11 The problem of the heavy mass

After presenting the low-energy expansion of the counterterms, we would like to present the low-energy expansion of the loops, as done for the pions. We would like to say that a given loop has a "chiral dimension", dictating its low-momentum behaviour,

$$D = 4L - 2I_\pi - I_N + \sum_i n_i d_i, \tag{1}$$

where L is the number of loops, I_π the number of pion internal lines (propagators), I_N the number of nucleon propagators and n_i the number of vertices of type i, each of dimension d_i in the chiral counting. This would follow from counting the nucleon propagators as $O(p^{-1})$, the inverse of $i\not{D} - m_N \sim O(p)$. Another way to put it is to say that the virtual nucleons always carry momentum $q = m_N v + k$ with v the four-velocity, $v^2 = 1$ and $k \sim O(p)$, so

$$\frac{1}{\not{q} - m_N + i\epsilon} = \frac{\not{q} + m_N}{q^2 - m_N + i\epsilon} \sim \frac{\not{q} + m_N}{2m_N v \cdot k + i\epsilon} \sim O(p^{-1}). \tag{2}$$

If that was the case then, using the already discussed topological identity which relates the number of loops L, the number of total internal lines $I = I_\pi + I_N$ and the number of vertices $V = \sum_i n_i$,

$$L = I_\pi + I_N - V + 1, \tag{3}$$

to remove the number of pion propagators, and the relation

$$2I_N + E_N = \sum_i n_i f_i, \tag{4}$$

which counts the total number of nucleon lines (external E_N or internal I_N) attached to the vertices, where to vertices of type i are attached f_i nucleon lines, we would arrive at

$$D = 2L + 2 - \frac{1}{2}E_N + \sum_i n_i \left(d_i - 2 + \frac{1}{2} f_i \right), \tag{5}$$

whence we would observe that D is bounded from below, for a given process, since chiral symmetry ensures that, for each vertex $d_i - 2 + f_i/2 \geq 0$. We would then have a well defined loop expansion as in the purely pionic

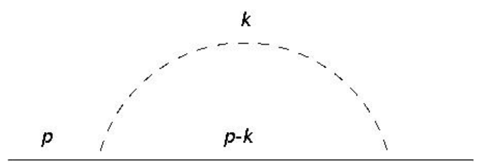

Figure 1: Pion loop contribution to the nucleon self-energy.

case. Unfortunately, the presence of the nucleon mass m_N, which is a hard scale entering in the loop integrals, not suppressed in the chiral counting, complicates the life a bit. Indeed, you can prove as exercise # 6, that the nucleon self energy diagram depicted in Fig. 1, which should be of order $O(p^3)$ on the basis of the above couunting, scales instead, in ordinary dimensional regularization, as m_N^3, so it is $O(1)$. This is the consequence of the presence of a hard scale in the integrand, which was not the case for the pions, when using a mass-independent regularization scheme. Thus, it doesn't happen anymore that loop renormalize the higher order LECs. They also renormalize the lower order LECs! One way to cure these drawbacks is to use the so-called heavy-baryon formalism. This was originally introduced for the heavy quarks, but it works in the same way. The idea is that, in the limit of infinite mass, the four-velocity of each baryon is fixed, it will never be changed by processes that happen at low momenta. In other words, if we start from some (on-mass-shell) state with momentum

$$P_\mu = m_N v_\mu, \quad v^2 = 1, \tag{6}$$

then every soft transition will lead to a momentum

$$P'_\mu = m_N v_\mu + k_\mu = m_N v'_\mu, \quad v'^2 = 1, \tag{7}$$

so that in the limit $m_N \to \infty$ we have $v_\mu = v'_\mu$, So, each baryon has a definite 4-velocity in this limit, that we can keep track of, and that will never change. Then, in defining the theory, we can introduce velocity-dependent heavy fermion fields. Once a given velocity is picked up, it will not change. Formally the velocity-dependent fields are defined in terms of the original nucleon field N like

$$N = \mathrm{e}^{-im_N v\cdot x} \left(H_v + h_v\right), \tag{8}$$

with H_v and h_v representing eigenspinors of $\not{v} = \gamma^\mu v_\mu$,

$$\not{v} H_v = H_v, \quad \not{v} h_v = -h_v. \tag{9}$$

Notice that, since $\not{v}^2 = v^2 = 1$, then the eigenvalues of $\not{v}$ can either be $+1$ or -1. The eigenspinors can be obtained by the action of the projectors

$$P^{\pm} = \frac{1 \pm \not{v}}{2}. \tag{10}$$

In the nucleon rest frame, in which $v = (1, 0, 0, 0)$, the H_v and h_v fields are expressed, respectively, in terms of the large and small components of the nucleon Dirac spinors, whence the notation. The phase factor in Eq. (8) is meant to absorb most of the time-dependence of H_v due to the heavy mass. The relation (8) can be used to express the Lagrangian in terms of the heavy fermion fields. Using the defining properties (9) one obtains, e.g.,

$$\begin{aligned} & \bar{H}_v \mathrm{e}^{im_N v\cdot x} \left(i\not{D} - m_N + \frac{1}{2} g_A \not{u} \gamma_5 \right) \mathrm{e}^{-im_N v\cdot x} N \qquad (11) \\ = & \frac{1}{2} \bar{H}_v \mathrm{e}^{im_N v\cdot x} \left[\not{v} \left(i\not{D} - m_N + \frac{1}{2} g_A \not{u} \gamma_5 \right) + \left(i\not{D} - m_N + \frac{1}{2} g_A \not{u} \gamma_5 \right) \not{v} \right] \mathrm{e}^{-im_N v\cdot x} N \end{aligned}$$

and using the Clifford algebra the above expression is equal to

$$\bar{H}_v \left(iv \cdot D + g_A S \cdot u \right) \tag{12}$$

where the spin four-vector,

$$S^\mu = \frac{i}{2} \sigma^{\mu\nu} \gamma_5 v_\nu, \quad \sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]. \tag{13}$$

Notice that the heavy scale, the nucleon mass m_N, has disappeared from the Lagrangian involving H_v, which was the purpose of the formalism. The leading order Lagrangian becomes,

$$\begin{aligned} \mathcal{L}^{(1)}_{\pi N} &= \bar{H}_v \left(iv \cdot D + g_A S \cdot u \right) + \left[\bar{h}_v \left(2iS \cdot D + \frac{g_A}{2} v \cdot u \right) H_v + \text{h.c.} \right] \\ &\quad + \bar{h}_v \left(-2m_N + iv \cdot D + g_A S \cdot u \right) h_v. \end{aligned} \tag{14}$$

One can then "integrate out" the field h_v, by using iteratively the equations of motion, so that the dependence on m_N is reduced to additional vertices representing relativistic $1/m_N$ corrections. Notice however that, despite being formally Lorentz invariant, the dependence on v introduces a preferred reference frame. One way to restore the appropriate relativistic properties is to impose the so-called "reparametrization invariance", i.e. the freedom to relabel the four velocity by addition of small terms,

$$(v, k) \to (v + q/m_N, k - q), \quad \text{with } (v + q/m_N)^2 = 1, \tag{15}$$

which puts non-trivial constraints on the construction of the Lagrangian. We will not pursue this formalism, since in the case of more nucleons, as will be explained in the following lectures, it is natural, following Weinberg, to abandon the Lorentz-invariant perturbation theory, or Feynman diagrams, and use the old-fashioned time-ordered perturbation theory (TOPT).

12 Recoil-corrected TOPT

Indeed, TOPT can also be profitably used in the 1-nucleon sector. To understand this, let's come back to the example of the nucleon self energy diagram, Fig. 1, which involves the scalar integral

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 - M^2 + i\epsilon]\,[(p-k)^2 - m_N^2 + i\epsilon]}, \tag{16}$$

which should count as $O(p)$ in the naïf counting. We can calculate thihs integral by first doing the integration over the temporal component k_0. This procedure actually generates the TOPT diagrams. In the complex k_0 plane the integrand has 4 poles located at

$$k_0^2 - \mathbf{k}^2 - M^2 + i\epsilon = 0 \implies k_0 = \pm\left(\sqrt{\mathbf{k}^2 + M^2} - i\epsilon\right), \tag{17}$$

$$(k_0 - p_0)^2 - (\mathbf{k} - \mathbf{p})^2 - m_N^2 + i\epsilon = 0 \implies k_0 = p_0 \pm \left(\sqrt{(\mathbf{k} - \mathbf{p})^2 + m_N^2} - i\epsilon\right) \tag{18}$$

two in the upper and two in the lower complex plane. We can close the contour at infinity, since the integrand goes like k_o^{-4}, and use Cauchy theorem. Choosing to complete the contour counterclockwise at positive imaginary infinity, we get contribution from the two poles at

$$k_0 \sim \sqrt{\mathbf{k}^2 + M^2} \equiv \omega_{\mathbf{k}}, \quad k_0 \sim p_0 - \sqrt{(\mathbf{k} - \mathbf{p})^2 + m_N^2} \sim O(p^2). \tag{19}$$

Neglecting $O(p^2)$ contributions the contour integration gives

$$2\pi i \frac{1}{4 m_n \omega_{\mathbf{k}}^2}, \tag{20}$$

which, after integration over the spatial $\mathbf{k}$, restores the proper counting $\sim O(p)$.

This procedure, of doing the k_0 integration first and then doing the $1/m_N$ expansion, is the same procedure which was used in the late 90's by Kaiser,

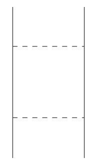

Figure 2: Planar box Feynman diagram contributing to the NN potential.

Brockmann and Weise, to single out the iterated one pion exchange (OPE) NN potential and subtract it from the two-pion exchange planar box diagram, depicted in Fig. 2. In that case the scalar integral contains 4 propagators and 8 poles. One can do the same exercise as done for the nucleon self-energy diagram, and discover an enhanced contribution which is the iteration of the OPE plus a remainder, that is interpreted as an irreducible TPE. Notice that the result is not simply the same as when neglecting the reducible TOPT diagrams tout-court. In order to obtain the correct result we have to include the recoil corrections to the reducible topologies, which scale, in the low-energy counting, as the contributions from the irreducible topologies. We can call this calculational scheme "recoil-corrected TOPT".

There is a further subtlety in using the TOPT. It is the fact that what is needed is not the Lagrangian, but the Hamiltonian. For derivative couplings, as it is the case in the effective theory, the interaction Hamiltonian is not simply the negative of the interaction Lagrangian. We need to go carefully through the canonical formalism, as done in a paper by Gerstein, Jackiw, Lee and Weinberg of the early 70's. Indeed, suppose we start from a (Lorentz invariant) Lagrangian density for interacting pions

$$\mathcal{L} = \frac{1}{2}\partial^\mu \pi^a G_{ab} \partial^b_\mu, \tag{21}$$

where the matrix G in isospin space depends on the pion fields, and so generates interactions. The canonical momentum Π is, in vector notation

$$\Pi = G\partial_0 \pi \Longrightarrow \partial_0 \pi = G^{-1}\Pi, \tag{22}$$

and the Hamiltonian density $\mathcal{H}$ is

$$\mathcal{H} = \Pi G^{-1}\Pi - \mathcal{L} = \frac{1}{2}\Pi Pi - \frac{1}{2}\partial^i \pi G \partial_i \pi + \frac{1}{2}\Pi\left(G^{-1} - 1\right)\Pi - \frac{1}{2}\partial^i \pi (G-1)\partial_i \pi \tag{23}$$

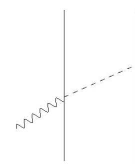

Figure 3: A time-ordered diagram contributing to the two-body nuclear axial charge.

can be divided into a free part and an interaction part. Reexpressing it in terms of $\partial_0\pi$ we get

$$\mathcal{H} = \frac{1}{2}\Pi\Pi - \frac{1}{2}\partial^i\pi\partial_i\pi - \mathcal{L}_{\rm int} - \frac{1}{2}\partial_0\pi\,(G-1)^2\,\partial_0\pi, \tag{24}$$

where

$$\mathcal{L}_{\rm int} = \frac{1}{2}\partial^\mu\pi(G-1)\partial_\mu\pi. \tag{25}$$

It is clear that we have extra interaction terms, which are not Lorentz invariant, and compensate non-invariant contributions arising from the propagator of the Π field. Notice that Eq. (24) is not the final expression, since we have to reexpress it in terms of Π. In the case of the coupling to the (temporal component of the) external axial source, a_0, we have

$$\mathcal{L} \sim \frac{1}{2}(\partial_0\pi\partial_0\pi - 2\partial_0\pi a_0) + \bar{N}i\gamma^\mu\Gamma_\mu N + ..., \tag{26}$$

so the canonical momentum Π depends both on a_0 and on a $\bar{N}\pi N$ vertex coming from

$$\Gamma_\mu \sim \frac{i}{4F_0^2}\partial_\mu\boldsymbol{\tau}\cdot\boldsymbol{\tau}\times\boldsymbol{\pi}, \tag{27}$$

so that

$$\Pi \sim \partial_0 - a_0 - \frac{1}{4F_0^2}\bar{N}\gamma^0\boldsymbol{\tau}\times\boldsymbol{\pi}N. \tag{28}$$

It is possible to show (exercise # 7) that the canonical formalism entails a direct coupling of the axial charge to $NN\pi$, which contributes in the diagram shown in Fig. 3, a vertex which is absent in the original Lagrangian density.

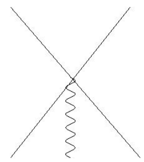

Figure 4: Two-nucleon contact current.

13 Chiral symmetry constraints on the two-body nuclear electroweak currents

As a last subject we discuss, related to the chiral effective field theory calculation of nuclear electroweak currents, of which we have only given the main ingredients in these lectures, the so-called "contact terms". Those are depicted diagrammatically in Fig. 4, and contribute to the two-body electroweak charge and current operators. Two-body currents have contributions from tree diagrams, as in Fig. 3, and also from pion loop diagrams. But in this kind of diagrams the couplings are not free. They acn be fixed in principle in the 1-nucleon sector, because there is always one nucleon participating at most. Therefore we can say that these contributions are constrained by chiral symmetry. On the contrary, interactions like the contact ones of Fig. 4 are free. They are genuinely 2-body operators, and subsume the contribution of short-distance physics, beyond the pion exchanges. In this theory, such couplings have to be fixed from experiment. Of course there will be an infinity of terms, but they can be ordered in the low-energy expansion. At each order, only a finite number of those contributes. It is therefore crucial to know the exact number of these couplings, and to make use of all possible constraints from chiral symmetry. The effective Lagrangian technique is devised exactely for that, and all the ingredients have been given. We have to build all possible chiral invariant operators with two nucleon bilinears, which respect all the symmerties of the underlying theory,

$$\bar{N} O_1 N \bar{N} O_2 N. \tag{29}$$

One possibility is to build them out of products of individually chirally invariant nucleon bilinears, but actually this is not the only possibility. In any way, we can ask questions like, e.g.: how can the axial current external source enter in these interactions? It has to enter through the building blocks, u_μ, and $D_\mu N$, besides $\mathcal{L}_{\mu\nu}$ and $\mathcal{R}_{\mu\nu}$ which, however, involve one extra deriva-

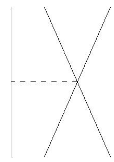

Figure 5: One short-range contribution to the three-nucleon interaction.

tive, so we can disregard them initially. By parity, the axial source can only enter inside $D_\mu N$ accompanied by an odd number of pion fields, therefore it doesn't contribute to the contact terms we are interested in. Instead, a_μ enters inside u_μ, and it enters in exactely the same way as $\partial_\mu \pi$. So the aial coupling to nucleons are the same as the pion coupling to the nucleons. We didn't discover anything, it is just the statement of PCAC, that we saw already at work for the Goldberger-Treiman relation. So a contribution to the two-nucleon axial current like the one represented in Fig. 4 will come with the same coupling constant as a πNN vertex, a vertex that is crucial because it contributes to the leading expression of the three-nucleon interaction, cfr. Fig. 5. The three nucleon force can therefore be related to a 2-nucleon weak process. This is almost a paradigmatic example of the utility of the effective Lagrangian approach in implementing the constraints that chiral symmetry puts on different processes.

14 Exercises

6. Show by explicit calculation of the Feynman diagram, in dimensional regularization, that the pion loop contribution to the nucleo self-energy of Fig. 1 contains contributions of order $O(m_N^3)$ which therefore are not suppressed in the chiral counting.

7. Starting from the Lagrangian (26), and carrying the canonical formalism to obtain the Hamiltonian density, show that there exists a direct coupling of the axial charge to $NN\pi$, which is absent in the Lagrangian density.