LECTURE 3

3.0 Introduction

When matter density becomes comaparable to the central density of atomic nuclei, $\rho_0 \sim = 2 \times 10^{14}$ g/cm^3, interactions between nucleons can no longer be neglected. Understandig the properties of matter in this density regime is made difficult by *both* the complexity of nuclear forces *and* the approximations implied in the theoretical description of quantum mechanical many particle systems. The situation gets even worse as we enter the region of *supernuclear* density, corresponding to $\rho > \rho_0$, as the available empirical information is scarce, and one has to unavoidably resort to a mixture of extrapolation and speculation.

The approach based on nonrelativistic quantum mechanics and phenomenological nuclear hamiltonians, while allowing for a rather satisfactory description of nuclear bound states and NN scattering data, fails to fulfill the contraint of causality (i.e. predicts a speed of sound in matter that exceeds the speed of light) as $\rho \to \infty$.

On the other hand, the approach based on quantum field theory, that incorporates the requirement of causality by construction, has not yet been developed to the point of providing a complete quantitative account of hadron phenomenology at low density.

In this lecture we will briefly describe the current understanding of the nucleon-nucleon (NN) interaction, as well as the nonrelativistic approaches that have been employed to study the structure of neutron star matter at nuclear and supernuclear density. As anticipated in the previous lecture, in this region the star is believed to consist of a uniform fluid of neutrons, protons and electrons in β-equilibrium. The relativistic approach will be the subject of Lecture 4.

3.1 Main features of the nucleon-nucleon interaction

The main features of the NN interaction, that can be inferred from the analysis of nuclear systematics, may be summarized as follows.

- The *saturation* of nuclear density, i.e. the fact that density in the interior of atomic nuclei is nearly constant and independent of the mass number A, tells us that nucleons cannot be packed together too tightly. Hence, at short distance the NN force must be repulsive. Assuming that the interaction can be described by a nonrelativistic potential v depending on the interparticle distance, $\mathbf{r}$, we can write:

$$v(\mathbf{r}) > 0 \quad , \quad |\mathbf{r}| < r_c \ ,$$

  r_c being the radius of the repulsive core.

- The fact that the nuclear binding energy per nucleon is roughly the same for all nuclei with A$\geq$ 20, its value being

$$\frac{B(\mathrm{Z},\mathrm{A})}{\mathrm{A}} \sim 8.5 \ \mathrm{MeV} \ ,$$

  suggests that the NN interaction has a finite range r_0, i.e. that

$$v(\mathbf{r}) = 0 \quad , \quad |\mathbf{r}| > r_0 \ .$$

- The spectra of the so called *mirror nuclei*, i.e. pairs of nuclei having the same A and charges differing by one unit (implying that the number of protons in a nucleus is the same as the number of neutrons in its mirror companion), e.g. $^{15}_{7}$N (A = 15, Z = 7, N = 8) and $^{15}_{8}$O (A = 15, Z = 8, N = 7), exhibit striking similarities. The energies of the levels with the same parity and angular momentum are the same up to small electromagnetic corrections, showing that protons and neutrons have similar nuclear interactions, i.e. that nuclear forces are *charge symmetric*.

  Charge symmetry is a manifestation of a more general property of the NN interaction, called *isotopic invariance*. The proton and the neutron can be viewed as two states of the same particle, the *nucleon*, carrying *isospin* t = 1/2. Formally, isospin can be treated as a quantum mechanical angular momentum, and proton and neutron belong to a doublet corresponding to isospin projections t_3=+1/2 and $-1/2$, respectively. Proton-proton and neutron-neutron pairs always have total isospin T=1, whereas a proton-neutron pair may have either T=0 or T=1, depending on the symmetry of the isospin state.

The interaction between two nucleons only depends upon their relative distance **r**, total spin S and total isospin T. For example, the potential $v(\mathbf{r})$ acting between two protons with spins coupled to S=0 is the same as the potential acting between a proton and a neutron with spins and isospins coupled to S=0 and T=1.

3.2 The two-nucleon system and the NN potential

The details of the NN interaction can be best studied in the two-nucleon system. There is *only one* NN bound state, the nucleus of deuterium, or deuteron (^{2}H), consisting of a proton and a neutron coupled to total spin and isospin S=1 and T=0, respectively. This fact is a clear manifestation of the *spin dependence* of nuclear forces.

Another important piece of information can be inferred from the observation that the deuteron exhibits a nonvanishing electric quadrupole moment, implying that its charge distribution is not spherically symmetryc. This is a manifestation of the fact that the NN interaction is *noncentral*.

Besides the properties of the two-nucleon bound state, the large data base of phase shifts measured in NN scattering experiments ($\sim$ 4000 data points corresponding to energies up to 350 MeV in the lab frame) provides valuable additional information on the nature of NN forces.

The theoretical description of the NN interaction was first attempted by Yukawa in 1935. He made the hypotesis that nucleons interact through the exchange of a particle, whose mass μ can be related to the interaction range r_0 according to

$$r_0 \sim \frac{\hbar}{\mu c}.$$

Using $r_0 \sim 1$ fm, the above relation yields $\mu \sim 200$ MeV.

Yukawa's idea has been successfully implemented identifying the exchanged particle with the π meson (or *pion*), discovered in 1947, whose mass is $\sim$ 140 MeV. Experiments show that the pion is a spin zero pseudoscalar particle † (i.e. it has spin-parity 0^-) that comes in three charge states, denoted π^+, π^- and π^0. Hence, it can be regarded as an isospin t=1 triplet, the charge states being associated with isospin projections t_3=+ 1, 0 and -1, respectively.

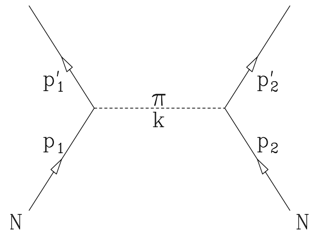

FIG. 1. Feynman diagram describing the one-pion-exchange process between two nucleons. The corresponding amplitude is given by eq.(1).

The simplest π-nucleon coupling compatible with the observation that nuclear interactions conserve parity has the pseudoscalar form $ig\gamma^5\boldsymbol{\tau}$ ‡ , where g is a coupling constant and $\boldsymbol{\tau}$ is the operator describing the isospin of the nucleon. With this choice for the interaction vertex, the amplitude of the process depicted in fig. 1 can readily be written, using standard Feynman's diagram techniques, as

$$\langle f|M|i\rangle = \frac{g^2}{4\pi}\ \frac{\bar{u}(p_2', s_2')\gamma_5 u(p_2, s_2)\bar{u}(p_1', s_1')\gamma_5 u(p_1, s_1)}{k^2-\mu^2}\ (\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2) \ , \tag{1}$$

where $k^2 = k_\mu k^\mu = k_0^2 - |\vec{k}|^2$ and $u(p, s)$ denotes the Dirac spinor associated with a nucleon of four momentum $p \equiv (\vec{p}, E)$ (E=$\sqrt{\mathbf{p}^2+m^2}$, m being the nucleon mass) and spin s.

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†The pion spin has been deduced from the balance of the reaction $\pi^+ + ^2\mathrm{H} \leftrightarrow p + p$, while its intrinsic parity was determined observing the π^- capture from the K shell of the deuterium atom, leading to the appearance of two neutrons: $\pi^- + d \to n + n$.
$^\ddagger\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\boldsymbol{\tau} \equiv (\tau^1, \tau^2, \tau^3)$, γ^μ ($\mu = 0, 1, 2, 3$) and τ^α ($\alpha = 1, 2, 3$) being the Dirac and Pauli matrices, respectively.

To extract a static potential from the above amplitude we first carry out a nonrelativistic reduction, implying the following approximations ($i = 1, 2$):

$$E_i = \sqrt{|\mathbf{p}_i|^2 + m^2} = E_i' = \sqrt{|\mathbf{p}_i'|^2 + m^2} \sim m \tag{2}$$

$$k^2 = (p_i' - p_i)^2 \sim -(\mathbf{p}_i' - \mathbf{p}_i)^2 \sim -|\mathbf{k}|^2 \tag{3}$$

$$\bar{u}(p_i', s_i')\gamma_5 u(p_i, s_i) \sim \ \chi^\dagger(s_i') \ \frac{\boldsymbol{\sigma}_i \cdot (\mathbf{p}_i - \mathbf{p}_i')}{2m} \ \chi(s_i) \ = \chi^\dagger(s_i') \ \frac{(\boldsymbol{\sigma}_i \cdot \mathbf{k})}{2m} \ \chi(s_i) \ , \tag{4}$$

where $\chi(s_i)$ is now a Pauli spinor. The resulting amplitude reads

$$\langle f|M|i\rangle = -\frac{g^2}{4\pi} \ \frac{1}{4m^2} \ \chi^\dagger(s_1')\chi^\dagger(s_2') \ \frac{(\boldsymbol{\sigma}_1 \cdot \mathbf{k})(\boldsymbol{\sigma}_2 \cdot \mathbf{k})}{|\mathbf{k}|^2 + \mu^2} \ (\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2) \ \chi(s_1)\chi(s_2) \ , \tag{5}$$

and the corresponding potential can be written in momentum space as

$$\widetilde{v}_\pi = -\frac{g^2}{4\pi} \ \frac{1}{4m^2} \ \frac{(\boldsymbol{\sigma}_1 \cdot \mathbf{k})(\boldsymbol{\sigma}_2 \cdot \mathbf{k})}{|\mathbf{k}|^2 + \mu^2} \ (\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2) \ . \tag{6}$$

Transformation of the above equation to coordinate space finally yields

$$\begin{aligned} v_\pi &= \frac{g^2}{(4\pi)^2} \ \frac{1}{4m^2} \ (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\nabla})(\boldsymbol{\sigma}_2 \cdot \boldsymbol{\nabla}) \ \frac{e^{-\mu r}}{r} \\ &= \frac{g^2}{(4\pi)^2} \ \frac{\mu^3}{4m^2} \ \frac{1}{3}(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2) \left\{ \left[(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2) + S_{12} \left(1 + \frac{3}{x} + \frac{3}{x^2} \right) \right] \frac{e^{-x}}{x} - \frac{4\pi}{\mu^3} (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2) \delta^{(3)}(\mathbf{r}) \right\} \ , \end{aligned} \tag{7}$$

with $x = \mu|\mathbf{r}|$ and

$$S_{12} = \frac{3}{r^2}(\boldsymbol{\sigma}_1 \cdot \mathbf{r})(\boldsymbol{\sigma}_2 \cdot \mathbf{r}) - (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2) \ . \tag{8}$$

For $g^2/(4\pi) = 14$, the above potential provides an accurate description of the long range part ($|\mathbf{r}| \ > 1.5$ fm) of the NN interaction, as shown by the very good fit of the NN scattering phase shifts in states of high angular momentum (note that in these states, due to the appearance of a strong centrifugal barrier, the probability of finding the two nucleons at small relative distances becomes negligibly small).

At medium- and short-range other more complicated processes, involving the exchange of two or more pions (possibly interacting among themselves) or heavier particles (like the ρ and the ω mesons, whose masses are $m_\rho = 770$ MeV and $m_\omega = 782$ MeV, respectively), have to be taken into account. Moreover, when their relative distance becomes very small ($|\mathbf{r}| \lesssim 0.5$ fm) nucleons, being composite and finite in size, are expected to overlap. In this regime, NN interactions should in principle be described in terms of interactions between nucleon constituents, i.e. quarks and gluons, as dictated by *quantum chromodynamics* (QCD), which is believed to be the underlying theory of strong interactions.

The phenomenological potential describing the *full* NN interaction is generally written in the form

$$v_{ij} = \sum_n v^{(n)}(r_{ij}) O^{(n)}_{ij} + v_{\pi,ij} \ , \tag{9}$$

where $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$ denotes the internucleon distance and v_π is the one pion exchange potential, defined by eqs.(7) and (8), stripped of the δ-function contribution. The radial shapes of the functions $v^{(n)}(r_{ij})$, accounting for the medium and short range behavior, and the set of operators $O^{(n)}_{ij}$ are chosen in such a way as to reproduce the available NN data (i.e. deuteron binding energy, charge radius and quadrupole moment and the NN scattering phase shifts). A widely employed set consists of the six operators

$$O^{(n)}_{ij} = 1, \ (\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j), \ (\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j), \ (\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j)(\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j), \ S_{ij}, \ S_{ij}(\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j) \ . \tag{10}$$

The potentials written as in eq.(9) using the operators of eq.(10) are referred to as v_6 models. The best available fits of NN scattering data have been obtained including up to eighteen operators, to take into account spin-orbit interactions as well as small charge simmetry breaking effects.

The typical shape of the NN potential in the state of relative angular momentum $\ell = 0$ and total spin and isospin $S = 0$ and $T = 1$ is shown in fig. 2.

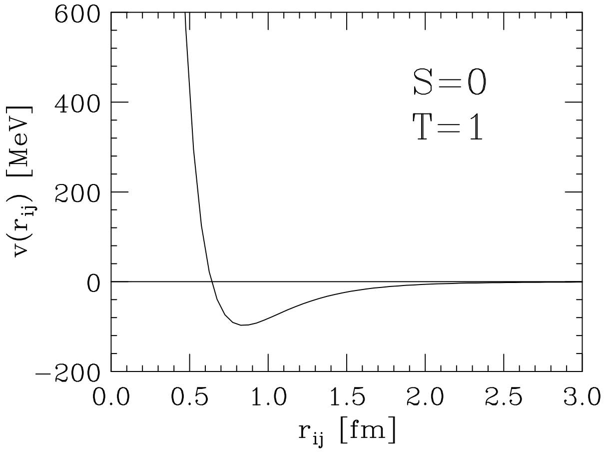

FIG. 2. Radial dependence of the NN potential describing the interaction between two nucleons in the state of relative angular momentum $\ell = 0$, and total spin and isospin $S = 0$ and $T = 1$.

The short range repulsive core, to be ascribed to heavy meson exchange or to more complicated mechanisms involving nucleon constituents, is followed by an intermediate range attractive region, which is believed to be largely due to two-pion exchange processes. Finally, at large interparticle distance the one-pion-exchange mechanism dominates.

3.3 The nonrelativistic nuclear many-body problem

Within nonrelativistic many-body theory, nuclear systems are described as a collection of pointlike nucleons interacting through the hamiltonian

$$H = \sum_{i=1}^{A} \frac{\mathbf{p}_i^2}{2m} + \sum_{j>i=1}^{A} v_{ij} + \ldots \ , \tag{11}$$

where $\mathbf{p}_i$ denotes the momentum carried by the i-th nucleon, v_{ij} is a two-body potential describing NN interactions the and the ellipsis refers to the possible existence of interactions involving more than two nucleons.

Let us begin our discussion of this approach providing arguments that support the validity of the main underlying assumptions. Two important questions need to be addressed:

1. *Can nucleons be teated as pointlike particles ?*

   As anticipated at the end of the previous section, one may argue that, as the nucleon density becomes large enough, nucleons start overlapping and no longer behave as individual particles. The typical interparticle separation in a nuclear system at constant number density n can be estimated computing the radius of a sphere containing only one nucleon:

   $$R_0 = \left(\frac{3}{4\pi n} \right)^{1/3} \ .$$

   The resulting values or R_0 range between 1.12 fm and .71 fm as n ranges between the central density of atomic nuclei $n_0 \sim .16$ fm^{-3} and $4n_0$.

   The role played by the finite size of the nucleons in NN interactions can be gauged using the existing parametrizations of the proton charge distribution, obtained from the analysis of electron-proton elastic scattering data. Figure 3, showing that the charge distributions of two protons whose centers are sitting 1 fm apart have little overlap, suggests that treating the nucleons as pointlike objects is quite reasonable up to densities $\sim 4n_0$.

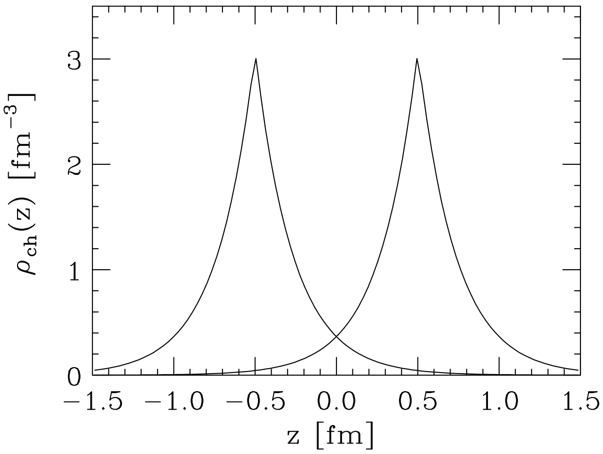

FIG. 3. Charge density distributions of two protons whose centers are sitting at relative distance of 1 fm. The curves displayed in the figure have been obtained from the dipole parametrization of the proton form factor measured in elastic electron-proton scattering at momentum transfer q: $F(q) = [1 + (q/q_0)^2]^{-2}$, with $q_0 = 0.84$ GeV/c.

2. *Is the nonrelativistic approximation reasonable ?*

Consider the average kinetic energy per particle of a degenerate neutron gas at number density n:

$$\langle T \rangle = \frac{3}{5} \frac{\hbar^2}{2m_n} (3\pi^2 n)^{1/3} .$$

It turns out that, as the density ranges between the central density of atomic nuclei $n_0 \sim .16$ fm^{-3} and $4n_0$, the neutron kinetic energy is much smaller than the neutron rest mass, the ratio $\langle T \rangle/m_n$ ranging between $\sim 4\%$ and $\sim 9\%$. Hence, the nonrelativistic approximation is certainly plausible, and relativistic effets are likely to be treatable as small corrections. Realistic many-body calculations have provided *a posteriori* evidence that the above argument holds true even in presence of NN interactions.

Having established the premises of the nonrelativistic many body approach, we are now confronted with the problem of their implementation. Unfortunately, solving the Schrödinger equation

$$H|\Psi_0\rangle = E_0|\Psi_0\rangle \tag{12}$$

for the ground state of a nucleus, using the hamiltonian (11) and the NN potential of eqs.(9) and (10), is only possible for A<8 †. In the case of neutron stars, correponding to A $\sim 10^{57}$, approximations need to be made.

The simplest approximation amounts to replace the complicated NN potential with a *mean field*, i.e. to substitute

$$\sum_{j>i=1}^{\rm A} v_{ij} \to \sum_{i=1}^{\rm A} U_i \ , \tag{13}$$

in eq.(11), the potential U being chosen in such a way that the *single particle* hamiltonian

$$H_0 = \frac{p^2}{2m} + U \tag{14}$$

be diagonalizable. Within this scheme the nuclear ground state wave function reduces to a Slater determinant, constructed using the A lowest energy eigenstates of H_0:

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†The numerical solution is trivial for A=2 only. For A=3 eq.(12) can still be solved using deterministic approaches, while for A>3 sthocastic methods, such as the Green Function Monte Carlo method, have to be employed. The fact that the dimensionality of the problem is a steeply increasing function of A currently limits the applicability of this approach to A≤8.

$$|\Psi_0\rangle = \frac{1}{\sqrt{A!}} \det\{\phi_i\} \ , \tag{15}$$

the ϕ_i's (i = 1, 2,..., A) being solutions of the Schrödinger equation

$$H_0|\phi_i\rangle = \epsilon_i|\phi_i\rangle \ . \tag{16}$$

It follows that the ground state energy is given by

$$E_0 = \sum_{i=1}^{A} \epsilon_i \ . \tag{17}$$

This procedure is at the basis of the nuclear shell model, that has been succesfully applied to explain many nuclear properties. Matter in the neutron star interior, however, can be modeled as a uniform, dense nuclear fluid, whose single particle wave functions are known to be plane waves, as dictated by translational invariace. Shell effects are not expected to play a major role in such a system. On the other hand, strong correlations between nucleons, induced by the NN potential, become more and more important as the density increases, and must be explicitely taken into account.

The main difficulty associated with the use of the full nuclear hamiltonian of eq.(11) in infinite nuclear matter calculations is the presence of the strong repulsive core of the NN potential, that can not be handled using ordinary perturbation theory. Among the approaches that have been developed to overcome this problem the most widely and successfully employed are: i) perturbation theories built on single particle basis functions, like *Brückner-Bethe-Goldstone* (BBG) perturbation theory and ii) *correlated basis function* (CBF) perturbation theory.

BBG perturbation theory uses a single particle basis obtained by diagonalizing a mean field hamiltonian written as in eq.(14), with a single particle potential chosen to either simplify numerical calculations (a most popular choice leads to cancellations between selected classes of terms in the perturbative expansion) or satisfy analitycity properties. The quantity $H_I = (H - H_0)$ is then treated as a perturbation. The problem of the strong range repulsion is circumvented replacing the bare NN interaction v with the G-matrix, obtained summing up ladder diagrams to all orders in v, as shown in fig. 4.

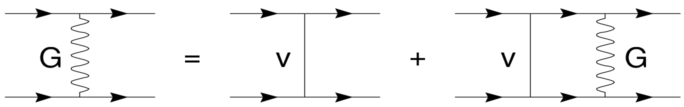

FIG. 4. Diagrammatic representation of eq.(18), definig the G-matrix in terms of the bare NN interaction v.

This procedure leads to the definition

$$G(\omega) = v + v\frac{Q}{\omega - H_0}v + v\frac{Q}{\omega - H_0}v\frac{Q}{\omega - H_0}v + \ldots = v + v\frac{Q}{\omega - H_0}G(\omega) \ , \tag{18}$$

where Q is an operator that enforces Pauli exclusion principle by projecting onto the subspace of unoccupied single particle states. The resulting perturbative series can be rearranged in terms of linked diagrams containing the operator G, that turns out to be well behaved even in case of NN potentials exhibiting an infinite repulsive core. Unfortunately, while most of the existing applications of BBG theory have been carried out including only the lowest order term, the rate of convergence of the expansion is not well established from the theoretical point of view. In fact, the available estimates of higher order terms indicate that convergence is already slow at nuclear density, and becomes even worse at $\rho > \rho_0$.

CBF theory exhibits several formal similarities to BBG theory, in that it also uses the standard machinery of many body perturbation theory. However, in CBF non perturbative effects associated with the strong shortrange repulsion are incorporated into the basis function. The basic ingredient of CBF is the set of correlated states

$$|n\rangle = \frac{F|n_0\rangle}{\langle n_0|F^\dagger F|n_0\rangle^{1/2}} \ , \tag{19}$$

where $|n_0\rangle$ is the n-th eigenstate of an independent particle hamiltonian (for example the Fermi gas hamiltonian in the case of nuclear matter) and F is a *correlation operator*, whose structure reflects the structure of the NN interaction. A widely used form of F is

$$F = \mathcal{S} \prod_{J>i=1}^{A} f_{ij} \ , \tag{20}$$

with

$$f_{ij} = \sum_n f^{(n)}(r_{ij}) O_{ij}^{(n)} \ , \tag{21}$$

where $\mathcal{S}$ is the symmetrization operator and the $O_{ij}^{(n)}$ are defined in eq.(10).

The correlated states of eq.(20) form a complete set but are *not* orthogonal. However, they can be othogonalized using standard techniques. The radial shape of the functions $f^{(n)}(r)$ are determined minimizing the expectation value of the hamiltonian of eq.(11) in the correlated ground state, satisying the inequality

$$\langle H \rangle = \langle 0|H|0 \rangle \geq E_0 \ . \tag{22}$$

As $r \to \infty$ the scalar correlation function, $f^{(1)}(r)$, goes to unity, while all other components, $f^{(n>1)}(r)$, go to zero. As a result, the quantity

$$f_{ij}^\dagger H_{ij} f_{ij} = f_{ij}^\dagger \left(\frac{p_i^2}{2m} + \frac{p_i^2}{2m} + v_{ij} \right) f_{ij} \tag{23}$$

is *well behaved* as $r \to 0$ and *heals* to H_{ij} at large interparticle distances, as does the G-matrix of BBG theory.

Once the correlated basis has been defined, the nuclear hamiltonian can be split into two pieces according to

$$H = H_V + H_I \ , \tag{24}$$

H_V and H_I being its diagonal and off-diagonal part, respectively

$$\langle m|H_V|n \rangle = \delta_{mn} \langle m|H|n \rangle \tag{25}$$

$$\langle m|H_I|n \rangle = (1 - \delta_{mn}) \langle m|H|n \rangle \ . \tag{26}$$

If the form of the correlation function is properly chosen, i.e. if $\langle H \rangle$ is close to E_0, the correlated states have large overlaps with the eigenstates of the hamiltonian. It follows that H_I is a small quantity and perturbative expansions of any observable in powers of H_I rapidly converge.

The results obtained applying CBF perturbation theory to the calculation of different nuclear matter properties, like the binding energy and equilibrium density, the nucleon self-energy and the momentum distribution, are in good agreement with the available empirical data.

For any given matter density ρ, the ground state energy of nuclear matter can be evaluated as a function of the proton fraction x (x=0 and 1/2 correspond to pure neutron matter and symmetric nuclear matter, consisting of equal number of protons and neutrons, respectively). The energy density of matter and the equation of state can then be obtained from

$$\epsilon(\rho, x) = \rho \left(\frac{E_0}{\mathrm{A}} + m \right) \tag{27}$$

and

$$P(\rho) = \rho^2 \frac{\partial}{\partial \rho} \left(\frac{E_0}{A} \right) \ . \tag{28}$$

3.4 Neutron star matter at nuclear and supernuclear density

As anticipated in section 2.3, at density $\rho \geq \rho_0$ the neutronization process eventually leads to the appearance of a uniform fluid of protons, neutrons and electrons in equilibrium with respsct to the process

$$n \leftrightarrow p + e^- \ . \tag{29}$$

Equilibrium and charge neutrlity require

$$\mu_n - \mu_p = \mu_e \tag{30}$$

and

$$\rho_p = \rho_e \ , \tag{31}$$

where μ_α and ρ_α denote the chemical potential and density of the species α ($\alpha = n, p, e$).

We have already discussed eq.(30) in Lecture 2, when studying the case of noninteracting particles. In the density regime we are focusing on in this Lecture, however, nuclear interactions can non longer be neglected, and the proton and neutron chemical potentials must be evaluated from

$$\mu_p = \frac{1}{\Omega} \frac{\partial}{\partial \rho_p} E(\rho, x) \quad , \quad \mu_n = \frac{1}{\Omega} \frac{\partial}{\partial \rho_n} E(\rho, x) \ , \tag{32}$$

where Ω is the normalization volume, $\rho_p = x\rho$ and $\rho_n = (1-x)\rho$, and the ground state energy of matter $E(\rho, x)$ can be obtained using CBF perturbation theory, as discussed in the previous section.

Substitution of eq.(32) into eq.(30) shows that, for any given value of matter density ρ, equilibrium and charge neutrality determine the ptoton fraction x.

As ρ increases (typically just above ρ_0), the electron chemical potential exceeds the muon mass (i.e. $\mu_e > m_\mu$= 105 MeV), and the process

$$n \leftrightarrow p + \mu^- \tag{33}$$

is also energetically allowed. The corresponding equilibrium condition is

$$\mu_n - \mu_p = \mu_\mu = \mu_e \ , \tag{34}$$

with

$$\mu_\mu = \sqrt{(p_{F,\mu} c)^2 + (m_\mu c^2)^2} \ , \tag{35}$$

the muon Fermi momentum being related to the muon number density through $n_\mu = (3\pi^2 n_\mu)^{1/3}$. In presence of muons, charge neutrality condition (eq.(31) obvioulsy becomes

$$\rho_p = \rho_e + \rho_\mu \ . \tag{36}$$