LECTURE 4

4.0 Introduction

The theoretical approach described in the previous lecture is based on the assumption that the degrees of freedom associated with the carriers of the NN interaction can be eliminated in favor of a static NN potential. While this procedure appears to be most succesful at $\rho \sim \rho_0$, as matter density (and therefore the nucleon Fermi momentum) increases the relativistic propagation of the nucleons, as well as the retarded propagation of the virtual meson fields giving rise to nuclear forces, are expected to become more and more important.

In principle, relativistic quantum field theory provides a well definied theoretical framework in which relativistic effects can be taken into account in a fully consistent fashion. Due to the complexity and nonperturbative nature of the interaction, however, the *ab initio* approach to the nuclear many problem, based on the QCD lagrangian, involves prohibitive difficulties. In fact, even the structure of individual hadrons, like the proton or the π meson, is not yet understood at a fully quantitative level in terms of QCD degrees of freedom. Let alone the structure of highly condensed hadronic matter at supernuclear densities.

It has to be pointed out, however, that when dealing with condensed matter it is often convenient to replace the lagrangian describing the interactions between elementary constituents, be it solvable or not, with properly constructed *effective interactions*. For example, the properties of highly condensed system bound by electromagnetic interactions are most successfully explained using effective interatomic potentials. In spite of the fact that the lagrangian of quantum electrodynamics is very well known and can be treated in perturbation theory, nobody in his right mind would ever use it to carry out explicit calculations in solid state physics.

The fact that most of the time nucleons in nuclear matter behave as individual particles interacting through boson exchange (see Lecture 3), suggests that the fundamental degrees of freedom of QCD, quarks and gluons, may indeed replaced by nucleons and mesons, to be regarded as the degrees of freedom of an *effective* field theory.

In this section we will describe a simple model in which nulcear matter is viewed as a static uniform system of nucleons, described by Dirac spinors and interacting through exchange of a scalar and a vector meson, called σ and ω, respectively. Note that throughout this Lecture we will use a system of units in which $\hbar = c = 1$.

4.1 The σ-ω model

The basic ingredient of the σ-ω model is the lagrangian

$$\mathcal{L} = \mathcal{L}_N + \mathcal{L}_B + \mathcal{L}_{int} \ , \tag{1}$$

where $\mathcal{L}_N$, $\mathcal{L}_B$ and $\mathcal{L}_{int}$ describe free nucleons and mesons and their interactions, respectively. The dynamics of the free nucleon field is dictated by the Dirac Lagrangian (we use the standard notation $\not{a} = \gamma^\mu a_\mu$, γ^μ and a_μ being the Dirac gamma matrix and the component of a generic four-vector, respectively)

$$\mathcal{L}_N(x) = \overline{\psi}(x)\,(i\not{\partial} - m)\,\psi(x) \ , \tag{2}$$

where the nucleon field, denoted by $\psi(x)$, combines the two four-component Dirac spinors describing proton and neutron, $\psi_p(x)$ and $\psi_n(x)$, into a single eight-component spinor according to

$$\psi(x) \ = \ \begin{pmatrix} \psi_p(x) \\ \psi_n(x) \end{pmatrix} \ . \tag{3}$$

The meson lagrangian reads

$$\mathcal{L}_B(x) = \mathcal{L}_\omega(x) + \mathcal{L}_\sigma(x) = -\frac{1}{4}F^{\mu\nu}(x)F_{\mu\nu}(x) + \frac{1}{2}m_\omega^2 V_\mu(x)V^\mu(x) + \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{1}{2}m_\sigma^2\phi(x)^2 \tag{4}$$

where

$$F_{\mu\nu}(x) = \partial_\mu V_\nu(x) - \partial_\nu V_\mu(x) \ , \tag{5}$$

$V_\mu(x)$ and $\sigma(x)$ are the vector and scalar meson fields, respectively, and m_ω and m_σ the corresponding masses.

In specifying the form of the interaction lagrangian we will require that, besides being a Lorentz scalar, $\mathcal{L}_{int}(x)$ give rise to a Yukawa-like meson exchange potential in the static limit. Hence, we write

$$\mathcal{L}_{int}(x) = g_\sigma \phi(x)\overline{\psi}(x)\psi(x) - g_\omega V_\mu(x)\overline{\psi}(x)\gamma^\mu\psi(x) \ , \tag{6}$$

where the g_σ and g_ω are coupling constants and the choice of signs reflect the fact that the NN interaction contains both attractive and repulsive contributions.

The equations of motion for the fields follow from the Euler-Lagrange equations associated with the lagrangian of eq.(1). The meson fields satisfy

$$\left(\Box + m_\sigma^2\right)\phi(x) = g_\sigma\ \overline{\psi}(x)\psi(x) \tag{7}$$

and

$$\left(\Box + m_\omega^2\right)V_\mu(x) - \partial_\mu\partial^\nu V_\nu = g_\omega\ \overline{\psi}(x)\gamma_\mu\psi(x) \ , \tag{8}$$

while the evolution of the nucleon field is dictated by the equation

$$\left[(\partial\!\!\!/ - g_\omega\gamma_\mu V^\mu(x)) - (m - g_\sigma\phi(x))\right]\psi(x) = 0 \ . \tag{9}$$

The above coupled equations are fully relativistic and Lorentz covariant. However, their solution is extremely difficult. Here we will restrict ourselves to the discussion of an approximation scheme widely used to solve eqs.(7)-(9), known as *mean field* approximation, that essentially amounts to treat $\phi(x)$ and $V_\mu(x)$ as classical fields.

We replace the meson field with their mean values in the ground state of static and uniform nuclear matter

$$\phi(x) \to \langle\phi(x)\rangle \ , \quad V_\mu(x) \to \langle V_\mu(x)\rangle \ , \tag{10}$$

where $\langle\phi(x)\rangle$ and $\langle V_\mu(x)\rangle$ must be computed from the equations of motion. In static and uniform nuclear matter the baryon and scalar densities, $n_B = \psi^\dagger\psi$ and $n_s = \overline{\psi}\psi$, as well as the current $j_\mu = \overline{\psi}\gamma_\mu\psi$, are constants, independent of x. As a consequence, the mean values of the meson fields are also constants satisfying the relations

$$m_\sigma^2\ \langle\phi\rangle = g_\sigma\langle\overline{\psi}\psi\rangle \tag{11}$$

$$m_\omega^2\ \langle V_0\rangle = g_\omega\langle\psi^\dagger\psi\rangle \tag{12}$$

$$m_\omega^2\ \langle V_i\rangle = g_\omega\langle\overline{\psi}\gamma_i\psi\rangle \ , \quad i = 1,2,3 \ . \tag{13}$$

The nucleon equation of motion, rewritten in terms of the mean values of the meson fields, reads

$$\left[(\partial\!\!\!/ - g_\omega\gamma_\mu\langle V^\mu\rangle) - (m - g_\sigma\langle\phi\rangle)\right]\psi(x) = 0 \ . \tag{14}$$

In static and uniform matter, the nucleon states must be four-momentum eigenstates, and the correponding field can be written

$$\psi(x) = \psi_{\mathbf{k}}\mathrm{e}^{ikx} = \psi_{\mathbf{k}}\mathrm{e}^{ik_\mu x^\mu} = \psi_{\mathbf{k}}\mathrm{e}^{i(k_0 t - \mathbf{k}\cdot\mathbf{r})} \ . \tag{15}$$

Substitution into eq.(14) yields

$$\left[(k\!\!\!/ - g_\omega\gamma_\mu\langle V^\mu\rangle) - (m - g_\sigma\langle\phi\rangle)\right]\psi_{\mathbf{k}} = \left[\gamma_\mu\left(k^\mu - g_\omega\langle V^\mu\rangle\right) - (m - g_\sigma\langle\phi\rangle)\right]\psi_{\mathbf{k}} = 0 \ . \tag{16}$$

The above equation can be recast in a form reminiscent of the Dirac equation defining

$$K_\mu = k_\mu - g_\omega\langle V^\mu\rangle \tag{17}$$

$$m^* = m - g_\sigma\langle\phi\rangle \ . \tag{18}$$

In terms of the above quantities we can write

$$(K\!\!\!/ - m^*)\,\psi_{\mathbf{k}} = 0. \tag{19}$$

The corresponding energy eigenvalues can be found from

$$(\not{K} + m^*)\,(\not{K} - m^*) = \not{K}\not{K} - {m^*}^2 = K_\mu K_\nu \gamma^\mu \gamma^\nu - {m^*}^2 = K_\mu K_\nu \frac{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu}{2} - {m^*}^2 = K_\mu K^\mu - {m^*}^2 \ , \tag{20}$$

implying

$$\left(K_\mu K^\mu - {m^*}^2 \right) \psi_{\mathbf{k}} = 0 \ , \tag{21}$$

leading to

$$\left(K_\mu K^\mu - {m^*}^2 \right) = 0 \tag{22}$$

and

$$K_0 = E_{\mathbf{k}} = k_0 - g_\omega \langle V_0 \rangle = \sqrt{|\mathbf{K}|^2 + {m^*}^2} = \sqrt{|\mathbf{k} - g_\omega \langle \mathbf{V} \rangle|^2 + (m - g_\sigma \langle \phi \rangle)^2} \ . \tag{23}$$

It follows that the energy eigenvalues associated with nucleons and antinucleons can be written

$$e_{\mathbf{k}} = E_{\mathbf{k}} + g_\omega \langle V_0 \rangle \tag{24}$$

and

$$\overline{e}_{\mathbf{k}} = E_{\mathbf{k}} - g_\omega \langle V_0 \rangle \ , \tag{25}$$

respectively. The above equations give the nucleon (and antinucleon) energies in terms of the mean meson fields, which are in turn defined in terms of the ground state expectation values of the nucleon densities and current, according to eqs.(11)-(13).

The ground state expectation value of an operator $\overline{\psi}\Gamma\psi$ can be evaluated exploiting the fact that each nucleon state is specified by its momentum, $\mathbf{k}$, and spin and isospin projections. Denoting the average of $\overline{\psi}\Gamma\psi$ in a single particle state by $\langle \overline{\psi}\Gamma\psi \rangle_{\mathbf{k}\alpha}$, where the index α labels the spin-isospin state, we can write the ground state expectation value as

$$\langle \overline{\psi}\Gamma\psi \rangle = \sum_\alpha \int \frac{d^3k}{(2\pi)^3} \ \langle \overline{\psi}\Gamma\psi \rangle_{\mathbf{k}\alpha} \ \theta(e_F - e_{\mathbf{k}}) \ , \tag{26}$$

where the θ-function restricts the momentum integration to the region corresponding to energies lower than the Fermi energy e_F. To obtain the single particle average $\langle \overline{\psi}\gamma_\mu\psi \rangle_{\mathbf{k}\alpha}$, we use eq.(19), implying

$$k_0 = \gamma_0 \left(\gamma \cdot \mathbf{k} + g_\omega \gamma_\mu \langle V^\mu \rangle + m^* \right) \ . \tag{27}$$

The quantity defined by the above equation can be regarded as the single nucleon hamiltonian, whose eigenvalues are given by (compare to eq.(24))

$$\langle k_0 \rangle_{\mathbf{k}\alpha} = \langle \psi^\dagger k_0 \psi \rangle_{\mathbf{k}\alpha} = E_{\mathbf{k}} + g_\omega \langle V_0 \rangle \ . \tag{28}$$

The ground state expectation value of the baryon density can be readily evaluated from eqs.(27) and (28) noting that

$$\frac{\partial}{\partial \langle V_0 \rangle} \langle \psi^\dagger k_0 \psi \rangle_{\mathbf{k}\alpha} = \frac{\partial}{\partial \langle V_0 \rangle} \left(E_{\mathbf{k}} + g_\omega \langle V_0 \rangle \right) = g_\omega = \langle \psi^\dagger \frac{\partial k_0}{\partial \langle V_0 \rangle} \psi \rangle_{\mathbf{k}\alpha} = g_\omega \langle \psi^\dagger \psi \rangle_{\mathbf{k}\alpha} \ , \tag{29}$$

implying

$$\langle \psi^\dagger \psi \rangle_{\mathbf{k}\alpha} = 1 \ . \tag{30}$$

It follows that n_B can be obtained using eq.(26), leading to

$$n_B = \langle \psi^\dagger \psi \rangle = \nu \int \frac{d^3k}{(2\pi)^3} \theta(e_F - e_{\mathbf{k}}) \ , \tag{31}$$

where ν is the degeneracy of the momentum eigenstate ($\nu = 2$ and 4 for pure neutron matter and symmetric nuclear matter, respectively). Note that Eq.(31) yields the familiar result $n_B = 2\ k_F^3/(3\pi^2)$ in the case of symmetric nuclear matter and spherical Fermi surface.

The same procedure can be applied to calculate the ground state expectation value $\langle\overline{\psi}\gamma^i\psi\rangle$ ($i = 1, 2, 3$). Taking the derivative with respect to k_i we find

$$\frac{\partial}{\partial k_i}\langle\psi^\dagger k_0\psi\rangle_{\mathbf{k}\alpha} = \frac{\partial}{\partial k_i}\left(E_{\mathbf{k}} + g_\omega\langle V_0\rangle\right) = \frac{\partial E_{\mathbf{k}}}{\partial k_i} = \langle\psi^\dagger\frac{\partial k_0}{\partial k_i}\psi\rangle_{\mathbf{k}\alpha} = \gamma^0\langle\psi^\dagger\gamma^i\psi\rangle_{\mathbf{k}\alpha} = \langle\overline{\psi}\gamma^i\psi\rangle_{\mathbf{k}\alpha}\ , \tag{32}$$

leading to

$$\langle\overline{\psi}\gamma^i\psi\rangle = \nu\int\frac{d^3k}{(2\pi)^3}\left(\frac{\partial E_{\mathbf{k}}}{\partial k_i}\right)\ \theta(e_F - e_{\mathbf{k}}) = \nu\int\frac{dk_j dk_k}{(2\pi)^3}\int dE_{\mathbf{k}}\ \theta(e_F - e_{\mathbf{k}}) = 0\ . \tag{33}$$

The above result follows from the fact that, by definition, $e_{\mathbf{k}} \equiv e_F - g_\omega\langle V_0\rangle$ everywhere on the boundary of the integration region. The vanishing of the baryon current, that could have been anticipated, as we are dealing with uniform matter in its ground state, implies that the space components of the mean values of the vector field also vanish, i.e. that $\langle V_i\rangle = 0$. As a consequence, the energy eigenvalues depend upon the magnitude of the nucleon momentum only, according to

$$e_{\mathbf{k}} = e_k = \sqrt{|\mathbf{k}|^2 + (m - g_\sigma\langle\phi\rangle)^2} + g_\omega\langle V_0\rangle\ , \tag{34}$$

and the occupied region of momentum space is sphere. Eq.(31) then shows that in symmetric nuclear matter, with Z=(A−Z)=A/2, the baryon density takes the familiar form $n_B = 2k_F^3/(3\pi^2)$, k_F being the Fermi momentum.

Finally, the scalar density $n_s = \langle\overline{\psi}\psi\rangle$ can be evaluated from the derivative of $\langle\psi^\dagger k_0\psi\rangle_{\mathbf{k}\alpha}$ with respect to m:

$$\frac{\partial}{\partial m}\langle\psi^\dagger k_0\psi\rangle_{\mathbf{k}\alpha} = \frac{\partial e_{\mathbf{k}}}{\partial m} = \langle\psi^\dagger\frac{\partial k_0}{\partial m}\psi\rangle_{\mathbf{k}\alpha} = \langle\psi^\dagger\gamma_0\psi\rangle_{\mathbf{k}\alpha} = \langle\overline{\psi}\psi\rangle_{\mathbf{k}\alpha}\ , \tag{35}$$

yielding

$$\langle\overline{\psi}\psi\rangle_{\mathbf{k}\alpha} = \frac{(m - g_\sigma\langle\phi\rangle)}{\sqrt{|\mathbf{k}|^2 + (m - g_\sigma\langle\phi\rangle)^2}} \tag{36}$$

and

$$\langle\overline{\psi}\psi\rangle = \frac{\nu}{2\pi^2}\int_0^{k_F} k^2 dk\ \frac{(m - g_\sigma\langle\phi\rangle)}{\sqrt{|\mathbf{k}|^2 + (m - g_\sigma\langle\phi\rangle)^2}} \tag{37}$$

Collecting together the results of eqs.(31), (33) and (37) we can rewrite the equations of motion (7)-(9) in the form:

$$g_\sigma\langle\phi\rangle = \left(\frac{g_\sigma}{m_\sigma}\right)^2\ \frac{\nu}{2\pi^2}\int_0^{k_F}|\mathbf{k}|^2 d|\mathbf{k}|\ \frac{(m - g_\sigma\langle\phi\rangle)}{\sqrt{|\mathbf{k}|^2 + (m - g_\sigma\langle\phi\rangle)^2}} \tag{38}$$

$$g_\omega\langle V_0\rangle = \left(\frac{g_\omega}{m_\omega}\right)^2\ \nu\ \frac{k_F^3}{6\pi^2} \tag{39}$$

$$m_\omega^2\langle V_i\rangle = 0,\quad i = 1, 2, 3\ . \tag{40}$$

Note that, while the eqs.(39) and (40) are trivial, eq.(38) implies a self-consistency requirement on the mean value of the scalar field, whose value has to satisfy a transcendental equation.

4.2 The nuclear matter equation of state within the σ-ω model

To obtain the equation of state, i.e. the relation between pressure and density (or energy density) of matter, in quantum field theory we start from the energy-momentum tensor, that for a generic Lagrangian $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu\phi)$ can be written

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - g^{\mu\nu}\mathcal{L} \ , \tag{41}$$

$g^{\mu\nu}$ being the metric tensor.

In a uniform system the expectation value of $T^{\mu\nu}$, is directly related to the energy density, ϵ, and pressure, P, through

$$\langle T_{\mu\nu}\rangle = u_\mu u_\nu \left(\epsilon + P\right) - g_{\mu\nu}P \ , \tag{42}$$

where u denotes the four velocity of the system, satisfying $u_\mu u^\mu = 1$. It follows that in the reference frame in which matter is at rest $\langle T_{\mu\nu}\rangle$ is diagonal and

$$\epsilon = \langle T_{00}\rangle = -\langle\mathcal{L}\rangle + \langle\overline{\psi}\gamma_0 k_0\psi\rangle \tag{43}$$

$$P = \frac{1}{3}\langle T_{ii}\rangle = \langle\mathcal{L}\rangle + \frac{1}{3}\langle\overline{\psi}\gamma_i k_i\psi\rangle. \tag{44}$$

Within the mean field approximation discussed in the previous section, the lagrangian of the σ-ω model reduces to

$$\mathcal{L}_{MF} = \overline{\psi}\left[i\partial\!\!\!/ - g_\omega\gamma^0\langle V_0\rangle - (m - g_\sigma\langle\phi\rangle)\right]\psi - \frac{1}{2}m_\sigma^2\langle\phi\rangle^2 + \frac{1}{2}m_\omega^2\langle V_0\rangle^2 \ , \tag{45}$$

implying

$$T^{\mu\nu}_{MF} = i\overline{\psi}\gamma^\mu\partial^\nu\psi - g^{\mu\nu}\left[-\frac{1}{2}m_\sigma^2\langle\phi\rangle^2 - \frac{1}{2}m_\omega^2\langle V_0\rangle^2\right] \ . \tag{46}$$

As a consequence, eqs.(43) and (44) become

$$\epsilon = -\langle\mathcal{L}\rangle + \langle\overline{\psi}\gamma_0 k_0\psi\rangle \tag{47}$$

$$P = \langle\mathcal{L}\rangle + \frac{1}{3}\langle\overline{\psi}\gamma_i k_i\psi\rangle, \tag{48}$$

where (use eqs.(28), (34) and (39))

$$\begin{aligned}\langle\overline{\psi}\gamma_0 k_0\psi\rangle &= \frac{\nu}{2\pi^2}\int_0^{k_F}|\mathbf{k}|^2 d|\mathbf{k}|\left[\sqrt{|\mathbf{k}|^2 + (m - g_\sigma\langle\phi\rangle)^2} + g_\omega\langle V_0\rangle\right] \\ &= g_\omega\langle V_0\rangle n_B + \frac{\nu}{2\pi^2}\int_0^{k_F}|\mathbf{k}|^2 d|\mathbf{k}| \ \sqrt{|\mathbf{k}|^2 + (m - g_\sigma\langle\phi\rangle)^2} \\ &= \frac{g_\omega^2}{m_\omega^2}n_B^2 + \frac{\nu}{2\pi^2}\int_0^{k_F}|\mathbf{k}|^2 d|\mathbf{k}| \ \sqrt{|\mathbf{k}|^2 + (m - g_\sigma\langle\phi\rangle)^2} \ ,\end{aligned} \tag{49}$$

and (use eq.(33))

$$\langle\overline{\psi}\gamma_i k_i\psi\rangle = \langle\overline{\psi}(\boldsymbol{\gamma}\cdot\mathbf{k})\psi\rangle = \frac{\nu}{2\pi^2}\int_0^{k_F}d|\mathbf{k}| \ \frac{|\mathbf{k}|^4}{\sqrt{|\mathbf{k}|^2 + (m - g_\sigma\langle\phi\rangle)^2}} \ . \tag{50}$$

Substitution of the above equations into eqs.(47)-(48) finally yields (use eq.(45) and the equation of motion for the nucleon field)

$$\epsilon = \frac{1}{2}\frac{m_\sigma^2}{g_\sigma^2}(m - m^*)^2 + \frac{1}{2}\frac{g_\omega^2}{m_\omega^2}n_B^2 + \frac{\nu}{2\pi^2}\int_0^{k_F}|\mathbf{k}|^2 d|\mathbf{k}| \ \sqrt{|\mathbf{k}|^2 + m^{*2}} \tag{51}$$

$$P = -\frac{1}{2}\frac{m_\sigma^2}{g_\sigma^2}(m - m^*)^2 + \frac{1}{2}\frac{g_\omega^2}{m_\omega^2}n_B^2 + \frac{\nu}{2\pi^2}\int_0^{k_F}d|\mathbf{k}| \ \frac{|\mathbf{k}|^4}{\sqrt{|\mathbf{k}|^2 + m^{*2}}} \tag{52}$$

The first two contributions to the right hand side of the above equations arise from the mass terms associated with the vector and scalar fields, while the remaining term gives the energy density and pressure of a relativistic Fermi gas of nucleons of mass m^* given by (see eq.(38)

$$
\begin{aligned}
m^* &= m - \frac{g_\sigma^2}{m_\sigma^2}\frac{\nu}{2\pi^2}\int_0^{k_F} |\mathbf{k}|^2 d|\mathbf{k}| \, \frac{m^*}{\sqrt{|\mathbf{k}|^2 + m^{*2}}} \\
&= m - \frac{g_\sigma^2}{m_\sigma^2}\frac{m^{*2}}{\pi^2}\left[k_F e_F^* - m^{*2}\ln\left(\frac{k_F + e_F^*}{m^*}\right)\right] \ ,
\end{aligned}
\tag{53}
$$

with $e_F^* = \sqrt{k_F^2 + m^{*2}}$. Eqs.(51)-(53) yield energy density and pressure of nuclear matter as a function of the baryon number density n_B (recall: $k_F = (6\pi^2 n_B/\nu)^{(1/3)}$). The values of the unknown coefficients (m_σ^2/g_σ^2) and (m_ω^2/g_ω^2) can be determined by a fit to the empirical saturation properties of nuclear matter, i.e. requiring

$$
\frac{B}{A} = \frac{\epsilon(n_0)}{n_0} - m = -16 \text{ MeV}
\tag{54}
$$

with $n_0 = .16 \text{ fm}^{-3}$. THis procedure leads to the result

$$
\frac{g_\sigma^2}{m_\sigma^2}\, m^2 = 267.1 \quad , \quad \frac{g_\omega^2}{m_\omega^2}\, m^2 = 195.9 \ .
\tag{55}
$$

Fig.1 shows the binding energies of symmetric nuclear matter (solid line) and pure neutron matter (dashed line) predicted by the $\sigma - \omega$ model, plotted against the Fermi momentum k_F. Note that pure neutron matter is always unbound.

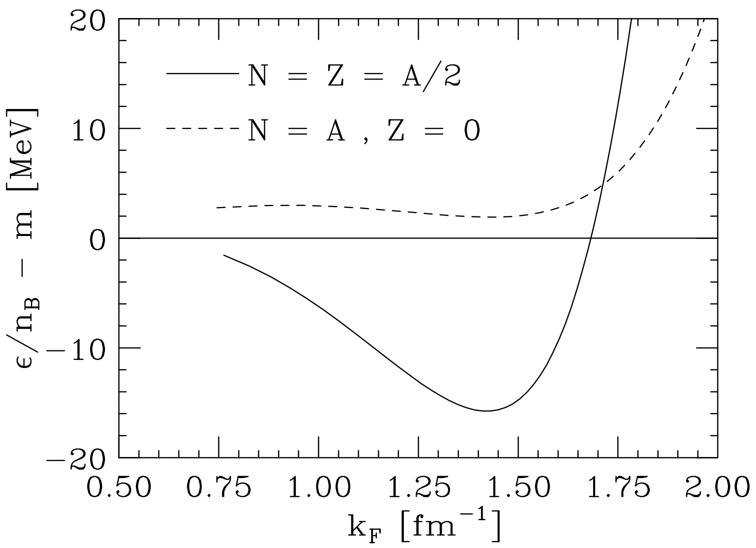

FIG. 1. Fermi momentum dependence of the binding energy per nucleon of symmetric nuclear matter (solid line) and pure neutron matter (dashed line) evaluated using the $\sigma - \omega$ model and the mean field approximation.