Modeling Charge-changing and Neutral-current Neutrino Reactions with Nuclei

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Trento

Connecting scaling with short-range correlations

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Introduction

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First kind: indepedence of the momentum transfer q

Second kind: independence of nuclear species (independence of k_F)

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Second kind: independence of nuclear species (independence of k_F)

In the present study we focused on

Introduction

We restrict our attention to an interacting, infinite, homogeneus, non-relativistic ensemble of nucleons, enclosed in a large volume V.

Our aim is not to provide a detailed numerical study of scaling phenomena but develop an analytic model as long as possible.

Our goal:

- To explore with a simple model what role short- and long-range correlations play in the scaling function.

- To study several properties of scaling (scaling violations, asymmetry, shift, how scaling is approached for large momenta).

The model

The basic formula we start with reads:

$$\frac{E}{A} = \frac{4V}{A}\int \frac{d\vec{k}}{(2\pi)^3}\frac{k^2}{2m}n(k) + \frac{1}{2A}\int d\vec{r}_1 d\vec{r}_2 v(\vec{r}_1-\vec{r}_2)C(\vec{r}_1-\vec{r}_2)$$

(see: K. Gottfried, Ann. of Phys., **21**, 29 (1963)).

Where:

n(k)	is the momentum distribution
$C(\vec{r}_1-\vec{r}_2)$	is the two-body force between the nucleons (potential)
$C(\vec{r}_1-\vec{r}_2)$	is the pair correlation function (related by a F.T. to the n(k))

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We applied it to an interacting, infinite, homogeneous, non-relativistic system of nucleons.

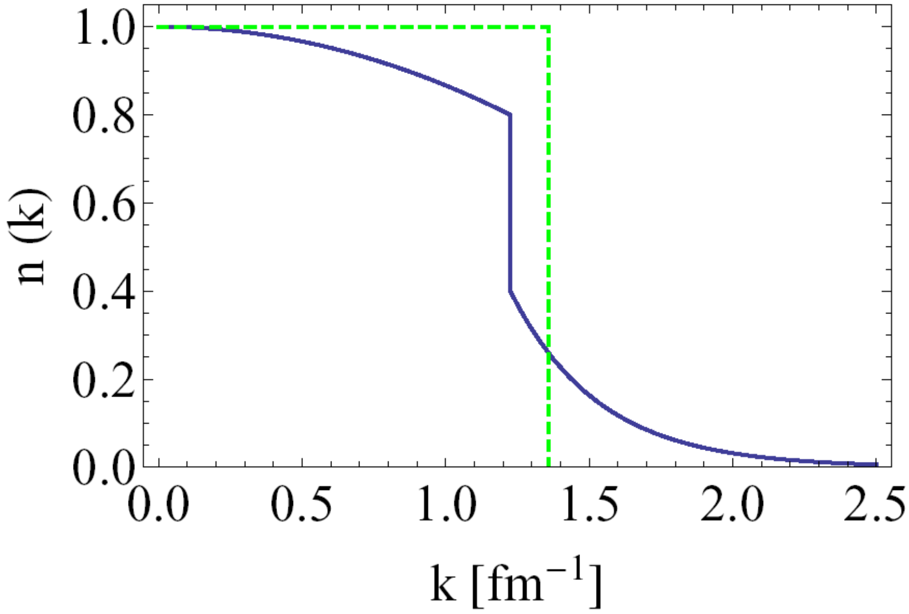
The model

So, we *assume* a momentum distribution, parametrized as:

$$n(k) = \theta(k_F - k)\left(1 - \alpha\frac{k^2}{k_F^2}\right) + \theta(k - k_F)\beta_1 e^{-\beta_2\left(\frac{k}{k_F}-1\right)}$$

that takes in account:

- the existence of a high-momentum tail (as suggested by experimental data)
- the existence of a Fermi surface (as the Luttinger theorem says about a "normal" Fermi system)



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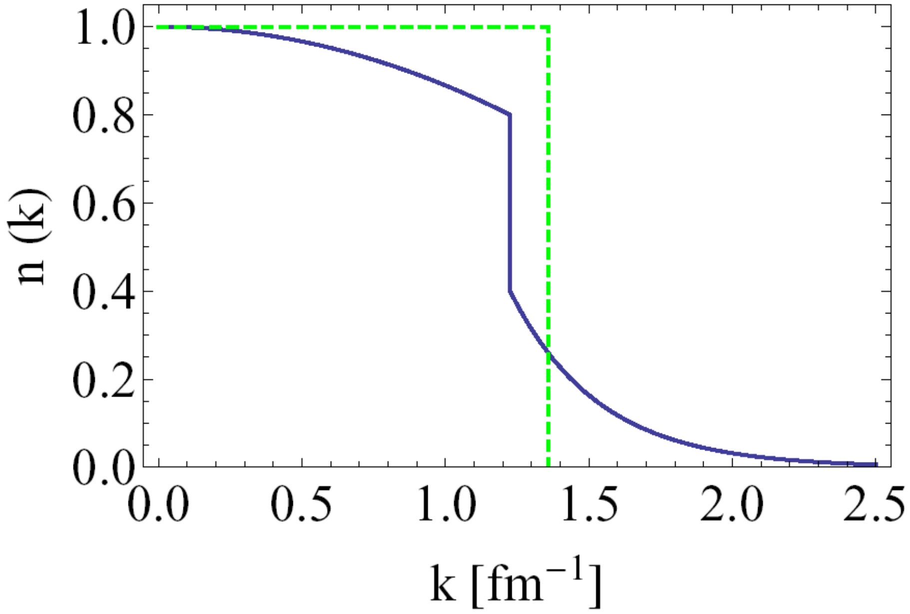


$$\alpha = 0.2 \quad \beta_1 = 0.4 \quad \beta_2 = 4$$

n(k) is a very simple parametrization but able to capture both short- and long-range correlations in a simple form that allows for analytic calculations.

The model

From this momentum distribution:

$$n(k) = \theta(k_F - k)\Big(1 - \alpha\frac{k^2}{k_F^2}\Big) + \theta(k - k_F)\beta_1 e^{-\beta_2(\frac{k}{k_F} - 1)}$$

the pair correlation function is obtained according to the definition:

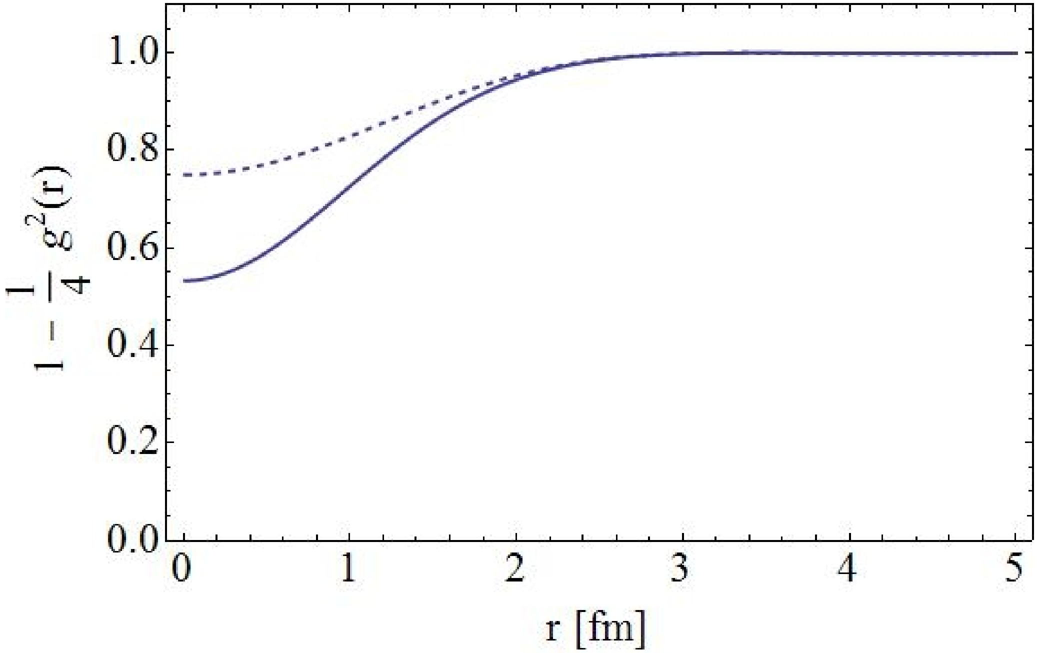
$$\begin{aligned} C(\vec{r}_1 - \vec{r}_2) &= \sum_{\gamma,\delta} < \Psi_0|\hat{\Psi}^\dagger_\gamma(\vec{r}_1)\hat{\Psi}^\dagger_\delta(\vec{r}_2)\hat{\Psi}_\delta(\vec{r}_2)\hat{\Psi}_\gamma(\vec{r}_1)|\Psi_0 > \\ &= \Big(\frac{A}{V}\Big)^2 - 4\Big[h(\alpha,\beta_1,\beta_2)\int\frac{d\vec{k}}{(2\pi)^3}e^{-i\vec{k}\cdot\vec{r}}n(k)\Big]^2 = n_0^2\Big\{1 - \frac{1}{4}g^2(r)\Big\} \end{aligned}$$

and one gets:

$$\begin{aligned} C(r) &= n_0^2\Bigg\{1 - \frac{1}{4}\Bigg\{\frac{3}{k_F r}\Bigg[j_1(k_F r) - \frac{\alpha}{(k_F r)^4}\Big(3((k_F r)^2 - 2)\sin(k_F r) - \\ &-k_F r((k_F r)^2 - 6)\cos(k_F r)\Big) + \beta_1\Big(\frac{k_F r}{(k_F r)^2 + \beta_2^2}\Big)^2 \times \\ &\times\Big(\sin(k_F r)\Big(\beta_2 + \frac{\beta_2^2 + \beta_2^3}{(k_F r)^2} - 1\Big) + \cos(k_F r)\Big(k_F r + \frac{2\beta_2 + \beta_2^2}{k_F r}\Big)\Big)\Bigg]\Bigg\}^2\Bigg\} \end{aligned}$$

The model

$$C(r) = n_0^2\Big\{1 - \frac{1}{4}\Big\{\frac{3}{k_F r}\Big[j_1(k_F r) - \frac{\alpha}{(k_F r)^4}\Big(3((k_F r)^2 - 2)\sin(k_F r) - $$
$$-k_F r((k_F r)^2 - 6)\cos(k_F r)\Big) + \beta_1\Big(\frac{k_F r}{(k_F r)^2 + \beta_2^2}\Big)^2 \times$$
$$\times\Big(\sin(k_F r)\Big(\beta_2 + \frac{\beta_2^2 + \beta_2^3}{(k_F r)^2} - 1\Big) + \cos(k_F r)\Big(k_F r + \frac{2\beta_2 + \beta_2^2}{k_F r}\Big)\Big)\Big]\Big\}^2\Big\}$$



In the figure you can see:

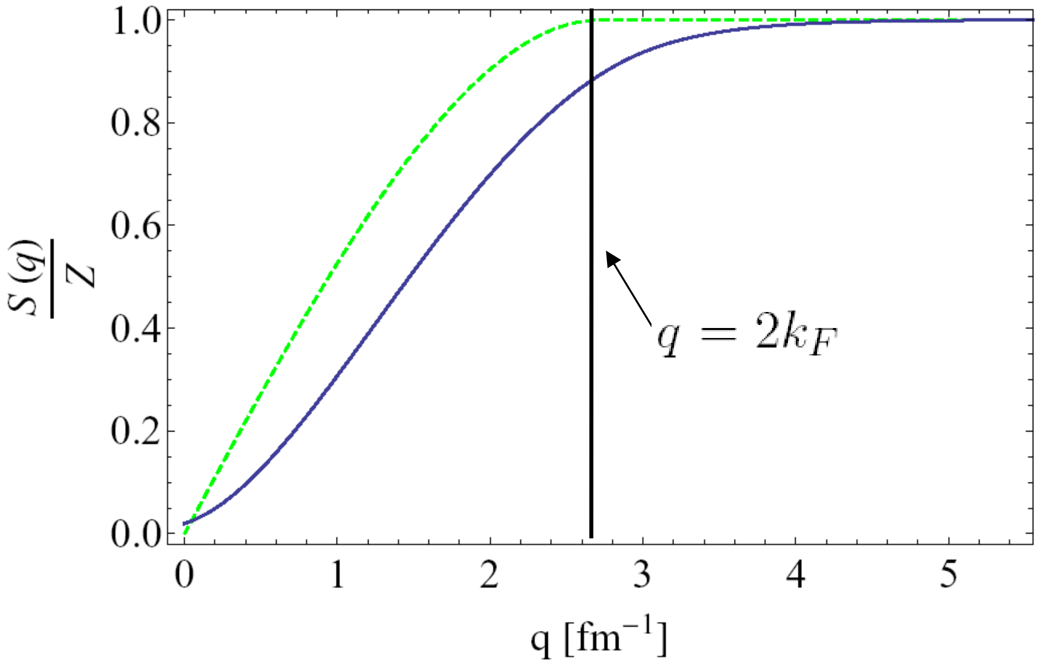
the marked difference between the two correlation functions at short distances, while they coincide at large distances.

This behaviour nicely illustrates the role of short-range correlations.

The model

Here we plot the Coulomb Sum Rule, a quantity depending only upon the n(k) through the following equation:

$$S(q) = Z - n_0^2 V \frac{1}{2} \int d\vec{r} e^{-i\vec{q}\cdot\vec{r}} g^2(r)$$



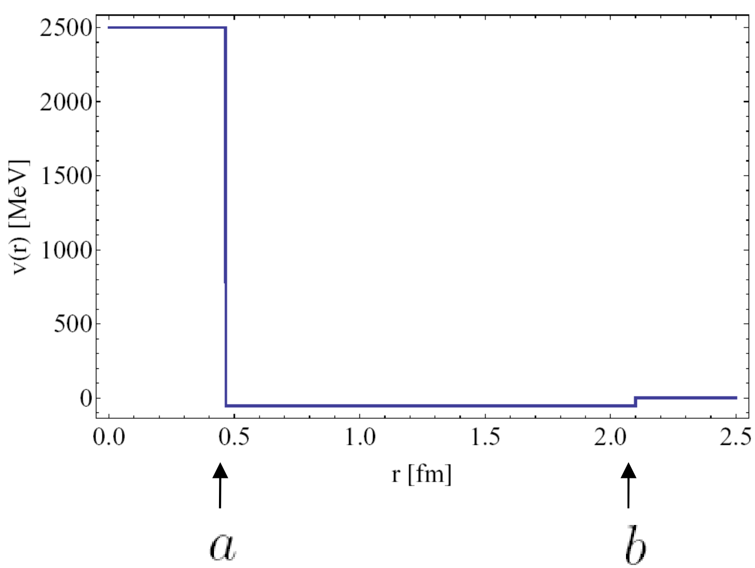
You can see that the two lines coincide at large momenta (uncorrelated fermions);
at medium momenta they differ due to the action of correlations among the fermions;
for small momenta, this difference tends to disappear.

The model

Now, we employ a mixture of a Wigner and Majorana force, namely

$$v(r) = u(r)[1-\gamma+\gamma P_x] = \begin{cases} +U_0 & \gamma = 0 & r \leq a \\ -V_0 & \gamma = \frac{1}{2} & a \leq r \leq b \\ 0 & & b \leq r, \end{cases}$$

where P_x is the space exchange operator and γ a parameter varying over the range $0 < \gamma < 1$.



This very simple, instantaneous potential is meant to represent an effective NN interaction in the medium arising from the ladder diagrams summed up via the Bethe-Goldstone equation.

The model

Starting from

$$\frac{E}{A} = \frac{4V}{A}\int \frac{d\vec{k}}{(2\pi)^3}\frac{k^2}{2m}n(k) + \frac{1}{2A}\int d\vec{r}_1 d\vec{r}_2 v(\vec{r}_1-\vec{r}_2)C(\vec{r}_1-\vec{r}_2)$$

and the normalization of the momentum distribution

$$\begin{aligned}\frac{A}{V} &= 4\int \frac{d\vec{k}}{(2\pi)^3}n(k) = \frac{2}{\pi^2}\left\{\int_0^{k_F} k^2dk\left(1-\alpha\frac{k^2}{k_F^2}\right) + \beta_1\int_{k_F}^{\infty} k^2dke^{-\beta_2(\frac{k}{k_F}-1)}\right\}\\ &= \frac{2k_F^3}{3\pi^2}\left(1-\alpha\frac{3}{5}+3\frac{\beta_1}{\beta_2^3}(\beta_2^2+2\beta_2+2)\right) = \frac{2k_F^3}{3\pi^2}h(\alpha,\beta_1,\beta_2) = n_0,\end{aligned}$$

we want to reproduce some properties of nuclear matter (binding energy, density, compressibility).

So we start by choosing "reasonable" values of the parameters for n(k) and v(r).

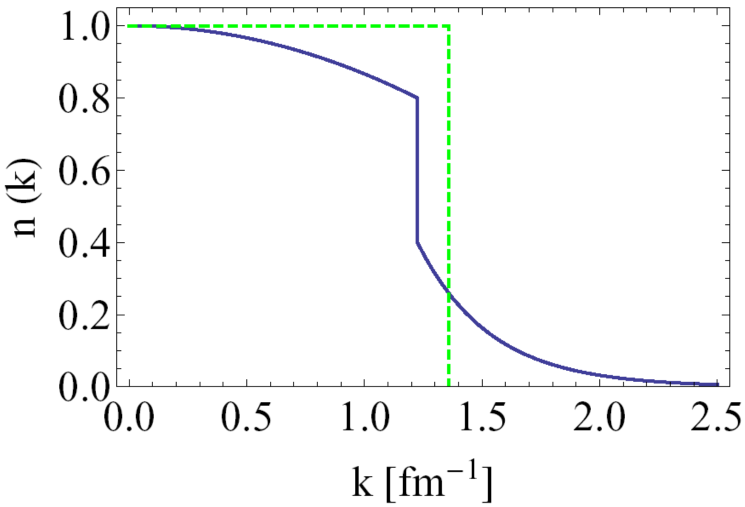
The model

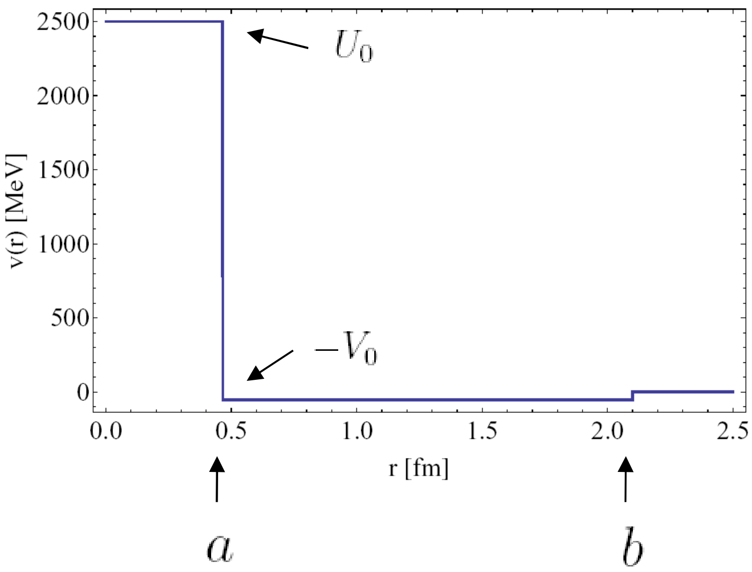
Choosing:

$\alpha = 0.2 \quad \beta_1 = 0.4 \quad \beta_2 = 4$

$U_0 = 2.5 \text{ GeV}, \quad V_0 = 53 \text{ MeV},$

$a = 0.465 \text{ fm}, \quad b = 2.10 \text{ fm}$

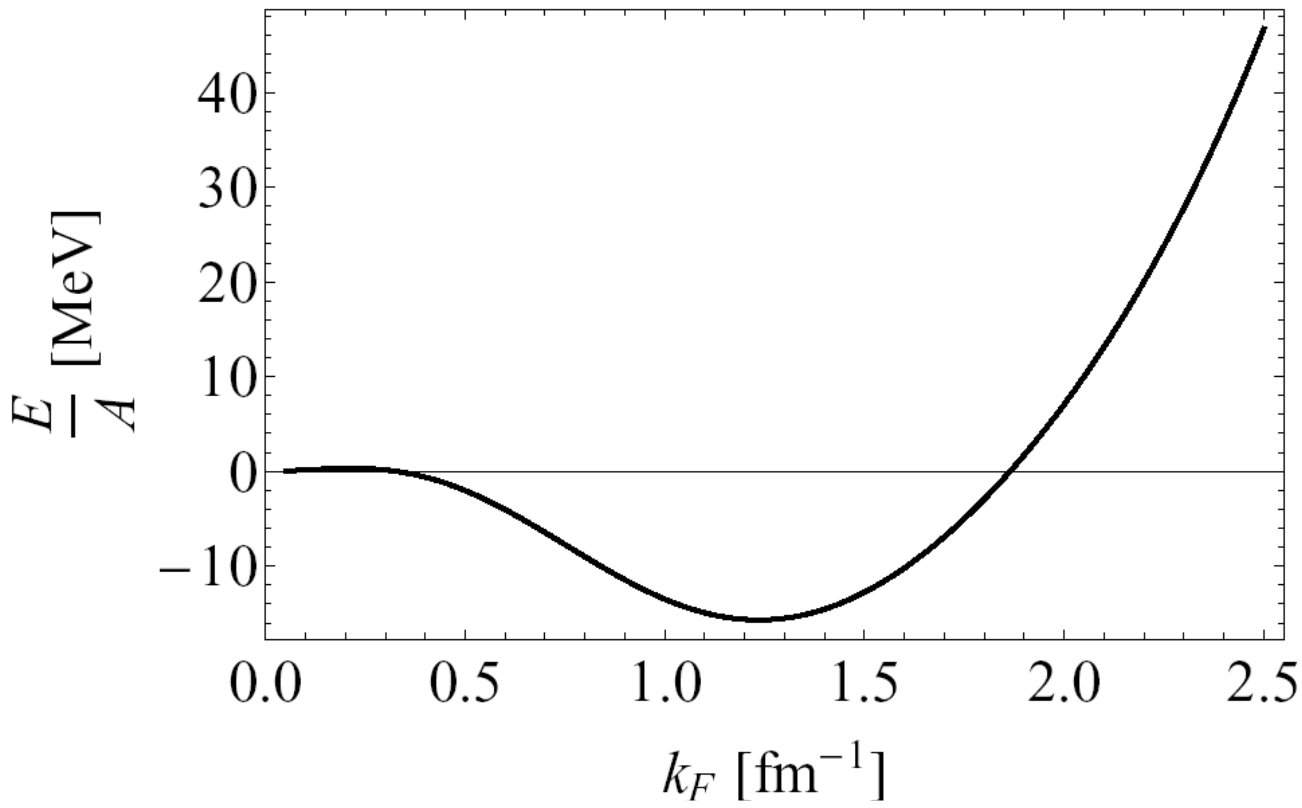




Of course, this set of parameters is far from unique.

The model

We obtain the binding energy versus k_F :



$$\left(\frac{E}{A}\right)_{\min} = -15.68 \text{ MeV}, \quad (k_F)_{\min} = 1.23 \text{ fm}^{-1}, \quad (\kappa)_{\min} = 13.8 \text{ MeV}$$

$$n_0 = 0.17 \text{ fm}^{-3}$$

Setting up the propagator

To calculate the scaling function, starting from our n(k) and v(r), we need the single particle propagator G(k, ω).

So, we can build the two-particle propagator (or density-density correlation function).

$$\Pi(q,\omega) = -\frac{i}{2}\int \frac{d^4k}{(2\pi)^4} G(k+q)G(k)$$

The scaling function is simply related to the Π(q, ω) by

$$F(q,\omega) = -\frac{q}{m}\frac{V}{\pi}\mathrm{Im}\Pi(q,\omega)$$

Setting up the propagator

To do this, we start from the equation of the binding energy:

$$\frac{E}{A} = \frac{4}{n_0}\int \frac{d\vec{k}}{(2\pi)^3}\frac{k^2}{2m}n(k) + \frac{n_0}{2}\left[\int d\vec{r}v_D(r) - \frac{1}{4}\int d\vec{r}v_E(r)g^2(r)\right]$$

Doing a Fourier Transform of the potential:

$$\tilde{v}_D(k) = \int d\vec{r}e^{-i\vec{k}\cdot\vec{r}}v_D(r) = \frac{4\pi}{3}U_0 a^3\frac{3j_1(ka)}{ka} - \frac{\pi}{2}V_0\left(b^3\frac{3j_1(kb)}{kb} - a^3\frac{3j_1(ka)}{ka}\right)$$

$$\tilde{v}_E(k) = \int d\vec{r}e^{-i\vec{k}\cdot\vec{r}}v_E(r) = \frac{4\pi}{3}\left[a^3\frac{3j_1(ka)}{ka}\left(U_0 - \frac{3}{2}V_0\right) + \frac{3}{2}V_0 b^3\frac{3j_1(kb)}{kb}\right]$$

We obtain:

$$\frac{E}{A} = \frac{4}{n_0}\int\frac{d\vec{k}}{(2\pi)^3}n(k)\left[\frac{k^2}{2m} + \frac{n_0}{2}\tilde{v}_D(0) - \frac{h^2(\alpha,\beta_1,\beta_2)}{2}\int\frac{d\vec{q}}{(2\pi)^3}n(q)\tilde{v}_E(|\vec{k}+\vec{q}|)\right] = \frac{4}{n_0}\int\frac{d\vec{k}}{(2\pi)^3}\epsilon_k^{(h)}$$

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We obtain:

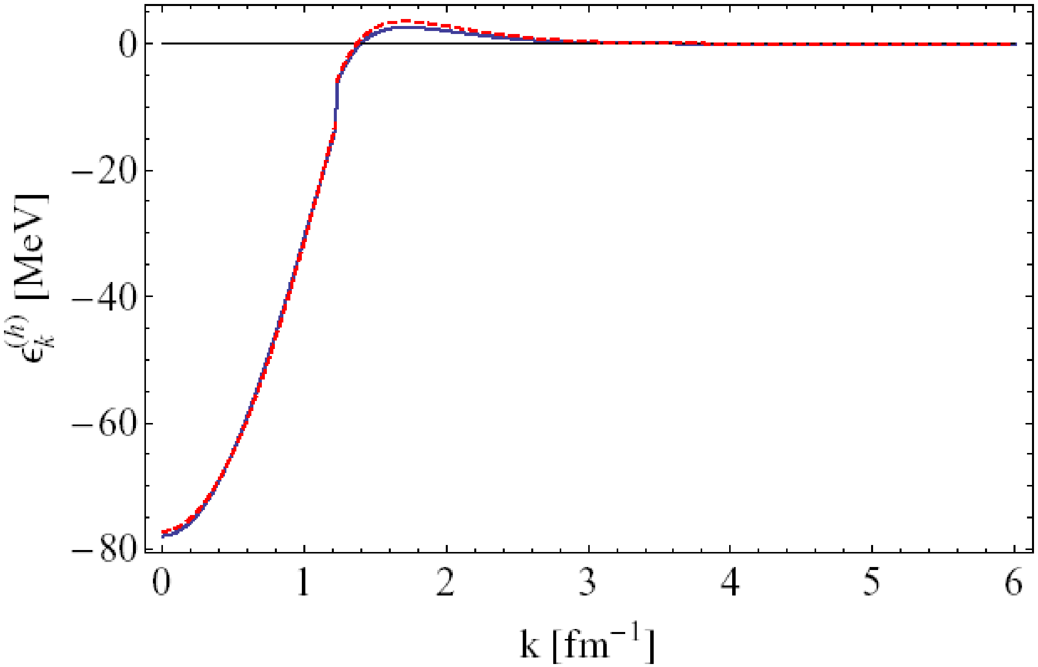
$$\frac{E}{A} = \frac{4}{n_0}\int \frac{d\vec{k}}{(2\pi)^3}\boxed{n(k)\left[\frac{k^2}{2m} + \frac{n_0}{2}\tilde{v}_D(0) - \frac{h^2(\alpha,\beta_1,\beta_2)}{2}\int \frac{d\vec{q}}{(2\pi)^3}n(q)\tilde{v}_E(|\vec{k}+\vec{q}|)\right]} = \frac{4}{n_0}\int \frac{d\vec{k}}{(2\pi)^3}\boxed{\epsilon_k^{(h)}}$$

Expression for the single particle energy

Setting up the propagator

So the single-particle energy (hole) reads:

$$\epsilon_k^{(h)} = n(k)\left[\frac{k^2}{2m} + \frac{n_0}{2}\tilde{v}_D(0) - \frac{h^2(\alpha,\beta_1,\beta_2)}{2(2\pi)^2}\int_0^\infty dpp^2\tilde{v}_E(p)\int_{-1}^{1} dxn(|\vec{p}-\vec{k}|)\right]$$



Note the discontinuity (~ 6.5 MeV) due to the discontinuity in n(k).

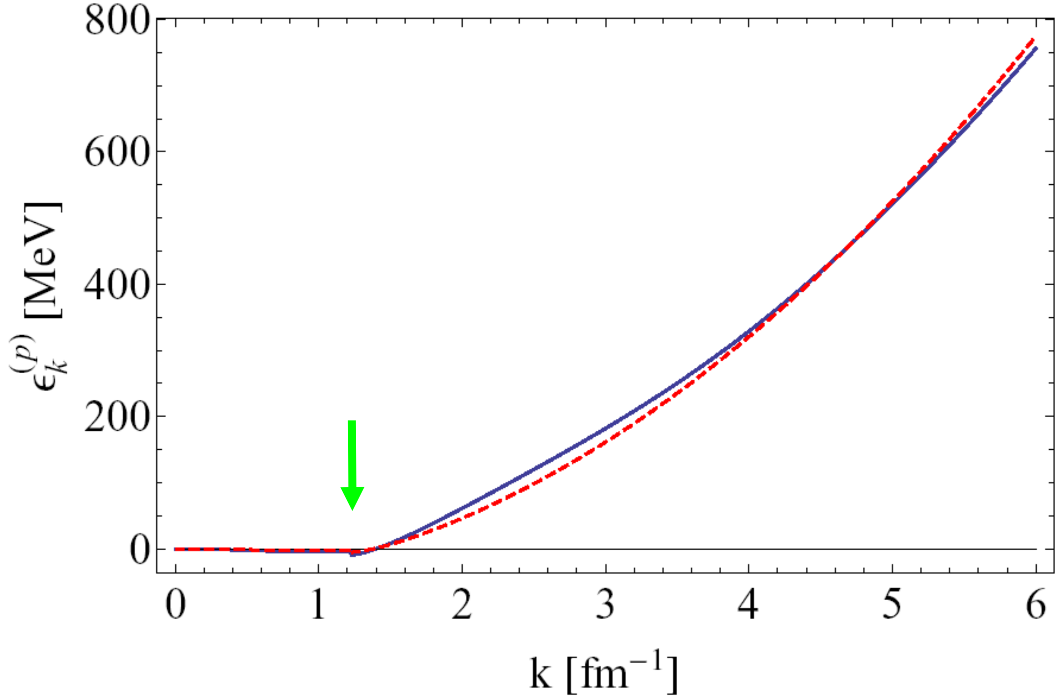
$$\epsilon^{(h)}(k) = n(k)(A_{(h)} + B_{(h)}k^2) = n(k)\left(A_{(h)} + \frac{k^2}{2m^\star_{(h)}}\right)$$

$A_{(h)} = -77.16$ MeV $\quad$ $B_{(h)} = 41.10$ MeV fm^2 $\quad$ $m^\star_{(h)} = 0.50\, m$

Setting up the propagator

The single-particle energy (particle) is related to the previous equation:

$$\epsilon_k^{(p)} = \frac{1-n(k)}{n(k)}\epsilon_k^{(h)}$$



Note, again, the discontinuity (~ 6.5 MeV) due to the discontinuity in n(k).

$$\epsilon^{(p)}(k) = (1-n(k))(A_{(p)} + B_{(p)}k^2) = (1-n(k))\left(A_{(p)} + \frac{k^2}{2m^\star_{(p)}}\right)$$

$$A_{(p)} = -43.09 \text{ MeV} \qquad B_{(p)} = 22.72 \text{ MeV fm}^2 \quad m^\star_{(p)} = 0.91\, m$$

Setting up the propagator

With the previous elements, we propose a form for the single fermion propagator that reads as follows:

$$G(k,\omega) = \frac{n(k)}{\omega - n(k)[\frac{k^2}{2m} + \frac{n_0}{2}\tilde{v}_D(0) - \frac{h^2(\alpha,\beta_1,\beta_2)}{2(2\pi)^2}\int_0^\infty dpp^2\tilde{v}_E(p)\int_{-1}^1 dx n(|\vec{p}-\vec{k}|)] - i\eta}$$
$$+ \frac{1-n(k)}{\omega - [1-n(k)][\frac{k^2}{2m} + \frac{n_0}{2}\tilde{v}_D(0) - \frac{h^2(\alpha,\beta_1,\beta_2)}{2(2\pi)^2}\int_0^\infty dpp^2\tilde{v}_E(p)\int_{-1}^1 dx n(|\vec{p}-\vec{k}|)] + i\eta}$$

of course, for an infinite, homogeneous, interacting many-body system of nucleons.

We propose a kind of mean-field approximation of G, that provides the correct system energy and momentum distribution.

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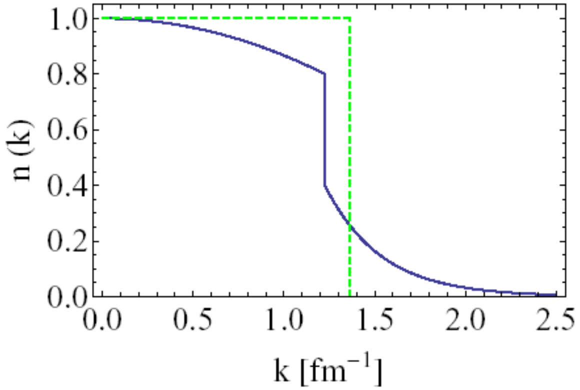
$$+ \frac{1-n(k)}{\omega - [1-n(k)][\frac{k^2}{2m} + \frac{n_0}{2}\tilde{v}_D(0) - \frac{h^2(\alpha,\beta_1,\beta_2)}{2(2\pi)^2}\int_0^\infty dp p^2 \tilde{v}_E(p)\int_{-1}^1 dx n(|\vec{p}-\vec{k}|)] + i\eta}$$

This structure of the propagator tells us that in our model:

the holes exist below, but also above, the Fermi surface and

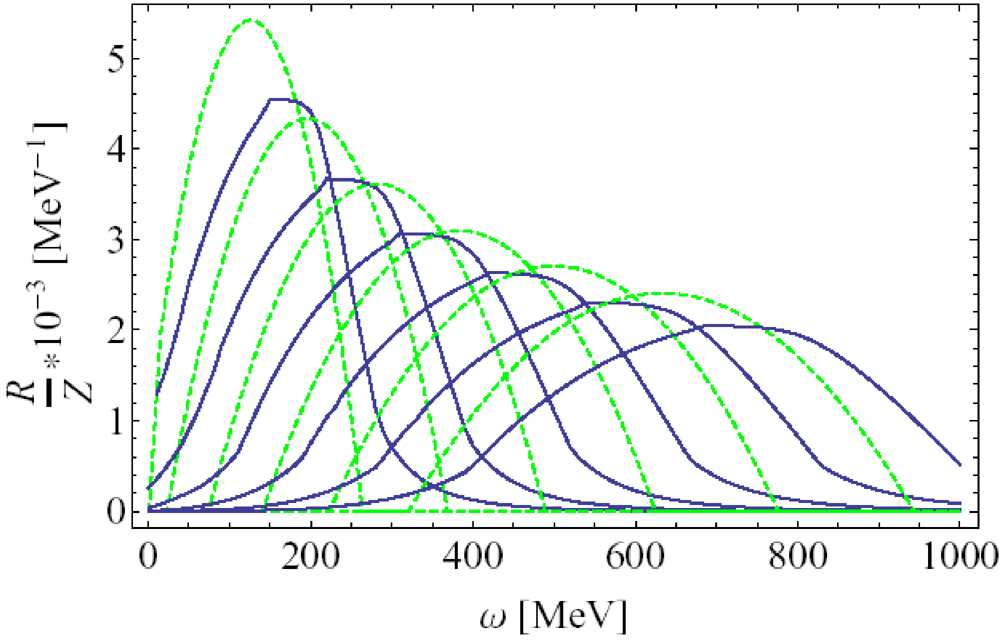
the particles exist above, but also below, the Fermi surface.



Results

Starting from G, one can compute Π and so the response function is easily derived:

$$R(q,\omega) = -\frac{V}{\pi}\mathrm{Im}\Pi(q,\omega)$$



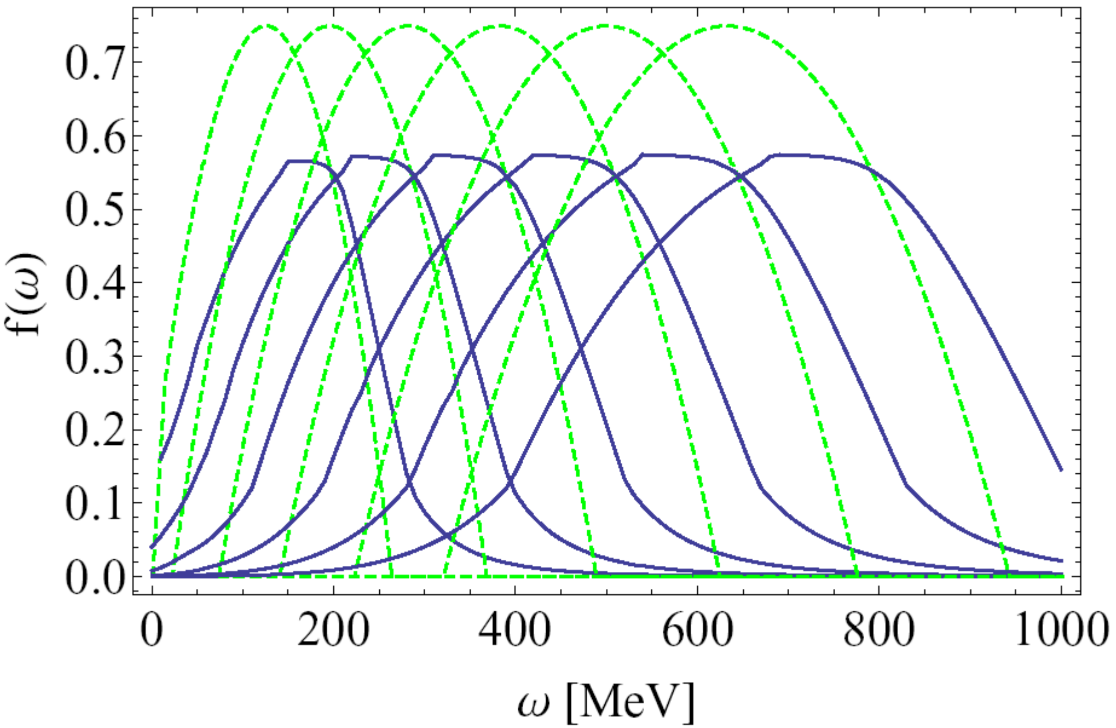
The response function of our model (blue line) and the response function of the Free Fermi gas (green line) plotted versus ω for $q = 2k_F = 2.46\ \mathrm{fm}^{-1}$ up to $q = 4.5k_F = 5.53\ \mathrm{fm}^{-1}$ in step of $0.5k_F$.

Results

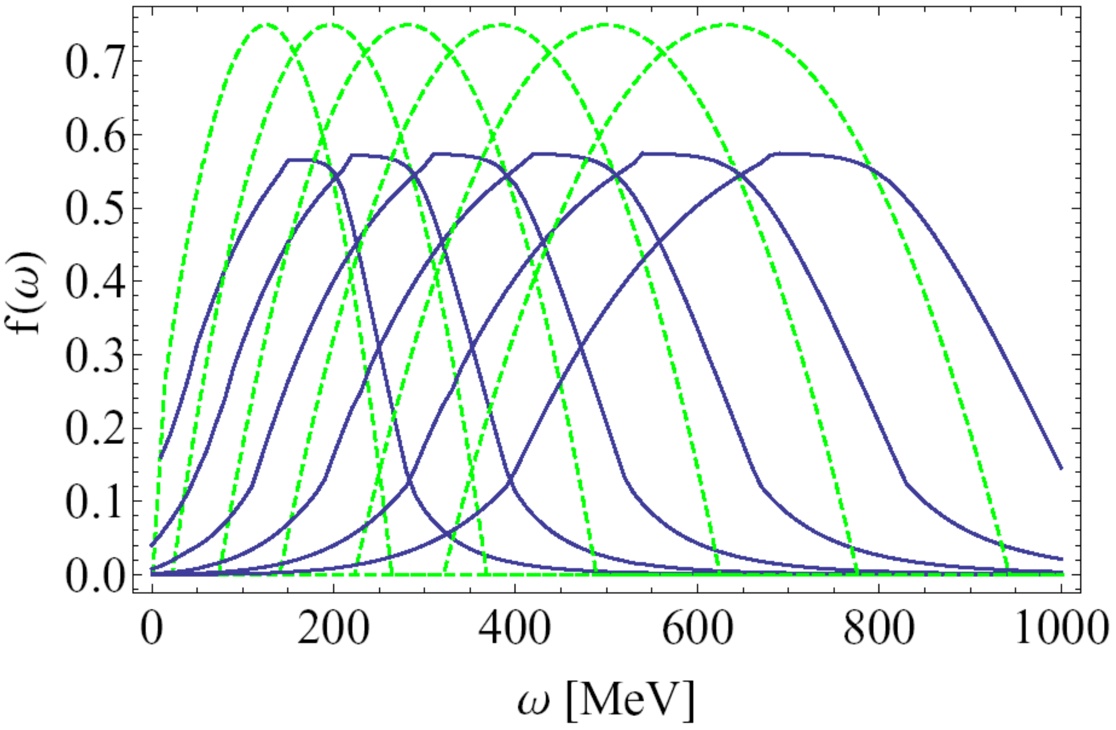
From the response, one can obtain the scaling function per proton according to:

$$
\begin{aligned}
F(q,\omega) &= \frac{q}{m}\frac{R(q,\omega)}{Z} \\
&= \frac{q}{m}\frac{1}{n_0}\frac{1}{\pi^2}\int_0^\infty dk k^2 n(k)\int_{-1}^{1} dx[1-n(|\vec{k}+\vec{q}|)]\delta[\omega-\epsilon^{(p)}(|\vec{k}+\vec{q}|)+\epsilon^{(h)}(k)]
\end{aligned}
$$

Using the usual dimensionless scaling function $f(q,\omega)\equiv k_F\times F(q,\omega)$



Results

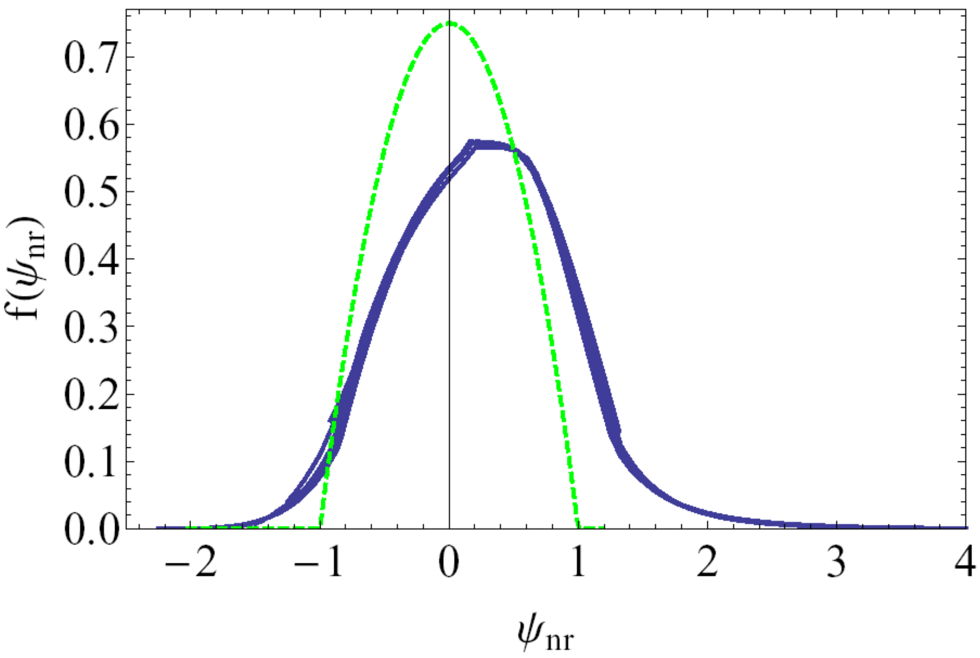


- The scaling function obtained spans a range of energy loss that extends to larger values than that seen for the Fermi gas model (a clear indication of the role of correlations among the nucleons).
- The widths seen in our model are somewhat larger than those of the Fermi gas and the peak heights are somewhat lower.
- The peak positions in our model are shifted to higher energy loss than for the Fermi gas.

Results

To investigate better the scaling behaviour of our model we follow the usual procedures and display f versus the well-known scaling variable

$$\psi_{nr} = \frac{1}{k_F}\left(\frac{m\omega}{q} - \frac{q}{2}\right)$$



The scaling functions for different q tend to group together very closely when displayed versus the scaling variable, that is they scale.

Noting that the coalescence occurs at a peak value ψ_{nr}>0, it is interesting to investigate that…

Unlike for the Fermi gas in our model the scaling functions are no longer perfectly symmetric around their maxima (as observed experimentally).

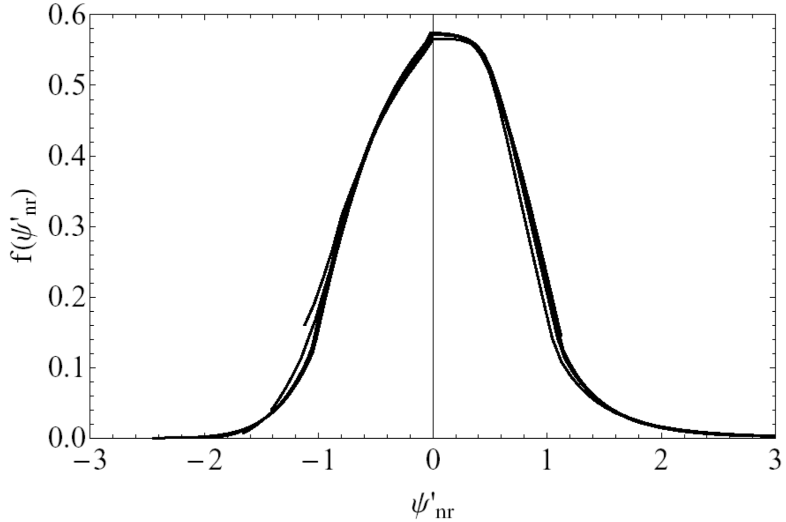
Results

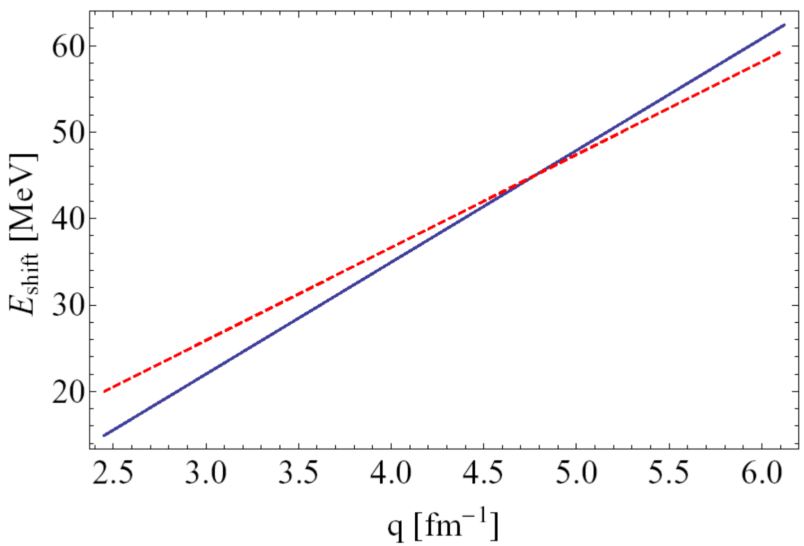
With a simple modification of the scaling variable, one can move the peak positions to zero:

$$\psi'_{nr} \equiv \frac{1}{k_F}\left(\frac{m\omega'}{q} - \frac{q}{2}\right) \qquad \omega' = \omega - E_{shift}(q)$$

where $E_{shift}(q) = E_0 + E_1(q/k_F)$ with

$$\begin{aligned} E_0 &= -17.4 \ \text{MeV} \\ E_1 &= 15.9 \ \text{MeV} \end{aligned}$$



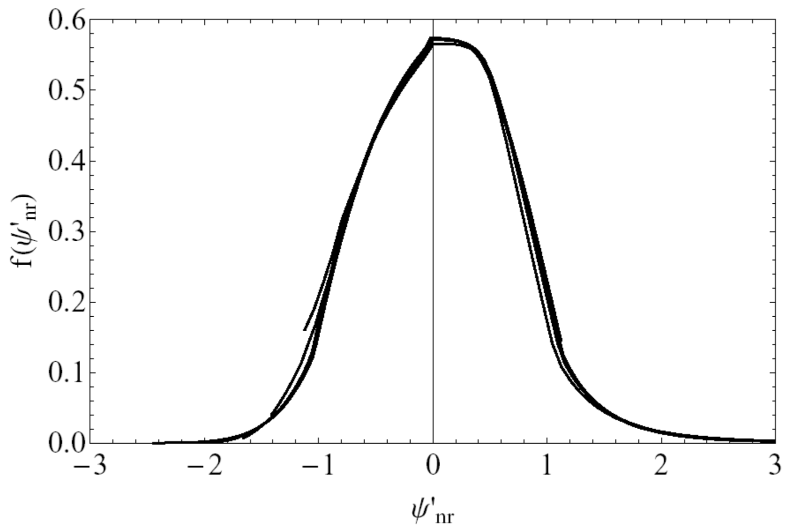


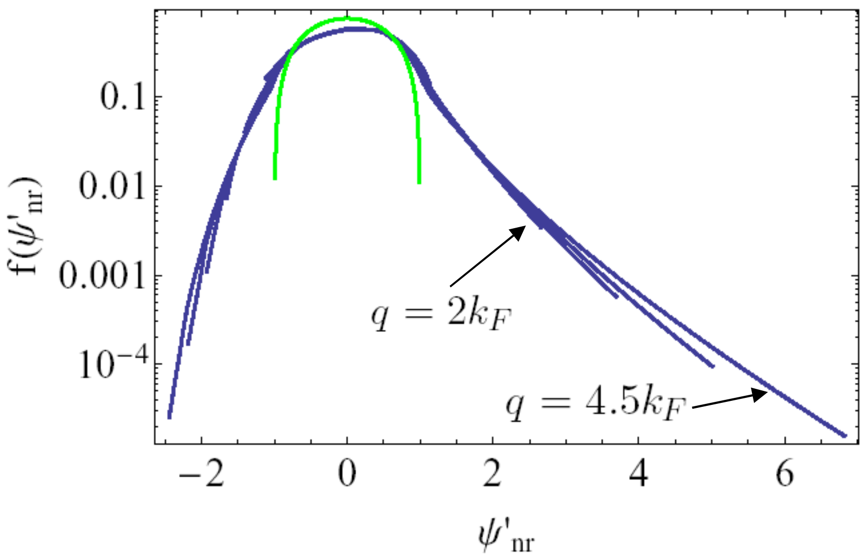
Solid line: q-dependent energy shift

Dashed line: the energy shift obtained in a RMF studies of ^{12}C .

Results

To see better the asymmetry of the scaling function, we plotted the above on a semilog scale:





Here the asymmetry, while small, is clearly apparent.

Unfortunately it is not enough so to agree with the experiment.

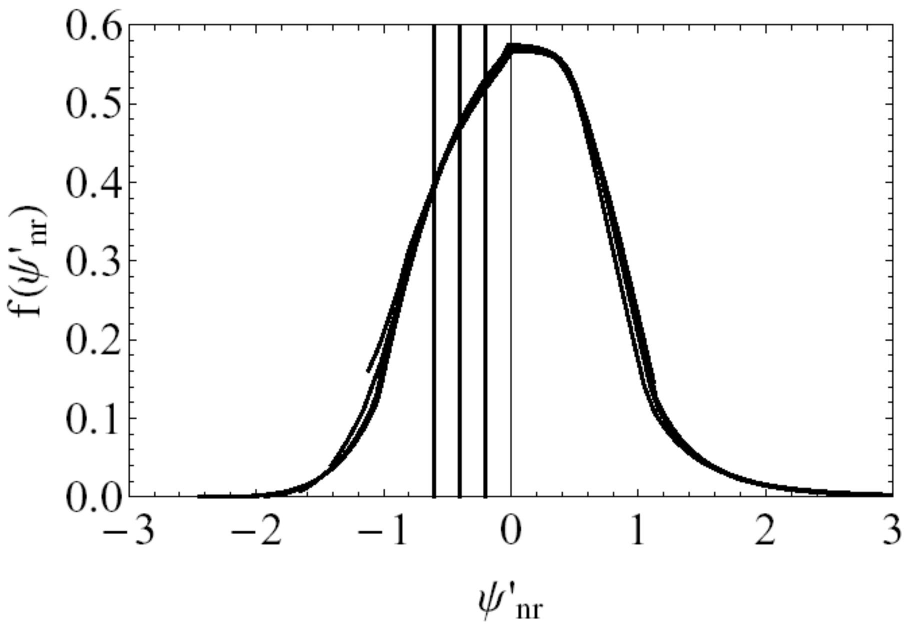
Results

In concluding this research we have plotted the scaling function versus q, in the scaling region, for $[\psi'_{nr}]_0 = -0.6$, -0.4 and -0.2, according this formula:

$$\rho(q) \equiv f(q,[\psi'_{nr}]_0)/f(2k_F,[\psi'_{nr}]_0)$$

where $[\psi'_{nr}]_0 = \frac{1}{k_F}\left(\frac{m\omega'}{q} - \frac{q}{2}\right)$ and ω' = ω – 30 MeV

A constant energy shift



Basically, we have fixed a value of scaling function and we have explored the behaviour of f vs q.

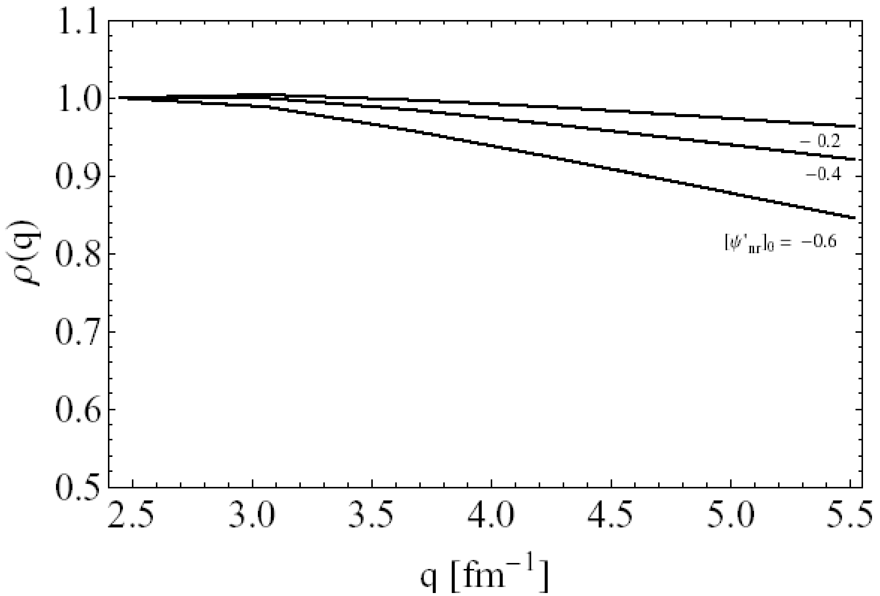
Results

In concluding this research we have plotted the scaling function versus q, in the scaling region, for $[\psi'_{nr}]_0 = -0.6$, -0.4 and -0.2, according this formula:

$$\rho(q) \equiv f(q, [\psi'_{nr}]_0) / f(2k_F, [\psi'_{nr}]_0)$$

where $[\psi'_{nr}]_0 = \frac{1}{k_F}\left(\frac{m\omega'}{q} - \frac{q}{2}\right)$ and ω' = ω – 30 MeV

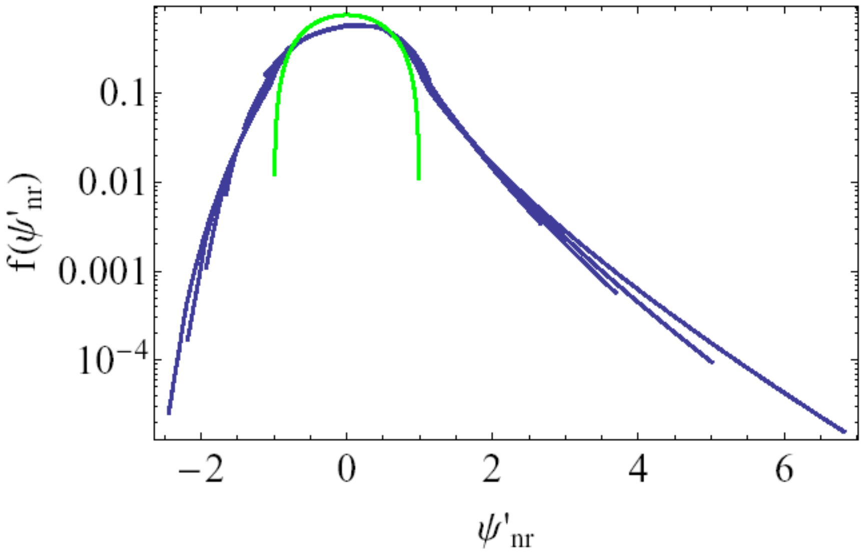
A constant energy shift

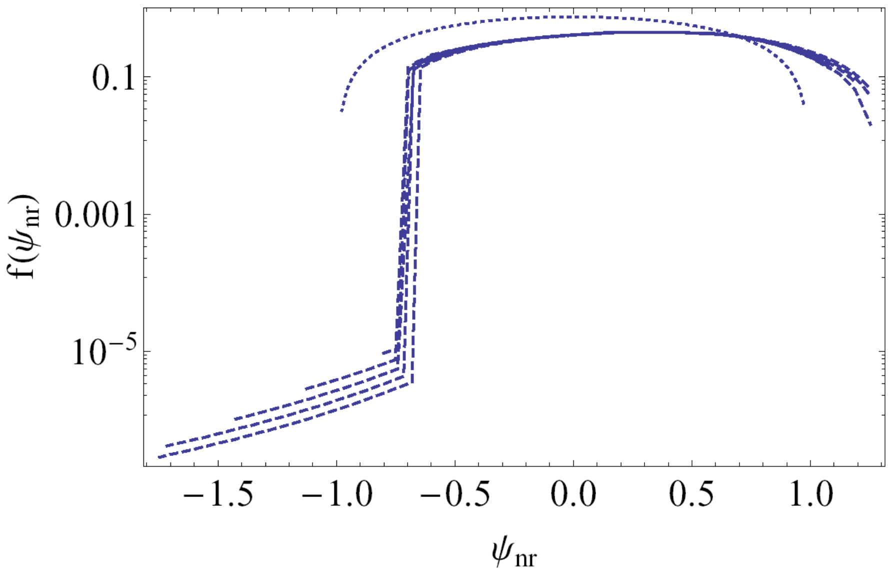


It is clear that the scaling regime is approached from above, an occurrence which is in accord with the experimental data.

Results

If the high momentum tail in n(k) is set to 0, then the contributions extending to large $|\psi'_{nr}|$ essentially disappear.



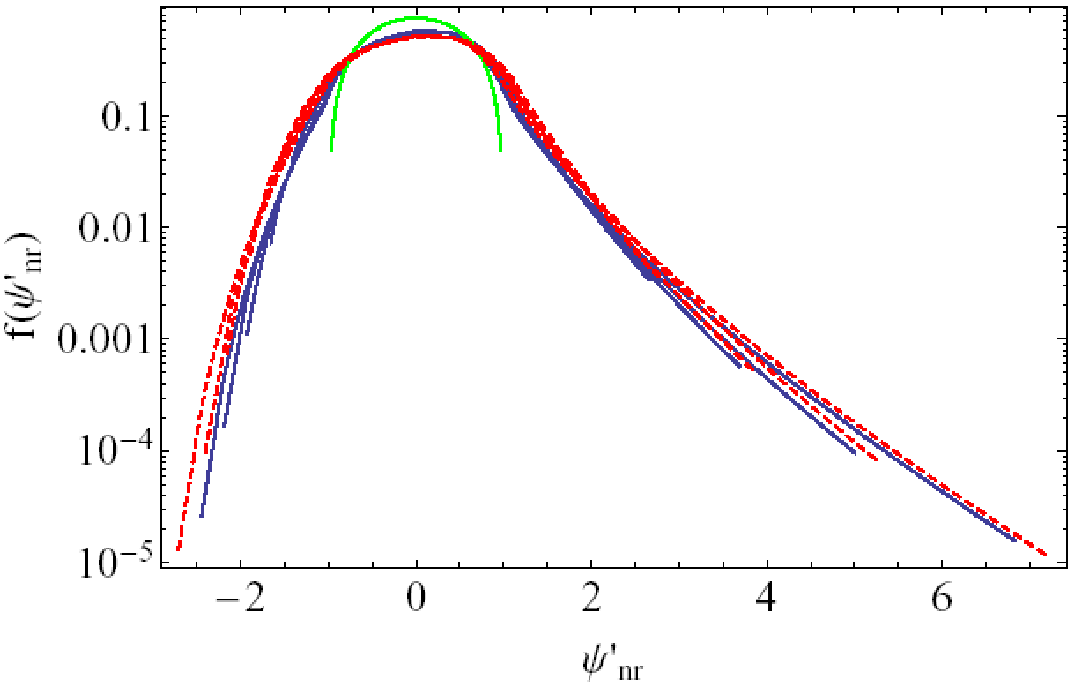


It is very suggestive that in the present model the origin of the tails, in the scaling function, is principally due to the short-range physics, as is often assumed to be the case.

Results

To understand better the role of the tail of n(k), we have repeated the entire calculation using a stronger high momentum tail.

Also in this case we are able to reproduce the density, binding energy and compressibility and we find scaling.



The scaling function occurring in this case is found to be more asymmetric than the previous one.

Summary

- A momentum distribution has been chosen with low- and high-k components.

- We have restricted our study to the infinite, homogeneous, non-relativistic nuclear matter.

- We have devised a single-particle Green function that leads to the known properties of nuclear matter.

- From the Green function we have obtained the electron scattering response function.

- For the latter we have explored several aspects of scaling.

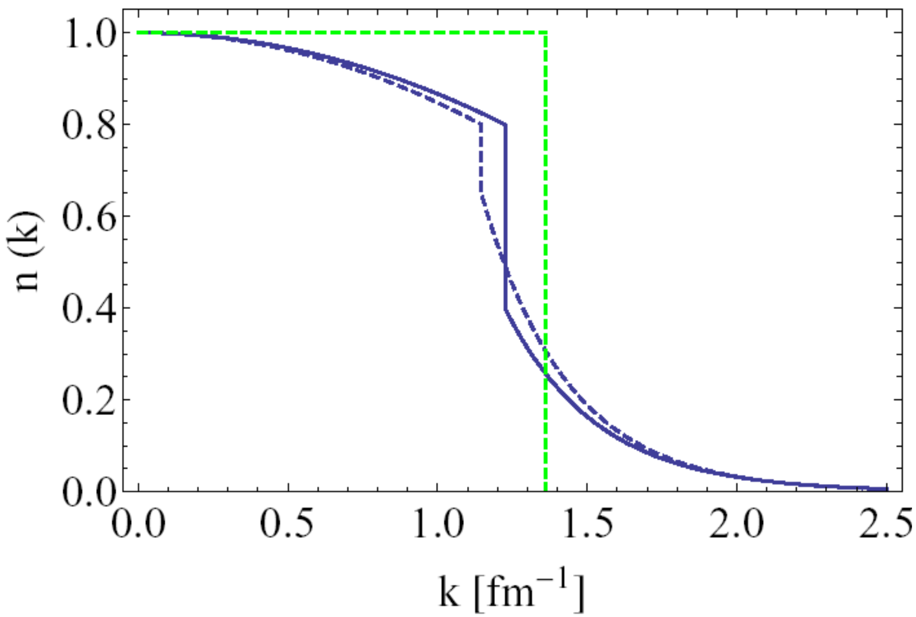
The End

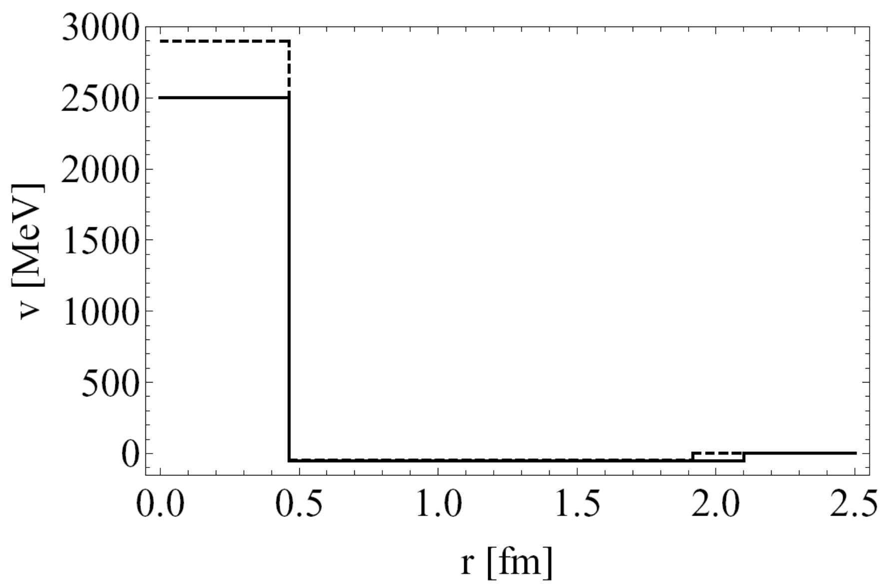
Thanks to all!!!

Results

$$\alpha = 0.2, \boxed{\beta_1 = 0.65,} \beta_2 = 4$$

$$\boxed{U_0 = 2.9 \text{ GeV},} V_0 = 48 \text{ MeV},$$
$$a = 0.464 \text{ fm}, \boxed{b = 1.917 \text{ fm}}$$





The dashed lines represent the momentum distribution with a stronger tail and the potential with a bigger repulsion and lower range attraction.