1 Chiral symmetry and its consequences

There is a very fortunate connection between the electroweak sector of the Standard Model and the world of strong interactions. The link is provided by chiral symmetry, which is the approximate symmetry exhibited by QCD due to the small values of the light quark masses, as compared to the intrinsic scale of the theory, $\Lambda_{\rm QCD}$. Indeed, if we set the light quark masses to zero, then we can mix independently the left and right chiralities of the quarks,

$$\psi_L = \begin{pmatrix} u \\ d \\ \vdots \end{pmatrix}_L \to V_L \psi_L, \quad \psi_r = \begin{pmatrix} u \\ d \\ \vdots \end{pmatrix}_R \to V_L \psi_R, \tag{1}$$

where $\psi_{R/L} = (1 \pm \gamma_5)/2\psi$ and the transformation matrix $V_{R/L}$ $\in$SU(N_f), N_f being the number of light flavours, in the case of interest $N_f = 2$. Without quark masses, this transformation leaves the Lagrangian invariant, since the kinetic term does not mix chiralities,

$$\bar{\psi} i \not{D} \psi = \bar{\psi}_R i \not{D} \psi_R + \bar{\psi}_R i \not{D} \psi_R. \tag{2}$$

In the absence of quark masses the left and right chiralities live their life independently.

This is a global symmetry, since if we allow V_L and V_R to depend on the spacetime point, then we could not shift the transformation matrix through the (covariant) derivative. It is also a continuous symmetry, characterized by parameters $\alpha^a_{R/L}$, such that the SU(N) transformation matrices can be written

$$V_{R/L} = \mathrm{e}^{i \sum_a \frac{\lambda^a}{2} \alpha^a_{R/L}}, \tag{3}$$

in terms of the group generators λ^a in the defining representation (in the $N_f = 2$ case $\lambda^a = \tau^a$, the Pauli matrices, $a = 1, 2, 3$). As all continuous symmetries, chiral symmetry implies the existence of conserved Noether currents and associated charges. It is an easy first exercise to derive them as

$$J^a_{\mu R/L} = \bar{\psi}_{R/L} \gamma_\mu \frac{\lambda^a}{2} \psi_{R/L} \tag{4}$$

with corresponding charges

$$Q^a_{R/L} = \int d^3\mathbf{x} J^a_{0R/L} \tag{5}$$

which, in force of the current conservation equation $\partial^\mu J_\mu = 0$, are time-independent. And here is the link that was mentioned before: these Noether currents are the same as the electroweak currents, that is, to these currents are coupled weakly interacting particles.

It is customary to define the vector and axial vector currents as the appropriate combinations

$$V^a_\mu = J^a_{\mu R} + J^a_{\mu L}, \quad A^a_\mu = J^a_{\mu R} - J^a_{\mu L}, \quad , \tag{6}$$

with corresponding vector and axial vector charges denoted as Q^a and Q^a_5 respectively. At this point it is worthwile to observe that actually the symmetry group of the Lagrangian is $\mathrm{U}(N_f)\times\mathrm{U}(N_f)$, so that the index a runs over the U(1) component too, say $a = 0$. But this is only possible for the vector symmetry, because the axial U(1) is affected by the QCD anomaly. The charges are the group generators and they satisfy the group algebra. Also the currents, by covariance, satisfy what is called the "current algebra",

$$[Q^a, V^b_\mu] = if^{abc}V^c_\mu, \quad [Q^a, A^b_\mu] = if^{abc}A^c_\mu, \tag{7}$$
$$[Q^a_5, V^b_\mu] = if^{abc}A^c_\mu, \quad [Q^a_5, A^b_\mu] = if^{abc}V^c_\mu. \tag{8}$$

On the basis of these relations a number of results were obtained in the '60, at a time when the theory of strong interaction was not yet known.

2 Chiral Ward identities

The symmetry, which at the classical level is simply expressed by the current conservation relation (or partial conservation), implies, at the quantum level, a whole hierarchy of relations among Green functions, i.e. vacuum correlation functions of time-ordered products of local operators involving the Noether currents themselves, e.g.,

$$q^\mu \int dx \mathrm{e}^{iq\cdot x} \langle 0|T\left\{J_\mu(x)O_1(x_1)...O_n(x_n)\right\}|0\rangle \tag{9}$$
$$\sim \int dx \mathrm{e}^{iq\cdot x} \langle 0|T\left\{\partial^\mu J_\mu(x)O_1(x_1)...O_n(x_n)\right\}|0\rangle + \langle 0|[Q, O_1(x_1)]...O_n(x_n)|0\rangle + \tag{10}$$

When we are in the presence of a spontaneously broken symmetry, as in the case of chiral symmetry, these identities lead to the appearance of the order parameters of the spontaneous breaking, since the commutator terms do not vanish in general if the charge doesn't annihilate the vacuum.

There is a very convenient way to resume all these Ward identities. It consists of promoting the global symmetry to a local one. This means that the symmetry transformation parameters $\alpha^a_{R/L}$ are allowed to depend on the spacetime. Under such local transformations,

$$\psi_R \to V_R(x)\psi_R, \quad \psi_L \to V_L(x)\psi_L, \tag{11}$$

the Lagrangian is not left invariant, unless we equip it with external fields that transform as gauge fields, in order to absorb the non-invariant terms. We will thus write,

$$\bar{\psi}_R i \not{D} \psi_R + \bar{\psi}_R r^\mu \gamma_\mu \psi_R, \tag{12}$$

with the external field r^μ transforming under the symmetry transfornation as

$$r^\mu \to V_R r^\mu V_R^\dagger - u\partial^\mu V_R V_R^\dagger, \tag{13}$$

and, to gauge the left transformations, we have to introduce a corresponding external field ℓ^μ, transforming analogously. The external fields r^μ and ℓ^μ are matrices in flavor space, they belong to the group algebra, so that we can expand them in the basis of the generators,

$$r_\mu = \sum_a r^a_\mu \frac{\lambda^a}{2}, \quad \ell_\mu = \sum_a \ell^a_\mu \frac{\lambda^a}{2}. \tag{14}$$

We have thus promoted the global symmetry to a local one. We can also add scalar and pseudoscalar sources that absorb the non invariant contributions issuing from the quark mass terms and write

$$\bar{\psi}_R(s+ip)\psi_L + \bar{\psi}_L(s-ip)\psi_R, \tag{15}$$

where, under chiral transformations,

$$s+ip \to V_R(s+ip)V_L^\dagger, \quad s-ip \to V_L(s-ip)V_R^\dagger. \tag{16}$$

What we have done is to couple bilinear quark operators to external vector, axial, scalar and pseudoscalar sources,

$$\mathcal{L}_{\rm QCD}[v_\mu, a_\mu, s, p] = \mathcal{L}^0_{\rm QCD} + \bar{\psi}\left[\gamma^\mu\left(v_\mu + \gamma_5 a_\mu\right) - s + ip\gamma_5\right]\psi, \tag{17}$$

where $\mathcal{L}^0_{\rm QCD}$ is the QCD Lagrangian in the chiral limit and $v_\mu = r_\mu + \ell_\mu$, $a_\mu = r_\mu - \ell_\mu$ are the external vector and axial vector sources. Besides rendering the theory chiral gauge invariant, the external sources are alsto useful because they can generate the Green functions of quark bilinears, through the generating functional $W[v_\mu, a_\mu, s, p]$,

$$\mathrm{e}^{iW[v_\mu,a_\mu,s,p]} = \int \mathcal{D}\mu_{\mathrm{QCD}} \mathrm{e}^{i\int dx \mathcal{L}_{\mathrm{QCD}}[v_\mu,a_\mu,s,p]}. \tag{18}$$

By differentiating W with respect to its arguments and evaluating the result at zero external sources, we get all connected Green functions of the quark bilinears. If, instead of evaluating the functional derivatives at zero value of the external sources we put $V_\mu = A_\mu = P = 0$ and $s = \mathcal{M}$, where $\mathcal{M}$ is the light quark mass matrix, we obtain the physical correlation functions, away from the chiral limit.

Now, what happens if we subject the sources to the chiral gauge transformations Eqs.(13)-(16) that we introduced?

$$\mathrm{e}^{iW[v'_\mu,a'_\mu,s',p']} = \int \mathcal{D}\mu_{\mathrm{QCD}} \mathrm{e}^{i\int dx \mathcal{L}_{\mathrm{QCD}}[v'_\mu,a'_\mu,s',p']} = \int \mathcal{D}\mu_{\mathrm{QCD}} \mathrm{e}^{i\int dx \tilde{\mathcal{L}}_{\mathrm{QCD}}[v_\mu,a_\mu,s,p]}, \tag{19}$$

where $\tilde{\mathcal{L}}$ is written in terms of the antitransformed quark fields,

$$\psi_{R/L} = V_{R/L}\tilde{\psi}_{R/L}, \tag{20}$$

using the invariance of the complete Lagrangian. But the quark fields are mere functional integration variables, dummy variables. Provided that the QCD functional measure is invariant under the transformation, say $\mathcal{D}\mu_{\mathrm{QCD}} = \mathcal{D}\tilde{\mu}_{\mathrm{QCD}}$, we can conclude that

$$\mathrm{e}^{iW[v'_\mu,a'_\mu,s',p']} = \mathrm{e}^{iW[v_\mu,a_\mu,s,p]}. \tag{21}$$

which express the invariance of the generating functional under local chiral transformation. It should be remembered that, for axial transformation, the integration measure is not invariant, nevertheless the Jacobian can be expressed in closed form, and this has been done by Bardeen. So the correct equation is

$$W[v'_\mu, a'_\mu, s', p'] = W[v_\mu, a_\mu, s, p] + \Delta[v_\mu, a_\mu, s, p; V_L V_R^\dagger]. \tag{22}$$

It is called the chiral anomaly, but we need not discuss it here. By focusing on infinitesimal transformations, characterized by infinitesimal vector and axial parameters $\alpha^a = \alpha^a_R + \alpha^a_L$, $\beta^a = \alpha^a_R - \alpha^a_L$, and taking a derivative with respect to α^a and β^a one obtains two functional equations which contain all the constraints from the chiral Ward identities for quark bilinears. To obtain such equations is left as the second exercise.

Of course these are just formal manipulations, as we don't know how to calculate the functional integral. Perturbation theory in powers of the QCD coupling constants will certainly not work at large distances, due to the asymptotic freedom. Nevertheless, the spontaneous breakdown of chiral symmetry ensures two things: *i*) that there exist massless particles, the Goldstone bosons, to be identified with the pions; *ii*) that these particles interact weakly at low energy. The latter is also a consequence of the Goldstone theorem, something that entails what is called "soft pions theorems". These two facts allow to establish a calculational scheme for this functional. It is based on a representation in terms of the lightest vacuum excitation of the theory, the pions. So the same functional $W[v_\mu, a_\mu, s, p]$ is written as a result of a theory of interacting pions,

$$\mathrm{e}^{iW[v_\mu,a_\mu,s,p]} = \int \mathcal{D}U \mathrm{e}^{i\int dx \mathcal{L}_{\mathrm{eff}}[U;v_\mu,a_\mu,s,p]}. \tag{23}$$

If we want it to respect the chiral Ward identities, we have to demand that the effective Lagrangian be invariant under the local chiral transformations of the sources. But first we should ask: how to the pions transform under the chiral symmetry?

3 Non linear symmetry realization

Let us denote by π the pion fields. Under a transformation g of the chiral group $g =$SU(2)$_L\times$SU(2)$_R$, it will be transformed as

$$\pi \xrightarrow{g} \pi' = f(\pi, g), \tag{24}$$

with a certain function f which must satisfy the group composition law, i.e.

$$\pi \xrightarrow{g_1} \pi' = f(\pi, g_1) \xrightarrow{g_2} \pi'' = f(\pi', g_2) = f(f(\pi, g_1), g_2) = f(\pi, g_2 g_1). \tag{25}$$

Consider now a transformation $h \in G$ that leaves the origin of the pion field manifold invariant, i.e.

$$f(0, h) = 0. \tag{26}$$

It can readily be checked that all such transformations form a subgroup $H \subset G$. Moreover, for any $g \in G$ and $h \in H$,

$$f(0, g) = f(0, gh), \tag{27}$$

in force of the group composition law (25). Thus, for any $\bar{g}$,

$$f(0,\bar{g}) = f(0,g), \quad \forall g \in \bar{g}H, \tag{28}$$

where $\bar{g}H$ denotes what is called a left coset of the subgroup H. We can view $f(0,...)$ as a function which takes from the set of all the left cosets of H to the manifold of the pion fields. In addition, the correspondence is invertible, since, if $f(0,g) = f(0,g')$ then

$$f(0,g^{-1}g') = f(f(0,g'),g^{-1}) = f(f(0,g),g^{-1}) = f(0,g^{-1}g) = 0 \tag{29}$$

which implies that $g^{-1}g' \in H$ so that there is a $h \in H$ such that

$$g^{-1}g' = h \Longrightarrow g' = gh, \tag{30}$$

i.e. g and g' belong to the same left coset. So there is a one-to-one correspondence between the left cosets of H and the pion field manifold. Each pion field corresponds to some left coset of H. The set of all left cosets of H is called the (left) coset space and it is denoted by G/H. The pion fields live in this space, they can be viewed as coordinates of this space. In the case of chiral symmetry breakdown $G =$SU$(2)_L\times$SU$(2)_R \sim$SU$(2)_V\times$SU$(2)_A$ and $H =$SU$(2)_V$, and the coset space is a group itself, the group SU$(2)_A$. The pion field is thus essentially an SU(2) matrix. To set the correspondence between the coset space and SU(2) transformations we have to choose a representative in each left coset, say

$$g = (g_L, g_R) \to (1, g_R g_L^{-1}). \tag{31}$$

So, to every group transformation g, there corresponds a unique element of the coset space, represented by a SU(2) matrix $U = g_R g_L^{-1}$. We know that group transformation transform under group operation by the group composition law (by definition), so that

$$g = (g_L, g_R) \stackrel{(V_L,V_R)}{\longrightarrow} (V_L g_L, V_R g_R) \tag{32}$$

which corresponds to the coset space element U',

$$U' = V_R g_R g_L^{-1} V_L^{-1} = V_R U V_L^{-1} \tag{33}$$

so that the SU(2) matrix U, which can be taken to represent the pion field, transforms as

$$U \to U' = V_R U V_L^{\dagger}. \tag{34}$$

Different choices of the group representatives of the coset space elements (for instance $g \to (g_L g_R^{-1}, 1)$ would correspond to different transformation properties of the pion field matrix, which however lead to the same physical consequences, it would just amount to a change of coordinates of the coset space. The canonical choice for the parametrization of the SU(2) matrix U in terms of the pion field is

$$U = \mathrm{e}^{\frac{i}{F_0}\sum_a \pi^a \tau^a}, \tag{35}$$

where the constant F_0 sets the scale of the pion field. Another frequently used parametrization is the σ-model one

$$U = \sigma + \frac{i}{F_0}\sum_a \pi^a \tau^a, \quad \sigma^2 + \frac{\boldsymbol{\pi}^2}{F_0^2} = 1, \tag{36}$$

You may prove, as exercise number 3, that the most general possible parametrization is

$$U = f_0(\boldsymbol{\pi}^2) + i\left[1 - f_0(\boldsymbol{\pi}^2)\right]\sum_a \pi^a \tau^a, \tag{37}$$

with a real scalar function f_0, using the fact that π^a is an isotriplet, the known transformation properties of U under isospin ($V_L = V_R = V$), and the requirement of definite transformation laws for U under the discrete symmetries. Show also that, up to four powers of the pion fields, the most general parametrization depends on only two arbitrary parameters, one of which sets the overall scale of the pion field.

Nothing must depend on the choice of the pion field, since U is a mere functional integration variable. Or, better said, the QCD Green functions generated by the functional W must be independent of the choice of the pion field, and the same is true for on-shell pion amplitudes, which constitute pole residues of those correlation functions. Off the mass-shell there can be differences though. At the level of the effective Lagrangian, a change of field variables φ induces additional terms in the action, which are proportional to the equations of motion, since, if the equation of motions are fulfilled, then the action is stationary with respect to change of fields,

$$\frac{\delta S}{\delta \varphi} = 0. \tag{38}$$

Such "equations of motion terms" will affect off-shell amplitudes, renormalization constants, but not physical quantities. They are therefore irrelevant.

4 Exercises

As exercise #0, please point out the mistakes you will spot in these notes!

1. Derive the Noether currents of the chiral symmetry of QCD

2. Find how the vector, axial, scalar and pseudoscalar sources transform under infinitesimal local vector and axial transformations, with (infinitesimal) parameters $\alpha(x)$ and $\beta(x)$. Impose the invariance of the generating functional $W[V_\mu, A_\mu, s, p]$ of connected Green functions, with respect to such transformation to derive two functional equations (ignore the chiral anomaly for simplicity).