{\vec{h}\}_n\,)=\frac{1}{x_{n+1}}\log\mathbb{E}_{\vec{h},\mathbb{P}^{\otimes(2/c)}}\left[\exp\left(\,x_{n+1}\phi_{n+1}^{(e/v)}(\vec{J},\{\vec{h}\}_{n+1})\right)\,\middle|\{\vec{h}\}_n\right]\text{for}1\le n\le r-1\end{aligned} \tag{21}$$

and finally

$$\phi_0^{(e/v)}(\vec{J},\vec{h}^{(0)}\,)=\frac{1}{x_1}\log\mathbb{E}_{\vec{h},\mathbb{P}^{\otimes(2/c)}}\left[\exp\left(\,x_1\phi_1^{(e/v)}(\vec{J},\vec{h}^{(0)},\vec{h}^{(1)})\right)\,\middle|\vec{h}^{(0)}\right]. \tag{22}$$

The process $\vec{h}$ is the vector field process (with 2 or c respectively for the edge or the vertex contribution). The probability measure of $\vec{h}$, indicated with $\mathbb{P}^{\otimes(2/c)}$, is given by the product measure amongst the probabilities $\mathbb{P}_i$ of the single components the number of which is 2 or c. As in (18), we denote with $\{\vec{h}\}_n$ the realization of the vector field process $\vec{h}$ up to the level $n\le r$.

The expectation value $\mathbb{E}_{\vec{h},\mathbb{P}^{\otimes(2/c)}}[\cdot|\{\vec{h}\}_n]$ is with respect the probability $\mathbb{P}^{\otimes(2/c)}$, taking $\{\vec{h}\}_n$ fixed.

We will refer to the components of the process $\vec{h}$ as cavity components.

It is worth noting that the function $\Delta^{(e/v)}$, as defined in (5), depends only on the value $\vec{h}_r$ of the process $\vec{h}$ at the last level r,(14), and not on the whole process $\vec{h}$.

The functional $\phi_n^{(e/v)}$ depends on the realization of the process $\vec{h}$ up to the level n only, so we say that the process $\phi^{(e/v)}(\vec{J},\{\vec{h}\}):=\{\phi_n^{(e/v)}(\vec{J},\{\vec{h}\}_n\,),1\le n\le r\}$ is adapted [26](or non-anticipative) to (the natural filtration of) the process $\vec{h}$; in particular it is a supermartingale [26].

Let us define also the $r-$steps free energy stochastic process ϕ such that:

$$\phi=\phi^{(v)}-\frac{c}{2}\phi^{(e)} \tag{23}$$

The tricky functional dependence of ϕ on the particular realizations of the process $\vec{h}$ is basically due to the form of the conditional expectation.

In the Parisi solution of the SK model, the field process is Markovian, so one can define the Kolmogorov equation for the expectation value and averaged away the randomness.

In the following, the dependence of the process $\phi^{(e)}$ and $\phi^{(v)}$ on the random couplings will be omitted for convenience; all the equations below will refer to a single realization of the random couplings.

3.3 Variational representation of the r-RSB free energy functional

In this section, we get a variational representation of the recursive law (21). The variational representation turns to be a powerful tool to get the $r\to\infty$ limit.

In this subsection, and in the rest of this paper, the martingale formalism [26] is deeply used. Let first introduce some notation.

Let $\mathcal{D}_r^{(e/v)}$ be the set of the stochastic processes adapted to the r-steps vector field process $\vec{h}$, either for the edge or the vertex free energy contribution.

Let $\mathcal{M}_r^{(e/v)}\subset\mathcal{D}_r^{(e/v)}$ the subspace of $\vec{h}$-adapted martingales and $\mathcal{M}_{r,1,>}^{(e/v)}\subset\mathcal{M}_r^{(e/v)}$ the subset of strictly positive $\vec{h}$-adapted martingales with average equal to 1.

Furthermore, for each level $0 < n \leq r$, let define the set $\mathcal{R}^{(e/v)}_{n,1,>}$ of strictly positive random variables, depending on the random process $\vec{h}$ up to the level n, with expectation value over $\vec{h}^{(n)}$, conditionally to a fixed realization $\{\vec{h}\}_{n-1}$, equal to 1.

Recursive composition of the Gibbs variational principle. We start with a variational representation of the recursion formula (21), for each level $0 \leq n \leq r$ separately.

The second member of the recursion formula (21) has the form of the usual Helmotz free energy in the canonical ensemble, with x_{n+1} as inverse temperature and $-\phi_{n+1}$ as Hamiltonian. As a consequence, it can be represented via the Gibbs variational principle:

$$\phi^{(e/v)}_n\big(\{\vec{h}\}_n\big) = \max_{\rho^{(e/v)}_n \in \mathcal{R}^{(e/v)}_{n,1,>}} \Bigg\{ \mathbb{E}_{\vec{h},\mathbb{P}^{\otimes(2/c)}} \Big[\rho^{(e/v)}_{n+1}\big(\{\vec{h}\}_{n+1}\big) \phi^{(e/v)}_{n+1}\big(\{\vec{h}\}_{n+1}\big) \Big| \{\vec{h}\}_n \Big] - \frac{1}{x_{n+1}} \mathbb{E}_{\vec{h},\mathbb{P}^{\otimes(2/c)}} \Big[\rho^{(e/v)}_{n+1}\big(\{\vec{h}\}_{n+1}\big) \log \rho^{(e/v)}_{n+1}\big(\{\vec{h}\}_{n+1}\big) \Big| \{\vec{h}\}_n \Big] \Bigg\}, \quad (24)$$

and the maximum is attained by:

$$\rho^{(e/v)*}_{n+1}\big(\{\vec{h}\}_{n+1}\big) = \frac{1}{Z^{(e/v)}_{n+1}\big(\{\vec{h}\}_{n+1}\big)} \exp\Big(x_{n+1} \phi^{(e/v)}_{n+1}\big(\{\vec{h}\}_{n+1}\big) \Big), \quad \text{with} \quad Z_{n+1}\big(\{\vec{h}\}_{n+1}\big) = \mathbb{E}_{\vec{h},\mathbb{P}^{\otimes(2/c)}} \Big[\exp\Big(x_{n+1} \phi^{(e/v)}_{n+1}\big(\{\vec{h}\}_{n+1}\big) \Big) \Big| \{\vec{h}\}_n \Big]. \quad (25)$$

Because of the monoticity of the expectation value on $\mathcal{D}^{(e/v)}_r$ and the tower property [26] [31], the representations (24), for each level, can be unified in a unique variational representation for the entire map $\phi^{(e/v)}_r = \log \Delta^{(e/v)}_r \mapsto \phi^{(e/v)}_0$:

$$\phi^{(e/v)}_0(\vec{h}^{(0)}) = \max_{R^{(e/v)} \in \mathcal{M}^{(e/v)}_{r,1,>}} \Bigg\{ \mathbb{E}_{\vec{h},\mathbb{P}^{\otimes(2/c)}} \Big[R^{(e/v)}_r\big(\{\vec{h}\}\big) \log\Big(\Delta^{(e/v)}_r\big(\vec{h}^{(r)}\big) \Big) \Big| \vec{h}^{(0)} \Big] - \sum_{n=1}^{r} \frac{1}{x_n} \mathbb{E}_{\vec{h},\mathbb{P}^{\otimes(2/c)}} \left[R^{(e/v)}_n\big(\{\vec{h}\}_n\big) \log\left(\frac{R^{(e/v)}_n\big(\{\vec{h}\}_n\big)}{R^{(e/v)}_{n-1}\big(\{\vec{h}\}_{n-1}\big)} \right) \Bigg| \vec{h}^{(0)} \right] \Bigg\}. \quad (26)$$

The martingale $R^{(e/v)}$ is a strictly positive martingale of average 1, related to the random variables $\{\rho^{(e/v)}_1, \rho^{(e/v)}_2, \cdots, \rho^{(e/v)}_r\}$ in such a way:

$$\begin{aligned} R^{(e/v)}_n\big(\{\vec{h}\}_n\big) &= \prod_{m=1}^{n} \rho^{(e/v)}_m\big(\{\vec{h}\}_m\big), \text{for } 1 < n \leq r, \\ R^{(e/v)}_0\big(\{\vec{h}\}_n\big) &= 1. \end{aligned} \quad (27)$$

The maximum is attained substituting the solutions (25) in (27), for each level n.

Change of measure. In this paragraph we propose a suitable change of probability measure and rephrase the functional (26) in a different formalism that can be easily extended to the continuous limit.

The random variable $R^{(e/v)}_r$ can be considered as the density probability function of a new probability distribution that is absolutely continuous with respect $\mathbb{P}^{\otimes(2/c)}$. All the new conditional probability distributions, given the realization $\{\vec{h}\}_n$ for any level $0 \leq n \leq r$, are then generated by the whole martingale $R^{(e/v)}$. It is possible, then, to define a change of variable at any level, induced by the martingale $R^{(e/v)}$ [26].

By definition, we have:

$$\mathbb{E}_{\vec{h},\mathbb{P}^{\otimes(2/c)}} \left[\frac{R^{(e/v)}_r\big(\{\vec{h}\}\big)}{R^{(e/v)}_n\big(\{\vec{h}\}_n\big)} \Bigg| \{\vec{h}\}_n \right] = \mathbb{E}_{\vec{h},\mathbb{P}^{\otimes(2/c)}} \left[\prod_{m=n+1}^{r} \rho^{(e/v)}_m\big(\{\vec{h}\}_m\big) \Bigg| \{\vec{h}\}_n \right] = 1, \quad \forall 0 \leq n \leq r, \quad (28)$$

then, the "entropic" part in (26) can be reshuffled in such a way:

$$\mathbb{E}_{\vec{h},\mathbb{P}^{\otimes(2/c)}}\left[R_n^{(e/v)}\big(\{\vec{h}\}_n\big)\log\left(\frac{R_n^{(e/v)}\big(\{\vec{h}\}_n\big)}{R_{n-1}^{(e/v)}\big(\{\vec{h}\}_{n-1}\big)}\right)\,\middle|\,\vec{h}^{(0)}\right]=$$
$$\mathbb{E}_{\vec{h},\mathbb{P}^{\otimes(2/c)}}\left[R_n^{(e/v)}\big(\{\vec{h}\}_n\big)\mathbb{E}_{\vec{h},\mathbb{P}^{\otimes(2/c)}}\left[\frac{R_r^{(e/v)}\big(\{\vec{h}\}\big)}{R_n^{(e/v)}\big(\{\vec{h}\}_n\big)}\,\middle|\,\{\vec{h}\}_n\right]\log\left(\frac{R_n^{(e/v)}\big(\{\vec{h}\}_n\big)}{R_{n-1}^{(e/v)}\big(\{\vec{h}\}_{n-1}\big)}\right)\,\middle|\,\vec{h}^{(0)}\right]=$$
$$\mathbb{E}_{\vec{h},\mathbb{P}^{\otimes(2/c)}}\left[R_r^{(e/v)}\big(\{\vec{h}\}\big)\Big(\log\Big(R_n^{(e/v)}\big(\{\vec{h}\}_n\big)\Big)-\log\Big(R_{n-1}^{(e/v)}\big(\{\vec{h}\}_{n-1}\big)\Big)\Big)\,\middle|\,\vec{h}^{(0)}\right]\quad(29)$$

Since $R^{(e/v)}\in\mathcal{M}_{r,1,>}^{(e/v)}$, there exists a probability measure $\mathbb{R}^{(e/v)}$ on the product probability space $(\Omega_r,\mathcal{B})^{\otimes 2/c}$, defined at the end of subsection (3.1), such that:

$$\forall\, O\in\mathcal{D}_r^{(e/v)},\quad \mathbb{E}_{\vec{h},\mathbb{R}^{(e/v)}}\Big[O_n\big(\{\vec{h}\}_n\big)\Big]=\mathbb{E}_{\vec{h},\mathbb{P}^{\otimes(2/c)}}\Big[R_n^{(e/v)}\big(\{\vec{h}\}_n\big)O_n\big(\{\vec{h}\}_n\big)\Big]\quad(30)$$

and

$$\mathbb{E}_{\vec{h},\mathbb{R}^{(e/v)}}\Big[O_n\big(\{\vec{h}\}_n\big)\,\Big|\,\{\vec{h}\}_l\Big]=\mathbb{E}_{\vec{h},\mathbb{P}^{\otimes(2/c)}}\left[\frac{R_n^{(e/v)}\big(\{\vec{h}\}_n\big)}{R_l^{(e/v)}\big(\{\vec{h}\}_l\big)}O_n\big(\{\vec{h}\}_n\big)\,\middle|\,\{\vec{h}\}_l\right],\ \forall 0\le l\le n\le r,\quad(31)$$

where $\mathbb{E}_{\vec{h},\mathbb{R}^{(e/v)}}[\,\cdot\,]$ is the expectation value over the realization of the field process $\vec{h}$, under the probability measure $\mathbb{R}^{(e/v)}$. The symbol O, without subscript, stands for the whole process, whilst O_n is the value of the process O at the level n.

By definition, the probability measure $\mathbb{R}^{(e/v)}$ is equivalent to $\mathbb{P}^{\otimes(2/c)}$ (i.e. the two probability measures are absolutely continuous with respect each other [31]) with

$$\frac{d\mathbb{R}^{(e/v)}}{d\mathbb{P}^{\otimes(2/c)}}\big(\{\vec{h}\}\big)=R_r^{(e/v)}\big(\{\vec{h}\}\big)\quad(32)$$

and

$$\frac{d\mathbb{P}^{\otimes(2/c)}}{d\mathbb{R}^{(e/v)}}\big(\{\vec{h}\}\big)=\frac{1}{R_r^{(e/v)}\big(\{\vec{h}\}\big)}\,,\quad(33)$$

where the process $\frac{1}{R_r^{(e/v)}\left(\{\vec{h}\}\right)}$ is a strictly positive martingale with average 1 under the probability $\mathbb{R}^{(e/v)}$:

$$\mathbb{E}_{\vec{h},\mathbb{R}^{(e/v)}}\left[\frac{1}{R_n^{(e/v)}\big(\{\vec{h}_n\}\big)}\,\middle|\,\{\vec{h}\}_l\right]=\frac{1}{R_l^{(e/v)}\big(\{\vec{h}_l\}\big)}\,,\quad 0<l\le n\le r\,,\quad(34)$$

$$\mathbb{E}_{\vec{h},\mathbb{R}^{(e/v)}}\left[\frac{1}{R_n^{(e/v)}\big(\{\vec{h}_n\}\big)}\right]=1\,,\quad 0<n\le r\,.\quad(35)$$

Note that, in general, the process $\vec{h}$ is not a martingale under $\mathbb{R}^{(e/v)}$ and all the cavity components may be $\mathbb{R}^{(e/v)}-$correlated. Moreover, the two probabilities $\mathbb{R}^{(e)}$ and $\mathbb{R}^{(v)}$ are different.

Let introduce two vector processes $\vec{u}^{(e)}\in\mathcal{D}_r^{(e)}$ and $\vec{u}^{(v)}\in\mathcal{D}_r^{(v)}$, respectively with 2 and c cavity components, defined as:

$$\begin{aligned}&\vec{u}_0^{(e/v)}=\vec{h}^{(0)}\\&\vec{u}_n^{(e/v)}\big(\{\vec{h}\}_n\big)=\vec{h}^{(n)}-\left(\frac{1}{R^{(e/v)}}\cdot\Big\langle R^{(e/v)},\vec{h}\Big\rangle\right)_n,\quad 0<n\le r\end{aligned}\quad(36)$$

where the $\Big\langle R^{(e/v)},\vec{h}\Big\rangle$ is the covariation [26] between the two processes $R^{(e/v)}$ and $\vec{h}$, defined as:

$$\Big\langle R^{(e/v)},\vec{h}\Big\rangle_n=\sum_{l=1}^{n}\Big(R_l^{(e/v)}\big(\{\vec{h}\}_l\big)-R_{l-1}^{(e/v)}\big(\{\vec{h}\}_{l-1}\big)\Big)\,(\vec{h}_l-\vec{h}_{l-1})\,,\quad(37)$$

and

$$\left(\frac{1}{R^{(e/v)}}\cdot\left\langle R^{(e/v)},\vec{h}\,\right\rangle\right)_n=\sum_{l=1}^{n}\frac{R_l^{(e/v)}\big(\{\vec{h}\}_l\big)-R_{l-1}^{(e/v)}\big(\{\vec{h}\}_{l-1}\big)}{R_l^{(e/v)}\big(\{\vec{h}\}_l\big)}(\vec{h}_l-\vec{h}_{l-1})\,. \tag{38}$$

It can be proved that the process $\vec{u}_n^{(e/v)}$ is a $\mathbb{R}^{(e/v)}$ martingale, i.e. it satisfies:

$$\mathbb{E}_{\vec{h},\mathbb{R}^{(e/v)}}\left[\vec{u}_n^{(e/v)}\big(\{\vec{h}\}_n\big)\Big|\{\vec{h}\}_l\right]=\vec{u}_l^{(e/v)}\big(\{\vec{h}\}_l\big)\,,\quad 0<l\leq n\leq r\,. \tag{39}$$

In the following, we will refer to the vector process $\vec{u}^{(e/v)}$ as auxiliary field process.

Thank to the symmetric role between the measures $\mathbb{P}_i$ and $\mathbb{R}$, provided by equation (32), (33) and (34), the relation (36) between $\vec{u}^{(e/v)}$ and $\vec{h}$ can be inverted in such a way:

$$\vec{h}_n=\vec{h}_n^{(e/v)}\big(\{\vec{u}^{(e/v)}\}_n\big)=\vec{u}_n^{(e/v)}-\left(\frac{1}{\widetilde{R}^{(e/v)}}\left\langle\widetilde{R}^{(e/v)},\vec{u}^{(e/v)}\,\right\rangle\right)_n\,, \tag{40}$$

with

$$\widetilde{R}_m^{(e/v)}\left(\left\{\vec{u}^{(e/v)}\big(\,\{\vec{h}\}_\bullet\,\big)\right\}_m\right)=\frac{1}{R_m^{(e/v)}\big(\{\vec{h}\}_m\big)}\,. \tag{41}$$

Let now introduce the probability law $\mathbb{Q}^{(e/v)}$ of the process $\vec{u}^{(e/v)}$, induced by $\mathbb{R}^{(e/v)}$, defined as:

$$\forall\, O\in\mathcal{D}_r^{(e/v)},\quad \mathbb{E}_{\vec{h},\mathbb{R}^{(e/v)}}\left[O_n\big(\{\vec{h}\}_n\big)\right]=\mathbb{E}_{\vec{u}^{(e/v)},\mathbb{Q}^{(e/v)}}\left[O_n\left(\left\{\vec{h}\big(\,\{\vec{u}^{(e/v)}\}_\bullet\,\big)\right\}_n\right)\right]. \tag{42}$$

Using the equation (36) and the relation between the probability measures $\mathbb{P}^{\otimes 2/c}$ and $\mathbb{R}^{(e/v)}$, the functional (26) can be rewritten performing he change of variables $\vec{h}$ to $\vec{u}^{(e/v)}$:

$$\begin{aligned}\phi_0^{(e/v)}(\vec{h}^{(0)}\,)=&\\ \max_{\widetilde{R}^{(e/v)}\in\mathcal{M}^{(e/v)}_{r,1,>}}&\left\{\mathbb{E}_{\vec{u}^{(e/v)},\mathbb{Q}^{(e/v)}}\left[\log\left(\Delta^{(e/v)}\left(\vec{u}_r^{(e/v)}-\sum_{m=1}^{r}\frac{1}{\widetilde{R}_m^{(e/v)}\big(\{\vec{u}^{(e/v)}\}_m\big)}\left\langle\widetilde{R}^{(e/v)},\vec{u}^{(e/v)}\,\right\rangle_m\right)\right)\,\middle|\,\vec{h}^{(0)}\right]\right.\\ &\left.+\sum_{n=1}^{r}\frac{1}{x_n}\mathbb{E}_{\vec{u}^{(e/v)},\mathbb{Q}^{(e/v)}}\left[\log\Big(\widetilde{R}_n^{(e/v)}\big(\{\vec{u}^{(e/v)}\}_n\big)\Big)-\log\Big(\widetilde{R}_{n-1}^{(e/v)}\big(\{\vec{u}^{(e/v)}\}_{n-1}\big)\Big)\,\middle|\,\vec{h}^{(0)}\right]\right\},\end{aligned} \tag{43}$$

where $\widetilde{R}_n^{(e/v)}$ is a strictly positive $\mathbb{Q}^{(e/v)}-$martingale, adapted to $\vec{u}$ and with average 1.

It is worth stressing that the distributions $\mathbb{Q}^{(e)}$ and $\mathbb{Q}^{(v)}$ are different, but not independent. Actually, the two distributions are related to each other in a tricky way, through the distributions $\mathbb{P}_i$ of the field processes h_i. The distributions of the cavity fields $\mathbb{P}_i$ are the true order parameters of the system, and we will refer to it as cavity distributions. The distributions $\mathbb{Q}^{(e)}$ and $\mathbb{Q}^{(v)}$ will be called as auxiliary distributions.

In general, we will call *cavity frame* a probability space with respect to the field processes h_i are martingales and independent for different index i, and *auxiliary frame* a probability space where either $\vec{u}^{(e)}$ or $\vec{u}^{(v)}$ is a martingale (two different space for the edge and the vertex contribution).

The continuous limit will be obtained through the auxiliary frame, and then, at the end, we will go back to the cavity frame.

The equilibrium free-energy is obtained extremizing the free energy functional (43) with respect the field process distributions $\mathbb{P}_i$; the representations of the two free energy contributions, given by(43), constitute two independent variational problems inside a larger variational problem.

In the following, we will refer to (43) as auxiliary variational problem, whilst the extremization over the cavity field process distribution is the physical variational problem.

4 The continuous replica symmetry breaking

In the previous section, the r-RSB free energy (4) is represented as a variational functional of discrete time martingales.

In this section, we extend the auxiliary variational problem (43) to continuous time martingales that have also a continuous sample path [26] and get the $full$-RSB free energy functional.

From a rigorous mathematical point of view, such extension should not be considered as the $r \to \infty$ limit of the r-RSB theory, but rather a generalization of the previous stochastic analysis to another class of martingales.

In the first subsection, we present the formal derivation of the auxiliary variational problem in the continuous case.

In the second subsection, we reduce the analysis to Itô stochastic processes, for which the auxiliary variational problem is solvable, and the $full$-RSB free energy functional is finally achieved. We then address the physical variational problem: we define a suitable order parameter and derive a self-consistency equation from the stationary condition of the free energy.

We conclude by obtaining a set of coupled stochastic equations that allows the computation of the equilibrium free energy.

4.1 Continuous case: the generalized Chen-Auffinger variational representation

In this subsection, the variational auxiliary problem for continuous martingales is derived.

At the first, we rephrase the martingale approach of the previous section, with a suitable formalism, representing both continuous and discrete time martingales. Then we concentrate on path continuous martingales, getting a generalization of Auffinger-Chen variational representation of the Parisi solution of the SK models [29].

Continuous time formalism. Each $r-$steps process $O \in \mathcal{D}_r^{(e/v)}$, together with the increasing sequence $X = (0 = x_0 \leq x_1 \leq \cdots \leq x_{r+1} = 1)$, identifies a continuous-time stepwise bounded stochastic process $O := \{O(x);\, x \in [0,1]\}$, that we call continuous extension, defined as:

$$O(x) = O_0 \chi_{[0,x_1]}(x) + \sum_{i=1}^{r} O_n \chi_{(x_n, x_{n+1}]}(x), \quad x \in [0,1]. \tag{44}$$

In the following, we will use the same notation for $r-$step processes and their continuous-time extension.

In the $r \to \infty$ limit, the sequence X "fills" the $[0,1]$ segment, so this limit can be formally achieved enlarging the extremal point quest to a more generic class of continuous time bounded stochastic processes.

Let us remind some standard crucial mathematical tools from the continuous time stochastic process theory [26].

We call quadratic variation of a process O, the process $[O]$ defined as:

$$[O](x) = \lim_{\substack{M\to\infty \\ \delta\to 0}} \sum_{n=0}^{M} \left(O(x_{n+1}) - O(x_n)\right)^2, \text{ with } 0 = x_0 \leq x_1 \leq \cdots \leq x_M = x\,, \tag{45}$$

where δ is the mesh of the partition.

In the same manner, the covariation between two processes O and P is the process $\langle O, P\rangle$ such that:

$$\langle O, P\rangle(x) = \lim_{\substack{M\to\infty \\ \delta\to 0}} \sum_{n=0}^{M} \left(O(x_{n+1}) - O(x_n)\right)\left(P(x_{n+1}) - P(x_n)\right), \text{ with } 0 = x_0 \leq x_1 \leq \cdots \leq x_M = x\,. \tag{46}$$

Obviously, the quadratic variation and the covariation vanish for smooth functions, but they are not trivial for all continuous martingale.

For stepwise processes, as (44), these two quantities reduce to the sum over the discontinuity jumps; in particular the discrete-time covariation, defined on (41), between the $r-$ steps processes $R^{(e/v)}$ and $\vec{h}$ coincides with the continuous-time covariation (46) between the corresponding stepwise processes.

Note also that, given any bounded function $f : [0,1] \to \mathbb{R}$, the stepwise process (44) satisfies:

$$\sum_{n=1}^{r} f(x_n)(\, O_n - O_{n-1}\,) = \int_0^1 f(x) dO(x) \tag{47}$$

where the right member is the Lebesgue-Stieltjes integral [31] or is the generalized Itô integral [35] whether O is of bounded variation or of bounded quadratic variation respectively [26].

The auxiliary variational representation (43) can now be reformulated for generic continuous-time processes as:

$$\begin{aligned}\phi^{(e/v)}(x, \vec{h}^{(0)}\,) = \\ \max_{\widetilde{R}^{(e/v)} \in \mathcal{M}^{(e/v)}_{[0,1],1,>}} \Bigg\{ \mathbb{E}_{\vec{u}^{(e/v)}, \mathbb{Q}^{(e/v)}} \left[\log \left(\Delta^{(e/v)} \left(\vec{u}^{(e/v)}(1) - \int_0^1 \frac{d \left\langle \widetilde{R}^{(e/v)}, \vec{u}^{(e/v)} \right\rangle (x)}{\widetilde{R}^{(e/v)}\big(x, \big\{\vec{u}^{(e/v)}\big\}_x\big)} \right) \right) \,\middle|\, \vec{h}^{(0)} \right] \\ + \mathbb{E}_{\vec{u}^{(e/v)}, \mathbb{Q}^{(e/v)}} \left[\int_0^1 \frac{1}{x}\, d\left(\log\, \widetilde{R}^{(e/v)}\big(x, \big\{\vec{u}^{(e/v)}\big\}_x\big)\, \right) \,\middle|\, \vec{h}^{(0)} \right] \Bigg\}, \end{aligned} \tag{48}$$

where the range set $\mathcal{M}^{(e/v)}_{[0,1],1,>}$ of the variational parameter $R^{(e/v)}$ is the closure of the limit set $\bigcup_{r\geq 1} \mathcal{M}^{(e/v)}_{r,1,>}$.

The symbol $\big\{\vec{u}^{(e/v)}\big\}_x$ indicates that the functional $R^{(e/v)}$ depends on the whole specific realization of continuous vector process $\vec{u}^{(e/v)}$ up to the "time" x. In mathematical jargon [26], the process $R^{(e/v)}$ is said to be adapted to (the usual $\mathbb{Q}^{(e/v)}$-completion [2] of the natural filtration of) $\vec{u}^{(e/v)}$.

In the following, I shall assume that the $\mathbb{Q}^{(e/v)}-$martingales $\vec{u}^{(e/v)}$ and $R^{(e/v)}$ have a.s. continuous sample paths.

Path continuity assumption. The path continuity ensure the Itô lemma of the stochastic differential calculus [35]; in particular we have:

$$d\left(\log\, \widetilde{R}^{(e/v)}\Big(x, \big\{\vec{u}^{(e/v)}\big\}_x\Big)\, \right) = \frac{d\widetilde{R}^{(e/v)}\big(x, \big\{\vec{u}^{(e/v)}\big\}_x\big)}{\widetilde{R}^{(e/v)}\big(x, \big\{\vec{u}^{(e/v)}\big\}_x\big)} - \frac{1}{2} \frac{d[\widetilde{R}^{(e/v)}]\big(x, \big\{\vec{u}^{(e/v)}\big\}_x\big)}{\Big(\, \widetilde{R}\big(x, \big\{\vec{u}^{(e/v)}\big\}_x\big)\, \Big)^2}\,. \tag{49}$$

Let define the stochastic process $L^{(e/v)}$ such that:

$$L^{(e/v)}(x) = -\int_0^x \frac{1}{x'}\, \frac{d\widetilde{R}^{(e/v)}\big(x', \big\{\vec{u}^{(e/v)}\big\}_{x'}\big)}{\widetilde{R}^{(e/v)}\big(x', \big\{\vec{u}^{(e/v)}\big\}_{x'}\big)}\,. \tag{50}$$

Since $\widetilde{R}^{(e/v)}$ is a continuous $\vec{u}^{(e/v)}$-adapted martingale with average 1, then $L^{(e/v)}$ is also a $\vec{u}^{(e/v)}$-adapted martingale, with average 0. For convenience, the dependence of $L^{(e/v)}$ on $\vec{u}^{(e/v)}$ is omitted.

By definition, the quadratic variation of $L^{(e/v)}$ is given by:

$$[L^{(e/v)}](x) = \int_0^x \frac{1}{x'^2}\, \frac{d[\widetilde{R}^{(e/v)}]\big(x', \big\{\vec{u}^{(e/v)}\big\}_{x'}\big)}{\Big(\, \widetilde{R}^{(e/v)}\big(x', \big\{\vec{u}^{(e/v)}\big\}_{x'}\big)\, \Big)^2}\,, \tag{51}$$

[2] The usual completion of a continuous time filtration, with respect a given probability measure, is the smallest right-continuous filtration that contains the original one, enlarged with the set with probability 0 with respect the given probability measure. Usually, a rigorous treatment of continuous time stochastic processes requires the usual completion.

Combining the equations (49), (50) and (51) in (48), one get

$$\phi^{(e/v)}(0,\vec{h}^{(0)}) = \max_{L^{(e/v)}\in\mathcal{M}^{(e/v)}_{[0,1],C}} \left\{ \mathbb{E}_{\vec{u}^{(e/v)},\mathbb{Q}^{(e/v)}} \left[\log\left(\Delta^{(e/v)} \left(\vec{u}^{(e/v)}(1) + \int_0^1 dx\, x\, d\left\langle L^{(e/v)}, \vec{u}^{(e/v)} \right\rangle(x) \right) \right) \middle| \vec{h}^{(0)} \right] - \frac{1}{2}\mathbb{E}_{\vec{u}^{(e/v)},\mathbb{Q}^{(e/v)}} \left[\int_0^1 dx\, x\, d[\, L^{(e/v)}\,](x) \middle| \vec{h}^{(0)} \right] \right\}, \quad (52)$$

where $\mathcal{M}^{(e/v)}_{[0,1],C}$ is the set of $\vec{u}^{(e/v)}-$adapted martingales under the probability measure $\mathbb{Q}^{(e/v)}$, with continuous sample path and starting form 0.

The $full$-RSB free energy functional is then given by:

$$\Phi = \overline{\int \left(\prod_{i=1}^{c} dP_0(h_i^{(0)}) \right) \phi^{(v)}(0,\vec{h}^{(0)})} - \frac{c}{2}\overline{\int dP_0(h_1^{(0)}) dP_0(h_2^{(0)}) \phi^{(e)}(\,0,\,h_1^{(0)},h_2^{(0)}\,)}, \quad (53)$$

where the overline $\bar{\cdot}$ represents the average over the random couplings.

The representation (52), is a generalization of the Chen-Auffinger representation of the Parisi functional, defined for the SK model. Note that, in the Chen-Auffinger representation, the process $L^{(e/v)}$ is a Markov Gaussian process [13].

The free energy (53) actually provides a mathematical representation of the $full$-RSB free energy, then it may represent an interesting the starting point for further qualitative analysis about replica symmetry breaking on sparse graphs.

For the quantitative evaluation of the free energy, we reduce the present computation to Itô processes, for which the auxiliary variational problem can be explicitly solved.

Finally, we must take account of the fact that the free energy (53) is actually a functional of the field processes h_i, that, in the $full$-RSB case, are continuous stochastic process themselves. The equilibrium free energy is then obtained by representing the functional (53) with the cavity frame and solving the physical variational problem over the field processes.

4.2 The $full$-RSB equations

In this subsection, we solve the auxiliary variational problem, getting an explicit formula of the free energy as a functional of a suitable order parameter. The mean field equations of the model are then derived from the first variation of the free energy with respect the order parameter.

Throughout the sequel we shall consider the canonical probability space $(\Omega,\mathcal{B},\mathbb{W})$, where Ω is the space of continuous functions $\omega:[0,1]\to\mathbb{R}$, vanishing in 0, $\mathcal{B}$ is the Borel σ-algebra and $\mathbb{W}$ is the Wiener measure [26]. By definition, the coordinate map process $W(\omega) := \{W(x,\{\omega\}_x) = \omega(x);\, x\in[0,1]\}$, together with its natural filtration, is a standard Brownian motion with respect to $\mathbb{W}$. In the following, the Brownian motion will be indicated with ω.

Throughout the sequel we shall reduce the actual variational computation to the space of continuous stochastic process that are adapted to usual $\mathbb{W}$-completion of the natural the filtration generated by ω [26].

Solution of the auxiliary variational problem. The continuous processes $\vec{u}^{(e/v)}$ and $L^{(e/v)}$ are martingales with respect the auxiliary frame, then, by the Itô representation theorem [36] [37], they can be represented as Itô integrals [35]:

$$u_i^{(e/v)}(x,\vec{h}^{(0)},\{\vec{\omega}\}_x) = h_i^{(0)} + \sum_{j=1}^{(2/c)} \int_0^x \zeta_{ij}^{(e/v)}(x',\vec{h}^{(0)},\{\vec{\omega}\}_{x'}) d\omega_j(x') \quad (54)$$

$$L^{(e/v)}(x,\vec{h}^{(0)},\{\vec{\omega}\}_x)=\sum_{j=1}^{(2/c)}\int_0^x r_j^{(e/v)}(x',\vec{h}^{(0)},\{\vec{\omega}\}_{x'})d\omega_j(x')=\int_0^x \vec{r}^{(e/v)}(x',\vec{h}^{(0)},\{\vec{\omega}\}_{x'})\cdot d\vec{\omega}(x')\,, \tag{55}$$

where $\vec{\omega}$ the vectorial Brownian motion, with 2 or c independent components. The matrix elements $\zeta_{ij}^{(e/v)}$ and the components of the vector $\vec{r}^{(e/v)}$ are locally square integrable and predictable processes.

In the following, we will use the symbol $\widehat{\zeta}^{(e/v)}$ to denote the whole matrix with matrix elements $\zeta_{ij}^{(e/v)}$. We shall also use the shorthand notation $\widehat{\zeta}^{(e/v)}\cdot\vec{v}$ for the matrix-vector product with a generic vector $\vec{v}$.

In this frame, the field vector process $\vec{h}$ is a generic Itô process. From the equation (40), we get:

$$\vec{h}^{(e/v)}(x,\vec{h}^{(0)},\{\vec{\omega}\}_x)=\vec{u}^{(e/v)}(x,\vec{h}^{(0)},\{\vec{\omega}\}_x)+\int_0^x dx'\,x'\,d\,\langle\, L^{(e/v)},\vec{u}\,\rangle\,(x')=$$
$$\vec{h}^{(0)}+\int_0^x \widehat{\zeta}^{(e/v)}(x',\vec{h}^{(0)},\{\vec{\omega}\}_{x'})\cdot d\vec{\omega}(x')+\int_0^x dx'\,x'\,\widehat{\zeta}^{(e/v)}\big(x',\vec{h}^{(0)},\{\vec{\omega}\}_{x'}\big)\cdot\vec{r}^{(e/v)}\big(x',\vec{h}^{(0)},\{\vec{\omega}\}_{x'}\big) \tag{56}$$

The representations formulas (54) and (55), with the functional (52), gives:

$$\phi^{(e/v)}(\,0,\,\vec{h}^{(0)}\,)=\max_{\vec{r}^{(e/v)}\in\mathcal{M}^{(e/v)}_{[0,1],C}}\left\{\mathbb{E}_{\mathbb{W}^{\otimes(2/c)}}\left[\log\left(\Delta^{(e/v)}\left(\vec{h}^{(e/v)}(1,\vec{h}^{(0)},\{\vec{\omega}\}_1)\right)\middle|\,\vec{h}^{(0)}\right]\right.\right.$$
$$\left.\left.-\frac{1}{2}\mathbb{E}_{\mathbb{W}^{\otimes(2/c)}}\left[\int_0^1 dx\,x\,\left\|\,\vec{r}^{(e/v)}\big(x,\vec{h}^{(0)},\{\vec{\omega}\}_x\big)\,\right\|^2\,\middle|\,\vec{h}^{(0)}\right]\right\}, \tag{57}$$

where $\vec{h}(1,\vec{h}^{(0)},\{\vec{\omega}\}_1)$ is the end point of the stochastic process defined in (56) and the expectation values is with respect to the $(2/c)-$components Brownian motion $\vec{\omega}$.

It is also convenient to define the continuous time supermartingale

$$\phi^{(e/v)}\big(\,x,\vec{h}^{(0)},\{\vec{\omega}\}_x\,\big)=\max_{\vec{r}^{(e/v)}\in\mathcal{M}^{(e/v)}_{[0,1],C}}\left\{\mathbb{E}_{\mathbb{W}^{\otimes(2/c)}}\left[\log\,\Delta^{(e/v)}\Big(\,\vec{h}^{(e/v)}(1,\vec{h}^{(0)},\{\vec{\omega}\}_1)\,\Big)\middle|\,\vec{h}^{(0)},\{\vec{\omega}\}_x\right]\right.$$
$$\left.-\frac{1}{2}\mathbb{E}_{\mathbb{W}^{\otimes(2/c)}}\left[\int_x^1 dx'\,x'\,\left\|\,\vec{r}^{(e/v)}\big(x',\vec{h}^{(0)},\{\vec{\omega}\}_{x'}\big)\,\right\|^2\middle|\,\vec{h}^{(0)},\{\vec{\omega}\}_x\right]\right\}. \tag{58}$$

The symbol $\mathbb{E}\left[\,\cdot\,\middle|\,\vec{h}^{(0)},\{\vec{\omega}\}_x\right]$ is the expectation value conditionally to a fixed realization of the vectorial Brownian motion $\vec{\omega}$ up to the time x and the root field $\vec{h}^{(0)}$. In the limit $x\to 0$, the supermartingale (58) converges, in probability, to the free energy contribution (57). In the following, we will refer to the process (58) edge/vertex free energy supermartingale.

By analogy with the Auffinger-Chen work, we argue that the auxiliary variational problem (57) should be strictly concave, then the maximum can be obtained by taking the functional derivative over $\vec{r}^{(e/v)}$ and putting it equal to 0. The maximum is then attained by:

$$\vec{r}^{(e/v)}\big(x,\vec{h}^{(0)},\{\vec{\omega}\}_x\big)=\widehat{\zeta}^{(e/v)}(x,\vec{h}^{(0)},\{\vec{\omega}\}_x)\cdot\vec{m}^{(e/v)}\big(x,\vec{h}^{(0)},\{\vec{\omega}\}_x\big)\,, \tag{59}$$

with

$$\vec{m}^{(e/v)}\big(x,\vec{h}^{(0)},\{\vec{\omega}\}_x\big)=\mathbb{E}_{\mathbb{W}^{\otimes(2/c)}}\left[\overrightarrow{M}^{(e/v)}\left(\,\vec{h}^{(e/v)}(\,1,\vec{h}^{(0)},\{\vec{\omega}\}_1)\,\right)\,\middle|\vec{h}^{(0)},\{\vec{\omega}\}_x\right] \tag{60}$$

and

$$\overrightarrow{M}^{(e/v)}\,(\,\vec{y}\,)=\overrightarrow{\nabla}_{\vec{y}}\,\log\,\Delta^{(e/v)}(\,\vec{y}\,)\,, \tag{61}$$

where the symbol $\overrightarrow{\nabla}_{\vec{y}}$ is the gradient operator with respect to the 2 or c components of the vector $\vec{y}$.

The process $\vec{m}^{(e/v)}$ is a ω-adapted martingale.

Substituting such solution on the edge/vertex free energy supermartingale (57), one get:

$$\phi^{(e/v)}\big(\, x, \vec{h}^{(0)}, \{\vec{\omega}\}_x \,\big) = \mathbb{E}_{\mathbb{W}^{\otimes(2/c)}}\left[\log\, \Delta^{(e/v)}\left(\, \vec{h}^{(e/v)}(\,1,\, \vec{h}^{(0)}, \{\vec{\omega}\}_1)\,\right)\,\middle|\vec{h}^{(0)}, \{\vec{\omega}\}_x\,\right] - \frac{1}{2}\mathbb{E}_{\mathbb{W}^{\otimes(2/c)}}\left[\int_x^1 dx'\, x'\left\|\widehat{\zeta}^{(e/v)}(x', \vec{h}^{(0)}, \{\vec{\omega}\}_{x'}) \cdot \vec{m}^{(e/v)}\big(x', \vec{h}^{(0)}, \{\vec{\omega}\}_{x'}\big)\right\|^2 \middle|\vec{h}^{(0)}, \{\vec{\omega}\}_x\right] \quad (62)$$

with

$$\vec{h}^{(e/v)}(x, \vec{h}^{(0)}, \{\vec{\omega}\}_x) = \vec{h}^{(0)} + \int_0^x \widehat{\zeta}^{(e/v)}(x',\, \vec{h}^{(0)}, \{\vec{\omega}\}_{x'}) \cdot d\vec{\omega}(x') + \int_0^x dx'\, x'\, \left(\widehat{\zeta}^{(e/v)}(x',\, \vec{h}^{(0)}, \{\vec{\omega}\}_{x'})\right)^2 \cdot \vec{m}^{(e/v)}\big(x', \vec{h}^{(0)}, \{\vec{\omega}\}_{x'}\big)\,, \quad (63)$$

and

$$\phi^{(e/v)}\big(\, 0, \vec{h}^{(0)}\,\big) = \phi^{(e/v)}\big(\, x, \vec{h}^{(0)}, \{\vec{\omega}\}_x\,\big)\Big|_{x=0}\,. \quad (64)$$

It is worth noting that, according to the auxiliary variational problem solution, the following identity holds:

$$\overrightarrow{\nabla}_{\vec{h}^{(0)}}\phi^{(e/v)}(x, \vec{h}^{(0)}, \{\vec{\omega}\}_x\,) = \vec{m}^{(e/v)}\big(x, \vec{h}^{(0)}, \{\vec{\omega}\}_x\big). \quad (65)$$

A remarkable consequence of this identity is that the derivative of the edge/vertex free energy supermartingale is actually a martingale.

The free energy functional must be extremized with respect to the root field distributions P_0 and the two stochastic matrices $\widehat{\zeta}^{(v)}$ and $\widehat{\zeta}^{(e)}$. Such matrices are not independent, but they are related through the distributions $\mathbb{P}_i$ of the cavity field processes. In the next paragraph, we will go back to the cavity frame and then we derive the self-consistency equations for a suitable set of order parameters.

Cavity frame and self-consistency equation. Let us consider a change of probability measure $\mathbb{W} \longrightarrow \widetilde{\mathbb{W}}_i$, such as $(\Omega, \mathcal{B}, \widetilde{\mathbb{W}}_i)$ is a cavity frame for the field process h_i, for a given vertex index i. Let $\widetilde{\omega}_i$ be a process, adapted to the usual $\mathbb{W}$-completion of the natural filtration of ω, that is a (one dimensional) standard Brownian motion with respect to $\widetilde{\mathbb{W}}_i$.

By Itô representation theorem, the field process h_i can be represented as:

$$h_i(x) = h_i^{(0)} + \int_0^x \Xi(x', h_i^{(0)}, \{\widetilde{\omega}_i\}_{x'}) d\widetilde{\omega}_i(x')\,. \quad (66)$$

for some $\widetilde{\omega}-$adapted process Ξ, that we assumed to be independent on the vertex index i. With no loss of generality, we can assume that the process Ξ is almost surely nonnegative.

The process Ξ is related to the quadratic variation $[h_i]$ of the field process h_i in such a way:

$$[h_i](x) = Q(x, \{\widetilde{\omega}_i\}_x) = \int_0^x \Xi^2(x', \{\widetilde{\omega}_i\}_{x'}) \longrightarrow \Xi(x, \{\widetilde{\omega}_i\}_x) = \sqrt{\dot{Q}(x, \{\widetilde{\omega}_i\}_x)}\,, \quad (67)$$

where the dot symbol $\dot{}$ stands for the total derivative with respect x.

We will refer to the quadratic variation process Q as cavity fields overlap process.

The cavity fields overlap Q and the root field distributions P_0 are the actual variational order parameters of the system.

Comparing the expression (66) and (63) and using the Cameron-Martin-Girsanov theorem [38, 39], we get

$$\widetilde{\omega}_i(x) \overset{\mathbb{W}}{\sim} \omega_i(x) + \sum_{j=1}^{(2/c)} \int_0^x dx'\, x'\, \zeta_{ij}^{(e/v)}(x',\, \vec{h}^{(0)}, \{\vec{\omega}\}_{x'}) m_j^{(e/v)}\big(x', \vec{h}^{(0)}, \{\vec{\omega}\}_{x'}\big) \quad (68)$$

where the symbol $\overset{\mathbb{W}}{\sim}$ indicates the equivalence in probability with respect the probability measure $\mathbb{W}$. We must consider the processes m^e and $\widehat{\zeta}^e$ or the processes m^v and $\widehat{\zeta}^v$ respectively if the vertex index i refers to a spin of the edge or the vertex contribution. Note that the process $\widetilde{\omega}_i$ depends implicitly on the vertex index i and on the random couplings through $m_i^{(e/v)}$.

From the equation (68) we get the relation between the matrixes $\hat{\zeta}^{(e)}$ and $\hat{\zeta}^{(v)}$ and the order parameters

$$\zeta_{ij}^{(e/v)}\big(x,\vec{h}^{(0)},\{\vec{\omega}\}_x\big) \overset{\mathbb{W}}{\sim} \delta_{ij}\sqrt{\dot{Q}\left(x,\, h_i^{(0)},\, \{\widetilde{\omega}_i\}_x\right)}\,, \tag{69}$$

where the symbol $\cdot$ represents the dependence on the time of the process. Note that the process ζ_{ij} on the left hand depends on $\vec{\omega}$, whilst the process Q, on the right hand, depends on $\widetilde{\omega}_i$, which, in turn, depends implicitly on $\vec{\omega}$, according to the equation (68).

Using the equation (69) in the equations (62), (63) and (64), together with the equation (60), we can write the variational free energy as a functional over the process Q and the distribution P_0:

$$\begin{aligned}
\Phi = &\overline{\int\Big(\prod_{i=1}^{c} dP_0(h_i^{(0)})\Big)\phi^{(v)}(0,\vec{h}^{(0)})} - \frac{c}{2}\overline{\int dP_0(h_1^{(0)})dP_0(h_2^{(0)})\phi^{(e)}(\,0,\,h_1^{(0)},h_2^{(0)}\,)} = \\
&\overline{\mathbb{E}_{(P_0\times\mathbb{W})^{\otimes c}}\left[\log\Delta^{(v)}\left(\vec{h}^{(v)}\big(x,\vec{h}^{(0)},\{\vec{\omega}\}_x\big)\right)\right]} - \frac{c}{2}\overline{\mathbb{E}_{(P_0\times\mathbb{W})^{\otimes 2}}\left[\log\Delta^{(e)}\left(\vec{h}^{(e)}\big(x,\vec{h}^{(0)},\{\vec{\omega}\}_x\big)\right)\right]} \\
&- \frac{1}{2}\overline{\sum_{i=1}^{c}\mathbb{E}_{(P_0\times\mathbb{W})^{\otimes c}}\left[\int_0^1 dx\, x\,\dot{Q}\left(x,h_i^{(0)},\{\widetilde{\omega}_i^{(v)}\}_x\right)\left(\mathbb{E}_{\mathbb{W}^{\otimes c}}\left[M_i^{(v)}\left(\vec{h}^{(v)}\big(1,\vec{h}^{(0)},\{\vec{\omega}\}_1\big)\right)\,\middle|\,\vec{h}^{(0)},\{\vec{\omega}\}_x\right]\right)^2\right]} \\
&+ \frac{c}{4}\overline{\sum_{i=1}^{2}\mathbb{E}_{(P_0\times\mathbb{W})^{\otimes 2}}\left[\int_0^1 dx\, x\,\dot{Q}\left(x,h_i^{(0)},\{\widetilde{\omega}_i^{(e)}\}_x\right)\left(\mathbb{E}_{\mathbb{W}^{\otimes 2}}\left[M_i^{(e)}\left(\vec{h}^{(e)}\big(1,\vec{h}^{(0)},\{\vec{\omega}\}_1\big)\right)\,\middle|\,\vec{h}^{(0)},\{\vec{\omega}\}_x\right]\right)^2\right]},
\end{aligned} \tag{70}$$

where the symbol $\mathbb{E}_{(P_0\times\mathbb{W})^{\otimes(2/c)}}[\,\cdot\,]$ is the expectation value with respect the $2/c$-components vectorial Brownian motion $\vec{\omega}$ and the root cavity fields $\vec{h}^{(0)}$.

The processes $\widetilde{\omega}_i^{(e)}$ and $\widetilde{\omega}_i^{(v)}$ are given by using the equation (60) in (68):

$$\widetilde{\omega}_i^{(e)}(x) = \omega_i(x) + \int_0^x dx'\,x'\sqrt{\dot{Q}\left(x',h_i^{(0)},\,\{\widetilde{\omega}_i^{(e)}\}_{x'}\right)}\,\mathbb{E}_{\mathbb{W}^{\otimes 2}}\left[M_i^{(e)}\left(\vec{h}^{(e)}\big(1,\vec{h}^{(0)},\{\vec{\omega}\}_1\big)\right)\,\middle|\,\vec{h}^{(0)},\{\vec{\omega}\}_{x'}\right], \quad \text{with } i\in\{1,2\}\,, \tag{71}$$

$$\widetilde{\omega}_i^{(v)}(x) = \omega_i(x) + \int_0^x dx'\,x'\sqrt{\dot{Q}\left(x',h_i^{(0)},\,\{\widetilde{\omega}_i^{(v)}\}_{x'}\right)}\,\mathbb{E}_{\mathbb{W}^{\otimes c}}\left[M_i^{(v)}\left(\vec{h}^{(v)}\big(1,\vec{h}^{(0)},\{\vec{\omega}\}_1\big)\right)\,\middle|\,\vec{h}^{(0)},\{\vec{\omega}\}_{x'}\right], \quad \text{with } i\in\{1,\cdots,c\} \tag{72}$$

and the cavity field processes are defined as :

$$\begin{aligned}
h_i^{(e)}(x,\vec{h}^{(0)},\{\vec{\omega}\}_x) = &\\
&h_i^{(0)} + \int_0^x dx'\,x'\,\dot{Q}\left(x',h_i^{(0)}\,\{\widetilde{\omega}_i^{(e)}\}_{x'}\right)\,\mathbb{E}_{\mathbb{W}^{\otimes 2}}\left[M_i^{(e)}\left(\vec{h}^{(e)}\big(1,\vec{h}^{(0)},\{\vec{\omega}\}_1\big)\right)\,\middle|\,\vec{h}^{(0)},\{\vec{\omega}\}_x\right] \\
&\qquad + \int_0^x\sqrt{\dot{Q}\left(x',h_i^{(0)},\,\{\widetilde{\omega}_i^{(e)}\}_{x'}\right)}\,d\omega_i(x'), \quad \text{with } i\in\{1,2\}
\end{aligned} \tag{73}$$

and

$$h_i^{(v)}(x,\vec{h}^{(0)},\{\vec{\omega}\}_x) = h_i^{(0)} + \int_0^x dx'\, x'\, \dot{Q}\left(x',\, h_i^{(0)},\, \{\widetilde{\omega}_i^{(v)}\}_{x'}\right) \mathbb{E}_{\mathbb{W}^{\otimes c}}\left[M_i^{(v)}\left(\vec{h}^{(v)}\big(1,\vec{h}^{(0)},\{\vec{\omega}\}_1\big)\right)\,\middle|\,\vec{h}^{(0)},\{\vec{\omega}\}_x\right] + \int_0^x \sqrt{\dot{Q}\left(x',\, h_i^{(0)},\, \{\widetilde{\omega}_i^{(v)}\}_{x'}\right)}\, d\omega_i(x'), \qquad \text{with } i\in\{1,\cdots,c\}\,. \quad (74)$$

The self-consistency mean field equations are then derived from the first variation of the variational free energy with respect Q and P_0. After some manipulation we get:

$$\overline{\int dP_0\big(h_2^{(0)}\big)\ \phi^{(e)}(0,y,h_2^{(0)})} = \overline{\int\left(\prod_{i=2}^{c} dP_0\big(h_i^{(0)}\big)\right)\phi^{(v)}(0,y,h_2^{(0)},\cdots,h_c^{(0)})}\,, \quad \forall y\in\mathbb{R} \quad (75)$$

and

$$\frac{1}{2}\sum_{i=1}^{2}\overline{\mathbb{E}_{(P_0\times\mathbb{W})^{\otimes 2}}\left[\left(\mathbb{E}_{\mathbb{W}^{\otimes 2}}\left[M_i^{(e)}\left(\vec{h}^{(e)}(1,\vec{h}^{(0)},\{\vec{\omega}\}_1)\right)\,\middle|\,\vec{h}^{(0)},\{\vec{\omega}\}_x\right]\right)^2\delta_{[0,x]}\left[v(\cdot)-\widetilde{\omega}_i^{(e)}(\cdot)\right]\right]} = \frac{1}{c}\sum_{j=1}^{c}\overline{\mathbb{E}_{(P_0\times\mathbb{W})^{\otimes c}}\left[\left(\mathbb{E}_{\mathbb{W}^{\otimes c}}\left[M_j^{(v)}\left(\vec{h}^{(v)}(1,\{\vec{\omega}\}_2)\right)\,\middle|\,\vec{h}^{(0)},\{\vec{\omega}\}_x\right]\right)^2\delta_{[0,x]}\left[v(\cdot)-\widetilde{\omega}_j^{(v)}(\cdot)\right]\right]}\,, \quad \forall v\in\Omega \quad \text{and} \quad \forall x\in[0,1], \quad (76)$$

where $\delta_{[0,x]}[\,\cdot\,]$ is the functional Dirac delta over the realization of the considered process up to the time x.

Using the self-consistency equations and the variational free energy functional, the equilibrium free energy can be write as follow:

$$-\beta F = \frac{2-c}{2}\left\{\overline{\mathbb{E}_{(P_0\times\mathbb{W})^{\otimes c}}\left[\log\,\Delta^{(v)}\Big(\vec{h}^{(v)}(1,\vec{h}^{(0)},\{\vec{\omega}\}_1)\Big)\right]} - \frac{1}{2}\sum_{i=1}^{c}\overline{\mathbb{E}_{(P_0\times\mathbb{W})^{\otimes c}}\left[\int_0^1 dx\, x\,\dot{Q}\left(x,\, h_i^{(0)},\, \{\widetilde{\omega}_i^{(v)}\}_x\right)\left(\mathbb{E}_{\mathbb{W}^{\otimes c}}\left[M_i^{(v)}\left(\vec{h}^{(v)}(1,\vec{h}^{(0)},\{\vec{\omega}\}_1)\right)\,\middle|\,\vec{h}^{(0)},\{\vec{\omega}\}_x\right]\right)^2\right]}\right\}. \quad (77)$$

One must keep in mind that this expression for the free energy is correct only the equations (75) and (76) are satisfied, so it not represents a variational free energy.

5 Summary and conclusion

In this paper, we have obtained the first non-ambiguous and complete description of the $full$-RSB formalism for the Ising spin glass on random regular graphs.

We have formalized the ideas suggested by G. Parisi in [27] and got a proper new stochastic approach, that provides a powerful mathematical tool to study such class of models.

We have got the $full$-RSB free energy functional, together with the self-consistency mean field equations.

We believe that the solution of the presented mean field equations provides the right equilibrium free energy, but a rigorous proof is still missing. We stress, however, that the mathematical formalization of the $full$-RSB scheme, that we have proposed, is a necessary groundwork for a rigorous analysis of the RSB phenomenon-beyond the fully-connected models.

Unfortunately, the mean field equations are very difficult to solve: we have not solved explicitly the equations in this paper. Since the order parameter is a functional of a Brownian motion, we guess that the

self-consistency equation may be solved by a population dynamics over populations of Brownian motion paths.

The quantitative evaluation of the free energy and a deeper analysis of the physics properties will be investigated in next works.

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