Higher-order corrections to the effective potential close to the jamming transition in the perceptron model

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In view of the results achieved in a previously related work [A. Altieri, S. Franz, and G. Parisi, J. Stat. Mech. (2016) 093301], regarding a Plefka-like expansion of the free energy up to the second order in the perceptron model, we improve the computation here focusing on the role of third-order corrections. The perceptron model is a simple example of constraint satisfaction problem, falling in the same universality class as hard spheres near jamming and hence allowing us to get exact results in high dimensions for more complex settings. Our method enables to define an effective potential (or Thouless-Anderson-Palmer free energy), namely a coarse-grained functional, which depends on the generalized forces and the effective gaps between particles. The analysis of the third-order corrections to the effective potential reveals that, albeit irrelevant in a mean-field framework in the thermodynamic limit, they might instead play a fundamental role in considering finite-size effects. We also study the typical behavior of generalized forces and we show that two kinds of corrections can occur. The first contribution arises since the system is analyzed at a finite distance from jamming, while the second one is due to finite-size corrections. We nevertheless show that third-order corrections in the perturbative expansion vanish in the jamming limit both for the potential and the generalized forces, in agreement with the isostaticity argument proposed by Wyart and coworkers. Finally, we analyze the relevant scaling solutions emerging close to the jamming line, which define a crossover regime connecting the control parameters of the model to an effective temperature.

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I. INTRODUCTION

The anomalous properties of low-temperature structural glasses have been the object of intense studies for decades. By analyzing a vast class of materials with only repulsive contact interactions—for instance, emulsions, hard-sphere suspensions, and granular media—a new kind of transition has been detected [1–9], *the jamming transition*, consisting in the passage from a fluid phase to a regime characterized by a stiff arrangement of particles unable to move and flow. While the glass transition is generated by a rapid cooling down of the liquid in order to avoid crystallization, the jamming transition is induced by an increasing density protocol in the zero-temperature limit. This defines a purely geometric problem where thermal energy does not contribute to determining or facilitating the transition. In any case, the analytical investigation of the jamming transition turns out to be a very challenging issue, both in a mean-field scenario and in finite dimension.

Very recently a breakthrough has been achieved in the context of hard-sphere systems in the limit of infinite space dimensions [10–15]. In this context, the possibility of establishing a unifying framework for jamming, irrespective of microscopic details and specific numerical setups, looks very intriguing. Indeed, several properties of the jamming transition—such as the emergence of a power-law behavior in the distribution of the forces and gaps between particles, the nature and the shape of vibrational modes [3–5]—turn out to be independent of the protocol.

The underlying idea of a sort of universal behavior goes beyond mechanical considerations, involving a broader class of systems, known as continuous constraint satisfaction problems (CSPs), where a set of constraints is imposed on a set of continuous variables. Similarly, in a jammed system, the particle motion is hindered by neighboring particles, which induce geometrical and mechanical constraints in terms of force and torque balance. The connection between jammed systems and the CSP paradigm has been proposed in several works [16–20]. However, further developments in this field have been made possible once it was realized that sphere systems in high dimension belong to the same universality class as a simplified model, the perceptron, according to a new interpretation proposed by Franz and Parisi [21].

The perceptron model has been exploited as a linear signal classifier in computer science for many years [22,23]. It is nevertheless proposed here in a modified form [24–26], with a particular emphasis on a regime that gives rise to nonconvex properties in the space of allowed configurations. We shall clarify this point in more details later.

The Franz-Parisi model is a remarkable starting point for studying jamming in the infinite-dimensional limit. It essentially consists of M obstacles randomly distributed over a spherical surface in N dimensions. The positions of the particles must satisfy specific constraints, which affect the general properties of the model and the energy value, as for each violated constraint there is an associated energy cost to pay. The Hamiltonian of the model depends on $M = \alpha N$ random *gaps* $h_\mu(\vec{x})$ (where $\mu = 1, \ldots, M$) by a soft-constraint

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interaction:

$$\mathcal{H}[\vec{x}] = \frac{1}{2}\sum_{\mu=1}^{M} h_\mu^2(\vec{x})\theta[-h_\mu(\vec{x})], \quad (1)$$

where $\theta(x)$ is the Heaviside function. The gaps are functions of the system configuration $\vec{x} = \{x_1, \dots, x_N\}$, defined on a N-dimensional hypersphere, i.e., $\sum_{i=1}^N x_i^2 = N$. They satisfy the following relation:

$$h_\mu(\vec{x}) = \sum_{i=1}^{N} \frac{\xi_i^\mu x_i}{\sqrt{N}} - \sigma, \quad (2)$$

corresponding to the scalar product between the random obstacles ξ_i^μ ($i = 1, \dots, N$ and $\mu = 1, \dots, M$) and the reference particle position. Since the scalar product is a random variable of order $\sqrt{N}$, the factor $\sqrt{N}$ ensures that the gap is of order one. The components ξ_i^μ, which play the role of quenched disorder, are independent and identically distributed random variables according to a normal distribution $\mathcal{N}(0,1)$.

By conveniently varying the two tunable parameters, σ and $\alpha = M/N$, one might identify different regions of the phase diagram. In particular, the system might undergo a critical transition from a satisfiable region, the SAT phase (where at least one configuration $\vec{x}$ satisfies simultaneously all the constraints), to an unsatisfiable one, the UNSAT phase (where all the constraints are not verified at the same time). The former corresponds to the hard-sphere (HS) regime defined by a zero energy manifold, whereas the second scenario can be mapped to a soft-sphere (SS) problem described by a harmonic potential in $h_\mu(\vec{x})$ [see Eq. (1)]. Physically, this SAT-UNSAT transition coincides with the jamming transition at which the volume of the space of solutions satisfying the given assignments continuously shrinks to zero.

Depending on the positive or negative value of the control parameter σ, two different situations can occur. For positive σ the model defines the usual perceptron classifier, which gives rise to a convex optimization problem, whereas for negative σ the space of allowed configurations is no longer convex, losing its ergodicity properties and inducing new interesting features. In Fig. 1 a simple instance in the space of configurations is sketched in the presence of a single constraint $\vec{\xi}_1$. If $\sigma > 0$, then the scalar product between the space variable $\vec{x}$ and the constraint $\vec{\xi}_1$ defines a convex region. Adding one more constraint has the effect of reducing the space of allowed configurations, which remains nevertheless convex as it results from the intersection of two convex domains. The situation is completely different for $\sigma < 0$ because the scalar product defined in Eq. (2) should be bigger than a negative constant. Each constraint thus defines a nonconvex domain of allowed configurations with the possibility to observe disconnected *clusters* of solutions. This can indeed be regarded as the problem of a single *dynamical* sphere in a background of quenched obstacles $\vec{\xi}^\mu$, often called *patterns* in neural network notation: each of them prevents from reaching and ergodically exploring a given domain around it. It has been shown [21,26] that this second regime plays a fundamental role in the description of jamming and glassy phases. As a matter of fact, the perceptron model bridges the gap between generic CSPs and disordered sphere packings, thanks to the similarity between its phase diagram and that of hard spheres in high dimensions and to the emergence of a jamming transition belonging to the same universality class. Formally speaking, while for positive σ a replica symmetric description can be safely applied, for negative σ and sufficiently close to the jamming line, the ergodicity breaking translates into a full replica symmetry breaking (RBS) [24,26].

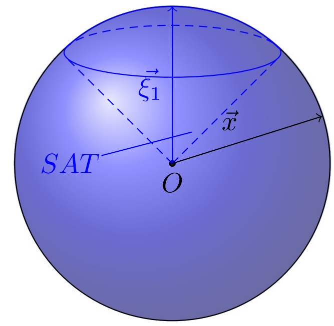

FIG. 1. Schematic illustration of the perceptron model for $\sigma > 0$, with only one constraint. The scalar product between $\vec{x}$ and $\vec{\xi}_1$ should be bigger than a positive threshold, hence excluding all the vectors in the region below the dashed blue cone. The resulting space of solutions is convex. In the presence of more constraints the space of allowed configurations would reduce but always preserving its convexity. By contrast, for $\sigma < 0$ a completely different situation occurs given by the intersection of nonconvex domains: to go from one set of allowed solutions to another one should contemplate the possibility to pass through a positive energy region.

We aim at determining an effective thermodynamic potential, which might properly describe the perceptron model. It turns out to be a central issue especially in the SAT phase, where the energy manifold is flat and unseemly to analyze small harmonic fluctuations around the metastable states of the systems, i.e., the minima of a suitable functional. Therefore, we need to introduce coarse-grained variables and formulate a systematic approach in order to study the free-energy landscape. At the end a detailed analysis of the most relevant features of the SAT phase and the jamming regime will be suitably addressed.

In Ref. [25] the definition of an effective potential as a function of local order parameters, namely of both the average particle positions and the *generalized forces*, has been proposed. Generalized in the sense that they result from the differentiation of the potential with respect to the effective gaps rather than to the particle positions. Our computation is focused on a formal coupling expansion of the free energy, which actually coincides with a perturbative diagrammatic expansion in $1/N$, valid both in the liquid phase and in the low-temperature regime. In our previous work only the first two moments of the expansion have been taken into account within a mean-field-like picture. The main goals of this paper are instead: (i) the computation of third-order corrections to the effective potential, which might also capture important features in the physics of low-temperature glasses in finite dimensions, and (ii) the estimation of subleading contributions to the generalized forces.

The paper is organized as follows. In Sec. II, we introduce the mathematical details to define the model and we briefly summarize the main steps to derive the effective potential in the SAT phase. In Sec. III we compute third-order corrections and we explain why their contribution reasonably vanishes at jamming. Once a suitably coarse-grained free energy as a function of both particle positions and contact forces is defined, in Sec. IV we evaluate the leading and subleading contributions of such forces. We highlight the emergent contributions of terms that can be incorporated in a generic scaling function, which accounts for distinguishing between the jamming limit and the more general case. Two kinds of corrections in the force expression are expected: one due to subleading terms in the asymptotic expansion of the potential near the jamming line, visible even in a mean-field framework, and another due to finite-size corrections in ordinary systems. Finally, in Sec. V we propose a scaling argument based on the full RSB ansatz, which provides a fairly accurate estimate of a crossover regime as a function of an effective temperature.

II. TAP FREE ENERGY FOR THE NEGATIVE PERCEPTRON

In Ref. [25] the effective thermodynamic potential $\Gamma(\vec{m},\vec{f})$, as a function of both particle positions and forces, has been derived for the perceptron model. In the following we shall give a hint of the analytical scheme to implement to calculate the effective potential in high-dimensional systems. We shall make use of a small coupling expansion according to the formalism proposed first by Plefka [27] and reformulated then by Georges and Yedidia [28]. Thanks to the fully connected structure of the model, the calculations lead to a simplified derivation and to a reasonable truncated expansion with only a finite number of terms. This approach provides a nonconvex free-energy functional that nevertheless gives access to the metastable states of the system. We start with the following definition:

$$e^{-G(\vec{m})} = \int d\vec{x} e^{-\beta H[\vec{x}]+\sum_{i=1}^N u_i(x_i-m_i)}, \tag{3}$$

where the integral is performed over the positions $\vec{x}$ in the presence of N Lagrange multipliers, which fix the average value $m_i = \langle x_i\rangle$. However, to address the problem from a broader perspective, given the definition of the gaps in Eq. (2), we also enforce that $h_\mu = h_\mu(x)$ in the partition function via M auxiliary variables $i\hat{h}_\mu$ conjugated to the gaps. As we did for the average positions, we introduce M other Lagrange multipliers v_μ which in turn enforce the average value of $i\hat{h}_\mu$ to be $\langle i\hat{h}_\mu\rangle = f_\mu$. Therefore:

$$e^{-\Gamma(\vec{m},\vec{f})} = \int d\vec{x}d\vec{h}d\vec{\hat{h}}\; e^{-\beta H[\vec{h}]+\sum_i(x_i-m_i)u_i+\sum_\mu(i\hat{h}_\mu-f_\mu)v_\mu+\sum_\mu i\hat{h}_\mu(h_\mu(x)-h_\mu)} = e^{J(\vec{u},\vec{v})-\vec{m}\cdot\vec{u}-\vec{f}\cdot\vec{v}}\,, \tag{4}$$

with $\frac{\partial J}{\partial u_i} = \frac{\partial J}{\partial v_\mu} = 0, \forall i,\mu$. The functional $\Gamma(\vec{m},\vec{f})$ thus reads:

$$\Gamma(\vec{m},\vec{f}) = \sum_{i=1}^N m_i u_i + \sum_{\mu=1}^M f_\mu v_\mu - \log\int d\vec{x}d\vec{h}d\vec{\hat{h}} e^{-\beta H[\vec{h}]+\sum_{i=1}^N x_i u_i+\sum_{\mu=1}^M i\hat{h}_\mu v_\mu+\sum_{\mu=1}^M i\hat{h}_\mu(h_\mu(x)-h_\mu)}, \tag{5}$$

and by definition $G(\vec{m})$ corresponds to

$$G(\vec{m}) = \Gamma(\vec{m},\vec{f}) \qquad \text{evaluated in} \quad \frac{\partial\Gamma(\vec{m},\vec{f})}{\partial f} = 0\,. \tag{6}$$

To clarify the meaning of f_μ in this formalism, we consider the total force acting on particle i and given by

$$F_i = -\frac{d\mathcal{H}}{dx_i} = \sum_{\mu=1}^M(-h_\mu\theta(-h_\mu))\frac{dh_\mu}{dx_i} = \sum_{\mu\in\mathcal{C}} f_\mu S_{\mu i}, \tag{7}$$

where $S_{\mu i} = dh_\mu/dx_i$ is usually called *dynamical matrix* [26]. The notation $\mu\in\mathcal{C}$ stands for those contacts such that $h_\mu < 0$, namely the set of unsatisfied constraints. Starting from this definition of the contact force and looking at the derivative of the functional $\Gamma(\vec{m},\vec{f})$ with respect to the gap, we recover back:

$$\frac{d\Gamma(\vec{m},\vec{f})}{dh_\mu} = \frac{d}{dh_\mu}\left[\frac{\beta}{2}\sum_{\mu=1}^M h_\mu^2\theta(-h_\mu)\right] + \langle i\hat{h}_\mu\rangle. \tag{8}$$

As we are interested in the SAT regime where the gaps are positive definite, the only surviving term is the ensemble average value $\langle i\hat{h}_\mu\rangle$, defined above as the *generalized forces* f_μ. They result from the differentiation of the free-energy functional with respect to the gaps rather than to the position variables.

Similarly to a spin-glass model where the free energy is a function of the overlap value, here the free energy depends on the *self-overlap* between two particle configurations, also called the Edwards-Anderson parameter, as well as on the first two moments of the forces:

$$q = \frac{1}{N}\sum_{i=1}^N m_i^2, \quad r = -\frac{1}{\alpha N}\sum_{\mu=1}^M f_\mu^2, \quad \tilde{r} = \frac{1}{\alpha N}\sum_{\mu=1}^M \langle\hat{h}_\mu^2\rangle. \tag{9}$$

Equation (5) can then be rewritten as:

$$\Gamma(\vec{m},\vec{f}) = \sum_{i=1}^N m_i u_i + \sum_{\mu=1}^M f_\mu v_\mu - \log\int d\vec{x}d\vec{h}d\vec{\hat{h}}\; e^{S_\eta(\vec{x},\vec{h},\vec{\hat{h}})}, \tag{10}$$

where

$$S_\eta(\vec{x},\vec{h},\vec{\hat{h}}) = \sum_{i=1}^N u_i x_i + \sum_{\mu=1}^M i v_\mu\hat{h}_\mu - \lambda\sum_{i=1}^N\left(x_i^2 - N\right) - \frac{\beta}{2}\sum_{\mu=1}^M h_\mu^2\theta(-h_\mu) - i\sum_{\mu=1}^M \hat{h}_\mu(h_\mu - \eta h_\mu(x)) - \frac{b}{2}\sum_{\mu=1}^M\left(\hat{h}_\mu^2 - \alpha N\tilde{r}\right). \tag{11}$$

Note that we have introduced two additional parameters compared to Eq. (5): λ guarantees the correct normalization on the N-dimensional sphere, while b enforces the second moment of $i\hat{h}_\mu$. As we shall clarify in the following, we also need to fix the average value of $(i\hat{h}_\mu)^2$ to handle a closed set of equations. Note that the value of the multiplier b is constrained to be $(1-q)$ by the saddle-point equation $\frac{\partial\Gamma}{\partial\tilde{r}}=0$.

The main goal of this paper is to study the low-energy phase of the perceptron model at zero temperature. In the zero-temperature limit we can distinguish two different behaviors. In the SAT phase several solutions are possible and the overlap parameter $q<1$. Conversely, in the UNSAT phase, the energy has one single minimum and the overlap parameter is always equal to 1. In the following, we will focus on the SAT phase in the $T\to 0$ limit in which the free energy corresponds to the configurational entropy of the system as a measure of the number of microstates $\Omega(v)$ with a given volume v. In other terms, $S\propto\log\Omega(v)$, where $\Omega(v)=\int d\vec{x}\delta(v-W(\vec{x}))\Theta_{\text{jamm}}$, the Θ function enforcing the excluded volume constraint [29,30].

The core of our computation lies in the definition of an auxiliary *effective Hamiltonian* $\mathcal{H}_{\text{eff}}=i\eta\sum_\mu\hat{h}_\mu h_\mu(\vec{x})$, where η represents the parameter in terms of which we perform a Plefka-like expansion [27,28]. Indeed, the original Hamiltonian in Eq. (1) is zero in the whole SAT phase and it only contributes to forcing the particles to stay close.

Our expansion in η actually coincides with a diagrammatic expansion in the inverse of the dimension $1/N$, benefitting from the fact that in a fully connected system in the large-N limit one can recover the mean-field predictions by considering the first two terms of the expansion only. Higher-order terms provide systematic corrections to the mean-field approximation, relevant for short-range interacting models or finite-dimensional ones. In general, we need to determine the following quantity:

$$\Gamma(\eta)=\sum_{n=0}\frac{1}{n!}\left.\frac{\partial^n\Gamma}{\partial\eta^n}\right|_{\eta=0}\eta^n, \tag{12}$$

where Γ is the free-energy functional, which, for the sake of convenience, we write here as a function of the parameter η only. We formally expand around $\eta=0$ and then we set $\eta=1$ without any loss of generality. From Eq. (11) we compute the first derivative of the free-energy functional with respect to η, which coincides with the average effective Hamiltonian:

$$\frac{\partial\Gamma}{\partial\eta}=-\langle H_{\text{eff}}\rangle=-\sum_{i,\mu}\frac{\xi_i^\mu m_i f_\mu}{\sqrt{N}}, \tag{13}$$

whereas the second derivative gives rise to the *Onsager reaction term* in the Thouless-Anderson-Palmer (TAP) formalism [31]. It consists of the connected part of the effective Hamiltonian plus mixing terms associated with the derivatives of the Lagrange multipliers u_i and v_μ:

$$\frac{\partial^2\Gamma}{\partial\eta^2}=-\left\{\left\langle H_{\text{eff}}^2\right\rangle-\langle H_{\text{eff}}\rangle^2+\left\langle H_{\text{eff}}\left[\sum_i\frac{\partial u_i}{\partial\eta}(x_i-m_i)+\sum_\mu\frac{\partial v_\mu}{\partial\eta}(i\hat{h}_\mu-f_\mu)\right]\right\rangle\right\}. \tag{14}$$

The resulting expression for the potential up to the second order in η thus reads:

$$\begin{aligned}\Gamma(\vec{m},\vec{f})&=\sum_{i=1}^N\phi(m_i)+\sum_{\mu=1}^M\Phi(f_\mu)+\left.\frac{\partial\Gamma}{\partial\eta}\right|_{\eta=0}\eta+\frac{1}{2}\left.\frac{\partial^2\Gamma}{\partial\eta^2}\right|_{\eta=0}\eta^2+\mathcal{O}(\eta^3)\\&\approx-\frac{N}{2}\log(1-q)+\sum_\mu\Phi(f_\mu)-\sum_{i,\mu}\frac{\xi_i^\mu m_i f_\mu}{\sqrt{N}}+\frac{\alpha N}{2}(\tilde{r}-r)(1-q).\end{aligned} \tag{15}$$

Note that, while in a fully connected ferromagnetic model the only relevant term is the first moment, as all couplings are $O(1/N)$ and all spins are equivalent [32], in a disordered system both the first and the second moments cannot be neglected. To obtain the last line of Eq. (15), we have simply evaluated via a saddle-point computation the integral over $\vec{x}$, which corresponds to the entropy of a noninteracting system constrained on a spherical manifold. The term $\phi(m_i)$ then turns out to be proportional to $\log(1-q)$, as expected for a spherical model. For more details, we refer the interested reader to Appendix A or Ref. [25].

Factorizing the terms that depend on the Lagrange multiplier v_μ and on $\hat{h}_\mu,\hat{h}_\mu^2$, respectively, the functional $\sum_\mu\Phi(f_\mu)$ can be rewritten in a more straightforward way. Note that while the integral over $\hat{h}_\mu$ is extended over all values in $(-\infty,\infty)$, the integral over the gaps h_μ can take only positive values in the SAT phase. Since $i\hat{h}_\mu$ is a real variable by definition, namely a physical force, the integration is actually performed in the complex plane and one looks at the values of h_μ and $\hat{h}_\mu$ for which the action is stationary. We then obtain:

$$\Phi(\vec{f})=\min_v\left[fv-\log H\left(\frac{\sigma-v}{\sqrt{1-q}}\right)\right], \tag{16}$$

where we indicated as $H(x)\equiv\frac{1}{2}\text{Erfc}\left(\frac{x}{\sqrt{2}}\right)$. Differentiating the above equation with respect to v_μ, we immediately obtain the expression for the forces f_μ. Their behavior as a function of $(\sigma-v_\mu)$ is shown in Fig. 2 by progressively varying the distance from the jamming threshold. It is worth noticing that both the method and the results discussed here for the negative perceptron can be safely generalized to sphere systems in high dimensions, where the effective potential takes roughly the same form.

Focusing now on the perceptron model, from Eq. (15) we can immediately write the following stationary equations for

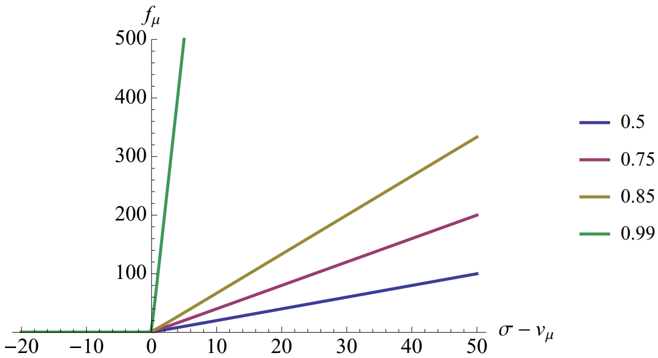

FIG. 2. Generalized forces as a function of $\sigma - v_\mu$ plotted for different values of the overlap q. In the jamming limit, as $q \to 1$, the function approaches the vertical axis (green line), in agreement with the expected divergence of the forces.

the local quantities m_i and f_μ to be solved iteratively:

$$\frac{\partial\Gamma}{\partial m_i} = 0 \quad \Rightarrow \quad m_i\left[\frac{1}{1-q} - \alpha(\tilde{r} - r)\right] = \sum_\mu \frac{\xi_i^\mu f_\mu}{\sqrt{N}}, \tag{17}$$

$$\frac{\partial\Gamma}{\partial f_\mu} = \Phi'(f_\mu) - \sum_i \frac{\xi_i^\mu m_i}{\sqrt{N}} + (1-q)f_\mu = 0. \tag{18}$$

An alternative procedure consists in deriving the belief propagation equations [33] for the x_i's and $i\hat{h}_\mu$'s and then assuming that they can be parametrized by Gaussian distributions. Assuming that the moments can be expanded up to order $O(1/N)$, we end up with the belief propagation equations in terms of single-site quantities associated with the nodes of a factor graph [33]. This procedure leads exactly to Eqs. (17) and (18). In an ordinary ferromagnet the solution of the equations above is very easy to find since the couplings are known and they do not depend on the space indices separately. In a spin glass or a generic disordered system the situation is much more complex since the ξ_i^μ's are random variables whose probability distribution is the only available information. However, for an infinite-range model in the $N \to \infty$ limit, it is possible to prove [32] that only a marginal modification is needed, namely considering an auxiliary system of $N-1$ and $M-1$ variables with the ith and the μth ones removed. Defining $\sum_{i=1}^N \frac{\xi_i^\mu m_i}{\sqrt{N}} \equiv h_\mu(\vec{m}) + \sigma$ and recalling that $\Phi'(f_\mu) = v_\mu$, we can rewrite Eq. (18) as:

$$\begin{aligned} h_\mu(\vec{m}) &= v_\mu - \sigma + (1-q)f_\mu \\ &= v_\mu - \sigma - \sqrt{1-q}\,\frac{H'\left(\frac{\sigma - v_\mu}{\sqrt{1-q}}\right)}{H\left(\frac{\sigma - v_\mu}{\sqrt{1-q}}\right)}. \end{aligned} \tag{19}$$

If the argument of the complementary error function $H(x)$ is much greater than 1, i.e., in the jamming limit, then the last term can be simplified and the resulting expression turns out to be linear in $(\sigma - v_\mu)/\sqrt{1-q}$. In this regime we exactly recover a logarithmic effective potential as a function of the average gaps. This characteristic behavior is independent of the actual dimension of the system and exactly derivable in infinite-dimensional systems [25].

The random gaps are thus written as the contribution of the so-called *cavity field*, in the parlance of spin-glass literature [32,33], and the Onsager reaction term, which provides the correction with respect to the naïve mean-field equation. This argument can be understood looking at Eq. (18) in which the value of v_μ is actually due to m_i in the absence of the μth contact. The reaction term, namely $(1-q)f_\mu$, represents instead the influence of the μth particle on the others. Hence, there is a subtle difference between the effective gap $h_\mu(\vec{m})$ and the cavity field $v_\mu - \sigma$, the field that neighboring particles would feel if one removes a single particle in the network. As mentioned above, the set of values for which $h_\mu < 0$ corresponds to the effective contacts at jamming and, since in the SAT phase the gaps are positive, the only possibility is to have negative values for the cavity field.

III. THIRD-ORDER CORRECTIONS TO THE EFFECTIVE POTENTIAL

In the previous section we showed the derivation of the TAP free energy by taking into account only the first two terms of the expansion, in a mean-field-like picture. One might be interested in defining a modified version of the perceptron—for instance, a diluted model with finite-connectivity patterns $\vec{\xi}^\mu$—or even a finite-dimensional system not exactly at jamming. In both cases, further order corrections would play a relevant role and provide a finite contribution to the perturbative expansion in the inverse of the dimension. Hence, we should take into account all the corrections to the potential coming from loopy structures by summing over triplets, quadruplets and generic combinations of links. The computation of the next order corrections to the TAP free energy turns out to be a useful tool to understand how the coarse-grained potential deviates from its critical trend. We then need to determine the following expression [28,34]:

$$\begin{aligned} \frac{\partial^3\Gamma}{\partial\eta^3} = \langle H_{\text{eff}}\rangle \frac{\partial\langle H_{\text{eff}}\rangle}{\partial\eta} + \langle H_{\text{eff}}\Upsilon_2\rangle \\ + \langle H_{\text{eff}}\,(H_{\text{eff}} - \langle H_{\text{eff}}\rangle + \Upsilon_1)^2\rangle, \end{aligned} \tag{20}$$

where Υ_n reads:

$$\Upsilon_n = \sum_i \frac{\partial}{\partial y_i}\left(\frac{\partial^n\Gamma}{\partial\eta^n}\right)(s_i - y_i)\,. \tag{21}$$

For simplicity, we indicated both derivatives, with respect to m_i and f_μ, as $(s_i - y_i)\frac{\partial}{\partial y_i}$. The resulting expression for the third-order correction thus reads:

$$\begin{aligned} \frac{\partial^3\Gamma}{\partial\eta^3} = {} & \langle H_{\text{eff}}^3\rangle + \langle H_{\text{eff}}\rangle\langle H_{\text{eff}}^2\rangle - 2\langle H_{\text{eff}}\rangle^3 \\ & - \langle H_{\text{eff}}\rangle\alpha N r(1-q) - \langle H_{\text{eff}}\rangle\alpha N q(\tilde{r} - r) \\ & + \left\langle H_{\text{eff}}\left(-\sum_{i,\mu}\frac{\delta x_i}{\sqrt{N}}\xi_i^\mu f_\mu\right)^2\right\rangle \\ & + \left\langle H_{\text{eff}}\left(-\sum_{i,\mu}\frac{\delta f_\mu}{\sqrt{N}}\xi_i^\mu m_i\right)^2\right\rangle \\ & - 2\left\langle H_{\text{eff}}^2\left(\sum_i \delta x_i \sum_\mu \frac{\xi_i^\mu f_\mu}{\sqrt{N}} + \sum_\mu \delta f_\mu \sum_i \frac{\xi_i^\mu m_i}{\sqrt{N}}\right)\right\rangle, \end{aligned} \tag{22}$$

where $\delta x_i = (x_i - m_i)$ and $\delta f_\mu = (i\hat{h}_\mu - f_\mu)$ are the relative deviations of the particle positions and the contact forces from their own mean value respectively. The first terms in Eq. (22) reminds an analogous expression for the Sherrington-Kirkpatrick (SK) model in the TAP approach [32]. The other terms are instead due to the variation of the additional parameters, on which the perceptron model actually depends. In principle, such finite-size corrections are not negligible. However, in the jamming limit, i.e., as $q \to 1$, most of these terms can be reexpressed in a more straightforward way. The fourth and the fifth term cancel with the next two terms with opposite sign, their squared moments being $(1-q)r$ and $(\tilde{r}-r)q$, respectively. We have only to deal with the first three terms and the very last one. As in the jamming limit the position and force variables tend to their coarse-grained values, i.e., $x_i \to m_i$ and $i\hat{h}_\mu \to f_\mu$, the last term can be neglected and then the most interesting contribution comes from the first three terms. In particular, the first two ones in Eq. (22) can be rewritten as:

$$\left\langle \sum_{i,\mu} \frac{\xi_i^\mu x_i i\hat{h}_\mu}{\sqrt{N}} \sum_{j,\nu} \frac{\xi_j^\nu x_j i\hat{h}_\nu}{\sqrt{N}} \sum_{k,\rho} \frac{\xi_k^\rho x_k i\hat{h}_\rho}{\sqrt{N}} \right\rangle + \left\langle \sum_{i,\mu} \frac{\xi_i^{\mu 2} x_i^2 (i\hat{h}_\mu)^2}{N} \right\rangle \left\langle \sum_{j,\nu} \frac{\xi_j^\nu x_j i\hat{h}_\nu}{\sqrt{N}} \right\rangle. \tag{23}$$

In our previous work [25] we gave an argument to support the fact the only contributing terms are those with equal indices, i.e., $i = j$ and $\mu = \nu$, as detailed in the Appendix. In the same spirit of the derivation proposed in Ref. [25] up to the second order, we can apply here the same reasoning. Furthermore, an underlying property of jamming is the presence of very weak correlations, which seem to connect the jamming transition to a mean-field-like scenario. The jamming limit corresponds indeed to a small-cage size expansion, which implies in turn that actual forces and positions can be safely replaced by their coarse-grained values. This is why the two terms above give roughly the same contribution in the jamming limit compensating the third term with opposite sign in Eq. (22). We can thus conclude that in this regime the first three terms as well give a vanishing contribution. This is in remarkable agreement with the fact that the jamming transition is well described in terms of binary interactions only [35]. We shall better clarify this point in the next section concerning the analysis of the typical scaling of the effective forces.

The idea supporting our computation is that all powers, except for the first two, actually vanish in the jamming limit. This means that, by considering the functional derivative $\frac{\partial^n \Gamma}{\partial \eta^n}\Big|_{\eta=0}$ evaluated at $\eta = 0$, the result should be identically zero. Moreover, the gaps are marginally satisfied at the transition [26] as we can immediately understand considering Eq. (19) above. We proved that as $q \to 1$ the complementary error function can be asymptotically expanded to obtain:

$$h_\mu(\vec{m}) = v_\mu - \sigma - \sqrt{1-q}\,\frac{H'\left(\frac{\sigma - v_\mu}{\sqrt{1-q}}\right)}{H\left(\frac{\sigma - v_\mu}{\sqrt{1-q}}\right)} \approx v_\mu - \sigma - \sqrt{1-q}\left[-\frac{\sigma - v_\mu}{\sqrt{1-q}} - \frac{\sqrt{1-q}}{\sigma - v_\mu}\right], \tag{24}$$

since $H'(x)/H(x) \approx -x - 1/x + O(1/x^3)$ at the leading order. The jamming condition is identified by $\frac{\sigma - v_\mu}{\sqrt{1-q}} \gg 1$, which allows us to neglect the last term in Eq. (24) and to conclude that the remaining term exactly cancels out with $v_\mu - \sigma$. As $h_\mu(\vec{m})$ approaches zero in the jamming limit, as a consequence, all further correlation functions defined from the random gaps are zero as well.

We also expect that fourth-order corrections to the effective potential do not change the disclosed behavior in the critical jamming region. The underlying reason is again related to isostaticity. At the jamming transition the number of degrees of freedom in the system exactly equals the number of contacts, each contact being shared at most by two particles. This condition allows to justify the presence of second-order corrections, and hence binary interactions, only. More work will be needed to give an analytical evidence of the vanishing trend of all corrections at any order greater than two. We leave this proposal for future research.

However, it is worth highlighting that in glassy systems a perturbative diagrammatic expansion of the correlation functions can be established if the cage of the particles is sufficiently small [36,37], namely in the high-pressure regime. The hypernetted chain (HNC) approximation [38,39] fails in the case of small-cage radius, but alternative approaches can be exploited. In particular, in Ref. [40] the authors proposed a method that allows us to rewrite the correlation functions of the glass as the correlation functions of the effective liquid. In that case, the contributions due to three-point correlators can be factorized and rewritten as a function of two-point correlators only. Our result, based on the determination of a well-defined potential exclusively in terms of the first two moments in the jamming limit, seems to be strictly correlated to this issue.

IV. LEADING AND SUBLEADING CONTRIBUTIONS TO THE FORCES NEAR JAMMING

The experimental determination of interparticle forces in glassy materials is generally a complicated task. Conversely, from an analytical point of view, the distribution of forces can be exactly reconstructed, at least in the jamming limit. In this section we show the connection between the effective forces and the gaps and we determine their leading and subleading contributions. The computation is performed in the perceptron model where only one *annealed* particle interacts with a quenched background of spherical obstacles. Anyway, the generalization to sphere systems is immediate. In that case, the gaps will depend on two labels $(\alpha\beta)$, which identify two interacting particles.

The main difficulty in determining the effective interactions in amorphous systems stems from the impossibility of writing a simple relation between the force and the gap as soon as one attempts to extend the formalism beyond jamming. Indeed, on decreasing the density, there is no reason to believe that the effective forces should remain binary. As we briefly mentioned in the Introduction, two kinds of corrections should emerge in the expression of the generalized forces, one related to finite-size effects and another due to the increasing distance from jamming. Let us focus on the first type. As in the jamming limit Eq. (22) reduces to zero, its derivative with respect to f_μ reflects the same behavior and confirms the validity of Eq. (18). Far from jamming we should instead take into account further contributions, in particular coming from $\frac{\partial}{\partial f_\mu}\frac{\partial^3\Gamma(\vec{m},\vec{f})}{\partial\eta^3}$. It requires a long and more careful analysis as, in addition to the first and second-order moments, nonlinear terms in f_μ and random contributions, like $\langle H_{\rm eff}\rangle(1-q)f_\mu$ or $\alpha Nq(\tilde{r}-r)\sum_{i=1}^N\frac{\xi_i^\mu m_i}{\sqrt{N}}$, must be embedded in the computation as well. Clearly, this much complicates Eq. (18) preventing us from writing a simple one-to-one relation between contact forces and gaps. Plotting in log-log scale the force scaling versus the gaps, we should expect a significant scatter around the linear function, more pronounced the greater the distance from jamming and then proportional to the amplitude of next many-body interactions.

Let us consider now the second kind of correction to investigate how the mutual relation between the forces and the gaps is modified on increasing the distance from jamming, even in a mean-field-like scenario. The starting point is the definition of the potential $\Phi(\vec{f})$:

$$\Phi(\vec{f}) = \min_v\left[fv - \log H\left(\frac{\sigma-v}{\sqrt{1-q}}\right)\right]$$
$$\approx \min_v\left\{f\cdot v+\theta(\sigma-v)\left[\frac{(\sigma-v)^2}{2(1-q)}+\log\left(\frac{\sigma-v}{\sqrt{1-q}}\right)\right]\right\}. \tag{25}$$

We aim at refining our estimate by taking into account also subleading terms in the asymptotic expansion of the complementary error function, provided that $(1-q)$ approaches zero in the jamming limit. Inserting the following expression in the potential:

$$H(x) = \frac{1}{2}\text{Erfc}\left(\frac{x}{\sqrt{2}}\right)\approx\frac{e^{-x^2/2}}{\sqrt{2\pi}x}\left[1+\sum_{n=1}^{\infty}(-1)^n\frac{(2n)!}{n!(\sqrt{2}x)^{2n}}\right], \tag{26}$$

we get:

$$\Phi(\vec{f})\approx\min_v\left[fv+\theta(\sigma-v)\left(\frac{(\sigma-v)^2}{2(1-q)}+\log\left(\frac{\sigma-v}{\sqrt{1-q}}\right)\right.\right.$$
$$\left.\left.-\log\left\{1-\frac{1}{[(\sigma-v)/\sqrt{1-q})]^2}\right\}\right)\right]. \tag{27}$$

Differentiating with respect to v as before, we obtain a better estimate for the generalized force:

$$f=\frac{\sigma-v}{1-q}+\frac{1}{\sigma-v}-\frac{2(1-q)}{(\sigma-v)^3\left[1-\frac{(1-q)}{(\sigma-v)^2}\right]}. \tag{28}$$

Assuming q to be not exactly one, but very close to it, we can expand the last term as a sum of odd powers of $\sigma-v$, which leads to:

$$f_\mu\approx\left(\frac{\sigma-v_\mu}{1-q}\right)\left[1+\frac{1-q}{(\sigma-v_\mu)^2}-\frac{2(1-q)^2}{(\sigma-v_\mu)^4}\right.$$
$$\left.-\frac{2(1-q)^3}{(\sigma-v_\mu)^6}+\cdots\right]$$
$$\approx\frac{1}{h_\mu(\vec{m})}\left[1+\frac{h_\mu(\vec{m})^2}{1-q}-\frac{2h_\mu(\vec{m})^4}{(1-q)^2}+\cdots\right]$$
$$f_\mu\approx\frac{1}{h_\mu(\vec{m})}\mathcal{G}\left[\frac{h_\mu(\vec{m})}{\sqrt{1-q}}\right]. \tag{29}$$

The intermediate expression in Eq. (29) is justified by the fact that at the leading order the term $\frac{\sigma-v}{1-q}$ coincides with the inverse gap, as largely explained in Ref. [25] and expected near jamming. Note that this scaling is valid only on approaching the critical transition from the SAT phase. Next terms, including odd powers of $\frac{1}{\sigma-v}$, seem to encode the effect of a rescaled inverse pressure, which typically vanishes in the SAT region. More precisely, a logarithmic interaction dominating the jamming regime emerges under the assumption to neglect terms of order $h_\mu^2/(1-q)$. They would instead contribute in the unjamming regime of the perceptron phase diagram as explained in Ref. [25]. This result, which has been proven for a fully connected system in the thermodynamic limit, is in remarkable agreement with the argument proposed by Wyart *et al.* [41,42] analyzing three-dimensional hard-sphere glass formers. It thus supports the idea of a kind of universal behavior at jamming, independent of the dimension and the microscopic details.

The last line of Eq. (29) can be easily understood by looking at the connected part of the average gap:

$$\langle h_\mu^2\rangle_c=\frac{1}{N}\sum_{ij}\xi_i^\mu\xi_j^\mu(\langle x_ix_j\rangle-m_im_j)=1-q. \tag{30}$$

Given this relation, Eq. (29) can be written in a more compact way in terms of a scaling function $\mathcal{G}$:

$$f_\mu\approx\frac{1}{\langle h_\mu\rangle}\mathcal{G}\left(\frac{\langle h_\mu\rangle}{\langle h_\mu^2\rangle_c^{1/2}},\alpha\right), \tag{31}$$

which gives rise to two different behaviors depending on the specific limit. As $\alpha\to\alpha_J$, where α_J is the critical jamming value, the scaling function $\mathcal{G}\to 1$, confirming that the only relevant scale is the interparticle gap, whereas for $\alpha<\alpha_J$ a full expression for $\mathcal{G}$ is needed. Indeed, the crossover regime, which determines where the logarithmic potential is no longer valid, is signaled by the condition: $h_\mu\sim\sqrt{1-q}$. Another way to better investigate this aspect is to consider directly the expression for the generalized forces f_μ:

$$f_\mu=-\frac{1}{\sqrt{1-q}}\frac{H'\left(\frac{\sigma-v_\mu}{\sqrt{1-q}}\right)}{H\left(\frac{\sigma-v_\mu}{\sqrt{1-q}}\right)}, \tag{32}$$

and to use Eq. (19), reported below, which highlights the relationship forces and gaps in a mean-field framework:

$$h_\mu(\vec{m})=v_\mu-\sigma+(1-q)f_\mu. \tag{33}$$

Inserting the above equation in (32), we obtain:

$$\sqrt{1-q}\,f_\mu=-\frac{H'\left(\frac{-h_\mu}{\sqrt{1-q}}+\sqrt{1-q}\,f_\mu\right)}{H\left(\frac{-h_\mu}{\sqrt{1-q}}+\sqrt{1-q}\,f_\mu\right)}. \tag{34}$$

By inverting this function with respect to f_μ, we can immediately recover the typical trend shown in Fig. 3,

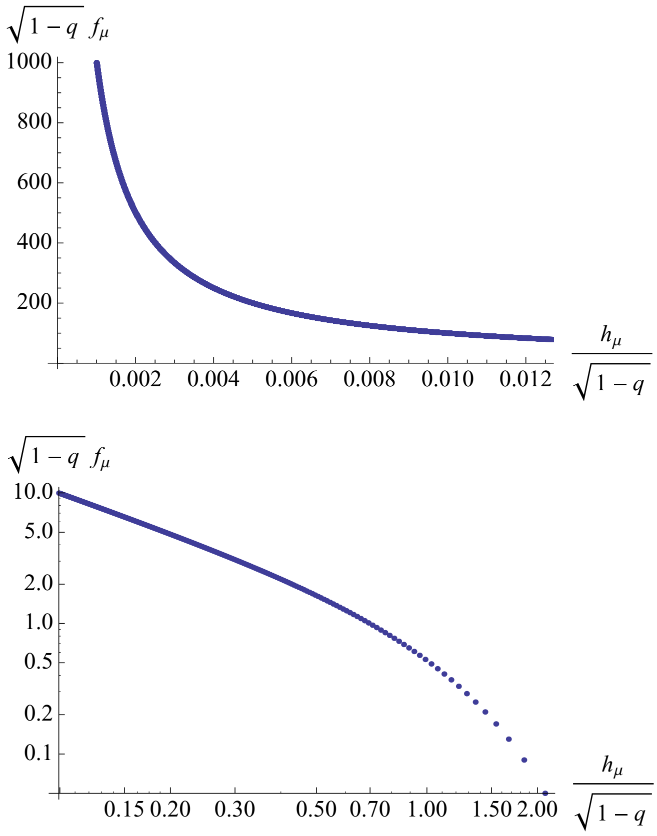

FIG. 3. Scaling function showing the generalized forces as a function of the gaps, in linear and log-log scales, well fitted via a power law $a + b\left(\frac{h_\mu}{\sqrt{1-q}}\right)^{-c}$ in the small-gap regime.

which is divergent as the gaps shrink to zero and finite otherwise.

Equation (29) and, consequently, Eq. (34) suggest a deep analogy between the free energy of a hard-sphere glass and the energy of an athermal network of logarithmic springs [35,41] if one looks at the dynamics on a time interval much greater than the collisional time but smaller than the structural relaxation time. This leads to the determination of a contact network and, thanks to the fact that all configurations are equiprobable at jamming [29,35,43], a one-to-one mapping between the particle displacements and the gaps can be established. The total number of contacts equals the number of degrees of freedom, according to the isostaticity condition. In this case, a simple relationship between the forces and the gaps can be determined as well.

One might wonder why this relation should be valid in dimensions higher than 1, where the mapping is no longer linear. The answer again lies in the underlying isostaticity condition, which characterizes the jamming transition. However, on increasing the distance from the jamming line this condition does not hold anymore and the forces are not only functions of h_μ but of a complex combination of random parameters. However, no analytical predictions about the typical scaling of the forces by taking into account also subleading terms are available.[1] Several numerical simulations have been carried out attempting to explain the observed behavior [41,44]. In particular, in Ref. [41] the authors showed that the deviation of the force from its leading behavior can be estimated numerically in molecular dynamics and the subleading term should be of order of the number of effective contacts $\delta z = z - z_c$.

Recently, a numerical analysis of both hard and soft-sphere systems has been proposed [45] to investigate the effect of multibody interactions below jamming. Concerning the hard-sphere regime, our results are in great agreement with the above-mentioned work in which the authors claim that the effective forces *trivialize to binary interactions* with no need to take care of cage fluctuations. The plot reported in their Supplementary Material, which shows the force behavior for hard disks, exactly maps onto the upper plot of Fig. 3 above. The only difference is in some spurious effects due to negative gaps as a consequence of their dynamical protocol. The other aspect pointed out in Ref. [45] concerns the non-mean-field nature of further many-body contributions, which seem to remain relevant also at jamming if one relaxes the hard-core constraint and considers softer potentials.

We caution the reader that our analysis is entirely concentrated on the SAT phase of the perceptron model, namely on the hard-sphere phase, where a robust mean-field-like behavior seems to emerge in the jamming limit. We can nevertheless study softer interactions within this framework provided some modifications. In the UNSAT phase, when $T \to 0$, the entropy is zero and the free energy exactly coincides with the energy of the system. The UNSAT phase minima are isolated and their properties can be directly studied. However, by contrast to the hard-sphere case, once the expression for the effective forces is known, we need to pay attention to correctly compute the zero-temperature limit and hence to make a suitable ansatz for the overlap parameter q and its scaling with temperature. This point will be better discussed in the following section.

V. SCALING BEHAVIORS AND CROSSOVER REGIME

We discussed above the leading behavior of the generalized forces near jamming and the emergence of a smooth logarithmic interaction in the perceptron model, computable for hard-sphere systems in high dimensions as well. However, as $h_\mu \sim \sqrt{1-q}$ the transition towards a logarithmic regime is progressively smeared out.

In this section we shall focus on the scaling functions describing the SAT and the UNSAT phase. In particular, our aim is to understand their matching in the crossover region. We shall make use of the main predictions known from full RSB solution and largely explained in Ref. [26] in order to determine

[1] Another interesting issue concerns the distribution of the effective forces acting on a single particle, instead of the total force distribution that is well known to scale near the jamming line as $P(f) \sim f^\theta$ with $\theta = 0.42311$. This computation can be presumably done in the cavity formalism [33] suggesting another interesting direction for future research.

the crossover temperature-dependent behavior between these two phases.

At low temperature in the UNSAT phase the overlap has a simple dependence on temperature given by:

$$1 - q = \chi T + O(T^2), \tag{35}$$

where χ is determined by the condition:

$$\left(1 + \frac{1}{\chi}\right) = \sqrt{\frac{\alpha}{\alpha_J(\sigma)}}. \tag{36}$$

The parameter $\alpha_J(\sigma)$ is the critical value on the jamming line, for which a generic expression was first derived by Gardner in the context of neural networks [23].

The zero-temperature limit should be carefully performed, sending T and $1 - q = \chi T$ to zero simultaneously. At jamming $\chi \to \infty$ and $q \to 1$, which determines two different scaling solutions depending on the values of q and q^*, the Edwards-Anderson parameter and the matching point, respectively. The matching point corresponds to the condition $\chi P(1,0)\sqrt{1-q^*} \sim 1$, where the probability distribution $P(q,h)$ is evaluated in $q = 1$ and $h = 0$ and it verifies the Parisi equation [32]. If $q \gg q^*$, then we recover the ordinary UNSAT phase behavior, while for $q \ll q^*$ the jamming solution occurs. We know that in the UNSAT phase the pressure is proportional to the first moment of the gap $[h]$, which in turn satisfies the following relation $[h] \equiv 1/N \sum_{\mu=1}^{M} h_\mu \theta(-h_\mu) \propto 1/\chi^2$. Using these relations we have [26]:

$$(1 - q^*) \sim \chi^{\frac{k}{1-k}}, \tag{37}$$

with an exponent $k \approx 1.41$. To make progress, we note that close to jamming the full RSB equations show a scaling regime. We focus on the regime in which the Edwards-Anderson parameter is close to the cut-off value, $q \sim q^*$, and we deduce the typical behavior in temperature. Let us suppose that temperature is raised by a finite amount, yielding:

$$(1 - q^*) \sim \chi^{\frac{k}{1-k}} \sim \chi T. \tag{38}$$

From this relation we also conclude that:

$$T^* \sim \chi^{\frac{2k-1}{1-k}}. \tag{39}$$

Given that in the soft-sphere regime (UNSAT) the pressure scales as $p \sim 1/\chi^2$ [24,26], we can write:

$$T^* \sim p^{\frac{2k-1}{2k-2}}. \tag{40}$$

For $T \sim T^*$ the UNSAT phase and the jamming solution cannot be distinguished. Note that the relation (40), connecting temperature and pressure, exactly coincides with the one proposed in Ref. [42] based on an effective medium theory argument.

Moreover, in the SAT phase $(1 - q) \sim \epsilon^k$, where ϵ stands for the linear distance from the jamming line. These two relations together lead to the condition:

$$T^* \sim \epsilon^{2k-1}. \tag{41}$$

Under these assumptions we should be able to define a scaling function of the form:

$$(1 - q) \sim \epsilon^k \mathcal{F}(T\epsilon^{1-2k}), \tag{42}$$

which guarantees the correct trend in each regime, either when its argument diverges or goes to zero. According to this simple argument, three different regimes can be highlighted: a HS-SAT regime, characterized by a zero energy manifold and studied in this paper by means of the TAP formalism; a SS-UNSAT regime, whose low-energy vibrational properties have been largely analyzed in Ref. [24]; and an *anharmonic* regime signaled by the crossover temperature T^*, which can be also related to the linear distance ϵ from jamming. Below T^* the system actually consists of an assembly of soft harmonic particles, whereas above it its vibrational properties turn out to be indistinguishable from those of a hard-sphere system.

VI. CONCLUSIONS

We presented a simple model of continuous CSP, the negative perceptron, which displays a critical jamming transition. According to whether the constraints are violated, this model displays two different phases: a SAT phase, corresponding to a hard-sphere (HS) regime, and an UNSAT one, which can be mapped to a soft-sphere (SS) problem, well described by a harmonic potential in the average gaps. This SAT-UNSAT transition exactly coincides with the jamming line.

Our main goal was to capture the most relevant features of the HS regime and to specialize then the analysis in the jammed phase. In line with the derivation proposed in Ref. [25] of the TAP free energy, which serves as a coarse-grained functional after integrating out fast degrees of freedom, we developed here the computation up to the third order. The analytical scheme is based on a formal Plefka-like expansion of the free energy, valid both in the high-temperature and in the low-temperature phases. The results obtained for the negative perceptron can be safely generalized to high-dimensional sphere models allowing to get to the same conclusions.

Our analysis showed that higher-order corrections do not contribute in the jamming limit, as correctly expected according to the isostaticity argument, a very general argument independent of the details and the dimension of the system looking into. These results suggest the idea of a close link between the jamming regime and a mean-field scenario. By contrast, third and higher-order corrections turn out to be relevant in accounting for finite-dimensional systems not exactly at jamming. They can be of great interest for numerical simulations and real glasses.

From the analysis of the effective potential near the jamming line, we also derived the leading and subleading contributions in the expression of contact forces, which correctly diverge at jamming and are finite away from the critical line. The subleading contributions can be embedded in a scaling function that depends on the average gaps and the distance from jamming. The behavior of such scaling function has been analyzed in this framework.

The discussion about the typical scaling laws dominating the jamming phase naturally led to the investigation of a crossover regime between the SAT and the UNSAT phases of the perceptron model. We determined a crossover temperature and connected it to other physical quantities of the model, such as the pressure and the linear distance from the jamming line. This crossover temperature plays a central role in defining two different regimes. Below that threshold, the system behaves like a zero-temperature assembly of soft particles, otherwise it enters the entropic-like regime.

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APPENDIX A: DERIVATION OF THE EFFECTIVE POTENTIAL IN THE SAT PHASE

To help the potential reader in deeply understanding the mathematical formalism and the results outlined in this paper, we repropose here some of the main equations already derived in Ref [25]. To define an effective potential as a function of mean particle position, we define:

$$e^{-G(\vec{m})} = e^{-\sum_{i=1}^N m_i u_i} \int d\vec{x}\; e^{-\beta H[\vec{x}]+\sum_{i=1}^N x_i u_i} = e^{-\sum_{i=1}^N m_i u_i + K[\vec{u}]}, \tag{A1}$$

which is evaluated in $\vec{u}$ such that $\vec{m} + \nabla_{\vec{u}} K[\vec{u}] = 0$. In other words, the potential $G(\vec{m})$ identifies the coarse-grained free energy once that fast degrees of freedom are integrated out. However, to get the expression for the potential $G(\vec{m})$ we find more convenient to write a more generic expression including in its definition generalized forces as well. We enforce $h_\mu = h_\mu(x)$ in the partition function via M auxiliary variables $i\hat{h}_\mu$ that are conjugated to the gaps, while the average values of the forces and the positions are enforced via N and M Lagrange multipliers, u_i and v_μ, respectively. We thus have:

$$e^{-\Gamma(\vec{m},\vec{f})} = \int d\vec{x} d\vec{h} d\vec{\hat{h}}\; e^{-\beta H[\vec{h}]+\sum_i (x_i - m_i)u_i + \sum_\mu (i\hat{h}_\mu - f_\mu)v_\mu + \sum_\mu i\hat{h}_\mu (h_\mu(x)-h_\mu)} = e^{J(\vec{u},\vec{v}) - \vec{m}\cdot\vec{u} - \vec{f}\cdot\vec{v}}, \tag{A2}$$

where $\frac{\partial J}{\partial u_i} = \frac{\partial J}{\partial v_\mu} = 0, \forall i,\mu$. From Eq. (A2) the functional $\Gamma(\vec{m},\vec{f})$ reads:

$$\Gamma(\vec{m},\vec{f}) = \sum_{i=1}^N m_i u_i + \sum_{\mu=1}^M f_\mu v_\mu - \log \int d\vec{x} d\vec{h} d\vec{\hat{h}} e^{-\beta H[\vec{h}] + \sum_i x_i u_i + \sum_\mu i\hat{h}_\mu v_\mu + \sum_\mu i\hat{h}_\mu (h_\mu(x) - h_\mu)}. \tag{A3}$$

Defining the self-overlap and the first two moments of the generalized forces, i.e.,

$$q = \frac{1}{N}\sum_{i=1}^N m_i^2, \quad r = -\frac{1}{\alpha N}\sum_{\mu=1}^M f_\mu^2, \quad \tilde{r} = \frac{1}{\alpha N}\sum_{\mu=1}^M \langle \hat{h}_\mu^2 \rangle, \tag{A4}$$

we can rewrite the potential as:

$$\Gamma(\vec{m},\vec{f}) = \sum_i m_i u_i + \sum_\mu f_\mu v_\mu - \log \int d\vec{x} d\vec{h} d\vec{\hat{h}}\; e^{S_\eta(\vec{x},\vec{h},\vec{\hat{h}})}, \tag{A5}$$

where the action $S_\eta(\vec{x},\vec{h},\vec{\hat{h}})$ reads:

$$S_\eta(\vec{x},\vec{h},\vec{\hat{h}}) = \sum_i u_i x_i + \sum_\mu i v_\mu \hat{h}_\mu - \lambda \sum_i \left(x_i^2 - N\right) - \frac{\beta}{2}\sum_\mu h_\mu^2 \theta(-h_\mu) - i \sum_\mu \hat{h}_\mu (h_\mu - \eta h_\mu(x)) - \frac{b}{2}\sum_\mu \left(\hat{h}_\mu^2 - \alpha N \tilde{r}\right). \tag{A6}$$

We introduced two additional Lagrange multipliers to fix the spherical constraint on the x_i's and the average moment of $\hat{h}_\mu^2$. Our analysis is based on a Plefka-like expansion of the free energy in terms of the coupling η, which the *effective Hamiltonian* $\mathcal{H}_{\text{eff}} = i\eta \sum_\mu \hat{h}_\mu h_\mu(\vec{x})$ actually depends on. The first derivative of the functional (A5) with respect to the parameter η, set to one at the end of the computation, coincides with the average effective Hamiltonian evaluated in the corresponding coarse-grained values, i.e.,

$$\frac{\partial \Gamma}{\partial \eta} = -\langle H_{\text{eff}} \rangle = -\sum_{i,\mu} \frac{\xi_i^\mu m_i f_\mu}{\sqrt{N}}, \tag{A7}$$

while the second-order derivative gives rise to a more involved expression, including both the connected part of the effective Hamiltonian and the partial derivatives of the additional Lagrange multipliers. As far as the second-order term is concerned, in principle one should consider several mixing terms in the second-order correction to the potential. We checked that only those with equal indices ($\mu = \nu, i = j$) provide a nonvanishing contribution. For more details we refer the interested reader to the

Appendix in Ref. [25]. Therefore, the second-order term reads:

$$\frac{\partial^2\Gamma}{\partial\eta^2} = -\left\{\langle H_{\text{eff}}^2\rangle - \langle H_{\text{eff}}\rangle^2 + \left\langle H_{\text{eff}}\left[\sum_i \frac{\partial u_i}{\partial\eta}(x_i - m_i) + \sum_\mu \frac{\partial v_\mu}{\partial\eta}(i\hat{h}_\mu - f_\mu)\right]\right\rangle\right\}, \tag{A8}$$

where the term in bracket can be decomposed as:

$$\frac{\partial u_i}{\partial\eta} = \frac{\partial^2\Gamma}{\partial\eta\partial m_i} = -i\sum_\mu \frac{\xi_i^\mu\langle\hat{h}_\mu\rangle}{\sqrt{N}} = -\sum_\mu \frac{\xi_i^\mu f_\mu}{\sqrt{N}}, \tag{A9}$$

$$\frac{\partial v_\mu}{\partial\eta} = \frac{\partial^2\Gamma}{\partial\eta\partial f_\mu} = -\sum_i \frac{\xi_i^\mu\langle x_i\rangle}{\sqrt{N}} = -\sum_i \frac{\xi_i^\mu m_i}{\sqrt{N}}. \tag{A10}$$

The expressions above allow us to rewrite the last terms in Eq. (A8) as:

$$\left\langle H\sum_i \frac{\partial u_i}{\partial\eta}(x_i - m_i)\right\rangle = \alpha N r(1-q), \tag{A11}$$

$$\left\langle H\sum_\mu \frac{\partial v_\mu}{\partial\eta}(i\hat{h}_\mu - f_\mu)\right\rangle = \alpha N q(\tilde{r} - r), \tag{A12}$$

and then the second moment of the potential as:

$$\frac{\partial^2\Gamma}{\partial\eta^2} = -\{\alpha N(-\tilde{r} + qr) + \alpha N r(1-q) + \alpha N q(\tilde{r} - r)\} = \alpha N\left[(\tilde{r} - r)(1-q)\right]. \tag{A13}$$

Gathering all relevant information above, the resulting expression up to the second order in η turns out to be:

$$\begin{aligned}\Gamma(\vec{m},\vec{f}) &= \sum_i \phi(m_i) + \sum_\mu \Phi(f_\mu) + \left.\frac{\partial\Gamma}{\partial\eta}\right|_{\eta=0}\eta + \frac{1}{2}\left.\frac{\partial^2\Gamma}{\partial\eta^2}\right|_{\eta=0}\eta^2 + \mathcal{O}(\eta^3)\\ &\approx -\frac{N}{2}\log(1-q) + \sum_\mu \Phi(f_\mu) - \sum_{i,\mu}\frac{\xi_i^\mu m_i f_\mu}{\sqrt{N}} + \frac{\alpha N}{2}(\tilde{r} - r)(1-q).\end{aligned} \tag{A14}$$

To obtain this expression we have simply evaluated by a saddle-point approximation the integral over $\vec{x}$, corresponding to the entropy of a noninteracting system constrained on a hypersphere, i.e.,

$$\phi(m) = \min_u\left[mu - \log\int dx e^{-\lambda(x^2-1)+ux}\right]. \tag{A15}$$

Using the integral representation of the δ function and neglecting irrelevant prefactors, this computation leads to a logarithmic contribution, as correctly expected in the presence of a spherical constraint:

$$\sum_i \phi(m_i) \approx -\frac{N}{2}\log(1-q). \tag{A16}$$

The functional $\sum_\mu \Phi(f_\mu)$ can also be written in a more straightforward way:

$$\Phi(f) = \min_v\left[fv - \log\int\frac{dhd\hat{h}}{2\pi}e^{-i\hat{h}(h+\sigma)+iv\hat{h}-\frac{b}{2}(\hat{h}^2-\tilde{r})}\right], \tag{A17}$$

where we actually neglected the original Hamiltonian term $\frac{\beta}{2}h^2\theta(-h)$, which is identically zero in the SAT phase. While the integral over $\hat{h}_\mu$ is extended over all values in $(-\infty,\infty)$, the integral over the gap variables h_μ is allowed to take positive valuesonly in the SAT phase. Reminding that at the saddle point the parameter b is equal to $(1-q)$, we finally obtain:

$$\Phi(\vec{f}) = \min_v\left[fv - \log H\left(\frac{\sigma - v}{\sqrt{1-q}}\right)\right], \tag{A18}$$

where we used the notation $H(x) \equiv \frac{1}{2}\mathrm{Erfc}\big(\frac{x}{\sqrt{2}}\big)$ for the complementary error function. As mentioned in the main text, by differentiating the expression above with respect to v_μ we immediately get the expression for the generalized forces f_μ, i.e.,

$$f_\mu = -\frac{1}{\sqrt{1-q}}\frac{H'\left(\frac{\sigma - v_\mu}{\sqrt{1-q}}\right)}{H\left(\frac{\sigma - v_\mu}{\sqrt{1-q}}\right)}. \tag{A19}$$

APPENDIX B: DETAILED COMPUTATION OF THE THIRD-ORDER CORRECTIONS TO THE TAP FREE ENERGY

We present here a detailed derivation of the third-order term in the TAP free energy, which was studied in the main text by using a Plefka-like expansion [27,28,34]. We have to evaluate the following expression:

$$\frac{\partial^3\Gamma}{\partial\eta^3} = \langle H_{\text{eff}}\rangle\frac{\partial\langle H_{\text{eff}}\rangle}{\partial\eta} + \langle H_{\text{eff}}\Upsilon_2\rangle + \langle H_{\text{eff}}\,(H_{\text{eff}} - \langle H_{\text{eff}}\rangle + \Upsilon_1)^2\rangle, \tag{B1}$$

where Υ_n reads:

$$\Upsilon_n = \sum_i \frac{\partial}{\partial y_i}\left(\frac{\partial^n\Gamma}{\partial\eta^n}\right)(s_i - y_i), \tag{B2}$$

and the average effective Hamiltonian $\langle H_{\text{eff}}\rangle = \sum_{i,\mu}\frac{\xi_i^\mu m_i f_\mu}{\sqrt{N}}$. We can rewrite Eq. (B1) as:

$$\begin{aligned}\frac{\partial^3\Gamma}{\partial\eta^3} &= -\langle H_{\text{eff}}\rangle\frac{\partial^2\Gamma}{\partial\eta^2} + \langle H_{\text{eff}}\Upsilon_2\rangle + \langle H_{\text{eff}}\,(H_{\text{eff}} - \langle H_{\text{eff}}\rangle + \Upsilon_1)^2\rangle\\ &= \langle H_{\text{eff}}\rangle\left[\left\langle H_{\text{eff}}^2\right\rangle - \langle H_{\text{eff}}\rangle^2 - \left\langle H_{\text{eff}}\sum_i(s_i - y_i)\frac{\partial\langle H_{\text{eff}}\rangle}{\partial y_i}\right\rangle\right] + \left\langle H_{\text{eff}}\sum_i(s_i - y_i)\frac{\partial}{\partial y_i}\frac{\partial^2\Gamma}{\partial\eta^2}\right\rangle\\ &\quad + \left\langle H_{\text{eff}}\left(H_{\text{eff}} - \langle H_{\text{eff}}\rangle - \sum_i\frac{\partial\langle H_{\text{eff}}\rangle}{\partial y_i}(s_i - y_i)\right)^2\right\rangle.\end{aligned} \tag{B3}$$

Expanding the last square term and differentiating explicitly with respect to m_i and f_μ, we obtain:

$$\begin{aligned}\frac{\partial^3\Gamma}{\partial\eta^3} &= \left\langle H_{\text{eff}}^3\right\rangle + \langle H_{\text{eff}}\rangle\left\langle H_{\text{eff}}^2\right\rangle - 2\langle H_{\text{eff}}\rangle^3 + \langle H_{\text{eff}}\rangle\left\langle H_{\text{eff}}\sum_i\frac{\partial\langle H_{\text{eff}}\rangle}{\partial m_i}(x_i - m_i)\right\rangle\\ &\quad + \langle H_{\text{eff}}\rangle\left\langle H_{\text{eff}}\sum_\mu\frac{\partial\langle H_{\text{eff}}\rangle}{\partial f_\mu}(i\hat{h}_\mu - f_\mu)\right\rangle + \left\langle H_{\text{eff}}\left(\sum_i\frac{\partial}{\partial m_i}\frac{\partial\Gamma}{\partial\eta}(x_i - m_i)\right)^2\right\rangle + \left\langle H_{\text{eff}}\left(\sum_\mu\frac{\partial}{\partial f_\mu}\frac{\partial\Gamma}{\partial\eta}(i\hat{h}_\mu - f_\mu)\right)^2\right\rangle\\ &\quad - 2\left\langle H_{\text{eff}}^2\left[\sum_i(x_i - m_i)\frac{\partial\langle H\rangle}{\partial m_i}\right\rangle + \sum_\mu(i\hat{h}_\mu - f_\mu)\frac{\partial\langle H_{\text{eff}}\rangle}{\partial f_\mu}\right]\right\rangle\\ &= \left\langle H_{\text{eff}}^3\right\rangle + \langle H_{\text{eff}}\rangle\left\langle H_{\text{eff}}^2\right\rangle - 2\langle H_{\text{eff}}\rangle^3 + \langle H_{\text{eff}}\rangle\left\langle\sum_{ij,\mu\nu}\frac{\xi_i^\mu\xi_j^\nu}{N}x_i(x_j - m_j)i\hat{h}_\mu f_\nu\right\rangle\\ &\quad + \langle H_{\text{eff}}\rangle\left\langle\sum_{ij,\mu\nu}\frac{\xi_i^\mu\xi_j^\nu}{N}x_i m_j i\hat{h}_\mu(i\hat{h}_\nu - f_\nu)\right\rangle + \left\langle H_{\text{eff}}\left(\sum_i(x_i - m_i)\sum_\mu\left(-\frac{\xi_i^\mu f_\mu}{\sqrt{N}}\right)\right)^2\right\rangle\\ &\quad + \left\langle H_{\text{eff}}\left(\sum_\mu(i\hat{h}_\mu - f_\mu)\sum_i\left(-\frac{\xi_i^\mu m_i}{\sqrt{N}}\right)\right)^2\right\rangle - 2\left\langle H_{\text{eff}}^2\left[\sum_i(x_i - m_i)\sum_\mu\frac{\xi_i^\mu f_\mu}{\sqrt{N}} + \sum_\mu(i\hat{h}_\mu - f_\mu)\sum_i\frac{\xi_i^\mu m_i}{\sqrt{N}}\right]\right\rangle.\end{aligned} \tag{B4}$$

Off-diagonal terms do not contribute in the computation, which means that we have to consider only diagonal terms, with $i = j$ and $\mu = \nu$. For more details, we refer the reader to the Appendix of Ref. [25]. The final expression reduces to:

$$\frac{\partial^3 \Gamma}{\partial \eta^3} = \langle H_{\text{eff}}^3 \rangle + \langle H_{\text{eff}} \rangle \langle H_{\text{eff}}^2 \rangle - 2\langle H_{\text{eff}} \rangle^3 - \langle H_{\text{eff}} \rangle \alpha N r(1-q) - \langle H_{\text{eff}} \rangle \alpha N q(\tilde{r} - r) + \left\langle H_{\text{eff}} \left(-\sum_{i,\mu} \frac{\delta x_i}{\sqrt{N}} \xi_i^\mu f_\mu \right)^2 \right\rangle$$
$$+ \left\langle H_{\text{eff}} \left(-\sum_{i,\mu} \frac{\delta f_\mu}{\sqrt{N}} \xi_i^\mu m_i \right)^2 \right\rangle - 2\left\langle H_{\text{eff}}^2 \left(\sum_i \delta x_i \sum_\mu \frac{\xi_i^\mu f_\mu}{\sqrt{N}} + \sum_\mu \delta f_\mu \sum_i \frac{\xi_i^\mu m_i}{\sqrt{N}} \right) \right\rangle. \tag{B5}$$

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