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Electron scattering

[F. Close: "An Introduction to quarks and partons", Chapter 9]

Consider the process $1+2 \rightarrow 3+4$

Exclusive cross section (cfr. for instance Bjorken - Drell, Appendix B)

$$d\sigma = \frac{1}{|\underline{v}_1 - \underline{v}_2|} \frac{1}{2E_1} \frac{1}{2E_2} |A|^2 \frac{d^3 p_3}{(2\pi)^3 2E_3} \frac{d^3 p_4}{(2\pi)^3 2E_4} \times (2\pi)^4 \delta^{(4)}(P_1 + P_2 - P_3 - P_4) \qquad (1)$$

[Normalization of the spinors $\bar{u}(p,s)\,u(p,s) = 2m$

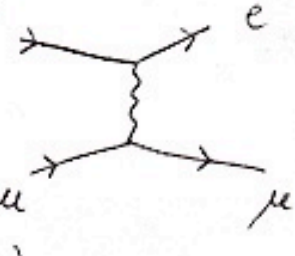
$\Rightarrow \sum_{\pm s} u_\alpha(p,s)\,\bar{u}_\beta(p,s) = (\not{p} + m)_{\alpha\beta}$]

* $\boxed{\text{Simple case } e^- \mu \text{ scattering} \Rightarrow \text{point coupling}}$ **

$$|A|^2 = \frac{1}{4} \sum_{s_1 s_2 s_3 s_4} \left| \bar{u}(k', s_3)\gamma^\mu u(k, s_1) \frac{e^2}{q^2} \bar{u}(p', s_4)\gamma_\mu u(p, s_2)\right|^2$$

average over initial spins

$k \equiv (E, \underline{k})$ $\quad$ $k' \equiv (E', \underline{k}')$



$p \equiv (M, \underline{0})$ $\quad$ $p' \equiv (E_4, \underline{p}_4)$

Using the projection operators $|A|^2$ can be rewritten in the form (The proof is in appendix A)

$$|A|^2 = \frac{e^4}{q^4} L^{(e)}_{\mu\nu} L^{\mu\nu}_{(\mu)}$$

where

$$L^{(e)}_{\mu\nu} = \frac{1}{2} \mathrm{Tr}\, (\not{k}' + m)\gamma_\mu (\not{k} + m)\gamma_\nu$$

$$= 2\left\{ k'_\mu k_\nu + k_\mu k'_\nu - g_{\mu\nu}\left[(k\cdot k') - m^2\right]\right\}$$

and m is the electron mass. The contraction of the electron and muon tensors yields

$$L^{(e)}_{\mu\nu} L^{\mu\nu}_{(\mu)} = 8\left[(k'\cdot p')(kp) + (p\cdot k')(k\cdot p') - m^2 (p\cdot p') - M^2 (k\cdot k') + 2m^2 M^2\right] \quad (2)$$

which, in the lab frame becomes, neglecting terms proportional to m^2

$$L^{(e)}_{\mu\nu} L^{\mu\nu}_{(\mu)} = 8\left[2M^2 EE' + \frac{q^2}{2} M(E'-E) + \frac{M^2 q^2}{2}\right] \quad (3)$$

Proof:

define $q = k - k'$.
one has:

$$(k' \cdot p') = E' E_4 - (\underline{k}' \cdot \underline{p}_4) \qquad (k \cdot p) = E M$$

$$(p \cdot k') = E' M \qquad (k \cdot p') = E E_4 - (\underline{k} \cdot \underline{p}_4)$$

$$(p \cdot p') = E_4 M \qquad (k \cdot k') = E E' - (\underline{k}\, \underline{k}')$$

Overall four momentum conservation implies

$$k - k' + p = p'$$

$$E_4 = E - E' + M$$

$$\underline{p}_4 = \underline{k} - \underline{k}'$$

hence using the relativistic limit for electrons $k \sim E$, $k' \sim E'$

$$(k' p')(k\, p) = \{ E' (E - E' + M) - [\underline{k}' (\underline{k} - \underline{k}')] \} E M$$

$$= E E' (E - E') M - E^2 E' M \cos\Theta + E (E')^2 M + E' E M^2$$

$$= E^2 E' M - E (E')^2 M - E^2 E' M \cos\Theta + E (E')^2 M + E' E M^2$$

$$= E^2 E' M (1 - \cos\Theta) + E' E M^2$$

$$(pk')(kp') = E'M\{E(E-E'+M) - [\underline{k}(\underline{k}-\underline{k}')]\}$$

$$= E'M\,E(E-E') - E'E^2M + (E')^2EM\cos\Theta + E'EM^2$$

$$= E'E^2M - (E')^2EM - E'E^2M + (E')^2EM\cos\Theta + E'EM^2$$

$$= -(E')^2EM(1-\cos\Theta) + E'EM^2$$

$$(k\cdot k') = EE' - (\underline{k}\underline{k}') = EE'(1-\cos\Theta)$$

Substituting in eq. (2) one gets

$$L^{(e)}_{\mu\nu}L^{\mu\nu}_{(\mu)} \sim 8\,[E^2E'M(1-\cos\Theta) - E(E')^2M(1-\cos\Theta)$$

$$+ 2E'EM^2 - M^2EE'(1-\cos\Theta)]$$

$$= 8\{2EE'M^2 + EE'(1-\cos\Theta)[M(E-E') - M^2]\}$$

Using now

(i) $$q^2 = (k-k')^2 = (E-E')^2 - |\underline{k}-\underline{k}'|^2$$

$$= (E-E')^2 - k^2 - k'^2 + 2\underline{k}\underline{k}'$$

$$= (E-E')^2 - E^2 - (E')^2 + 2EE'\cos\Theta$$

$$= -2EE'(1-\cos\Theta) = -4EE'\sin^2\Theta/2$$

$$\boxed{q^2 = -Q^2 = -4EE' \sin^2 \theta/2}$$

(ii)

$$P_4^2 = (E - E' + M)^2 - |\underline{q}|^2 = M^2$$

$$(E-E')^2 + M^2 - |\underline{q}|^2 + 2M(E-E') = M^2$$

$$q^2 + 2M(E-E') = 0$$

$$E - E' = -\frac{q^2}{2M} = \frac{Q^2}{2M}$$

$$\boxed{\nu = E - E' = -\frac{q^2}{2M} = \frac{Q^2}{2M}}$$

(iii)

$$2M\nu = Q^2 = 4\,EE' \sin^2 \theta/2$$

$$M(E-E') = EE'(1 - \cos\theta)$$

$$\boxed{M(E'-E) = EE'(\cos\theta - 1)}$$

one finally finds

$$L^{(e)}_{\mu\nu} L^{\mu\nu}_{(\mu)} = 8 \left\{ 2M^2 EE' + M(E-E') \left[M(E-E') - M^2 \right] \right\}$$

$$= 8 \left\{ 2M^2 EE' + M(E-E') \left[-\frac{q^2}{2} - M^2 \right] \right\}$$

$$= 8 \left\{ 2M^2 EE' + \frac{q^2}{2} M(E'-E) + \frac{M^2 q^2}{2} \right\}$$

$$= 8 \left\{ 2M^2 EE' + \left(\frac{q}{2}\right)^2{}^2 + M^2 \frac{q^2}{2} \right\}$$

$$= 8 \left\{ 2M^2 EE' + \left(\frac{4\, E\, E' \sin^2 \theta/2}{2} \right)^2 - \frac{M^2}{2} 4 EE' \sin^2 \theta/2 \right\}$$

$$= 8 \left\{ 2M^2 EE' + 4 E^2 (E')^2 \sin^4 \theta/2 - 2M^2 EE' \sin^2 \theta/2 \right\}$$

$$= 16\, M^2 EE' \left\{ 1 + 2 \frac{EE'}{M^2} \sin^4 \theta/2 - \sin^2 \theta/2 \right\}$$

$$= 16\, M^2 EE' \left\{ \cos^2 \theta/2 - \frac{q^2}{2M^2} \sin^2 \frac{\theta}{2} \right\}$$

$$\boxed{L^{(e)}_{\mu\nu} L^{\mu\nu}_{(\mu)} = 16\, M^2 EE' \left[\cos^2 \theta/2 - \frac{q^2}{2M^2} \sin^2 \theta/2 \right]}$$

Inserting now eq. (4) in eq. (1) yields:

$$d\sigma = \frac{1}{|v_1 - v_2|} \frac{1}{2E_1} \frac{1}{2E_2} \left\{ \frac{e^4}{q^4} \left[16 M^2 E E' \left(\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2} \sin^2\frac{\theta}{2} \right) \right] \right\}$$

$$\times \frac{d^3 p_3}{2(2\pi)^3 E_3} \frac{d^3 p_4}{(2\pi)^3 2 E_4} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4).$$

Putting $|\underline{v}_1 - \underline{v}_2| = |\underline{v}_1| = k/E \sim 1$, $p_2 = P$, $p_1 - p_3 = q$, $E_1 = E$, $E_2 = M$, $E_3 = E'$ one has

$$d\sigma = \frac{1}{2E} \frac{1}{2M} \left[\frac{e^4}{q^4} 16 M^2 E E' \left(\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2} \sin^2\frac{\theta}{2} \right) \right]$$

$$\times \frac{d^3 p_3}{(2\pi)^3 2E'} \frac{d^3 p_4}{(2\pi)^3 2E_4} (2\pi)^4 \delta^{(4)}(q + P - p_4)$$

$$= \frac{1}{2E} \frac{1}{2M} \left[\frac{e^4}{q^4} 16 M^2 E E' \left(\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2} \sin^2\frac{\theta}{2} \right) \right]$$

$$\frac{d\Omega\, p_3^2\, dp_3}{(2\pi)^3 2E'} \frac{d^3 p_4}{(2\pi)^3 2E_4} (2\pi)^4 \delta^{(4)}(q + P - p_4)$$

or

$$\frac{d\sigma}{d\Omega} = \frac{1}{2E}\,\frac{1}{2M}\left[\frac{e^4}{q^4}\,16\,M^2\,EE'\left(\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right)\right]$$

$$\frac{(E')^2\,dE'}{(2\pi)^3\,2E'}\;\frac{d^3p_4}{(2\pi)^3\,2E_4}\;(2\pi)^4\,\delta^{(4)}(q+p-p_4)$$

$$\frac{d^2\sigma}{dE'd\Omega} = \frac{1}{4}\,\frac{1}{EM}\left[\quad\right]\frac{E'}{2(2\pi)^3}\;\frac{1}{(2\pi)^3}\;(2\pi)^4\;\frac{d^3p_4}{2E_4}\,\delta^{(4)}(q+p-p_4)$$

$$= \frac{1}{4}\left(\frac{E'}{E}\right)\frac{1}{2M}\,\frac{1}{(2\pi)^2}\left[\quad\right]\;\frac{d^3p_4}{2E_4}\,\delta^{(4)}(q+p-p_4)$$

$$= \frac{1}{8}\,\frac{E'}{E}\,\frac{1}{M}\,\frac{(4\pi)^2}{(2\pi)^2}\,\frac{\alpha^2}{q^4}\,16\,M^2\,EE'\left(\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right)$$

$$\times\;\frac{d^3p_4}{2E_4}\,\delta^{(4)}(q+p-p_4)$$

$$= \frac{4\alpha^2(E')^2}{q^4}\left(\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right)2M\,\frac{d^3p_4}{2E_4}\,\delta^{(4)}(q+p-p_4)$$

with $\alpha = e^2/4\pi$

$$\frac{d\sigma}{dE'\,d\Omega} = \frac{4\alpha^2 (E')^2}{q^4}\left(\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right) 2M \frac{d^3p_4}{2E_4}\delta^{(4)}(q+p-p_4)$$

To get the inclusive x-section, one has to integrate over d^3p_4. One has

$$\int \frac{d^3p_4}{2E_4}\delta^{(4)}(q+p-p_4) = \int \frac{d^3p_4}{2E_4}\cdot\delta^{(3)}[(\underline{q}+\underline{p})-\underline{p}_4]\,\delta[(q+p)_0 - E_4]$$

$$= \int \frac{d^3p_4}{2E_4}\delta^{(3)}[(\underline{q}+\underline{p})-\underline{p}_4]\,\delta\left[\sqrt{(q+p)^2+|\underline{q}+\underline{p}|^2} - E_4\right]$$

$$= \frac{1}{2E_4}\delta\left[\sqrt{(q+p)^2+\underline{p}_4^2} - E_4\right] = \delta\left[(q+p)^2 - (E_4^2 - \underline{p}_4^2)\right]$$

Using now the relationship $\delta(x^2-a^2) = \frac{1}{2x}\delta(x-a)$, one finds

$$\int \frac{d^3p_4}{2E_4}\delta^{(4)}(q+p-p_4) = \delta\left[(q+p)^2 - M^2\right]$$

or, using $\delta(cx) = \delta(x)/c$, $p^2 = M^2$, $q^2 = -Q^2$, $qp = M\nu$

$$\int \frac{d^3p_4}{2E_4}\delta^{(4)}(q+p-p_4) = \delta(2M\nu - Q^2)$$

$$= \frac{1}{2M}\delta\left(\nu - \frac{Q^2}{2M}\right)$$

The inclusive x-section now reads

$$\boxed{\frac{d^2\sigma}{dE'd\Omega} = \frac{4\alpha^2(E')^2}{q^4}\left(\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right)\delta\left(\nu - \frac{Q^2}{2M}\right)} \quad (4)$$

Integrate now over dE' using again $\int dx\, \delta[f(x)]$ $= \int \left|\frac{df}{dx}\right|^{-1} \delta(x-a)$ with $f(a) = 0$.

$$\frac{d\sigma}{d\Omega} = \int \frac{4\alpha^2(E')^2}{q^4}\left(\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right)\delta\left(\nu - \frac{Q^2}{2M}\right) dE'$$

$$= \frac{4\alpha^2(E')^2}{q^4}\left(\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right)\left|\frac{d}{dE'}\left(E - E' - \frac{4EE'}{2M}\sin^2\frac{\theta}{2}\right)\right|^{-1}$$

$$= \frac{4\alpha^2(E')^2}{q^4}\left(1 + \frac{2E}{M}\sin^2\frac{\theta}{2}\right)^{-1}\left(\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right)$$

Since energy conservation implies that

$$1 + \frac{2E}{M}\sin^2\frac{\theta}{2} = 1 + \frac{1}{2E'M}Q^2 = \frac{1}{E'}\left(E' + \frac{Q^2}{2M}\right) = \frac{1}{E'}\left(E' + \nu\right) = \frac{E}{E'}$$

the x-section can finally be rewritten as

$$\frac{d\sigma}{d\Omega} = \frac{4\alpha^2(E')^2\cos^2\theta/2}{Q^4}\left(\frac{E'}{E}\right)\left(1 + \frac{Q^2}{2M^2}\tan^2\frac{\theta}{2}\right)$$

or

$$\boxed{\begin{aligned} \frac{d\sigma}{d\Omega} &= \left(\frac{d\sigma}{d\Omega}\right)_{Mott} \left(\frac{E'}{E}\right)\left(1 + \frac{Q^2}{2M^2}\tan^2\frac{\theta}{2}\right) \\ \left(\frac{d\sigma}{d\Omega}\right)_{Mott} &= \frac{4\alpha^2 (E')^2}{Q^4}\cos^2\frac{\theta}{2} = \frac{\alpha^2 \cos^2\theta/2}{4E^2 \sin^4\theta/2} \end{aligned}} \tag{5}$$

Note that, in conclusion, the integral

$$\int d^3p_4\, \delta^{(4)}(q+p-p_4)$$

yields a factor

$$\frac{E_4}{M}\,\delta\left(\nu - \frac{Q^2}{2M}\right)$$

which after integration over E' gives the additional factor

$$\frac{E'}{E}$$

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* Electron - Proton Scattering. General Case *

In this case the electron tensor is the same as in the previous example, whereas the hadronic tensor $W_{\mu\nu}$ reads as follows

this now means sum over final states and spin average in the initial state $$W_{\mu\nu} = \frac{1}{2}\left(\sum_n\right) \langle i | J_\mu^\dagger | f \rangle \langle f | J_\nu | i \rangle (2\pi)^3 \delta^{(4)}(p+q-p')$$

Conservation of the hadronic current implies

$$\frac{\partial}{\partial x_\mu} \langle f | J_\mu(x) | i \rangle = e^{i(p-p')x}\, i(p-p')_\mu \langle f | J_\mu(0) | i \rangle = 0$$

which is equivalent to

$$q_\mu \langle f | J_\mu(0) | i \rangle = 0$$

In conclusion, current conservation for the target proton can be expressed by the relation

$$q_\mu W_{\mu\nu} = W_{\mu\nu} q_\nu = 0$$

Exactly in the same way one can show the requirement implied by current conservation for the lepton. One has (due to the Dirac equation $(\not{k}+m)u(k)=0$)

$$\frac{\partial}{\partial x_\mu} \bar{u}(k')\gamma^\mu u(k) = \bar{u}(k')\not{q}\, u(k) = \bar{u}(k')(\not{k}-\not{k}')u(k) = 0$$

and

$$q_\mu \eta_{\mu\nu} = \eta_{\mu\nu} q_\nu = 0$$

Let's now come again to the hadronic tensor. We have two independent scalars at our disposal: q^2 and $q\cdot p = -Mq_0$. In fact, ~~we the scalar produces~~ $p^2 = M^2$ is fixed.

The most general tensor one can construct using q and p is

$$W_{\mu\nu} = W_1 g_{\mu\nu} + W_2 \frac{p_\mu p_\nu}{M^2} + W_3 \frac{q_\mu q_\nu}{M^2} +$$

$$+ W_4 \frac{(p_\mu q_\nu + p_\nu q_\mu)}{M^2} + W_5 \frac{(p_\mu q_\nu - p_\nu q_\mu)}{M^2}$$

use now current conservation

$$q_\mu W_{\mu\nu} = W_1 q_\nu + W_2 \frac{(p\cdot q)}{M^2} p_\nu + W_3 \frac{q^2}{M^2} q_\nu$$

$$+ W_4 \frac{[(p\cdot q) q_\nu + q^2 p_\nu]}{M^2} + W_5 \frac{[(pq) q_\nu - q^2 p_\nu]}{M^2} = 0$$

$$W_{\mu\nu} q_\nu = W_1 q_\mu + W_2 \frac{(p\cdot q)}{M^2} p_\mu + W_3 \frac{q^2}{M^2} q_\mu$$

$$+ W_4 \frac{[q^2 p_\mu + (pq) q_\mu]}{M^2} + W_5 \frac{[q^2 p_\mu - (pq) q_\mu]}{M^2} = 0$$

Since p and q are linearly independent, the first equation implies

$$W_1 + W_3 \frac{q^2}{M^2} + W_4 \frac{(p.q)}{M^2} + W_5 \frac{(p.q)}{M^2} = 0 \qquad (A)$$

$$W_2 \frac{(p.q)}{M^2} + W_4 \frac{q^2}{M^2} - W_5 \frac{q^2}{M^2} = 0 \qquad (B)$$

whereas the second one yields

$$W_1 + W_3 \frac{q^2}{M^2} + W_4 \frac{(p.q)}{M^2} - W_5 \frac{(p.q)}{M^2} = 0 \qquad (C)$$

$$W_2 \frac{(p.q)}{M^2} + W_4 \frac{q^2}{M^2} + W_5 \frac{q^2}{M^2} = 0 \qquad (D)$$

Eqs. (B) and (D) give

$$W_5 = 0$$

and

$$W_4 = - W_2 \frac{(p.q)}{q^2} .$$

Substitution in eq. (A) yields

$$W_1 + W_3 \frac{q^2}{M^2} - W_2 \frac{(p.q)^2}{M^2 q^2} = 0$$

or

$$W_3 = - \frac{M^2}{q^2} W_1 + \left(\frac{p.q}{q^2}\right)^2 W_2$$

In conclusion, the hadronic tensor has the following form

$$W_{\mu\nu} = W_1 g_{\mu\nu} + W_2 \frac{P_\mu P_\nu}{M^2} + \left[- W_1 \frac{M^2}{q^2} + \left(\frac{P\cdot q}{q^2}\right)^2 W_2 \right] \frac{q_\mu q_\nu}{M^2}$$

$$- W_2 \frac{(P\cdot q)}{q^2} \frac{(P_\mu q_\nu + P_\nu q_\mu)}{M^2}$$

$$= W_1 \left(g_{\mu\nu} - \frac{M^2}{q^2} \frac{q_\mu q_\nu}{M^2} \right) + W_2 \left[\frac{P_\mu P_\nu}{M^2} + \left(\frac{P\cdot q}{q^2}\right)^2 \frac{q_\mu q_\nu}{M^2} \right.$$

$$\left. - \frac{(P\cdot q)}{q^2} \frac{P_\mu q_\nu + P_\nu q_\mu}{M^2} \right]$$

$$= W_1 \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) + \frac{W_2}{M^2} \left[\left(P_\mu - \frac{P\cdot q}{q^2} q_\mu \right) \left(P_\nu - \frac{P\cdot q}{q^2} q_\nu \right) \right]$$

Changing now $W_1 \to -W_1$ (which is convenient to compare with the elastic scattering case, as we shall see later) one finds the general form for the hadronic tensor

$$W_{\mu\nu} = W_1 \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) + \frac{W_2}{M^2} \left[P_\mu - \frac{(P.q)}{q^2} q_\mu \right] \left[P_\nu - \frac{(P.q)}{q^2} q_\nu \right]$$

where W_1 and W_2 are function of the two available scalar quantities q^2 and $p \cdot q$. Recalling that, in the lab system $p \cdot q = M\nu$ one can also use

$$W_1 = W_1(\nu, q^2) \qquad W_2 = W_2(\nu, q^2)$$

Consider now the contraction $L_{\mu\nu} W^{\mu\nu}$ (L is given on page 2)

$$L_{\mu\nu} W^{\mu\nu} = 2 \left[k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu} (kk') \right]$$
$$\times \left[W_1 \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) + \frac{W_2}{M^2} \left(P^\mu - \frac{P.q}{q^2} q^\mu \right) \left(P^\nu - \frac{P.q}{q^2} q^\nu \right) \right]$$

where the limit $m^2 \to 0$ for the squared e-mass has been taken. Due to conservation of the lepton current ($L_{\mu\nu} q_\nu = 0$) one has

$$L_{\mu\nu} W^{\mu\nu} = 2\left[k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}(kk') \right] \cdot \left[-W_1 g^{\mu\nu} + \frac{W_2}{M^2} p^\mu p^\nu \right]$$

$$= 2\left[-2 W_1 (kk') + 4 W_1 (kk') + 2\frac{W_2}{M^2}(pk)(pk') - \frac{W_2}{M^2} p^2 (kk') \right]$$

Using now $kk' = EE'(1-\cos\theta) = 2EE' \sin^2\theta/2$, $(pk) = ME$, $(pk') = ME'$ and $p^2 = M^2$ one gets

$$L_{\mu\nu} W^{\mu\nu} = 4 W_1\, 2 EE' \sin^2\theta/2 + \frac{W_2}{M^2}\left(4EE'M^2 - 4M^2 EE' \sin^2\frac{\theta}{2} \right)$$

$$= 4EE' \left[2W_1(\nu, q^2) \sin^2\theta/2 + W_2(\nu, q^2) \cos^2\theta/2 \right]$$

$$\boxed{L_{\mu\nu} W^{\mu\nu} = 4EE' \left[2W_1(\nu, q^2)\sin^2\theta/2 + W_2(\nu, q^2)\cos^2\theta/2 \right]}$$

The inelastic x-section can now easily be obtained using eq. (1) and taking into account the fact that now $W^{\mu\nu}$ includes the sum over the allowed hadronic final states. One has:

$$d\sigma = \frac{1}{2E} \frac{d^3k'}{(2\pi)^3 2E'} (2\pi) \frac{e^4}{Q^4} L_{\mu\nu} W^{\mu\nu}$$

or

$$\frac{d^2\sigma}{d\Omega dE'} = \frac{\alpha^2}{Q^4} \frac{E'}{E} L_{\mu\nu} W^{\mu\nu}$$

$$\boxed{\frac{d^2\sigma}{d\Omega\, dE'} = \frac{\alpha^2}{Q^4} \frac{E'}{E} L_{\mu\nu} W^{\mu\nu}}$$

Using the explicit form of $L_{\mu\nu} W^{\mu\nu}$ this becomes

$$\frac{d^2\sigma}{d\Omega\, dE'} = \frac{\alpha^2}{Q^4} \frac{E'}{E} 4 EE' \left[2 W_1(\nu, q^2) \sin^2\frac{\theta}{2} + W_2(\nu, q^2) \cos^2\frac{\theta}{2} \right]$$

$$= \frac{4\alpha^2 (E')^2}{Q^4} \cos^2\frac{\theta}{2} \left[W_2(\nu, q^2) + 2 W_1(\nu, q^2) \tan^2\frac{\theta}{2} \right]$$

$$\boxed{\frac{d^2\sigma}{d\Omega\, dE'} = \left(\frac{d\sigma}{d\Omega}\right)_{Mott} \left[W_2(\nu, q^2) + 2 W_1(\nu, q^2) \tan^2\frac{\theta}{2} \right]}$$

Consider now the case of elastic scattering to establish the relationship between W_1 and W_2 and the nucleon form factors F_1 and F_2 (or G_E and G_M).
In this case, to evaluate the hadronic tensor, one has to compute

$$F_{\mu\nu} = \frac{1}{2} \sum_{s_2 s_4} \bar{u}(p', s_4)\, \Gamma_\mu\, u(p, s_2)\, \bar{u}(p, s_2)\, \Gamma_\nu\, u(p', s_4)$$

with

$$\Gamma_\mu = \gamma_\mu F_1(q^2) + i\, \sigma_{\mu\lambda} q^\lambda \frac{\varkappa}{2M} F_2(q^2)$$

and, as usual, $\sigma_{\mu\nu} = i[\gamma_\mu, \gamma_\nu]/2$. F_1, F_2 and $\varkappa$ represent the Dirac and Pauli form factors and the proton anomalous magnetic moment.
One finally gets $(Q^2 = -q^2)$ (see appendix B)

$$W_1^{el}(\nu, Q^2) = \frac{Q^2}{4M^2} G_M^2(Q^2)\, \delta\left(\nu - \frac{Q^2}{2M}\right)$$

$$W_2^{el}(\nu, Q^2) = \frac{G_E^2(Q^2) + (Q^2/4M^2)\, G_M^2(Q^2)}{1 + Q^2/4M^2}\, \delta\left(\nu - \frac{Q^2}{2M}\right)$$

Or, recalling the relationship between the Sachs form factors G_E and G_M and the Dirac and Pauli form factors F_1 and F_2

$$G_M = F_1 + \varkappa F_2$$

$$G_E = F_1 - \frac{Q^2}{4M^2} \varkappa F_2$$

$$W_1^{el}(\nu, Q^2) = \frac{Q^2}{4M^2}\left(F_1 + \varkappa F_2\right)^2 \delta\left(\nu - \frac{Q^2}{2M}\right)$$

$$W_2^{el}(\nu, Q^2) = \left\{1 + \frac{Q^2}{4M^2}\right\}^{-1}\left\{\left(F_1 - \frac{Q^2}{4M^2}\varkappa F_2\right)^2 + \frac{Q^2}{4M^2}\left(F_1 + \varkappa F_2\right)^2\right\}$$

$$\times\, \delta\left(\nu - \frac{Q^2}{2M}\right)$$

$$= \left(1 + \frac{Q^2}{4M^2}\right)^{-1}\left\{ F_1^2 + \left(\frac{Q^2}{4M^2}\right)^2 \varkappa^2 F_2^2\right.$$

$$- \cancel{\frac{Q^2}{2M^2}\varkappa F_1 F_2} + \frac{Q^2}{4M^2} F_1^2 + \cancel{\frac{Q^2}{2M^2}\varkappa F_1 F_2}$$

$$\left. + \frac{Q^2}{4M^2}\varkappa^2 F_2^2 \right\} \delta\left(\nu - \frac{Q^2}{2M}\right)$$

$$= \left(1 + \frac{Q^2}{4M^2}\right)^{-1}\left[F_1^2\left(1 + \frac{Q^2}{4M^2}\right) + \frac{Q^2}{4M^2}\varkappa^2 F_2^2\left(1 + \frac{Q^2}{4M^2}\right)\right]$$

$$\times\, \delta\left(\nu - \frac{Q^2}{2M}\right)$$

$$= \left(F_1^2 + \frac{Q^2}{4M^2}\varkappa^2 F_2^2\right) \delta\left(\nu - \frac{Q^2}{2M}\right)$$

In conclusion we find

$$\boxed{\begin{aligned} W_1^{el.}(\nu, Q^2) &= \frac{Q^2}{4M^2}\left(F_1 + \kappa F_2\right)^2 \delta\left(\nu - \frac{Q^2}{2M}\right) \\ W_2^{el.}(\nu, Q^2) &= \left(F_1^2 + \frac{Q^2}{4M^2}\kappa^2 F_2^2\right)\delta\left(\nu - \frac{Q^2}{2M}\right) \end{aligned}}$$

Substitution in the definition of $d^2\sigma / d\Omega\, dE'$ of page 18 yields ($\tau = Q^2/4M^2$)

$$\frac{d^2\sigma}{d\Omega\, dE'} = \left(\frac{d\sigma}{d\Omega}\right)_{Mott} \left[\left(F_1^2 + \tau\, \kappa^2 F_2^2\right) + 2\tau\left(F_1 + \kappa F_2\right)^2\right]$$

$$\times\, \delta\left(\nu - \frac{Q^2}{2M}\right)$$

The integration over dE' using the δ function gives a factor E'/E (see page 10).

<u>The final result is the Rosenbluth x-section</u>

$$\boxed{\begin{aligned} \left(\frac{d\sigma}{d\Omega}\right)_R = \left(\frac{d\sigma}{d\Omega}\right)_{Mott} \frac{E'}{E} \Big\{ &\left[F_1^2(Q^2) + \frac{Q^2}{4M^2}\kappa^2 F_2^2(Q^2)\right] \\ &+ \frac{Q^2}{2M^2}\left[F_1(Q^2) + \kappa F_2(Q^2)\right]^2 \tan^2\frac{\theta}{2} \Big\} \end{aligned}}$$