From Yukawa's Theory to the One-Pion-Exchange Potential

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I. THE TWO-NUCLEON SYSTEM

Neglecting a $\sim$ 1‰ mass difference, proton and neutron can be viewed as two states of the same spin 1/2 particle, the nucleon (N), specified by an additional quantum number dubbed isospin.

In the absence of interactions, the nucleon is described by the equation of motion derived from the Lagrangian density

$$\mathcal{L}_0 = \bar{\psi}_N \left(i\gamma^\mu \partial_\mu - m\right) \psi_N \ . \tag{1}$$

where the γ^μ are Dirac's matrices satisfying $\{\gamma^\mu, \gamma^\mu\} = 2g^{\mu\nu}$, $g^{\mu nu}$ being the metric tensor of Minkowski space, and the field ψ_N is conveniently written in the form

$$\psi_N = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix} , \tag{2}$$

with ψ_p and ψ_n being the spinor fields associated with the proton and the neutron, respectively.

The lagrangian density of Eq.(1) is invariant under the SU(2) global phase transformation

$$U = \mathrm{e}^{i\alpha_j \tau_j} , \tag{3}$$

where the α_j are constants, the τ_j are Pauli matrices acting in isospin space and a sum over the index j is understood. Proton and neutron correspond to isospin projections +1/2 and −1/2, respectively.

Proton-proton and neutron-neutron pairs always have total isospin T=1 whereas a proton-neutron pair may have either $T = 0$ or $T = 1$. The two-nucleon isospin states $|T, T_3\rangle$ can be can be written in terms of proton and neutron degrees of freedom as

$$\begin{aligned} |1,1\rangle &= |pp\rangle \\ |1,0\rangle &= \frac{1}{\sqrt{2}} \left(|pn\rangle + |np\rangle\right) \\ |1,-1\rangle &= |nn\rangle \\ |0,0\rangle &= \frac{1}{\sqrt{2}} \left(|pn\rangle - |np\rangle\right) \ . \end{aligned}$$

Isospin invariance implies that the interaction between two nucleons coupled to total spin S depends on their total isospin T, but not on its projection T_3. For example, the force acting between two protons with total spin $S = 0$ is the same as that acting between a proton and a neutron with spins and isospins coupled to $S = 0$ and $T = 1$.

There is *only one* observed nucleon-nucleon (NN) bound state, the nucleus of deuterium, or deuteron (^{2}H), consisting of a proton and a neutron with total spin and isospin $S = 1$ and $T = 0$, respectively. This is itself a clear manifestation of the fact that *nuclear forces are strongly spin-isospin dependent*.

Another important piece of information is inferred from the observation that the deuteron exhibits a non vanishing electric quadrupole moment, which implies that its charge distribution is not spherically symmetric. Hence, *nuclear forces are non central*.

Besides the properties of the two-nucleon bound state, the large data base of phase shifts accurately measured in NN scattering experiments ($\sim$ 4000 data points at beam energies up to 350 MeV in the lab frame) provides valuable additional information on the nature of NN interactions.

II. THE TWO-NUCLEON INTERACTION

The theoretical description of the NN interaction within the framework of quantum field theory was first attempted by Yukawa in in the 1930s. He made the hypothesis that nucleons interact through the exchange of a particle whose mass, μ, is related to the range of the interaction, r_0, through $r_0 \sim 1/\mu$[1]. Using $r_0 \sim 1$ fm = 10^{-13} cm, this relation yields $\mu \sim 200$ MeV.

Yukawa's idea has been successfully implemented identifying the exchanged particle with the π meson (or *pion*), discovered in 1947, the mass of which is $m_\pi \approx 140$ MeV. Experimental data show that the pion is a pseudoscalar particle[2]—that is, it has spin-parity 0^-—which comes in three charge states, denoted π^0, π^+, and π^-. Hence, it can be regarded as an isospin triplet having t=1, the charge states being associated with the projections $t_3 = 0, \pm 1$.

The simplest π-nucleon interaction Lagrangian compatible with the requirement of Lorentz invariance and with the observation that nuclear interactions conserve parity involves a

[1] We adopt the system of units in which $\hbar = c = 1$, implying in turn that 1 fm^{-1} = 197.3 MeV.

[2] The pion spin has been deduced from the balance of the reaction $\pi^+ + ^2\mathrm{H} \leftrightarrow p + p$, while its intrinsic parity was determined observing π^- capture from the K shell of the deuterium atom leading to the appearance of two neutrons: $\pi^- + d \to n + n$.

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pseudoscalar coupling, and can be written in the form

$$\mathcal{L}_I = -ig\bar{\psi}_N \gamma^5 \tau^j \psi_N \pi^j \ , \tag{4}$$

where

$$\begin{aligned} \pi^1 &= \frac{1}{\sqrt{2}}(\pi^+ + \pi^-) \ , \\ \pi^2 &= \frac{i}{\sqrt{2}}(\pi^+ - \pi^-) \ , \\ \pi^3 &= \pi^0 \ , \end{aligned} \tag{5}$$

and a sum over the index j is understood. In Eq.(4), g is the pseudoscalar strong interactions coupling constant and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. The γ^μ are again Dirac matrices, and the Pauli matrices τ^j, acting in isospin space, are associated with the isospin of the nucleon. Note that, in the non relativistic limit, the pseudoscalar coupling, $-ig\gamma^5\boldsymbol{\tau}$, and the alternative pseudovector coupling, $ig'\gamma^5\gamma^\mu\boldsymbol{\tau}\partial_\mu$, yield the same NN potential. It is apparent that in both cases the isospin formalism allows to take into account in a concise fashion all interaction vertices, involving proton, neutrons, charged pions and neutral pions.

Let us now consider the NN scattering process

$$N(p_1 s_1) + N(p_2 s_2) \to N(p_1' s_1') + N(p_2' s_2')$$

depicted by the Feynman diagrams of Fig. 1.

The corresponding S-matrix element reads

$$\begin{aligned} S_{fi} = (-ig)^2 \frac{m^2}{\left(E_1 E_2 E_1' E_2'\right)^{1/2}} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_1' - p_2') \\ \times \left\{ \left[\eta_{1'}^\dagger \bar{u}_{1'} i\gamma_5 \boldsymbol{\tau} u_1 \eta_1\right] \frac{i}{k^2 - m_\pi^2} \left[\eta_{2'}^\dagger \bar{u}_{2'} i\gamma_5 \boldsymbol{\tau} u_2 \eta_2\right] - \left[(1',2') \rightleftharpoons (2',1')\right] \right\} \ , \end{aligned} \tag{6}$$

where m_π is the pion mass, $k = p_1 - p_1' = p_2' - p_2$, and η_i denotes the two-component Pauli spinor describing the isospin state of particle i.

Let us consider the direct term of Eq.(6). It can be rewritten in the form

$$\begin{aligned} S_{fi}^{(D)} &= ig^2 \frac{m^2}{\left(E_1 E_2 E_1' E_2'\right)^{1/2}} \\ &\times (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_1' - p_2') \frac{1}{k^2 - m_\pi^2} \\ &\times \eta_{2'}^\dagger \boldsymbol{\tau} \eta_2 \ \bar{u}_{2'}\gamma_5 u_2 \ \bar{u}_{1'}\gamma_5 u_1 \ \eta_{1'}^\dagger \boldsymbol{\tau} \eta_1 \ . \end{aligned} \tag{7}$$

Substituting the expression of the Dirac spinor describing a particle with momentum $\mathbf{p}$, energy $E = \sqrt{\mathbf{p}^2 + m^2}$, and spin projection s

$$u_s(\mathbf{p}) = \sqrt{\frac{E+m}{2E}} \begin{pmatrix} \chi_s \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} \chi_s \end{pmatrix} \ , \tag{8}$$

where χ_s denotes a Pauli spinor acting in spin space, and taking the non relativistic limit we find

$$\begin{aligned} \bar{u}_{2'}\gamma_5 u_2 &= \sqrt{\frac{\left(E_2' + m\right)(E_2 + m)}{4E_2' E_2}} \\ &\times \left(\chi_{2'}^\dagger \frac{\boldsymbol{\sigma}\cdot\mathbf{p}_2}{E_2 + m} \chi_2 - \chi_{2'}^\dagger \frac{\boldsymbol{\sigma}\cdot\mathbf{p}_2'}{E_2' + m} \chi_2 \right) \\ &\approx \chi_{2'}^\dagger \frac{\boldsymbol{\sigma}(\mathbf{p}_2 - \mathbf{p}_2')}{2m} \chi_2 = -\chi_{2'}^\dagger \frac{(\boldsymbol{\sigma}\cdot\mathbf{k})}{2m} \chi_2 \ , \end{aligned} \tag{9}$$

and the similar expression for $\bar{u}_{1'}\gamma_5 u_1$.

The non relativistic approximation also implies that (use $E_i \approx E_i' \approx m_\pi$ and $k^2 = (E_i - E_i')^2 - |\mathbf{k}|^2 \approx -|\mathbf{k}|^2$)

$$\frac{1}{k^2 - m_\pi^2} \approx \frac{-1}{|\mathbf{k}|^2 + m_\pi^2} \ . \tag{10}$$

Substituting the above results in the definition of S_{fi} we obtain

$$\begin{aligned} S_{fi}^{(D)} &\approx -i\frac{g^2}{4m^2}(2\pi)^4\delta^{(4)}(p_1 + p_2 - p_1' - p_2') \\ &\times \langle f| \left[(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2) \frac{(\boldsymbol{\sigma}_1\cdot\mathbf{k})(\boldsymbol{\sigma}_2\cdot\mathbf{k})}{|\mathbf{k}|^2 + m_\pi^2} \right] |i\rangle \ , \end{aligned} \tag{11}$$

where $|i\rangle = \eta_1\eta_2 \ \chi_1\chi_2$. and $\langle f| = \eta_{1'}^\dagger \eta_{2'}^\dagger \ \chi_{1'}^\dagger \chi_{2'}^\dagger$ denote the initial and final states of the interacting particles, respectively.

Equation (11) suggest that the operator

$$\begin{aligned} \mathrm{v}_{\mathrm{OPE}}(\mathbf{k}) &= -\frac{g^2}{4m^2} \frac{(\boldsymbol{\sigma}_1\cdot\mathbf{k})(\boldsymbol{\sigma}_2\cdot\mathbf{k})}{|\mathbf{k}|^2 + m_\pi^2} \\ &= -\left(\frac{f}{m_\pi}\right)^2 \frac{(\boldsymbol{\sigma}_1\cdot\mathbf{k})(\boldsymbol{\sigma}_2\cdot\mathbf{k})}{|\mathbf{k}|^2 + m_\pi^2} \ , \end{aligned} \tag{12}$$

can be interpreted as the one-pion-exchange potential in momentum space. Note that in the second line of Eq.(12) we have replaced the pseudoscalar coupling constant g with the new dimensionless constant (use $g^2/4\pi \approx 14$)

$$f^2 = g^2 \frac{m_\pi^2}{4m^2} \approx 4\pi \times 14 \frac{(140)^2}{4 \times (939)^2} \approx 4\pi \times 0.08 \approx 1 \ . \tag{13}$$

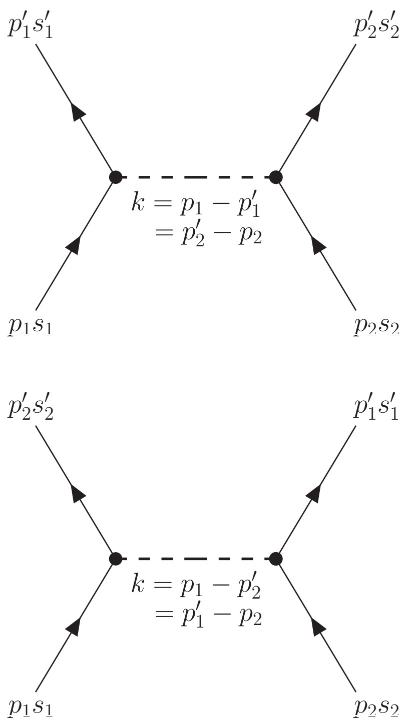

FIG. 1 Feynman diagrams describing direct (upper panel) and exchange (lower panel) contributions to one-pion-exchange between two nucleons. The corresponding amplitude is given by Eq. (6).

The coordinate-space potential is obtained from Fourier transformation according to

$$\mathrm{v}_{\mathrm{OPE}}(\mathbf{r}) = -\left(\frac{f}{m_\pi}\right) (\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2)\, (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\nabla})\, (\boldsymbol{\sigma}_2 \cdot \boldsymbol{\nabla}) \tag{14}$$

$$\times \int \frac{d^3k}{(2\pi)^3}\, \frac{1}{(|\mathbf{k}|^2 + m_\pi^2)}\, e^{-i\mathbf{k}\cdot\mathbf{r}} , \tag{15}$$

where

$$\int \frac{d^3k}{(2\pi)^3}\, \frac{1}{(|\mathbf{k}|^2 + m_\pi^2)}\, e^{-i\mathbf{k}\cdot\mathbf{r}} = \frac{1}{4\pi}\frac{e^{-m_\pi r}}{r} = \frac{1}{4\pi}\, y_\pi(r) . \tag{16}$$

The gradients appearing in Eq. (14) can be readily evaluated exploiting the relation

$$(-\nabla^2 + m_\pi^2)\; y_\pi(r) = 4\pi\delta(\mathbf{r}) , \tag{17}$$

and rewriting

$$(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\nabla})\, (\boldsymbol{\sigma}_2 \cdot \boldsymbol{\nabla}) y_\pi(r) = \left[(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\nabla})\, (\boldsymbol{\sigma}_2 \cdot \boldsymbol{\nabla}) - \frac{1}{3}(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\, \nabla^2 \right] y_\pi(r) + \frac{1}{3}(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\, \nabla^2\; y_\pi(r) . \tag{18}$$

The δ-function contribution to $\nabla^2 y_\pi(r)$, arising from Eq.(17), does not appear in the first term, yielding

$$\left[(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\nabla})\, (\boldsymbol{\sigma}_2 \cdot \boldsymbol{\nabla}) - \frac{1}{3}\, (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\nabla^2 \right] y_\pi(r) = \left[(\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}})\, (\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}}) - \frac{1}{3}\, (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2) \right] \times \left(m_\pi^2 + \frac{3m_\pi}{r} + \frac{3}{r^2} \right) y_\pi(r), \tag{19}$$

where $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$. In the second term, it can be replaced with $m_\pi^2 y_\pi(r) - 4\pi\, \delta(\mathbf{r})$ using Eq. (17).

Carrying out the calculation of the derivatives in Eq. (14) we finally find

$$\mathrm{v}_{\mathrm{OPE}}(\mathbf{r}) = \frac{1}{3}\frac{1}{4\pi}\, f^2\, m_\pi\, (\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2) \left[T_\pi(r) S_{12} + \left(Y_\pi(r) - \frac{4\pi}{m_\pi^3}\delta(\mathbf{r}) \right) (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2) \right] , \tag{20}$$

with

$$Y_\pi(r) = \frac{e^{-m_\pi r}}{m_\pi r} , \tag{21}$$

and

$$T_\pi(r) = \left(1 + \frac{3}{m_\pi r} + \frac{3}{m_\pi^2 r^2} \right) Y_\pi(r) . \tag{22}$$

Note that due to the presence of a contribution involving the operator

$$S_{12} = \frac{3}{r^2}(\boldsymbol{\sigma}_1 \cdot \mathbf{r})(\boldsymbol{\sigma}_2 \cdot \mathbf{r}) - (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2) , \tag{23}$$

reminiscent of the operator describing the interaction between two magnetic dipoles, the above potential is *not* spherically symmetric.

The above potential provides a good description of the long range part ($|\mathbf{r}| > 1.5$ fm) of the NN interaction, as shown by the fit to the NN scattering phase shifts in states of high angular momentum. Note that in these states, due to the strong centrifugal barrier, the probability of finding the two nucleons at small relative distances becomes negligibly small.