5 Construction of the effective Lagrangian

We determined the required transformation properties of the SU(2) pion field matrix U. Remember that V_R and V_L are spacetime dependent, so the derivatives of the pion field matrix will lead to extra contributions

$$\partial^\mu U \to \partial^\mu V_R U V_L^\dagger + V_R \partial^\mu U V_L^\dagger + V_R U \partial^\mu V_L^\dagger \tag{1}$$

which can be eliminated by the use of the covariant derivative

$$D^\mu U = \partial^\mu - i r^\mu U + i U \ell^\mu, \tag{2}$$

which transform homogeneously under chiral symmetry

$$D^\mu U \to V_R D^\mu U V_L^\dagger. \tag{3}$$

The same can be done for derivatives of the χ source,

$$D^\mu \chi \to V_R D^\mu \chi V_L^\dagger. \tag{4}$$

We may form invariant structures by taking flavour traces (denoted as $\langle ... \rangle$) of alternate products of operators transforming covariantly like U or $U^\dagger$. In addition we may also use the left and right source curvatures

$$\mathcal{R}_{\mu\nu} \to V_R \mathcal{R}_{\mu\nu} V_R^\dagger, \quad \mathcal{L}_{\mu\nu} \to V_L \mathcal{L}_{\mu\nu} V_L^\dagger, \tag{5}$$

to form invariant structures like e.g.

$$\langle \mathcal{R}_{\mu\nu} U \mathcal{L}^{\mu\nu} U^\dagger \rangle. \tag{6}$$

Of course we can build an infinite variety of chiral invariant operators, but they can be ordered according to the number of (covariant) derivatives, and/or external sources involved. Since each derivatives brings down one power of momenta, this ordering corresponds to a low-momentum expansion. An important fact is that there are no possible invariants without derivatives or external sources at all: the only one would be $\langle U^\dagger U \rangle$ which is however constant since $U \in$SU(2). The first non trivial operator contains two derivatives

$$\langle D^\mu U^\dagger D_\mu U \rangle, \tag{7}$$

and belong to the leading order Lagrangian

$$\mathcal{L}^{(2)} = \frac{F_0^2}{4} \left[\langle D^\mu U^\dagger D_\mu U \rangle + 2 B_0 \langle U^\dagger \chi + \chi^\dagger U \rangle \right], \tag{8}$$

together with a linear term in the scalar/pseudoscalar source $\chi = s + ip$, which provides a mass term for the pions, once evaluated at $s = \mathcal{M}$ the quark mass matrix. The superscript $^{(2)}$ in the Lagrangian specifies the so called "chiral power", according to the standard chiral counting

$$\partial^\mu \sim D^\mu \sim p^\mu \sim M_\pi \sim O(p), \quad \chi \sim O(p^2). \tag{9}$$

The constant F_0 which already appeared in the expression of U, is there to ensure a properly normalized pionic kinetic term. The other constant that appears at leading order (LO) is B_0, which is related to the vacuum quark-antiquark condensate in the SU(2) chiral limit. Notice that this Lagrangian already contains an infinite tower of pion self-interactions, which are always of derivative type, in the chiral limit. This is the manifestation of the good old soft pion theorems. Such interactions are also not renormalizable. But this causes no harm, as the renormalizability is recovered order by order in the chiral counting. This was Weinberg's original insight in the '70s, according to which a given diagram with L loops, I internal lines and n_i vertices of type i, each with chiral dimension d_i, will scale like p^ν with

$$\nu = 4L - 2I + \sum_i n_i d_i. \tag{10}$$

Using the topological identity relating L, I and $V = \sum_i n_i$, the total number of vertices,

$$L = I - V + 1, \tag{11}$$

we have that

$$\nu = 2L + 2 + \sum_i n_i (d_i - 2). \tag{12}$$

This implies that the loops are more and more suppressed in the chiral counting, since, as we have already noticed, $d_i \geq 2$. Notice that this depends crucially on the adoption of a mass-independent regularization scheme for the loop integrals, such as dimensional regularization, and on the fact that no hard scale is present in the integrands. This will change when there are nucleon propagators in the loops. But in this case we see that, up to a given chiral power ν only a finite number of diagrams needs to be calculated, and the divergences can be absorbed in the coefficients of the higher-order Lagrangians, which contain by construction all possible chiral invariant terms. What is the higher-order Lagrangian? Lorentz covariance and the adopted chiral counting implies that it may contain up to 4 (covariant) derivatives, or

2 derivatives and a χ source or source curvature, or two χ sources or source curvatures. The complete list of operators is

$$
\begin{aligned}
\mathcal{L}^{(4)} \quad = \quad & \frac{\ell_1}{4}\langle D^\mu U^\dagger D_\mu U\rangle^2 + \frac{\ell_2}{4}\langle D^\mu U^\dagger D^\nu U\rangle\langle D_\mu U^\dagger D_\nu U\rangle \\
& +\ell_3 B_0^2\langle U^\dagger\chi + \chi^\dagger U\rangle^2 + \ell_4\frac{B_0}{2}\langle D^\mu U^\dagger D_\mu\chi + D^\mu\chi^\dagger D_\mu\chi\rangle \\
& +\ell_5\langle\mathcal{R}^{\mu\nu}U\mathcal{L}_{\mu\nu}U^\dagger\rangle + \frac{i}{2}\ell_6\langle\mathcal{R}^{\mu\nu}D_\mu U D_\nu U^\dagger + \mathcal{L}^{\mu\nu}D_\mu U^\dagger D_\nu U\rangle \\
& -\ell_7\frac{B_0^2}{4}\langle U^\dagger\chi - \chi^\dagger U\rangle^2
\end{aligned} \tag{13}
$$

Whenever possible the terms with a single trace have been expressed as products of traces, using the Cayley-Hamilton relations satisfied by any 2×2 matrix X,

$$X^2 - X\langle X\rangle + \det X = 0, \tag{14}$$

that, applied to $X = A + B$ implies also

$$AB + BA - A\langle B\rangle - B\langle A\rangle - \langle AB\rangle + \langle A\rangle\langle B\rangle. \tag{15}$$

Also, terms with squared covariant derivatives, like D^2U can be eliminated using the equation of motion, according to the discussion above. The constants ℓ_i are "low-energy constants" (LECs), whence their initial, while other pure source contact terms like $\langle\chi^\dagger\chi\rangle$ are multiplied by "high-energy constants" denoted by h_i.

6 M_π and F_π to one loop

With this Lagrangian it is already possible to determine the chiral expansion of the pion mass, by considering the pion propagator, This could be the subject of the afternoon hands-on activity. The pion propagator up to one loop is made up of the diagrams depicted in Fig. 1, where the first term is the free propagator, the second term is the tadpole calculated with the vertices from $\mathcal{L}^{(2)}$ and the third one is the tree contribution from the subleading $\mathcal{L}^{(4)}$. The iteration of the Dyson series leads to

$$\frac{i}{p^2(1+A) - M^2 + B} \tag{16}$$

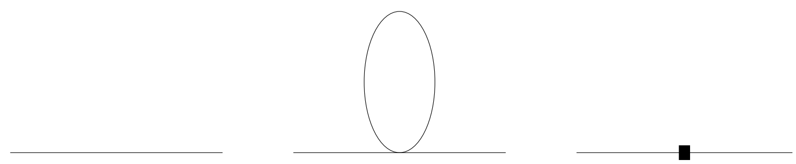

Figure 1: Pion propagator up to one loop.

with calculable coefficients A and B. from the pole of the dressed propagator we read the physical pion mass,

$$M_\pi^2 = \frac{M^2 - B}{1 + A}. \tag{17}$$

Explicit calculation in dimensional regularization gives

$$A = \frac{M^2}{8\pi^2 F_0^2} \left(\frac{1}{d-4} + ... \right) \tag{18}$$

where the dots stand for finite contributions for $d = 4$, and

$$B = -2\ell_3 \frac{M^2}{F_0^2} - \frac{3M^4}{16\pi^2 F_0^2} \left(\frac{1}{d-4} + ... \right), \tag{19}$$

so that the physical pion mass is

$$M_\pi^2 = M^2 \left[1 - 2\ell_3 \frac{M^2}{F_0^2} + \frac{M^2}{16\pi^2 F_0^2} \left(\frac{M^{d-4}}{d-4} + ... \right) \right], \tag{20}$$

where in all these equations $M = 2B\hat{m}$, with $\hat{m}$ the average light quark mass, is the leading order pion mass squared. The loop divergence shows up as a pole in dimension d, and it can be absorbed by the renormalization of the ℓ_3. The divergent contribution also specifies the so-called chiral log, as coming from expanding the non-analytic M-dependence in powers of $d - 4$. The renormalized result is

$$M_\pi^2 = M^2 \left(1 + 2\ell_3^r(\mu) \frac{M^2}{F_0^2} + \frac{M^2}{16\pi^2 F_0^2} \log \frac{M}{\mu} \right) \tag{21}$$

and the renormalized LEC becomes dependent on the scale μ, related to the choice of subtraction. The pion decay constant is found by calculating the diagrams in Fig. 2, in addition to the ones with the dressed pion propagators,

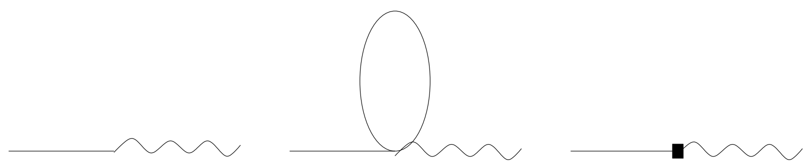

Figure 2: 1PI contributions to F_π up to one loop.

that we know already how to treat. The result is

$$\frac{i}{(1+A)(p^2-M_\pi^2)} i p_\mu F_0 \left[1+\ell_4\frac{M^2}{F_0^2}-\frac{4M^2}{16\pi^2}\left(\frac{M^{d-4}}{d-4}+...\right)\right]. \tag{22}$$

We have still to remember that there is a Z in the LSZ formula, so we have to multiply the result by $Z^{-1/2}$. Who is Z? The renormalization constant appears in the definition of the interpolating field φ, as, e.g.,

$$\varphi(x) \to Z^{1/2}\varphi_{\rm in}(x), \tag{23}$$

such that

$$\langle 0|\varphi(x)|\pi(\mathbf{p})\rangle = Z^{1/2}{\rm e}^{-ip\cdot x}. \tag{24}$$

The same Z appears in the $\pi \to \pi$ amplitude (the "dressed propagator"), or in the matrix element

$$\langle 0|\varphi(x)|\pi(\mathbf{p})\rangle = Z^{-1/2}\left(\frac{i}{p^2-M_\pi^2}\right)^{-1}\left(\frac{i}{(1+A)(p^2-M_\pi^2)}\right){\rm e}^{-ip\cdot x}, \tag{25}$$

so that

$$Z^{-1} = 1+A. \tag{26}$$

At the end we end up with

$$F_\pi = F_0\left[1+\ell_4\frac{M^2}{F_0^2}-2\frac{M^2}{16\pi^2F_0^2}\left(\frac{M^{d-4}}{d-4}+...\right)\right], \tag{27}$$

which can be made finite upon renormalization of the LEC ℓ_4. Notice that the pion field renormalization constant, as well as the individual contributions to Feynman diagrams, depend on the choice of the pion field. They would differ if another parametrization than the σ-model one was used. There would be differences also if equations of motion were used to change the Lagrangian $\mathcal{L}^{(4)}$ (as it is sometimes done to rewrite the term proportional to ℓ_4). Nevertheless, the physical results, and the anomalous dimensions of the LECs, would remain the same.

7 The physics of the LECs

There is another aspect to discuss concerning the LECs, i.e. the fact that they contain information about the short-distance physics that is not explicitly included in the effective theory. This is called “resonance saturation”. Indeed, heavier mesons can be included rather easily in the same picture, similarly to what will be explained for nucleons in the next lecture. Suppose, for simplicity, that there existed a scalar-isoscalar meson, described by a field ϕ, with mass m_S. Due to the particularly simple transformation properties, it is straightforward to write chiral invariant coupling of this field to the pions. One such coupling would be

$$h\phi\langle D^\mu U^\dagger D_\mu U\rangle, \tag{28}$$

with a coupling h to be determined from the phenomenology. In the functional integral we will have to integrate over ϕ as well. A given term of the perturbative expansion will have e.g.

$$\frac{1}{2}i^2\int d^4z_1 h\phi(z_1)\langle D^\mu U^\dagger D_\mu U\langle(z_1)\int d^4z_2 h\phi(z_2)\langle D^\nu U^\dagger D_\nu U\langle(z_2) \tag{29}$$

which will yield the ϕ propagator

$$\int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m_S^2+i\epsilon}\mathrm{e}^{-ip\cdot(z_1-z_2)}. \tag{30}$$

If the flowing momentum p is such that $p^2 << m_S^2$ the ϕ propagator becomes a Dirac δ function and its contribution amounts to a term

$$\frac{i^2}{2}h^2\left(\frac{-i}{m_S^2}\right)\int d^4z_1\langle D^\mu U^\dagger D_\mu U\rangle^2(z_1), \tag{31}$$

the same as would be given by the vertex proportional to ℓ_1. Therefore the heavy meson entails a contribution to the LEC ℓ_1

$$\ell_1^{(\phi)} = \frac{h^2}{2m_S^2}. \tag{32}$$

So, in general we can say that the LECs mimic the effect of virtual heavier particles, which have been “integrated out” from the theory. This works in practice much better for the vector meson (vector meson dominance), the principle is the same, the calculation a little more involved.

8 Extension to nucleons

In extending the effective chiral Lagrangian to include nucleons, the first thing we have to establish is how nucleons transform under the chiral symmetry. We know that they form an isodoublet under isospin, the vectorial SU(2),

$$N = \begin{pmatrix} p \\ n \end{pmatrix}, \tag{33}$$

but this leaves open many possibilities. One such possibility is that the respective chiralities transform linearly under the corresponding SU(2) transformations, e.g.,

$$N_L \to V_L N_L, \quad N_R \to V_R N_R, \tag{34}$$

where $N_{R/L} = (1 \pm \gamma_5)/2N$. When restricted to vector transformation $V_R = V_L = V$ then the isospin transformation properties are satisfied. However, the same is true for other choices, e.g.

$$N \to V_L N, \tag{35}$$

or

$$N \to V_R N. \tag{36}$$

It would seem that there is ample freedom in the choice of the representation. However, all the different choices are equivalent, since one can pass from one representation to the other by using the Goldstone bosons' field U. For instance, if we start from the transformation law (35), then the field $N' = UN$ transform according to (36),

$$N' = UN \to V_R U V_L^\dagger V_L N = V_R U N = V_R N'. \tag{37}$$

As we saw in the case of the pions, a field redefinition does not affect the physical consequences of the theory. Nevertheless, among all possible choices of transformation properties, there is a particularly convenient one. The reason is that all of the above choices lead to non-derivative couplings with pions, a fact that renders the crucial property at the basis of soft pion theorems (that would nevertheless arise), not immediately transparent from the power counting. This is clear, e.g. using the (parity respecting) representation (34), since in this case the nucleon mass term would take the chiral invariant form

$$m_N(\bar{N}_R U N_L + \bar{N}_L U^\dagger N_R) = m_N \left(\bar{N} \frac{U + U^\dagger}{2} N + \bar{N} \frac{U^\dagger - U}{2i} i\gamma_5 N \right), \tag{38}$$

leading to a tower of non-derivative pion-nucleon interactions, in addition to a pseudoscalar coupling that was discussed in the previous lectures at this School. Notice that a pseudoscalar pion-nucleon coupling is a rather reasonable choice. A choice which is "equivalent" to the axial vector coupling, as stated by so-called "equivalence theorems. These theorems involve precisely some field redefinition (or equations of motion) to demonstrate the equivalence. For instance, it is left as the 4th exercise to prove that, starting from the πN interaction Lagrangian

$$\bar{N}(i\partial\!\!\!/ - m_N)N - ig\bar{N}\gamma_5 \boldsymbol{\tau} \cdot \boldsymbol{\pi} N, \tag{39}$$

the nucleon field redefinition,

$$N \to N' = \mathrm{e}^{-i\frac{g}{2m_N}\gamma_5 \boldsymbol{\tau}\cdot\boldsymbol{\pi}} N \tag{40}$$

leads to the replacement of the pseudoscalar coupling with the axial vector one,

$$\frac{g}{2m_N}\bar{N}\gamma^\mu\gamma_5\partial_\mu \boldsymbol{\tau} \cdot \boldsymbol{\pi} N, \tag{41}$$

in addition to more many-pion couplings. The same results can be obtained by partial integration of the axial vector coupling and by using the nucleon equation of motion. We see that, in order to eliminate the unwanted non-derivative pion couplings issuing from $\bar{N}_R U N_L + \bar{N}_L U^\dagger N_R$, we have to split the SU(2) matrix $U = uu$ and assign it partially to the left and right-handed nucleon fields, i.e.,

$$N'_L = uN_L, \quad N'_R = u^\dagger N_R, \tag{42}$$

so that the mass term does not involve the pion field U anymore. We have now to determine the transformation properties of the transformed fields, and therefore of the square root $u = \sqrt{U}$. We know that

$$u^2 \to u'^2 = V_R u^2 V_L^\dagger, \tag{43}$$

and require that there is a SU(2) matrix h such that

$$V_R u h^\dagger = h u V_L, \tag{44}$$

so that

$$u' = V_R u h^\dagger = h u V_L^\dagger, \quad u'^\dagger = h u^\dagger V_R^\dagger = V_L u^\dagger h^\dagger. \tag{45}$$

h is called the compensator field and we can give an explicit expression for it from

$$u' = \sqrt{V_R U V_L^\dagger} = h u V_L^\dagger \Longrightarrow h = \sqrt{V_R U V_L^\dagger} V_L \sqrt{U^\dagger}. \tag{46}$$

In spite of the apparent uglyness, the transformation properties of the redefined nucleon fields are very simple,

$$N_L' = u N_L \to h u V_L^\dagger V_L N = h u N_L = h N_L', \tag{47}$$

and the same for N_R'. Then finally the redefined nucleon field N transforms homogeneously,

$$N \to hN. \tag{48}$$

You can easily prove that the set of transformations

$$U \to V_R U V_L^\dagger, \quad N \to h\psi, \tag{49}$$

with the above definition of the compensator field h, defines a (non-linear) representation of the chiral group, in the sense that it respects the group composition law. This is the exercise #5. We have disposed of the (non-derivative) pion interaction. What about the derivative one? We saw that they must come from the covariant derivative of the pion field $D_\mu U$, which, however, transforms as U itself, $D_\mu U \to V_R D_\mu U V_L^\dagger$. We can put it inside a nucleon bilinear if we multiply it by appropriate factors of the u fields, e.g.

$$u^\dagger D_\mu U u^\dagger \to h u^\dagger V_R^\dagger V_R D_\mu U V_L V_L^\dagger u^\dagger h^\dagger = h (u^\dagger D_\mu U u^\dagger) h^\dagger, \tag{50}$$

and the same happens with $u D_\mu U^\dagger u$. It is convenient to define the object

$$u_\mu = i u^\dagger D_\mu U u^\dagger = -i u D_\mu U^\dagger u \to h u_\mu h^\dagger, \tag{51}$$

where the i ensures its hermiticity. An invariant operator is e.g.

$$\bar{N} \gamma^\mu \gamma_5 u_\mu N, \tag{52}$$

which gives a πNN derivative coupling of axial vector type. The presence of γ_5 is dictated by the parity invariance. Indeed, the pions inherits its properties under the discrete symmetries from the fact that it couples to the Noether axial current,

$$\langle 0 | \bar{\psi} \gamma^\mu \gamma_5 \frac{\tau^a}{2} \psi(x) | \pi^b(\mathbf{p}) \rangle = i p^\mu \mathrm{e}^{-ip\cdot x} F_\pi, \tag{53}$$

therefore it is pseudoscalar and even under charge conjugation. So, e.g., under parity

$$U \xrightarrow{P} U^\dagger, \quad u^\mu \xrightarrow{P} -u_\mu. \tag{54}$$

So the pions can basically enter only through the field u_μ. Notice that, up to one spacetime derivative, we cant' have further invariant operator, as u_μ is traceless, so that e.g.

$$\bar{N}\gamma^\mu\gamma_5 N\langle u_\mu\rangle = 0. \tag{55}$$

We can also have derivatives of the nucleon fields, but we have to construct a chiral covariant derivative, since for local chiral transformation

$$N \to hN \Longrightarrow \partial^\mu N \to h\partial^\mu N + \partial^\mu hN. \tag{56}$$

This is done as usual, by introducing a connection with the duty to absorb the extra piece,

$$D^\mu N = (\partial^\mu + \Gamma^\mu)N, \tag{57}$$

where we require that

$$\Gamma^\mu \to h\Gamma^\mu h^\dagger - \partial^\mu h h^\dagger. \tag{58}$$

Now, we know that

$$u \to huV_L^\dagger \Longrightarrow \partial^\mu \to \partial^\mu huV_L^\dagger + h\partial^\mu V_L^\dagger + hu\partial^\mu V_L^\dagger, \tag{59}$$

so the field $\partial^\mu u$ can serve the purpose, but it has to be combined with $u^\dagger$,

$$\partial^\mu uu^\dagger \to h(\partial^\mu u^\dagger)h^\dagger + \partial^\mu hh^\dagger + hu\partial^\mu V_L^\dagger V_L u^\dagger h^\dagger \tag{60}$$

the unwanted term depending on $\partial^\mu V_L^\dagger$ can be compensated by the inclusion of the external source ℓ^μ whose transformation properties involves precisely that term. Finally parity requires that also the right handed source be included. At the end we find, for the chiral connection

$$\Gamma^\mu = \frac{1}{2}\left(u^\dagger\partial^\mu u - \partial^\mu uu^\dagger\right) - \frac{i}{2}u\ell^\mu u^\dagger - \frac{i}{2}u^\dagger r^\mu r, \tag{61}$$

which ensures that

$$D^\mu N = (\partial^\mu + \Gamma^\mu)N \to hD^\mu N h^\dagger. \tag{62}$$

Also the scalar/pseudoscalar sources can be used to build homogeneously transforming building blocks, as

$$u^\dagger\chi u^\dagger, \quad u\chi^\dagger u, \tag{63}$$

and the curvatures,

$$u^\dagger \mathcal{R}_{\mu\nu} u, \quad u \mathcal{L}_{\mu\nu} u^\dagger, \tag{64}$$

that transform as u^μ. The chiral counting is modified, due to the fact that the nucleon mass m_N is not protected by chiral symmetry, it must be counted as order $O(1)$. Only the space part of nucleon four-momenta must be counted as a small parameter, therefore also covariant derivatives of nucleon fields must count as $O(1)$, while $\not{D} - m_N \sim O(p)$. The leading order πN Lagrangian is therefore of order $O(p)$,

$$\mathcal{L}_{\pi N}^{(1)} = \bar{N}\left(i\not{D} - m_N + \frac{1}{2} g_A \not{u}\gamma_5\right) N, \tag{65}$$

with a single LEC, g_A, that determines the nucleon coupling to the pion and also to the axial current. This is the celebrated Goldberger-Treiman relation, which is authomatically built in in the effective theory. At the following order more LECs appear, it is, in its full glory,

$$\begin{aligned} \mathcal{L}_{\pi N}^{(2)} \quad = \quad & \bar{N}\Big\{ 2B_0 c_1 \langle U^\dagger \chi + \chi^\dagger U\rangle - \frac{c_2}{4m_N^2}\langle u_\mu u_\nu\rangle (D^\mu D^\nu + \text{h.c.}) \\ & + \frac{c_3}{2}\langle u^\mu u_\mu\rangle + \frac{i}{4} c_4 \sigma^{\mu\nu}[u_\mu, u_\nu] \\ & + 2B_0 c_5 \left(u\chi^\dagger u + u^\dagger \chi u^\dagger - \langle U^\dagger \chi + \chi^\dagger U\rangle \right) \\ & + \frac{c_6}{8m_N}\sigma^{\mu\nu}(u^\dagger \mathcal{R}_{\mu\nu} u + u\mathcal{L}_{\mu\nu}u^\dagger) \\ & + \frac{c_7}{8m_N}\sigma^{\mu\nu}\langle \mathcal{R}_{\mu\nu} + \mathcal{L}_{\mu\nu}\rangle \Big\} N. \end{aligned} \tag{66}$$

The LEC c_1 is related to the πN σ-term, i.e. the light-quark condensate inside the nucleon, which also dictates the chiral expansion of the nucleon mass. c_2, c_3 and c_4 can be measured in πN scattering, and the first two play an important role since they get large contributions from Δ-resonance saturation, by a similar mechanism to what we have seen in the previous lecture for the ℓ_i. Other constants describe the structure of the nucleon, like the anomalous magnetic moment of the protono and neutron.

9 Exercises

3. Show that the most general parametrization of the SU(2) matrix U in terms of the isovector pion field $\boldsymbol{\pi}(x)$ is the following

$$U = f_0(\boldsymbol{\pi}^2) + i\left[1 - f_0(\boldsymbol{\pi}^2)\right]\sum_a \pi^a \tau^a, \tag{67}$$

with a real scalar function f_0. Expand up to four powers of the pion fields, and give the most general expansion in terms of one parameter, besides F_0.

4. Show that, starting from the πN interaction Lagrangian

$$\bar{N}(i\partial\!\!\!/ - m_N)N - ig\bar{N}\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}N, \tag{68}$$

the nucleon field redefinition,

$$N \to N' = \mathrm{e}^{-i\frac{g}{2m_N}\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}}N \tag{69}$$

leads to the replacement of the pseudoscalar coupling with the axial vector one,

$$\frac{g}{2m_N}\bar{N}\gamma^\mu\gamma_5\partial_\mu\boldsymbol{\tau}\cdot\boldsymbol{\pi}N, \tag{70}$$

in addition to more many-pion couplings.

5. Show that the transformation

$$N \to hN, \quad U \to V_R U V_L^\dagger, \tag{71}$$

with h defined in Eq. (46), realizes a (non-linear) representation of the chiral group, in the sense that it respects the group composition law. Show also that, when restricted to vector transformations, you recover the proper isospin transformation law.

10 Hands-on activity

1. Derive the four-pion vertices from $\mathcal{L}^{(2)}$ and the two pion vertices from $\mathcal{L}^{(4)}$ in the σ-model representation of U.

2. Use the above vertices to compute the pion propagator up to one-loop order, $O(p^4)$ of the low-momentum expansion. Deduce the pion field renormalization constant Z and the chiral expansion of M_π up to that order.

3. Find the linear coupling of the axial source to one and three pions from $\mathcal{L}^{(2)} + \mathcal{L}^{(4)}$ and $\mathcal{L}^{(2)}$ respectively.

4. Determine the matrix element of the quark axial current between one pion and the vacuum. Deduce the chiral expansion of F_π up to $O(p^4)$ using the definition

$$\langle 0|\bar{\psi}\gamma_\mu\gamma_5\frac{\tau^a}{2}\psi(x)|\pi^b(\mathbf{p})\rangle = ip_\mu F_\pi \mathrm{e}^{-ip\cdot x}. \tag{72}$$