1 Two-neutrino oscillations

In the case of two-neutrino mixing between the flavors α and β, we deal with only one mixing angle θ, such that

$$\begin{pmatrix} |\nu_\alpha(t)\rangle \\ |\nu_\beta(t)\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} |\nu_1(t)\rangle \\ |\nu_2(t)\rangle \end{pmatrix}, \tag{1}$$

where ν_i is the mass eigenstate associated with the mass m_i.

The probability that a neutrino produced in the flavor eigenstate α at the time $t = 0$ is detected in the same flavor at t is given by

$$P_{\alpha\to\alpha} = \left|\langle\nu_\alpha(0)|\nu_\alpha(t)\rangle\right|^2. \tag{2}$$

From Eq. (1), the state ν_α at t is

$$|\nu_\alpha(t)\rangle = \cos\theta|\nu_1(t)\rangle + \sin\theta|\nu_2(t)\rangle,$$

and, therefore, the considered amplitude is

$$\begin{aligned} \langle\nu_\alpha(0)|\nu_\alpha(t)\rangle &= \cos^2\theta\langle\nu_1(0)|\nu_1(t)\rangle + \sin^2\theta\langle\nu_2(0)|\nu_2(t)\rangle \\ &= \cos^2\theta e^{-iE_1t} + \sin^2\theta e^{-iE_2t}, \end{aligned}$$

because the mass eigenstates are orthonormal, $\langle\nu_i(t)|\nu_j(t)\rangle = \delta_{ij}$, and the time evolution of the state $|\nu_i(t)\rangle$ is given by $|\nu_i(t)\rangle = e^{-iE_it}|\nu_i(0)\rangle$.

Now we are ready to calculate the probability from Eq. (2),

$$\begin{aligned} P_{\alpha\to\alpha} &= (\cos^2\theta e^{-iE_1t} + \sin^2\theta e^{-iE_2t})(\cos^2\theta e^{+iE_1t} + \sin^2\theta e^{+iE_2t}) \\ &= \cos^4\theta + \sin^4\theta + \sin^2\theta\cos^2\theta(e^{-i(E_2-E_1)t} + e^{+i(E_2-E_1)t}) \\ &= (\cos^2\theta + \sin^2\theta)^2 - 2\sin^2\theta\cos^2\theta + 2\sin^2\theta\cos^2\theta\cos[(E_2-E_1)t] \\ &= 1 - 2\sin^2\theta\cos^2\theta\{1 - \cos[(E_2-E_1)t]\} \\ &= 1 - 4\sin^2\theta\cos^2\theta\sin^2\frac{(E_2-E_1)t}{2} \\ &= 1 - \sin^2 2\theta\sin^2\frac{(E_2-E_1)t}{2}, \end{aligned}$$

In the above equations, we have used the identities $2\cos x = e^{+ix} + e^{-ix}$, $2\sin^2 x = 1 - \cos 2x$, and $\sin 2\theta = 2\sin\theta\cos\theta$.

The phase of the neutrino mass eigenstate $|\nu_1(0)\rangle$ is

$$tE_i = t\sqrt{\mathbf{p}^2 + m_i^2} = t|\mathbf{p}|\sqrt{1 + m_i^2/\mathbf{p}^2} \approx t|\mathbf{p}|\left(1 + \frac{m_i^2}{2\mathbf{p}^2}\right) \approx t|\mathbf{p}| + \frac{m_i^2 L}{2E_\nu},$$

where L is the distance between the neutrino source and the detected event, and E_ν is the neutrino energy. As a consequence,

$$\frac{1}{2}(E_2 - E_1)t = \frac{\Delta m^2 L}{4E_\nu},$$

with $\Delta m^2 = m_2^2 - m_1^2$. Therefore, the probability that a neutrino of the flavor α retains its identity is

$$P_{\alpha\to\alpha}(L, E_\nu) = 1 - \sin^2 2\theta \sin^2 \frac{\Delta m^2 L}{4E_\nu}. \tag{3}$$

2 Kinematic energy reconstruction

Charged-current (CC) quasielastic (QE) scattering,

$$\begin{aligned} \nu_\ell + n &\to \ell^- + p, \\ \bar{\nu}_\ell + p &\to \ell^+ + n, \end{aligned} \tag{4}$$

is the dominant mechanism of neutrino interactions in the ∼1-GeV energy region. Can we determine the neutrino energy from the kinematics of the interaction products?

2.1 Free nucleon at rest

Consider CC QE scattering off a free nucleons at rest. The four-momentum conservation then reads

$$k + p = k' + p' \tag{5}$$

where $k = (E_\nu, \mathbf{k})$ is the four-momentum of the neutrino that scatters off the initial nucleon of the four-momentum $p = (M, \mathbf{p} = 0)$, with M being its mass. The interaction produces the charged lepton of the four-momentum $k' = (E_\ell, \mathbf{k}_\ell)$ and the final nucleon of the four momentum $p' = (E_{p'}, \mathbf{p}')$ and the mass M'.

Equation (5) gives for p' the condition

$$p' = k + p - k',$$

the square of which reads

$$p'^2 = M'^2 = (k + p - k')^2 = k^2 + 2k \cdot (p - k') + (p - k')^2.$$

As at the considered kinematics, neutrinos can be treated as massless, so

$$2k \cdot (p - k') = M'^2 - (p - k')^2.$$

This relation can be simplified making use of vanishing momentum of the initial nucleon,

$$2E_\nu M - 2E_\nu E_\ell + 2E_\nu |\mathbf{k}_\ell| \cos\theta = M'^2 - M^2 + 2ME_\ell - m_\ell^2,$$

where θ and m_ℓ are the charged-lepton's production angle and its mass, respectively.

As a consequence, the neutrino energy in CC QE scattering off a free nucleon can be exactly determined knowing only the kinematics of the charged lepton,

$$E_\nu = \frac{2ME_\ell + M'^2 - M^2 - m_\ell^2}{2(M - E_\ell + |\mathbf{k}_\ell| \cos\theta)}. \tag{6}$$

2.2 Nucleon bound in a nucleus

In the case of CC QE interaction leading to a single nucleon knockout from a nucleus, the four momentum conservation reads

$$k + p_A = k' + p_{A-1} + p',$$

with $p_A = (M_A, \mathbf{p}_A = 0)$ and $p_{A-1} = (E_{A-1}, \mathbf{p}_{A-1})$ denoting the four momenta of the initial and final nuclei, respectively.

Similarly as in Sec. 2.1, let us start from the p' condition,

$$p' = k + p_A - p_{A-1} - k'.$$

Its square gives

$$p'^2 = M'^2 = (k + p - k')^2 = k^2 + 2k \cdot (p - k') + (p - k')^2,$$

with the short-hand notation $p = p_A - p_{A-1} = (E_p, \mathbf{p})$. Rearranging the terms, we obtain for massless neutrinos (i.e., when $k^2 = 0$),

$$2k \cdot (p - k') = M'^2 - p^2 + 2p \cdot k' - k'^2,$$

and explicitly, with θ_h being the angle between the vectors $\mathbf{p}$ and $\mathbf{k}$,

$$2E_\nu(E_p - E_\ell) - 2E_\nu\big(|\mathbf{p}| \cos\theta_h - |\mathbf{k}_\ell| \cos\theta\big) = M'^2 - p^2 + 2E_pE_\ell - 2\mathbf{p} \cdot \mathbf{k}_\ell - m_\ell^2.$$

Therefore, to determine the true value of the neutrino energy in CC QE single nucleon knockout,

$$E_\nu = \frac{2E_pE_\ell - 2\mathbf{p} \cdot \mathbf{k}_\ell + M'^2 - E_p^2 + \mathbf{p}^2 - m_\ell^2}{2(E_p - |\mathbf{p}| \cos\theta_h - E_\ell + |\mathbf{k}_\ell| \cos\theta)}, \tag{7}$$

one would need to know the three-momenta of the charged lepton and the residual nucleus, and the energy of the residual nucleus.

In practice, an application of the above formula requires (i) neglecting the unmeasured recoil momentum $\mathbf{p}_{A-1} = -\mathbf{p}$ and (ii) approximating the energy of the residual nuclear system by a constant or, equivalently, setting $E_p = M - \epsilon$. These simplifications lead to the approximation

$$E_\nu^{\rm rec} = \frac{2ME_\ell + M'^2 - (M-\epsilon)^2 - m_\ell^2}{2(M - \epsilon - E_\ell + |\mathbf{k}_\ell| \cos\theta)}, \tag{8}$$

which amounts to treating the nucleus as a constant potential in which all nucleons are at rest, compare Eqs. (6) and (8). Note that the appropriate value of ϵ, in general, depends on kinematics [PoS(NUFACT2014)004], and that the accuracy of this energy reconstruction formula is much worse for the the processes in which more nucleons are knocked out from the nucleus.

3 Coulomb correction

Consider CC QE interaction of neutrino (antineutrino) with the neutron (proton) bound in a nucleus. Each charged particle produced in the interaction is subject to a modification of its energy in the Coulomb field of the nucleus. When the proton is treated as a point particle, the absolute value of its Coulomb energy, V_C is equal to that of the charged lepton. What is the average value of that energy?

3.1 Simple estimate of the Coulomb energy

In general, the Coulomb energy of the charge e inside the nucleus of the charge Ze and the radius R can be expressed as

$$V_C(r) = \frac{Ze^2}{4\pi\varepsilon_0} \int d^3r' \frac{\rho_{\rm ch}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}, \tag{9}$$

with the fine structure constant $\alpha = e^2/(4\pi\varepsilon_0)$ and $\rho_{\rm ch}$ being the charge density. For a spherically symmetric nucleus, this expression simplifies to

$$V_C(r) = 4\pi Z\alpha \left[\frac{1}{r} \int_0^r dr'\, r'^2 \rho_{\rm ch}(r') + \int_r^R dr'\, r' \rho_{\rm ch}(r') \right]. \tag{10}$$

Proof: Selecting the z axis along the vector $\mathbf{r}$, we can express $V_C(r)$ as

$$V_C(r) = \frac{Ze^2}{4\pi\varepsilon_0} 2\pi \int_0^R dr'\, r'^2 \rho_{\rm ch}(r') \int_{-1}^{+1} d\cos\theta \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}}.$$

Using the identity

$$\int dx \frac{1}{\sqrt{a+bx}} = \frac{2}{b}\sqrt{a+bx},$$

we obtain

$$\begin{aligned}\int_{-1}^{+1} d\cos\theta \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta}} &= -\frac{1}{rr'}\sqrt{r^2+r'^2-2rr'\cos\theta}\Big|_{-1}^{+1} \\ &= \frac{1}{rr'}\big(|r+r'|-|r-r'|\big),\end{aligned}$$

which gives for the potential

$$\begin{aligned}V_C(r) &= \frac{Ze^2}{2\varepsilon_0}\int_0^R dr'\, r'^2 \rho_{\rm ch}(r')\frac{1}{rr'}\big(|r+r'|-|r-r'|\big) \\ &= 4\pi Z\alpha\left(\frac{1}{r}\int_0^r dr'\, r'^2\rho_{\rm ch}(r') + \int_r^R dr'\, r'\rho_{\rm ch}(r')\right),\end{aligned}$$

where $\alpha = e^2/(4\pi\epsilon)$.

The simplest—but widely used—approach is to treat the nucleus as a uniformly charged sphere of the charge density $\rho_{\rm ch} = 3/(4\pi R^3)$. Then

$$V_C(r) = \frac{3Z\alpha}{R^3}\left[\frac{r^2}{3}+\frac{R^2}{2}-\frac{r^2}{2}\right] = \frac{3Z\alpha}{R^3}\left[\frac{R^2}{2}-\frac{r^2}{6}\right] = \frac{Z\alpha}{2R}\left[3-\frac{r^2}{R^2}\right], \quad (11)$$

and the average potential energy is

$$V_C = \int d^3r V_C(r)\rho(r) = \frac{3Z\alpha}{2R} - \frac{Z\alpha}{2R^3}\frac{3}{R^3}\frac{R^5}{5} = \frac{6Z\alpha}{5R}, \quad (12)$$

with ρ denoting the point density.

The nuclear radius is customarily expressed as $R = r_C\sqrt[3]{A}$, with r_C being the reduced Coulomb radius. The latter can be parametrized as

$$r_C = 1.198 + 0.697A^{-2/3} + 12.994A^{-5/3} \quad (13)$$

for a large enough A, see Koning and Delaroche [Nucl. Phys. A **713**, 231 (2003)]. To determine the range of validity of this expression, the root-mean-square radii, being $\sqrt{\langle r^2\rangle} = \sqrt{3/5}R$ in the uniformly charged sphere approximation, can be compared with experimental data. Such a comparison with the data from de Vries *et al.* [At. Data Nucl. Data Tables **36**, 495 (1987)] is presented in Fig. 1 for a broad range of nuclei. This comparison suggests that the accuracy of Eq. (13) is 5% for $16 \le A \le 208$, and 3% for $20 < A \le 40$.

For carbon, Eq. (13) gives 3.52 fm, compared with the experimental value 3.18 fm. The corresponding Coulomb energies are 2.9 and 3.3 MeV.

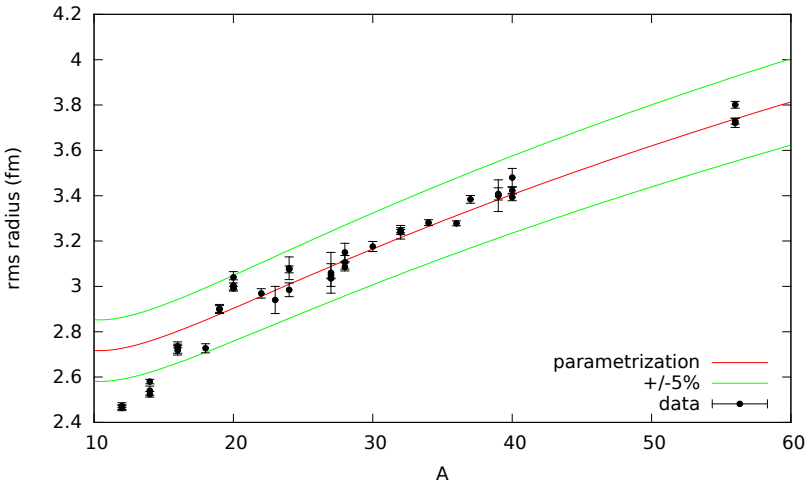

Figure 1: Comparison of the root-mean-square radii extracted from experimental data for electron scattering with the predictions of the expression $r_C\sqrt{3/5}\sqrt[3]{A}$, with r_C given by Eq. (13), as a function of the mass number A.

3.2 Refined estimate of the Coulomb energy

Applying to Eq. (10) a realistic charge density, one can improve the estimate of the Coulomb potential of the considered nucleus. For all the cases important for neutrino applications ($^{12}_{6}$C, $^{16}_{8}$O, $^{40}_{18}$Ar, $^{56}_{26}$Fe, $^{208}_{82}$Pb), the charge density in the article of de Vries *et al.* is available in the form

$$\rho_{\rm ch}(r) = \sum_{n=1}^{n_{\rm max}} a_n j_0(nkr), \tag{14}$$

with the spherical Bessel function $j_0(x)$ and $k = \pi/R$, R being the cutoff radius. Then, a simple analytical integration leads to the Coulomb energy expressed as

$$V_C(r) = 4\pi\alpha Z \sum \frac{a_n}{n^2k^2}\left[j_0(nkr) - (-1)^n\right]. \tag{15}$$

Using the point densities unfolded from the charge densities of de Vries *et al.* following to the procedure of Kelly [Phys. Rev. C **66**, 065203 (2002)], one can obtain the average Coulomb energies, $V_C = \int d^3r V_C(r)\rho(r)$, collected in Table 1.

Table 1: Average and center values of Coulomb energy for various nuclei.

nucleus	$^{12}_{6}$C	$^{16}_{8}$O	$^{40}_{18}$Ar	$^{40}_{20}$Ca	$^{48}_{22}$Ti	$^{56}_{26}$Fe	$^{204}_{82}$Pb	$^{208}_{82}$Pb
V_C (MeV)	3.5	4.2	7.3	8.0	8.5	9.6	20.2	20.1
$V_C^{\max}$ (MeV)	4.6	5.5	9.6	10.5	11.0	12.4	25.7	25.6

3.3 Coulomb corrections to the cross section

The Coulomb field of nucleus modifies the energies of charged particles and affects their wave functions. As a consequence, measurable kinematic variables, such as the energies of charged probe or knocked out proton, differ by the Coulomb correction from those which enter the energy conservation at the interaction vertex, and the effect of (de)focusing needs to be accounted for in calculations of the cross sections.

For electrons, the effect of the nuclear Coulomb potential on the cross section

$$\frac{d\sigma_{eA}}{dE_\ell d\Omega} = f_{eA}(E_\nu, \omega, |\mathbf{q}|)$$

may be approximated by the EMA$'$ prescription of Aste and Jourdan [Europhys. Lett. **67**, 753 (2004)],

(i) replacing the initial and final electron energy,

$$E_k \rightarrow E_k^{\rm eff} = E_k + V_C, \tag{16}$$
$$E_{k'} \rightarrow E_{k'}^{\rm eff} = E_{k'} + V_C, \tag{17}$$

what in turn changes the momenta

$$|\mathbf{k}| \rightarrow |\mathbf{k}_{\rm eff}| = \sqrt{(E_k + V_C)^2 - m^2}, \tag{18}$$
$$|\mathbf{k}'| \rightarrow |\mathbf{k}'_{\rm eff}| = \sqrt{(E_{k'} + V_C)^2 - m^2}, \tag{19}$$

(ii) multiplying the cross section by the focusing factor

$$\left(\frac{|\mathbf{k}_{\rm eff}^{\max}||\mathbf{k}_{\rm eff}'^{\max}|}{|\mathbf{k}_{\rm eff}||\mathbf{k}'_{\rm eff}|} \right)^2 . \tag{20}$$

Note that while this procedure leaves the energy transfer unaltered,

$$\omega_{\rm eff} = E_k^{\rm eff} - E_{k'}^{\rm eff} = E_k - E_{k'} = \omega,$$

this is not the case for the momentum transfer, the magnitude and direction of which are affected by the Coulomb field,

$$\mathbf{q}_{\rm eff} = \mathbf{k}\frac{|\mathbf{k}_{\rm eff}|}{|\mathbf{k}|} - \mathbf{k}'\frac{|\mathbf{k}'_{\rm eff}|}{|\mathbf{k}'|} \not\parallel \mathbf{k} - \mathbf{k}' = \mathbf{q}.$$

In summary, the (e, e') cross section corrected for the the Coulomb effects is

$$\frac{d\sigma_{eA}}{d\omega d\Omega} = \left(\frac{|\mathbf{k}^{\rm max}_{\rm eff}||\mathbf{k}'^{\rm max}_{\rm eff}|}{|\mathbf{k}_{\rm eff}||\mathbf{k}'_{\rm eff}|}\right)^2 f_{eA}(E^{\rm eff}_\nu, \omega, |\mathbf{q}_{\rm eff}|). \tag{21}$$

In CC (anti)neutrino scattering, the probe does not carry the electric charge and, therefore, accounting for the Coulomb corrections reduces to

(i) replacing the final muon energy and momentum by

$$E_\ell \to E_\ell + sV_C \quad \text{and} \quad |\mathbf{k}'| \to |\mathbf{k}'_{\rm eff}| = \sqrt{(E_\ell + sV_C) - m^2}. \tag{22}$$

with $s = +1$ for neutrinos and -1 for antineutrinos, and

(ii) multiplying the cross section by the focusing factor

$$\left(\frac{|\mathbf{k}'^{\rm max}_{\rm eff}|}{|\mathbf{k}'_{\rm eff}|}\right)^2. \tag{23}$$

This prescription leads to the CC cross section in the following form

$$\frac{d\sigma^{\rm CC}}{dE_\ell d\Omega} = \left(\frac{|\mathbf{k}'^{\rm max}_{\rm eff}|}{|\mathbf{k}'_{\rm eff}|}\right)^2 f^{\rm CC}(E_\nu, \omega_{\rm eff}, |\mathbf{q}_{\rm eff}|), \tag{24}$$

with

$$\omega_{\rm eff} = E_\nu - E^{\rm eff}_\ell, \tag{25}$$
$$\mathbf{q}_{\rm eff} = \mathbf{k} - \mathbf{k}'_{\rm eff}. \tag{26}$$

As the difference between the proton and neutron spectral functions increases the difference between antineutrino and neutrino interactions, it needs to be accounted for in calculations of the cross sections. Moreover, taking into account final-state interactions in neutrino and antineutrino scattering, one needs to keep in mind the difference between the proton and neutron spectrum produced by the Coulomb energy.

In Monte Carlo generators, the cross section is typically calculated as

$$\sigma^{\rm CC} = \int \frac{d^3k'}{E_\ell} \cdots = \int d|\mathbf{k}'|d\Omega \frac{|\mathbf{k}'|}{E_\ell}\frac{d\sigma^{\rm CC}}{dE_\ell d\Omega}. \tag{27}$$

Note that while the Coulomb effects modify the differential cross section $d\sigma^{\rm CC}/dE_\ell d\Omega$, they do not affect the factor $|\mathbf{k}'|/E_\ell$.

4 Point density

In the most general case, the charge density reads

$$\rho_{\rm ch}(r) = \int d^3r' \big[\rho_{\rm p}^{(p)}(r')\rho_{\rm ch}^{(p)}(r-r') + \rho_{\rm p}^{(n)}(r')\rho_{\rm ch}^{(n)}(r-r')\big], \tag{28}$$

where $\rho_{\rm p}^{(N)}$ is the point density of nucleons of isospin N ($N = p,\, n$) and $\rho_{\rm ch}^{(N)}$ is their charge distribution.

Assuming the scaling $N\rho_{\rm p}^{(n)}(r) = Z\rho_{\rm p}^{(p)}(r)$, we deal with the case of only one point density

$$\rho_{\rm p}(r) = \rho_{\rm p}^{(p)}(r) = \rho_{\rm p}^{(n)}(r)\frac{N}{Z},$$

and the charge density becomes

$$\rho_{\rm ch}(r) = \int d^3r' \rho_{\rm p}(r') \left[\rho_{\rm ch}^{(p)}(r-r') + \frac{N}{Z}\rho_{\rm ch}^{(n)}(r-r')\right], \tag{29}$$

Its Fourier transform,

$$\begin{aligned}
\rho_{\rm ch}(k) &= \int d^3r \rho_{\rm ch}(r) e^{i\mathbf{k}\cdot\mathbf{r}} \\
&= \int d^3r' \rho_{\rm p}(r') \int d^3r e^{i\mathbf{k}\cdot\mathbf{r}} \left[\rho_{\rm ch}^{(p)}(r-r') + \frac{N}{Z}\rho_{\rm ch}^{(n)}(r-r')\right] \\
&= \int d^3r' \rho_{\rm p}(r') e^{i\mathbf{k}\cdot\mathbf{r}'} \int d^3r e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \left[\rho_{\rm ch}^{(p)}(r-r') + \frac{N}{Z}\rho_{\rm ch}^{(n)}(r-r')\right] \\
&\equiv \rho_{\rm p}(k) \left[\rho_{\rm ch}^{(p)}(k) + \frac{N}{Z}\rho_{\rm ch}^{(n)}(k)\right],
\end{aligned} \tag{30}$$

allows us to find the point density in the momentum space

$$\rho_{\rm p}(k) = \frac{\rho_{\rm ch}(k)}{\rho_{\rm ch}^{(p)}(k) + \frac{N}{Z}\rho_{\rm ch}^{(n)}(k)}, \tag{31}$$

and in the coordinate space

$$\begin{aligned}
\rho_{\rm p}(r) &= \frac{1}{(2\pi)^3}\int d^3k \rho_{\rm p}(k) e^{-i\mathbf{k}\cdot\mathbf{r}} = \frac{1}{(2\pi)^3}\int d^3k e^{-i\mathbf{k}\cdot\mathbf{r}} \frac{\rho_{\rm ch}(k)}{\rho_{\rm ch}^{(p)}(k) + \frac{N}{Z}\rho_{\rm ch}^{(n)}(k)} \\
&= \frac{4\pi}{(2\pi)^3}\int dk\, k^2 j_0(kr) \frac{\rho_{\rm ch}(k)}{\rho_{\rm ch}^{(p)}(k) + \frac{N}{Z}\rho_{\rm ch}^{(n)}(k)}.
\end{aligned} \tag{32}$$

What is $\rho_{\rm ch}(k)$? In the article of de Vries *et al.* is available in the form

$$\rho_{\rm ch}(r) = \sum_{n=1}^{n_{\rm max}} a_n j_0(nkr), \tag{33}$$

with the spherical Bessel function $j_0(x)$ and $k = \pi/R$, R being the cutoff radius. Then,

$$\begin{aligned}\rho_{\rm ch}(k) &= \int d^3r e^{i\mathbf{k}\cdot\mathbf{r}}\rho_{\rm ch}(r) = \int d^3r j_0(kr)\rho_{\rm ch}(r) = 4\pi\int dr\, r^2 j_0(kr)\rho_{\rm ch}(r) \\ &= 4\pi\int dr\, r^2 j_0(kr)\sum_{n=1}^{n_{\rm max}} a_n j_0(n\pi r/R) = \frac{(2\pi)^2}{2k^2}\sum_{n=1}^{n_{\rm max}} a_n \delta\left(k - \frac{n\pi}{R}\right).\end{aligned}$$

In the last step we have used the orthogonality relation

$$\int dx\, x^2 j_\alpha(ux) j_\alpha(vx) = \frac{\pi}{2v^2}\delta(u-v),$$

valid for $\alpha \geq 0$.

Returning to the point density,

$$\begin{aligned}\rho_{\rm p}(r) &= \int dk\, j_0(kr)\sum_{n=1}^{n_{\rm max}} \frac{a_n \delta\left(k - \frac{n\pi}{R}\right)}{\rho_{\rm ch}^{(p)}(k) + \frac{N}{Z}\rho_{\rm ch}^{(n)}(k)} \\ &= \sum_{n=1}^{n_{\rm max}} \left.\frac{a_n j_0(kr)}{\rho_{\rm ch}^{(p)}(k) + \frac{N}{Z}\rho_{\rm ch}^{(n)}(k)}\right|_{k=\frac{n\pi}{R}}.\end{aligned} \tag{34}$$

In a naive nonrelativistic approximation, $\rho_{\rm ch}^{(N)}(k) = G_E^{(N)}(k^2)$, see e.g. Halzen and Martin, *Quarks and leptons* (Wiley, New York, 1984).

According to Kelly [Phys. Rev. C **66**, 065203 (2002)], more satisfactory results are obtained from

$$\rho_{\rm ch}^{(N)}(k) = G_E^{(N)}(Q^2)(1+\tau)^2,$$

with $k^2 = Q^2/(1+\tau)$ and $\tau = Q^2/(4M^2)$, which is equivalent to

$$\rho_{\rm ch}^{(N)}(k) = G_E^{(N)}\left(\frac{k^2}{1-k^2/(4M^2)}\right)\left(1 - \frac{k^2}{4M^2}\right)^{-2}.$$

5 Optical potential

The Dirac equation in the most general form reads

$$(\gamma^\mu p_\mu - M - S - \gamma^0 V)\psi = 0$$

where S and V are the scalar and vector potentials of complex values, respectively. It can be rewritten as

$$(-\gamma^i p_i + M + S)\psi = \gamma^0(E_{\rm tot} - V)\psi.$$

To obtain the energy of the particle in the potential, "square" the equation

$$(\gamma^j p_j + M + S)(-\gamma^i p_i + M + S)\psi = (\gamma^j p_j + M + S)\gamma^0(E_{\rm tot} - V)\psi.$$

The left-hand side gives

$$\begin{aligned}(\gamma^j p_j + M + S)\gamma^0(E_{\rm tot} - V)\psi &= (E_{\rm tot} - V)\gamma^0(-\gamma^j p_j + M + S)\psi \\ &= (E_{\rm tot} - V)\gamma^0\gamma^0(E_{\rm tot} - V)\psi = (E_{\rm tot} - V)^2\psi\end{aligned}$$

Introducing the shorthand notation $\tilde{M} = M + S = M + S_V + iS_W$, where S_V and S_W are the real and the imaginary part of the scalar potential, we can rewrite the right-hand side as follows

$$\begin{aligned}&(\gamma^j p_j + \tilde{M})(-\gamma^i p_i + \tilde{M})\psi = [-\gamma^j p_j \gamma^i p_i + \tilde{M}^2]\psi = [-\gamma^j p_j \gamma^i p_i + \tilde{M}^2]\psi \\ &= [-\gamma^j p_j \gamma^i p_i + \tilde{M}^2]\psi = \left[\frac{1}{2}(-\gamma^j p_j \gamma^i p_i - \gamma^i p_i \gamma^j p_j) + \tilde{M}^2\right]\psi \\ &= [-g^{ij} p_j p_i + \tilde{M}^2]\psi = [\mathbf{p}^2 + \tilde{M}^2]\psi\end{aligned}$$

Comparing the right- and left-hand side we obtain the relation

$$(E_{\rm tot} - V)^2 = \tilde{M}^2 + \mathbf{p}^2,$$

which can be used to define the (complex) optical potential U by means of the equation

$$E_p + U = \sqrt{\tilde{M}^2 + \mathbf{p}^2} + V.$$

The argument of the square root may be cast in the form

$$\tilde{M}^2 + \mathbf{p}^2 = a + ib$$

with

$$\begin{aligned}a &= E_p^2 + 2MS_V + S_V^2 - S_W^2, \\ b &= 2S_W(M + S_V).\end{aligned}$$

Therefore, from the identity

$$\sqrt{a + ib} = \sqrt{\frac{1}{2}\big(\sqrt{a^2 + b^2} + a\big)} + i\sqrt{\frac{1}{2}\big(\sqrt{a^2 + b^2} - a\big)},$$

it follows that the real part of the optical potential is

$$\Re U = \sqrt{\frac{1}{2}\sqrt{a^2+b^2}+\frac{1}{2}a} + V_V - E_p,$$

and the imaginary part equals to

$$\Im U = \sqrt{\frac{1}{2}\sqrt{a^2+b^2}-\frac{1}{2}a} + V_W.$$