A simplified Parisi Ansatz

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Abstract

Based on simple combinatorial arguments, we formulate a generalized cavity method where the Random Overlap Structure (ROSt) probability space of Aizenmann, Sims and Starr is obtained in a constructive way, and use it to give a simplified derivation of the Parisi formula for the free energy of the Sherrington-Kirckpatrick model

1 Introduction

Some decades ago a very sophisticated mean field theory has been developed by Parisi to compute the thermodynamic properties of the Sherrington-Kirckpatrick (SK) model in the low temperature phase [1, 2, 3, 4, 5]. In his theory, that is obtained within the larger framework of Replica Theory [2, 5], Parisi introduced many important concepts that are now standards of the field, like the overlap distribution as order parameter and the nontrivial hypothesis that the scalar products between independent replicas of the system (overlaps) concentrates on a numeric support that is ultrametrically organized [2, 3, 5, 7, 8, 9, 10, 13].

After many years Guerra [3] and Talagrand [4] showed that this remarkable mean field theory indeed provides the correct expression for the free energy of the SK model, while Panchenko proved that the SK Gibbs measure can be perturbed into a special cascade of Point Processes (Ruelle Cascade [9, 10]) that gives the same free energy and indeed satisfy the ultrametricity assumption [10, 13]. These mathematical milestones and many other theoretical and numerical tests (see [6] and references) contributed to form the idea that at least for mean-field models this ansatz provides the correct physical properties.

Following simple combinatorial arguments we show that the same results of the RSB theory can be obtained in a constructive way without relying on the replica trick, nor averaging on the disorder. After presenting a general analysis of the SK Hamiltonian, we will show that the usual assumptions associated to L levels of Replica Symmetry Breakings (RSB, see [2, 14, 8, 9, 10]) are consistent with a hierarchical mean field (MF) theory in which the states ensemble is charted according to a sigma algebra generated by a partition of the vertex set. The method provides a constructive derivation of the Random Overlap Structure (ROSt) probability space introduced in [8] by Aizenmann, Sims and Starr. We further tested by computing the corresponding incremental pressure that one obtains from the Cavity method [2, 8, 9] and it indeed provides the correct Parisi functional.

We start by introducing the basic notation. Let consider a spin system of N spins, we indicate the spins sites by the vertex set $V = \{1, 2, ..., N\}$, marked by the label i. To each vertex is associated a unique spin variable σ_i that can be plus or minus. Formally $\sigma_i \in \Omega$, hereafter we assume $\Omega = \{+, -\}$, although our argument holds for any size of $|\Omega|$ (for this paper a modulus $|\cdot|$ applied to a discrete set returns its cardinality, for example $|V| = N$). We collect the spins into the vector

$$\sigma_V = \{\sigma_i \in \Omega : i \in V\} \tag{1}$$

that is supported by the $N-$spin vector space Ω^V, we call these vectors magnetization states. Notice that we implicitly established an arbitrary reference frame on V by labeling the spins.

Let J be some matrix of entries $J_{ij} = O(1)$. Even if the arguments we are going to present are not limited to this case, in the following we also assume that the J_{ij} entries are random and normally distributed. Then the Sherrington-Kirckpatrick model without external field is described by the Hamiltonian

$$H(\sigma_V) = \frac{1}{\sqrt{N}} \sum_{(i,j)\in W} \sigma_i J_{ij} \sigma_j \tag{2}$$

where $W = V \otimes V$ is the edges set accounting for the possible spin-spin interactions and $\sqrt{N}$ is a normalization, that in mean field models can be $|V|-$dependent. In the case of the SK model the interactions are normally distributed and we have to take a normalization that is the square root of the number of spins $|V| = N$, but the same analysis can be repeated for any coupling matrix and its relative normalization.. As usual, we can define the partition function

$$Z_N = \sum_{\sigma\in\Omega^V} \exp\left[-\beta H\left(\sigma_V\right)\right] \tag{3}$$

and the associated Gibbs measure

$$\mu\left(\sigma_V\right) = \frac{\exp\left[-\beta H\left(\sigma_V\right)\right]}{Z_N}. \tag{4}$$

The free energy density is written in term of the pressure

$$p = \lim_{N\to\infty} \frac{1}{N} \log Z_N, \tag{5}$$

the free energy per spin is given by $-p/\beta$.

A variational formula for the pressure of the SK model has been found by Parisi [1]. Following this, and after [10, 2, 9, 8] it has been proven that the average pressure per spin can be computed from the relation

$$E\left(p\right) = \inf_{q,\lambda} A_P\left(q,\lambda\right) \tag{6}$$

where $A_P\left(q,\lambda\right)$ is the Parisi functional of the (asymmetric) SK model as defined in Eq. (3), hereafter for the noise average we use the special notation $E\left(\cdot\right)$.

The minimizer is taken over two non-decreasing sequences $q = \{q_0, q_1, \ldots, q_L\}$ and $\lambda = \{\lambda_0, \lambda_1, \ldots, \lambda_L\}$ such that $q_0 = \lambda_0 = 0$ and $q_L = \lambda_L = 1$. The Parisi functional is defined as follows

$$A_P\left(q,\lambda\right) = \log 2 + \log Y_0 - \frac{\beta^2}{2} \sum_{\ell\leq L} \lambda_\ell \left(q_\ell^2 - q_{\ell-1}^2\right), \tag{7}$$

where to obtain Y_0 we apply the recursive formula $Y_{\ell-1}^{\lambda_\ell} = E_\ell Y_\ell^{\lambda_\ell}$ to the initial condition

$$Y_L = \cosh\left(\beta \sum_{\ell\leq L} z_\ell \sqrt{2q_\ell - 2q_{\ell-1}}\right), \tag{8}$$

with z_ℓ i.i.d. normally distributed and $E_\ell\left(\cdot\right)$ normal average that acts on z_ℓ. Notice that we are using a definition where the temperature is rescaled by a factor $\sqrt{2}$ respect to the usual Parisi functional. This is because the Hamiltonian $H\left(\sigma_V\right)$ do not represent the original SK model, where in the coupling matrix the contribution between spins placed on the vertex pair (i, j) is counted only once, but the so called asymmetric version, that has independent energy contributions from both (i,j) and the commuted pair (j,i). The functional for the original SK model is recovered by substituting β with $\beta/\sqrt{2}$.

2 Martingale representation

Let partition the vertex set V into a number L of subsets V_ℓ, with ℓ from 1 to L. Notice that by introducing the partition V_ℓ we are implicitly defining the invertible map that establish which vertex i is placed in which subset V_ℓ, but, as we shall see, the relevant information is in the sizes $|V_\ell| = N_\ell$ and we don't need to describe the map in detail. The partition of V induces a partition of the state

$$\sigma_V = \left\{ \sigma_{V_\ell} \in \Omega^{V_\ell} : \ell \leq L \right\} \tag{9}$$

and its support. We call the sub-vectors σ_{V_ℓ} the local magnetization states of σ_V respect to V_ℓ , formally

$$\sigma_{V_\ell} = \{ \sigma_i \in \Omega : i \in V_\ell \}. \tag{10}$$

From the above definitions we can construct the sequence of vertex sets Q_ℓ that is obtained joining the V_ℓ sets in sequence, according to the label ℓ

$$Q_\ell = \bigcup_{t \leq \ell} V_t, \tag{11}$$

this sequence is such that $Q_\ell \setminus Q_{\ell-1} = V_\ell$, the terminal point is $Q_L = V$ by definition (we remark that the order is arbitrary). Hereafter we will assume that the sets Q_ℓ are of $O(N)$ in cardinality, the size of each set is given by $|Q_\ell| = q_\ell N$, the parameters are such that $q_L = 1$ and $q_{\ell-1} \leq q_\ell$. The associated sequence of states is obtained by joining the local magnetization states, one obtains

$$\sigma_{Q_\ell} = \bigcup_{t \leq \ell} \sigma_{V_t} \in \Omega^{Q_\ell} \tag{12}$$

composed by the first ℓ sub-states σ_{V_ℓ}. Also in this case hold the relations $\sigma_{Q_\ell} \setminus \sigma_{Q_{\ell-1}} = \sigma_{V_\ell}$ and $\sigma_{Q_L} = \sigma_V$. Notice that the sets V_ℓ are given the differences between consecutive Q_ℓ sets, then

$$|V_\ell| = |Q_\ell| - |Q_{\ell-1}| = (q_\ell - q_{\ell-1})N. \tag{13}$$

In this section we will show a martingale representation for the Gibbs measure $\mu(\sigma_V)$ where we interpret the full system as the terminal point of a sequence of subsystems of increasing size. Formally, we show that one can split $H(\sigma_V)$ into a sum of "layer Hamiltonians"

$$H(\sigma_V) = \sum_{\ell \le L} H_\ell\left(\sigma_{Q_\ell}\right), \tag{14}$$

each H_ℓ describing the layer of spins V_ℓ plus an external field that account for the interface interaction with the previous layer.

To prove this we first notice that the partition of the edges set W induced by that of V is into subsets W_ℓ that contains the edges with both ends in Q_ℓ minus those with both ends in $Q_{\ell-1}$, this is also shown in the diagram of Figure 1A where the edges (i, j) are represented as points on the square $V \otimes V$. The Hamiltonian $H(\sigma_V)$ can be written as a sum of layer Hamiltonians defined as follows

$$H_\ell\left(\sigma_{Q_\ell}\right) = \frac{1}{\sqrt{|V|}} \sum_{(i,j)\in W_\ell} \sigma_i J_{ij} \sigma_j, \tag{15}$$

each contains the energy contributions from $W_\ell = (Q_\ell \otimes Q_\ell) \setminus (Q_{\ell-1} \otimes Q_{\ell-1})$. The total number of energy contributions $\sigma_i J_{ij} \sigma_j$ given by W_ℓ is

$$|W_\ell| = |Q_\ell|^2 - |Q_{\ell-1}|^2 = \left(q_\ell^2 - q_{\ell-1}^2\right) N^2, \tag{16}$$

that already unveils a familiar coefficient of the Parisi formula. We can further rearrange the components of the layer contributions by noticing that

$$(Q_\ell \otimes Q_\ell) \setminus (Q_{\ell-1} \otimes Q_{\ell-1}) = (V_\ell \otimes V_\ell) \cup (V_\ell \otimes Q_{\ell-1}) \cup (Q_{\ell-1} \otimes V_\ell), \tag{17}$$

where the right side of the equation is also shown in Figure 1B. Then, the energy contributions coming from W_ℓ can be rewritten as follows

$$\sum_{(i,j)\in W_\ell} \sigma_i J_{ij} \sigma_j = \sum_{i\in V_\ell} \sum_{j\in V_\ell} \sigma_i J_{ij} \sigma_j + \sum_{i\in V_\ell} \sum_{j\in Q_{\ell-1}} \sigma_i \left(J_{ij} + J_{ji}\right) \sigma_j \tag{18}$$

and we can identify two components, one is the layer self-interaction, that depends only on the spins σ_{V_ℓ}

$$\sum_{i\in V_\ell} \sum_{j\in V_\ell} \sigma_i J_{ij} \sigma_j = \sqrt{|V_\ell|}\, H\left(\sigma_{V_\ell}\right). \tag{19}$$

The second contribution can be interpreted as the interface interaction between the layers. Let define the interface fields

$$h_{V_\ell}\left(\sigma_{Q_{\ell-1}}\right) = \left\{h_i\left(\sigma_{Q_{\ell-1}}\right) \in \mathbb{R} : i \in V_\ell\right\}, \tag{20}$$

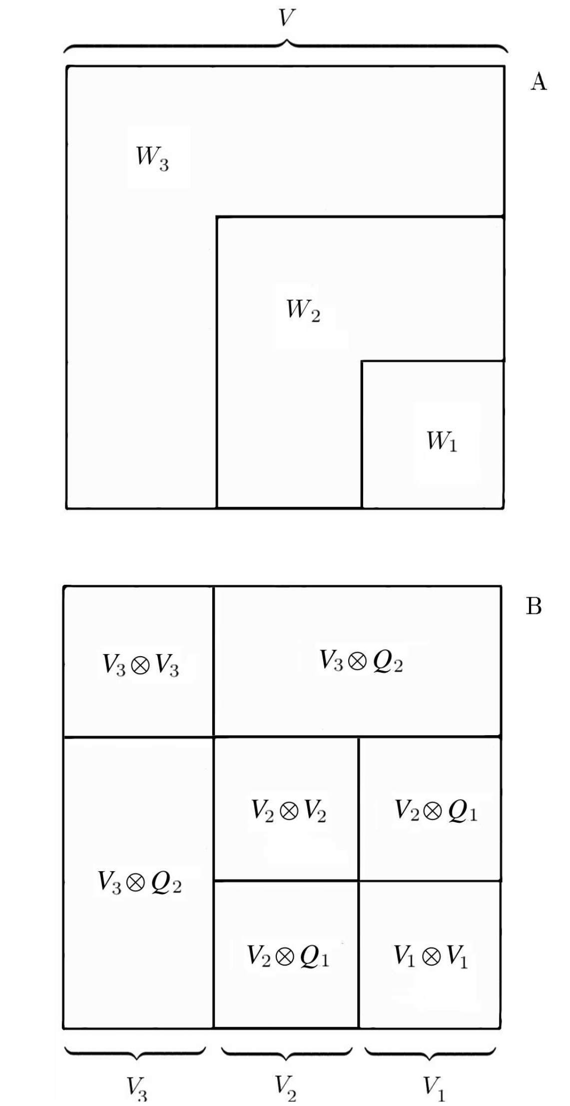

Figure 1: Figure A on top shows the partition of $V \otimes V$ following that of V for $L = 3$. The edges set is splitted into subsets W_ℓ containing all edges with both ends in Q_ℓ minus those with both ends in $Q_{\ell-1}$. The bottom figure B is intended to explain the structure of W_ℓ in terms of layers of spins: $V_\ell \otimes V_\ell$ contain the edges between the spins of V_ℓ while $V_\ell \otimes Q_{\ell-1}$ and $Q_{\ell-1} \otimes V_\ell$ contain the edges that make the interface between the layer V_ℓ and the rest of the system.

where the individual components are defined as follows

$$h_i\left(\sigma_{Q_{\ell-1}}\right)=\frac{1}{\sqrt{|Q_{\ell-1}|}}\sum_{j\in Q_{\ell-1}}\left(J_{ij}+J_{ji}\right)\sigma_j, \tag{21}$$

then the interface contributions can be written in terms of a perturbation depending on the preceding layers. Using these definitions into the previous equation we find that the SK Hamiltonian can be written as a sum of the layer energy contributions

$$H_\ell\left(\sigma_{Q_\ell}\right)=\sqrt{q_\ell-q_{\ell-1}}\,H\left(\sigma_{V_\ell}\right)+\sqrt{q_{\ell-1}}\,\sigma_{V_\ell}\cdot h_{V_\ell}\left(\sigma_{Q_{\ell-1}}\right). \tag{22}$$

Notice that the contributions of the $\ell-$th level only depend on the spins of V_ℓ and the previous V_t for $t<\ell$, but not on those for $t>\ell$, this is expression of the fact that the original system is reconstructed trough an adapted process, in which we start from the unperturbed seed $H\left(\sigma_{V_1}\right)$ of N_1 spins and then add layers of N_ℓ spins until reaching the size N. Also, notice the coefficient $\sqrt{q_\ell-q_{\ell-1}}$ in front of $H\left(\sigma_{V_\ell}\right)$ that is due to the $N-$dependent normalization of the SK Hamiltonian. This coefficient is special for fully connected random models, for a fully connected static model, like the Crurie-Weiss, would have been of order $q_\ell-q_{\ell-1}$, while for models with finite connectivity the coefficient is $O(1)$, as we shall see in short.

From the last equations we find the corresponding partition of the Gibbs measure. The partition function is obtained from the formula

$$Z_N=\sum_{\sigma_{V_1}\in\Omega^{V_1}}\exp\left[-\beta H_1\left(\sigma_{Q_1}\right)\right]\dots\sum_{\sigma_{V_\ell}\in\Omega^{V_\ell}}\exp\left[-\beta H_\ell\left(\sigma_{Q_\ell}\right)\right]\dots$$
$$\dots\sum_{\sigma_{V_L}\in\Omega^{V_L}}\exp\left[-\beta H_L\left(\sigma_{Q_L}\right)\right] \tag{23}$$

Let introduce the "layer distributions"

$$\xi_\ell\left(\sigma_{Q_\ell}\right)=\frac{\exp\left[-\beta\,H_\ell\left(\sigma_{Q_\ell}\right)\right]}{Z_{N_\ell}\left(\sigma_{Q_{\ell-1}}\right)} \tag{24}$$

with the layer partition functions given by

$$Z_{N_\ell}\left(\sigma_{Q_{\ell-1}}\right)=\sum_{\sigma_{V_\ell}\in\Omega^{V_\ell}}\exp\left[-\beta H_\ell\left(\sigma_{Q_\ell}\right)\right] \tag{25}$$

It is easy to verify that their products gives back the original Gibbs measure

$$\mu\left(\sigma_V\right) = \prod_{\ell \le L} \xi_\ell\left(\sigma_{Q_\ell}\right), \tag{26}$$

but notice that the relative weights $\xi_\ell\left(\sigma_{Q_\ell}\right)$ are measures themselves and sum to one in σ_{V_ℓ}

$$\sum_{\sigma_{V_\ell} \in \Omega^{V_\ell}} \xi_\ell\left(\sigma_{Q_\ell}\right) = 1, \; \forall\, \sigma_{Q_{\ell-1}} \in \Omega^{Q_{\ell-1}}. \tag{27}$$

We can finally write the martingale representation we where searching for. Consider the test function $f : \Omega^V \to \mathbb{R}$, then, applying the previous definitions the average $\langle f\left(\sigma_V\right)\rangle_\mu$ respect to μ is obtained through the following backward recursion. The initial condition is $f_L\left(\sigma_{Q_L}\right) = f\left(\sigma_{Q_L}\right)$, where $Q_L = V$, then we iterate the formula

$$f_{\ell-1}\left(\sigma_{Q_{\ell-1}}\right) = \sum_{\sigma_{V_\ell} \in \Omega^{V_\ell}} \xi_\ell\left(\sigma_{Q_\ell}\right) f_\ell\left(\sigma_{Q_\ell}\right) \tag{28}$$

backward until the first step $\ell = 0$ that gives the average of f respect to the Gibbs measure μ. This result is an expression of the Bayes rule and can be easily derived starting from the identity

$$\mu\left(\sigma_V\right) = \sum_{\tau_V \in \Omega^V} \mu\left(\tau_V\right) \prod_{i \in V} \left(\frac{1+\tau_i \sigma_i}{2}\right) \tag{29}$$

and substituting the definitions given before in that of $\langle f\left(\sigma_V\right)\rangle_\mu$ brings to the desired result. Notice that up to now our manipulations base on general principles and do not require any special assumption concerning the Hamiltonian.

Before going further we remark that these arguments are not limited to mean field models, for example, we can easily extend this description to the Ising Spin Glass in finite dimensions.

Let Λ be the adjacency matrix of the hyper-cubic lattice $\mathbb{Z}^d$ and substitute the Hadamard product $\Lambda \circ J$ on behalf of J and $\sqrt{g\left(\Lambda\right)}$ on behalf of $\sqrt{|V|}$, where the norm $g\left(\Lambda\right)$ is the average number of nearest-neighbors of a vertex according to Λ,

$$g_V\left(\Lambda\right) = \frac{1}{|V|} \sum_{i \in V} \sum_{j \in V} I\left(|\Lambda_{ij}| > 0\right). \tag{30}$$

If the adjacency matrix Λ is that of a fully connected graph we take $g\left(\Lambda\right) = |V|$ and recover the ASK model, otherwise for $\mathbb{Z}^d$ is $g\left(\Lambda\right) = 2d$. The result is the following generalized Hamiltonian

$$H_{\Lambda}\left(\sigma_{V}\right)=\frac{1}{\sqrt{g_{V}\left(\Lambda\right)}}\sum_{i\in V}\sum_{j\in V}\sigma_{i}\Lambda_{ij}J_{ij}\sigma_{j} \tag{31}$$

If the adjacency matrix is fully connected, which is the case of SK and other mean field models, there is no underlying geometry associated to V and we can grow the system the size we want. In finite dimensional models, however, we may have additional constraints. In the finite dimensional case, to grow an Ising spin glass on $\mathbb{Z}^d$ we should consider a cube that is enclosed in a larger cube and so on. To enclose an hyper-cubic region of $\mathbb{Z}^d$ of side lenght r and volume r^d into a larger region of side $r+k$ we need at least $(r+k)^d-r^d$ new sites to add, so the sizes of the V partition should satisfy the relation $|Q_\ell|=r_\ell^d$, or equivalently $|V_\ell|=r_\ell^d-r_{\ell-1}^d$, for some integer sequence r_ℓ.

Due to the presence of $g_V(\Lambda)$ nearest neighbors to each site, each layer contributes to the total energy with $|W_\ell|=g_V(\Lambda)|V_\ell|$ edges, each multiplied by its coupling J_{ij}. Apart from this, the partition works in the same way

$$H_{\ell}\left(\sigma_{Q_{\ell-1}},\sigma_{V_{\ell}}\right)=H_{\Lambda_{\ell}}\left(\sigma_{V_{\ell}}\right)+\sigma_{V_{\ell}}h_{V_{\ell}}\left(\sigma_{Q_{\ell-1}}\right), \tag{32}$$

where the local contributions are defined as follows

$$H_{\Lambda_{\ell}}\left(\sigma_{V_{\ell}}\right)=\frac{1}{\sqrt{g_{V}\left(\Lambda\right)}}\sum_{i\in V_{\ell}}\sum_{j\in V_{\ell}}\sigma_{i}\Lambda_{ij}J_{ij}\sigma_{j} \tag{33}$$

and the cavity fields again incorporate the interface interaction between the layers

$$h_{i}\left(\sigma_{Q_{\ell-1}}\right)=\frac{1}{\sqrt{g_{V}\left(\Lambda\right)}}\sum_{j\in Q_{\ell-1}}\left(\Lambda_{ij}J_{ij}+\Lambda_{ji}J_{ji}\right)\sigma_{j}. \tag{34}$$

For this paper we concentrate on the mean field description.

3 Incremental pressure

To make the previous formulas effective we need a way to express the pressure in terms of the Gibbs measure. This can be done by the Cavity Method [2, 14, 8], ie by relating the partition function of an $N-$spin system with that of a larger $(N+1)-$system and then computing the difference between the logarithms of the partition functions.

In this paper we follow a derivation in [7] originally due to Aizenmann et al. [8], see also [9, 10]. Define the cavity variables, ie the cavity field

$$\tilde{x}(\sigma_V) = \sqrt{\frac{2}{N}} \sum_{i\in V} \tilde{J}_{ii}\sigma_i \tag{35}$$

and the so called “fugacity term” (see[8])

$$\tilde{y}(\sigma_V) = \frac{1}{N} \sum_{i\in V} \sum_{j\in V} \sigma_i \tilde{J}_{ij}\sigma_j = \frac{1}{\sqrt{N}} \tilde{H}(\sigma_V) \tag{36}$$

that is proportional to the Hamiltonian in distribution, with a different noise matrix. First we apply the Gaussian summation rule

$$J_{ij}/\sqrt{N} \stackrel{d}{=} J_{ij}/\sqrt{N+1} + \tilde{J}_{ij}/\sqrt{N(N+1)} \tag{37}$$

to the Hamiltonian of the $N-$system to isolate the fugacity term. The matrix $\tilde{J}$ is a new noise independent from the J. The following relation holds in distribution

$$\begin{aligned} H(\sigma_V) = \frac{1}{\sqrt{N}} \sum_{i\in V} \sum_{j\in V} \sigma_i J_{ij} \sigma_j \stackrel{d}{=} \\ \stackrel{d}{=} \frac{1}{\sqrt{N+1}} \sum_{i\in V} \sum_{j\in V} \sigma_i J_{ij}\sigma_j + \frac{1}{\sqrt{N(N+1)}} \sum_{i\in V} \sum_{j\in V} \sigma_i \tilde{J}_{ij}\sigma_j, \end{aligned} \tag{38}$$

using the definition of $\tilde{y}(\sigma_V)$ the partition function is written as

$$Z_N \stackrel{d}{=} \sum_{\sigma_V\in\Omega^V} \exp\left(-\beta\sqrt{\tfrac{N}{N+1}} H(\sigma_V)\right) \cdot \exp\left(\beta\sqrt{\tfrac{N}{N+1}}\, \tilde{y}(\sigma_V)\right) \tag{39}$$

notice that the average is respect to a $N-$system at slightly shifted temperature. Now consider the system of $N+1$ spins, we separate the last spin to find

$$\begin{aligned} H\left(\sigma_{V\cup\{N+1\}}\right) = \frac{1}{\sqrt{N+1}} \sum_{i\in V\cup\{N+1\}} \sum_{j\in V\cup\{N+1\}} \sigma_i J_{ij}\sigma_j = \\ = \frac{1}{\sqrt{N+1}} \sum_{i\in V} \sum_{j\in V} \sigma_i J_{ij}\sigma_j + \frac{1}{\sqrt{N+1}} \sigma_{N+1} \sum_{i\in V} (J_{i,N+1} + J_{N+1,i})\,\sigma_i + \\ + O\left(\tfrac{1}{\sqrt{N+1}}\right). \end{aligned} \tag{40}$$

Since the sequence $J_{i,N+1}$ and its transposed are independent from the other J entries and also between themselves, we can write a more pleasant formula by using the diagonal terms of $\tilde{J}$ on behalf, ie we use again the Gaussian summation rule

$$J_{i,N+1} + J_{N+1,i} \stackrel{d}{=} \tilde{J}_{ii}\sqrt{2}, \tag{41}$$

where the superscript d specify that the equality holds in distribution. The noise relative to the vertex $N+1$ is written entirely in terms of the $\tilde{J}$ matrix. The associated partition function is computed by integrating the spin σ_{N+1}, one obtains

$$Z_{N+1} \stackrel{d}{=} \sum_{\sigma_V \in \Omega^V} \exp\left(-\beta\sqrt{\tfrac{N}{N+1}}H(\sigma_V)\right) \cdot 2\cosh\left(\beta\sqrt{\tfrac{N}{N+1}}\,\tilde{x}(\sigma_V)\right) \tag{42}$$

Now both partition functions are rewritten in terms of the $N-$system at rescaled temperature

$$\beta^* = \beta\sqrt{N/(N+1)}. \tag{43}$$

We distinguish the rescaled partition function from Z_N with a star in superscript

$$Z_N^* = \sum_{\sigma_V \in \Omega^V} \exp\left[-\beta^* H(\sigma_V)\right]. \tag{44}$$

Dividing by Z_N^* both Z_{N+1} and Z_N we can eventually write the incremental pressure in terms of the measure

$$\begin{aligned} \log Z_{N+1} - \log Z_N \stackrel{d}{=} & \\ &= \log \sum_{\sigma_V \in \Omega^V} \frac{\exp\left[-\beta^* H(\sigma_V)\right]}{Z_N^*} 2\cosh\left(\beta^*\,\tilde{x}(\sigma_V)\right) + \\ &\quad - \log \sum_{\sigma_V \in \Omega^V} \frac{\exp\left[-\beta^* H(\sigma_V)\right]}{Z_N^*} \exp\left(\beta^*\,\tilde{y}(\sigma_V)\right) = \\ &= \log \sum_{\sigma_V \in \Omega^V} \mu^*(\sigma_V)\, 2\cosh\left(\beta^*\tilde{x}(\sigma_V)\right) + \\ &\qquad - \log \sum_{\sigma_V \in \Omega^V} \mu^*(\sigma_V) \exp\left(\beta^*\tilde{y}(\sigma_V)\right). \end{aligned} \tag{45}$$

Then, apart from a rescaling $\beta^* \to \beta$ and other terms that are negligible in the thermodynamic limit the pressure can be bounded from below by the incremental pressure functional,

$$A(\tilde{x}, \tilde{y}, \mu) = \log\langle 2\cosh\left(\beta\tilde{x}(\sigma_V)\right)\rangle_\mu - \log\langle \exp\left(\beta\tilde{y}(\sigma_V)\right)\rangle_\mu. \tag{46}$$

because the pressure is always bounded from below by the limit inferior of the incremental pressure

$$p \geq \liminf_{N\to\infty} \log \frac{Z_{N+1}}{Z_N} \stackrel{d}{=} \liminf_{N\to\infty} A(\tilde{x},\tilde{y},\mu). \tag{47}$$

Until this point the analysis is well known. Let now apply some considerations from the previous section to the cavity variables. The cavity field is easy, as it is natural to split

$$\tilde{x}(\sigma_V) = \sqrt{\frac{2}{N}} \sum_{i\in V} \tilde{J}_{ii}\sigma_i = \sqrt{\frac{2}{N}} \sum_{\ell\leq L} \tilde{z}_\ell\left(\sigma_{V_\ell}\right)\sqrt{|V_\ell|} \tag{48}$$

into independent variables that are functions of the V_ℓ spins only

$$\tilde{z}_\ell\left(\sigma_{V_\ell}\right)\sqrt{|V_\ell|} = \sum_{i\in V_\ell} \tilde{J}_{ii}\sigma_i \tag{49}$$

The fugacity term is distributed like the Hamiltonian, and then we can use the same arguments before and write the decomposition

$$\tilde{y}(\sigma_V) = \frac{1}{N}\sum_{i\in V}\sum_{j\in V}\sigma_i\tilde{J}_{ij}\sigma_j = \frac{1}{N}\sum_{\ell\leq L}\sum_{(i,j)\in W_\ell}\sigma_i\tilde{J}_{ij}\sigma_j = \frac{1}{N}\sum_{\ell\leq L}\tilde{g}_\ell\left(\sigma_{Q_\ell}\right)\sqrt{|W_\ell|} \tag{50}$$

where we introduced the new variable

$$\tilde{g}_\ell\left(\sigma_{Q_\ell}\right)\sqrt{|W_\ell|} = \sum_{(i,j)\in W_\ell}\sigma_i\tilde{J}_{ij}\sigma_j, \tag{51}$$

Notice that both $\tilde{z}_\ell\left(\sigma_{V_\ell}\right)$ and $\tilde{g}_\ell\left(\sigma_{Q_\ell}\right)$ are normally distributed respect to σ_{Q_ℓ}, ie Gaussian instances and of unitary variance for all ℓ. In terms of these new variables the old cavity variables are

$$\tilde{x}(\sigma_V) = \sum_{\ell\leq L}\tilde{z}_\ell\left(\sigma_{V_\ell}\right)\sqrt{2q_\ell - 2q_{\ell-1}}, \tag{52}$$

$$\tilde{y}(\sigma_V) = \sum_{\ell\leq L}\tilde{g}_\ell\left(\sigma_{Q_\ell}\right)\sqrt{q_\ell^2 - q_{\ell-1}^2}. \tag{53}$$

and match that of the Random Overap Structure (ROSt) probability space first introduced in [8]. Indeed, this is precisely the point where the martingale representation before plays its role, as it allows to bridge between the Pure State distributions described in [2], that we can identify with the following products of layer distributions

$$\mu_\ell\left(\sigma_{Q_\ell}\right) = \prod_{k\leq\ell}\xi_k\left(\sigma_{Q_k}\right), \tag{54}$$

and the ROSt probability space given in [8], with all its remarkable mathematical features.

Putting together the functional becomes

$$A\left(q,\tilde{z},\tilde{g},\xi\right)\overset{d}{=}\log\left\langle \ldots\left\langle 2\cosh\left(\beta\sum_{\ell}\tilde{z}_{\ell}\left(\sigma_{V_{\ell}}\right)\sqrt{2q_{\ell}-2q_{\ell-1}}\right)\right\rangle_{\xi_L}\ldots\right\rangle_{\xi_1}+$$
$$-\log\left\langle \ldots\left\langle \exp\left(\beta\sum_{\ell}\tilde{g}_{\ell}\left(\sigma_{Q_{\ell}}\right)\sqrt{q_{\ell}^{2}-q_{\ell-1}^{2}}\right)\right\rangle_{\xi_L}\ldots\right\rangle_{\xi_1}. \quad (55)$$

In computing the previous formula we made the natural assumption that the partition used to split the Hamiltonian $H\left(\sigma_V\right)$ should be the same used to split the terms that appear in the cavity formula, then the dependence of A on q is both explicit and trough the distributions ξ_ℓ. It only remains to discuss the averaging properties of the layer distributions.

4 Simplified ansatz

We start by noticing that due to the vanishing coefficient $\sqrt{q_\ell - q_{\ell-1}}$ in front of $H\left(\sigma_{V_\ell}\right)$ this contribution in Eq. (22) can be actually neglected in the $L\to\infty$ limit. If we introduce the rescaled temperature parameter

$$\beta_\ell = \beta\sqrt{q_\ell - q_{\ell-1}}, \quad (56)$$

that can be made arbitrarily small in the $L\to\infty$ limit, then we can rewrite each layer in terms of an SK model of size N_ℓ at temperature β_ℓ

$$\beta H_\ell\left(\sigma_{Q_\ell}\right)\overset{d}{=}\beta_\ell\left[H\left(\sigma_{V_\ell}\right)+\sigma_{V_\ell}\cdot h^{*}_{V_\ell}\left(\sigma_{Q_{\ell-1}}\right)\right] \quad (57)$$

subject to the (strong) external field

$$h^{*}_{V_\ell}\left(\sigma_{Q_{\ell-1}}\right)=\frac{1}{\sqrt{|V_\ell|}}\sum_{j\in Q_{\ell-1}}\left(J_{ij}+J_{ji}\right)\sigma_j, \quad (58)$$

whose magnitude diverges in the $L\to\infty$ limit due to the $\sqrt{|V_\ell|}$ normalization. Then, for any finite temperature β we can make N and L large enough to have a q_ℓ sequence for which $\beta_\ell < \beta_c$ at any ℓ, and it is established since [11] and [12] that in the high temperature regime the annealed averages needed to compute Eq. (55) matches the quenched ones (the layers are Replica Symmetric).

To make this argument more precise let consider the Hamiltonian

$$\bar{H}_\ell\left(\sigma_{Q_\ell}\right) = \sqrt{q_{\ell-1}}\,\sigma_{V_\ell}\cdot h_{V_\ell}\left(\sigma_{Q_{\ell-1}}\right), \tag{59}$$

in the Thermodynamic Limit and for $L \to \infty$ one can compute the averages in Eq.(28) according to the Hamiltonian $\bar{H}_\ell\left(\sigma_{Q_\ell}\right)$ instead of $H_\ell\left(\sigma_{Q_\ell}\right)$, this will be shown at the end of this section. The partition function of the $\bar{H}_\ell$ model can be computed exactly and one finds

$$\begin{aligned}\bar{Z}_{N_\ell}\left(\sigma_{Q_{\ell-1}}\right) &= \sum_{\sigma_{V_\ell}\in\Omega^{V_\ell}} \exp\left[\beta\sqrt{q_{\ell-1}}\,\sigma_{V_\ell}\cdot h_{V_\ell}\left(\sigma_{Q_{\ell-1}}\right)\right] = \\ &= \prod_{i\in V_\ell} 2\cosh\left(\beta\sqrt{q_{\ell-1}}\,h_i\left(\sigma_{Q_{\ell-1}}\right)\right) = \exp\left[-\beta\bar{F}_{N_\ell}\left(\sigma_{Q_{\ell-1}}\right)\right].\end{aligned} \tag{60}$$

Moreover, following [2], from Boltzmann theory one can show that at equilibrium the logarithm of the associated Gibbs distribution is proportional to the fluctuations around the average internal energy

$$\bar{\xi}_\ell\left(\sigma_{Q_\ell}\right) \propto \exp\left[-\beta\Delta\bar{H}\left(\sigma_{Q_\ell}\right)\right] \tag{61}$$

where the fluctuations are defined as follows

$$\Delta\bar{H}\left(\sigma_{Q_\ell}\right) = \sqrt{q_{\ell-1}}\left[\sigma_{V_\ell}\cdot h_{V_\ell}\left(\sigma_{Q_{\ell-1}}\right) - \langle\sigma_{V_\ell}\cdot h_{V_\ell}\left(\sigma_{Q_{\ell-1}}\right)\rangle_{\bar{\xi}_\ell}\right]. \tag{62}$$

Notice that for the $\bar{H}_\ell$ model the energy overlap $\langle\Delta\bar{H}\left(\sigma_{Q_\ell}\right)\Delta\bar{H}\left(\tau_{Q_\ell}\right)\rangle_{\bar{\mu}\otimes\bar{\mu}}$ can be computed exactly, but this long algebraic work is not necessary in order to compute the Parisi functional. In fact, by Central Limit Theorem in the large N limit the fluctuations $\Delta\bar{H}\left(\sigma_{Q_\ell}\right)$ converge to a Gaussian set indexed by $\sigma_{Q_{\ell-1}}$, whose canonical variance is

$$\langle\Delta H\left(\sigma_{Q_\ell}\right)^2\rangle_{\bar{\xi}_\ell} = \frac{\partial}{\partial\beta^2}\bar{F}_{N_\ell}\left(\sigma_{Q_{\ell-1}}\right) = N\bar{\gamma}_\ell\left(\sigma_{Q_{\ell-1}}\right)^2. \tag{63}$$

Then, under the Gibbs measure $\bar{\xi}_\ell$ the energy fluctuations can be approximated in distribution by a Derrida's Random Energy Model (REM, see [10, 7])

$$H_\ell\left(\sigma_{Q_\ell}\right) = \varepsilon_\ell(N) + \Delta\bar{H}\left(\sigma_{V_\ell}\right) \overset{d}{=} \varepsilon_\ell(N) + \bar{\gamma}_\ell\left(\sigma_{Q_{\ell-1}}\right) g^*_\ell\left(\sigma_{V_\ell}\right)\sqrt{N} \tag{64}$$

where $g^*_\ell\left(\sigma_{V_\ell}\right)$ are i.i.d. normally distributed variables with covariance matrix

$$E\left[g^*_\ell\left(\sigma_{V_\ell}\right)g^*_\ell\left(\tau_{V_\ell}\right)\right]=\prod_{i\in V_\ell}I\left(\sigma_i=\tau_i\right). \tag{65}$$

Since the SK measure is weakly exchangeable, although $\bar{\gamma}_\ell\left(\sigma_{Q_{\ell-1}}\right)$ may depends on the spins of $\sigma_{Q_{\ell-1}}$ trough the cavity fields $h_{V_\ell}\left(\sigma_{Q_{\ell-1}}\right)$ the only way to enforce this invariance is to admit that eventually

$$\bar{\gamma}_\ell\left(\sigma_{Q_{\ell-1}}\right)^2\overset{d}{=}\bar{\gamma}_\ell^2 \tag{66}$$

under ξ_ℓ average for some positive number $\bar{\gamma}_\ell^2$. Notice that $\bar{\gamma}_\ell\left(\sigma_{Q_{\ell-1}}\right)^2=\bar{\gamma}_\ell^2$ independent of $\sigma_{Q_{\ell-1}}$doesn't mean that the sign of $\bar{\gamma}_\ell\left(\sigma_{Q_{\ell-1}}\right)$ is fixed, and under the full measure $\bar{\mu}$ one may have different correlations between the full states σ_{Q_ℓ} due to the averaging effect of $\bar{\mu}$ on the interface fields. The term $\varepsilon_\ell\left(N\right)$ in Eq. (64) is a constant that does not depend on the spins and we can interpret it as the deterministic component of $H_\ell\left(\sigma_{Q_\ell}\right)$ under Gibbs measure, for the SK model we expect $\varepsilon_\ell\left(N\right)=0$ for all ℓ but its exact value is not important in computing the Parisi functional because in the end it will washed out by the difference between the logarithms.

Before discussing the physical features let verify that the simplified ansatz provides the correct Parisi functional. As is shown in [9], the thermodynamic limit of a Gaussian REM of amplitude $\bar{\gamma}$ is proportional in distribution to a Poisson Point Process (PPP) of rate

$$\lambda_\ell=\frac{\sqrt{2\log 2}}{\bar{\gamma}_\ell}. \tag{67}$$

The system at equilibrium is then decomposed into a large (eventually infinite) number L of subsystems, one for each vertex set V_ℓ, whose Gibbs measure are proportional in distribution to a sequence of Poisson-Dirichlet (PD) point processes, ie the Gibbs measures that describe the layers are proportional in distribution to Poisson Point Processes (PPP) [9, 10] of rate λ_ℓ, that is a function of q but independent from the spins $\sigma_{Q_{\ell-1}}$.

By the special average property of PPP [9, 10] (see also the Little Theorem of [14]) for any positive test function $f:\Omega^N\to\mathbb{R}^+$ we have

$$\sum_{\sigma_{V_\ell}\in\Omega^{V_\ell}}\xi_\ell\left(\sigma_{Q_\ell}\right)f\left(\sigma_{Q_\ell}\right)\overset{d}{=}C_\ell\left(\sum_{\sigma_{V_\ell}\in\Omega^{V_\ell}}f\left(\sigma_{Q_\ell}\right)^{\lambda_\ell}\right)^{1/\lambda_\ell} \tag{68}$$

for some constant C_ℓ that may depend on β but not on the spins. Then the random average $\langle f(\sigma_V)\rangle_\mu$ is obtained through the following recursion

$$f_{\ell-1}\left(\sigma_{Q_{\ell-1}}\right)^{\lambda_\ell} \stackrel{d}{=} K_\ell \left(\frac{1}{2^{|V_\ell|}} \sum_{\sigma_{V_\ell} \in \Omega^{U_\ell}} f_\ell\left(\sigma_{V_\ell}\right)^{\lambda_\ell} \right), \tag{69}$$

that holds in distribution, with $K_\ell = 2^{|V_\ell|} C_\ell^{\lambda_\ell}$. This allows to compute the main contribution

$$\left\langle \dots \left\langle 2\cosh\left(\beta \sum_\ell \tilde{z}_\ell\left(\sigma_{V_\ell}\right)\sqrt{2q_\ell - 2q_{\ell-1}}\right)\right\rangle_{\xi_L} \dots \right\rangle_{\xi_1} \stackrel{d}{=} Y_0 \exp\left(\sum_{\ell \le L} \log K_\ell\right) \tag{70}$$

by applying the recursive relation

$$Y_{\ell-1}^{\lambda_\ell} = \frac{1}{2^{|V_\ell|}} \sum_{\sigma_{V_\ell} \in \Omega^{V_\ell}} Y_\ell^{\lambda_\ell} \tag{71}$$

to the initial condition

$$Y_L = 2\cosh\left(\beta \sum_{\ell \le L} \tilde{z}_\ell\left(\sigma_{V_\ell}\right)\sqrt{2q_\ell - 2q_{\ell-1}}\right) \tag{72}$$

down to the last $\ell = 0$. Notice that in the recursion the average over σ_{V_ℓ} is uniform, under uniform distribution both $\tilde{z}_\ell\left(\sigma_{V_\ell}\right)$ and $\tilde{g}_\ell\left(\sigma_{Q_\ell}\right)$ are normally distributed and independent from the previous spin layers $\sigma_{Q_{\ell-1}}$, then we can take

$$\tilde{z}_\ell\left(\sigma_{V_\ell}\right) \stackrel{d}{=} z_\ell,\ \tilde{g}_\ell\left(\sigma_{Q_\ell}\right) \stackrel{d}{=} g_\ell, \tag{73}$$

with z_ℓ and g_ℓ i.i.d. normally distributed, and change the uniform average over σ_{V_ℓ} into a Gaussian average E_ℓ acting on these new variables. We compute the fugacity term in the same way,

$$\left\langle \dots \left\langle \exp\left(\beta \sum_\ell \tilde{g}_\ell\left(\sigma_{Q_\ell}\right)\sqrt{q_\ell^2 - q_{\ell-1}^2}\right)\right\rangle_{\xi_L} \dots \right\rangle_{\xi_1} \stackrel{d}{=}$$
$$\stackrel{d}{=} \exp\left(\frac{\beta^2}{2}\sum_{\ell \le L}\lambda_\ell\left(q_\ell^2 - q_{\ell-1}^2\right) + \sum_{\ell \le L}\log K_\ell\right). \tag{74}$$

Putting together the contributions depending from K_ℓ cancel out and one finds

$$\exp A_P(q,\lambda) = Y_0 \exp\left(-\frac{\beta^2}{2}\sum_{\ell \le L}\lambda_\ell\left(q_\ell^2 - q_{\ell-1}^2\right)\right) \tag{75}$$

that is exactly the Parisi functional as is defined in the introduction. Notice that in this equation and in the previous we implicitly assumed that the sequences q and λ are exactly those that approximate the SK model. The lower bound in the variational formula can be easily obtained from the knowledge of the Parisi functional by minimizing on the possible sequences q and λ

$$A\left(q,\tilde{z},\tilde{g},\xi\right)\geq\inf_{q,\lambda}A_P\left(q,\lambda\right),\tag{76}$$

while the upper bound can be checked, at least for the SK model, by Guerra-Toninelli interpolation [3].

It remain to prove that one can compute the averages in Eq.(28) according to the Hamiltonian $\bar{H}_\ell\left(\sigma_{Q_\ell}\right)$ instead of $H_\ell\left(\sigma_{Q_\ell}\right)$, consider the full layer distribution

$$\xi_\ell\left(\sigma_{Q_{\ell-1}}\right)=\frac{1}{Z_{N_\ell}\left(\sigma_{Q_{\ell-1}}\right)}\exp\left[-\beta\,H_\ell\left(\sigma_{Q_\ell}\right)\right],\tag{77}$$

define its version without external field

$$\xi_\ell^*\left(\sigma_{V_\ell}\right)=\frac{1}{Z_{N_\ell}^*}\exp\left[-\beta_\ell H\left(\sigma_{V_\ell}\right)\right]\tag{78}$$

that is simply an SK model at (eventually high) temperature $\beta_\ell=\beta\sqrt{q_\ell-q_{\ell-1}}$. Now, if we assume the thermodynamic limit exists we can use the Boltzmann theory and express the thermodynamics fluctuations as they where a Gaussian set. Start from the general average formula

$$f_{\ell-1}\left(\sigma_{Q_{\ell-1}}\right)=\sum_{\sigma_{V_\ell}\in\Omega^{V_\ell}}\xi_\ell\left(\sigma_{Q_\ell}\right)f_\ell\left(\sigma_{Q_\ell}\right),\tag{79}$$

using the REM-PPP relation on $H\left(\sigma_{V_\ell}\right)$ and the PPP average properties one finds that the formula for the partition function is

$$\begin{aligned}
Z_{N_\ell}\left(\sigma_{Q_{\ell-1}}\right) = \sum_{\sigma_{V_\ell}\in\Omega^{V_\ell}} \exp\left[-\beta\, H_\ell\left(\sigma_{Q_\ell}\right)\right] = \\
= \sum_{\sigma_{V_\ell}\in\Omega^{V_\ell}} \exp\left[-\beta_\ell\, H\left(\sigma_{V_\ell}\right) - \beta \bar{H}_\ell\left(\sigma_{Q_\ell}\right)\right] = \\
= Z^*_{N_\ell} \sum_{\sigma_{V_\ell}\in\Omega^{V_\ell}} \xi^*_\ell\left(\sigma_{V_\ell}\right) \exp\left[-\beta \bar{H}_\ell\left(\sigma_{Q_\ell}\right)\right] = \\
= Z^*_{N_\ell} C^*_{N_\ell} \left[\frac{1}{2^{N_\ell}} \sum_{\sigma_{V_\ell}\in\Omega^{V_\ell}} \exp\left[-\beta \lambda^*_\ell \bar{H}_\ell\left(\sigma_{Q_\ell}\right)\right]\right]^{\frac{1}{\lambda^*_\ell}} = \\
= \left[\frac{Z^*_{N_\ell} C^*_{N_\ell}}{2^{N_\ell/\lambda^*_\ell}}\right] \bar{Z}^*_{N_\ell}\left(\sigma_{Q_{\ell-1}}\right)^{\frac{1}{\lambda^*_\ell}} \qquad (80)
\end{aligned}$$

where λ^*_ℓ is the rate of the associated PPP. In the last two steps we introduced the partition function $\bar{Z}^*_{N_\ell}\left(\sigma_{Q_{\ell-1}}\right)$ associated to the $\bar{H}_\ell$ model at slightly rescaled temperature $\bar{\beta}^* = \lambda^*_\ell \beta$. Let $\bar{\xi}^*_\ell\left(\sigma_{Q_{\ell-1}}\right)$ the associated Gibbs measure associated to the $\bar{H}_\ell$ model at rescaled temperature $\bar{\beta}^*$, then the formula for the average becomes

$$\begin{aligned}
f_{\ell-1}\left(\sigma_{Q_{\ell-1}}\right) = \sum_{\sigma_{V_\ell}\in\Omega^{V_\ell}} \xi_\ell\left(\sigma_{Q_\ell}\right) f_\ell\left(\sigma_{Q_\ell}\right) = \\
= \frac{1}{Z_{N_\ell}\left(\sigma_{Q_{\ell-1}}\right)} \sum_{\sigma_{V_\ell}\in\Omega^{V_\ell}} \exp\left[-\beta\, H_\ell\left(\sigma_{Q_\ell}\right)\right] f_\ell\left(\sigma_{Q_\ell}\right) = \\
= \frac{1}{Z_{N_\ell}\left(\sigma_{Q_{\ell-1}}\right)} \sum_{\sigma_{V_\ell}\in\Omega^{V_\ell}} \exp\left[-\beta_\ell H\left(\sigma_{V_\ell}\right) - \beta \bar{H}_\ell\left(\sigma_{Q_\ell}\right)\right] f_\ell\left(\sigma_{Q_\ell}\right) = \\
= \frac{Z^*_{N_\ell}}{Z_{N_\ell}\left(\sigma_{Q_{\ell-1}}\right)} \sum_{\sigma_{V_\ell}\in\Omega^{V_\ell}} \xi^*_\ell\left(\sigma_{V_\ell}\right) \exp\left[-\beta \bar{H}_\ell\left(\sigma_{Q_\ell}\right)\right] f_\ell\left(\sigma_{Q_\ell}\right) = \\
= \frac{Z^*_{N_\ell} C^*_{N_\ell}}{Z_{N_\ell}\left(\sigma_{Q_{\ell-1}}\right)} \left[\frac{1}{2^{N_\ell}} \sum_{\sigma_{V_\ell}\in\Omega^{V_\ell}} \exp\left[-\beta \lambda^*_\ell \bar{H}_\ell\left(\sigma_{Q_\ell}\right)\right] f_\ell\left(\sigma_{Q_\ell}\right)^{\lambda^*_\ell}\right]^{\frac{1}{\lambda^*_\ell}} = \\
= \left[\frac{Z^*_{N_\ell} C^*_{N_\ell}}{2^{N_\ell/\lambda^*_\ell}}\right] \frac{\bar{Z}^*_{N_\ell}\left(\sigma_{Q_{\ell-1}}\right)^{\frac{1}{\lambda^*_\ell}}}{Z_{N_\ell}\left(\sigma_{Q_{\ell-1}}\right)} \left[\sum_{\sigma_{V_\ell}\in\Omega^{V_\ell}} \bar{\xi}^*_\ell\left(\sigma_{Q_{\ell-1}}\right) f_\ell\left(\sigma_{Q_\ell}\right)^{\lambda^*_\ell}\right]^{\frac{1}{\lambda^*_\ell}} \qquad (81)
\end{aligned}$$

and putting together everything simplifies to

$$f_{\ell-1}\left(\sigma_{Q_{\ell-1}}\right)=\left[\sum_{\sigma_{V_\ell}\in\Omega^{V_\ell}}\bar{\xi}_\ell^*\left(\sigma_{Q_{\ell-1}}\right)f_\ell\left(\sigma_{Q_\ell}\right)^{\lambda_\ell^*}\right]^{\frac{1}{\lambda_\ell^*}} \tag{82}$$

This formula does not depend from N and then holds also in the thermodynamic limit $N\to\infty$, where we can actually take β_ℓ to zero, and applying to the recursion one would find

$$\lim_{\lambda_\ell^*\to 1}\left[\sum_{\sigma_{V_\ell}\in\Omega^{V_\ell}}\bar{\xi}_\ell^*\left(\sigma_{Q_{\ell-1}}\right)f_\ell\left(\sigma_{Q_\ell}\right)^{\lambda_\ell^*}\right]^{\frac{1}{\lambda_\ell^*}}=\sum_{\sigma_{V_\ell}\in\Omega^{V_\ell}}\bar{\xi}_\ell\left(\sigma_{Q_{\ell-1}}\right)f_\ell\left(\sigma_{Q_\ell}\right) \tag{83}$$

The idea is that for positive f_ℓ and in the limit $q_\ell-q_{\ell-1}\to 0$ one would have that the SK average is taken at infinite temperature, then equivalent to a PPP of rate $\lambda_\ell^*\to 1$.

5 Conclusive remarks

Even if we easily obtained the functional, from the physical point of view this short analysis still didn't clarified what is the proper approximation for $\Delta\bar{H}\left(\sigma_{Q_\ell}\right)$ under the full measure μ (see Figure 2). If one assume that the same approximation used under ξ_ℓ holds also under μ it would be equivalent to assert that

$$H_\ell\left(\sigma_{Q_\ell}\right)\to\sqrt{q_\ell-q_{\ell-1}}\,H\left(\sigma_{V_\ell}\right) \tag{84}$$

and then the model would be simply a sum of smaller independent systems at higher temperatures. By the way, we remark once again that for large L the coefficients $q_\ell-q_{\ell-1}$ vanish respect to $q_{\ell-1}$, and is unlikely that this ansatz can return stable solutions in any fully connected model.

In fact, this would be a quite orthodox mean-field ansatz [15] where the external field acting on the layer is irrelevant, although in SK the number of pairwise energy contributions from the interfaces is much larger than the energy contributions from the spins in the same layer, as already predicted in [16]. We expect that the proper approximation under μ would be the Generalized Random Energy model (GREM) [9, 10], where

$$H_\ell\left(\sigma_{Q_\ell}\right)\stackrel{d}{=}\varepsilon_\ell\left(N\right)+\sqrt{N}\,\bar{\gamma}_\ell g_\ell\left(\sigma_{Q_\ell}\right) \tag{85}$$

and λ_ℓ is the sequence of free parameters that controls the variance, and $g_\ell\left(\sigma_{Q_\ell}\right)$ is a

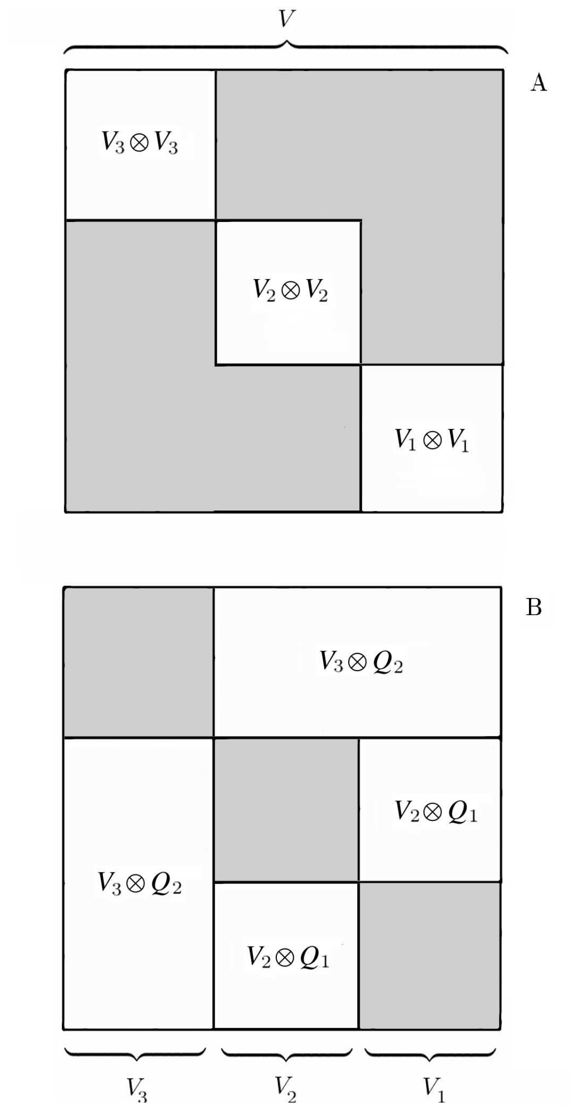

Figure 2: Two extreme pictures for the RSB ansatz for $L = 3$. The diagram shows the edges that contributes to the energy in the orthodox MF ansatz, top figure A, where the Hamiltonian operator is diagonal under Gibbs measure, and a situation where the interfaces dominate the total energy, lower figure B. We expect the second option to be much more likely for fully connected models, because in such models the interfaces are overwhelmingly large respect to the contribution from edges between spins of the same layer.

collection of normal random variables of covariance matrix

$$E\left[g_\ell\left(\sigma_{Q_\ell}\right)g_\ell\left(\tau_{Q_\ell}\right)\right]=\prod_{i\in Q_\ell}I\left(\sigma_i=\tau_i\right), \tag{86}$$

with $E(\cdot)$ representing the normal average that acts on the variables $g_\ell\left(\sigma_{Q_\ell}\right)$. The difference with the orthodox MF ansatz is in that by changing $g_\ell^*\left(\sigma_{V_\ell}\right)$ with $g_\ell\left(\sigma_{Q_\ell}\right)$ for any magnetization states σ_V and τ_V with $\sigma_{V_\ell}=\tau_{V_\ell}$, $\sigma_{Q_{\ell-1}}\neq\tau_{Q_{\ell-1}}$ now one has

$$E\left[H_\ell\left(\sigma_{Q_\ell}\right)H_\ell\left(\tau_{Q_\ell}\right)\right]=0 \tag{87}$$

instead of $\tilde{\gamma}_\ell^2 N$. This computing scheme is essentially a Guerra-Toninelli interpolation between the layers, a method first used by Billoire [17] to compute the finite size corrections to the SK model. One can easily verify that both ansatz gives the same recursive formula for the average, but this ansatz, that we interpret as fully equivalent to the RSB ansatz, bases on the fact that for large L the layer behavior is mostly dominated by the interface interaction from the previous layers,

$$H_\ell\left(\sigma_{Q_\ell}\right)\to\sqrt{q_{\ell-1}}\,\sigma_{V_\ell}\cdot h_{V_\ell}\left(\sigma_{Q_{\ell-1}}\right), \tag{88}$$

which seems the case indeed for any fully connected mean-field model, at least. Notice that in the termodynamic limit the associated Gibbs measure is distributed proportionally to a cascade of PPP, known as Ruelle Cascade [10, 9, 8, 13], that is known to have an ultrametric overlap support. For SK this property has been first proven in [13], where it is shown that the Gibbs measure of the SK model can be infinitesimally perturbed into a Ruelle Cascade.

In conclusion, it seems not possible to distinguish between the orthodox mean field ansatz (the Gibbs measure is a product measure) from the RSB ansatz (the measure is a Ruelle Cascade) by only looking at the Parisi Formula. Nonetheless, we argument that the orthodox mean field theory is unlikely to hold in SK, due to expected dominance of the interface contribution. Weather an orthodox mean-field ansatz is meaningful in some sense for the SK model we still cannot say, although it seems related to the replica trick. Despite this, we think it would naturally apply to many other disordered systems, like random polymers, or any other model with low connectivity between the layers.

6 Acknowledgments

I wish to thank Amin Coja-Oghlan (Goethe University, Frankfurt) for sharing his views on Graph Theory and Replica Symmetry Breaking, which deeply influenced this work. I also wish to thank Giorgio Parisi, Pan Liming, Francesco Guerra, Pietro Caputo, Nicola Kistler, Demian Battaglia, Francesco Concetti and Riccardo Balzan for interesting discussions and suggestions. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No [694925]).

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