Application of the S-matrix formalism to $e^+e^- \to e^+e^-$ and photon propagator

Omar Benhar & Valerio Casconi

Contents

1 e^+e^- scattering

Expansion of the S-matrix element between the initial state $|i\rangle$ and the final state $|f\rangle$. From the Dyson equation we get

$$\begin{aligned}\langle f|\, S\,|i\rangle &= \langle f| \sum_{n=0}^{\infty} S^{(n)}\,|i\rangle \\ &= \langle f| \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \ldots d^4x_n\, T\{\mathcal{H}_I(x_1)\ldots\mathcal{H}_I(x_n)\}\,|i\rangle \\ &= \delta_{i,f} + \langle f|\, S^{(1)}\,|i\rangle + \langle k|\, S^{(2)}\,|i\rangle + \ldots \end{aligned} \tag{1}$$

where, $\mathcal{H}_I$ is the QED Hamiltonian density

$$\mathcal{H}_I(x) = -\mathcal{L}_I(x) = -\,e\,N\{\bar{\Psi}(x)\gamma^\mu\Psi(x)A_\mu(x)\} \tag{2}$$

where N is the normal ordering of the operators and $A_\mu(x) = (\Phi, \vec{A})$ is the gauge potential.

We define the Coulomb gauge (*transverse* gauge) defined by:

$$\vec{\nabla}\cdot\vec{A} = 0 \tag{3}$$

From the Gauss' law we have $\vec{\nabla}\cdot\vec{E} = \rho$ and $\vec{E} = -\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}$. Thus, $\vec{E}$ must satisfy Poisson's equation:

$$\nabla^2\Phi = -\rho \tag{4}$$

eq. (4) can be solved using the Green function of ∇^2, leading to

$$\Phi(\vec{x},t) \;= \int d^3x' \, \frac{\rho(\vec{x}',t)}{4\pi|\vec{x}-\vec{x}'|} \tag{5}$$

In the process that we are considering the charge density generating the potential felt by the electron is:

$$e\,\bar{\Psi}\gamma^0\Psi \;= e\,\Psi^\dagger\Psi \tag{6}$$

hence, we obtain for $A^0(\vec{x},t) = \Phi(\vec{x},t)$

$$\Phi(\vec{x},t) \;= \frac{e}{4\pi}\int d^3x' \, \frac{\bar{\Psi}(\vec{x}',t)\gamma^0\Psi(\vec{x}',t)}{\mid \vec{x}-\vec{x}' \mid} \tag{7}$$

The first contribution to the S matrix expansion are $S^{(1)}$ and $S^{(2)}$

$$S^{(1)} = ie\int d^4xN\left[\bar{\Psi}(x)\not{A}(x)\Psi(x)\right] \tag{8}$$

$$S^{(2)} = -\frac{e^2}{2}\int d^4xd^4x'T\left[\bar{\Psi}(x)\not{A}(x)\Psi(x)\bar{\Psi}(x')\not{A}(x')\Psi(x')\right] \tag{9}$$

We want to evaluate the matrix element between the initial state $|i\rangle$ and the final state $|f\rangle$, where

$$|i\rangle = c_p^\dagger d_k^\dagger\,|0\rangle = |e^-(p)e^+(k)\rangle \tag{10}$$

$$|f\rangle = c_{p'}^\dagger d_{k'}^\dagger\,|0\rangle = |e^-(p')e^+(k')\rangle \tag{11}$$

Consider first the matrix element of $S^{(1)}$. The only non-vanishing contributions come from terms containing: $c_{p'}^\dagger$, c_p or $d_{k'}^\dagger$, d_k.

We are interested in the scattering terms.

The only part of $S^{(1)}$ yielding a non-vanishing contribution to the matrix element is the time component of A^μ[1]

$$\begin{aligned} ie\int d^4xN\{\bar{\Psi}(x)\not{A}(x)\Psi(x)\} = \\ ie^2\int d^4xN\left[\Psi^\dagger(\vec{x},t)\left(\int d^3x'\,\frac{\Psi^\dagger(\vec{x}',t)\Psi(\vec{x},t)}{4\pi|\vec{x}-\vec{x}'|}\right)\Psi(\vec{x},t)\right] \\ = ie^2\int d^4xN\left[\bar{\psi}(x)\gamma^0\left(\int\frac{d^3x'}{4\pi|\vec{x}-\vec{x'}|}\bar{\psi}(x')\gamma^0\psi(x')\right)\psi(x)\right] \end{aligned} \tag{12}$$

Note that this contribution to $S^{(1)}$ is order e^2.

The field ψ is written as

$$\psi(x) = \sum_{p,r}\frac{1}{N_p}\left(c_pu_r(p)e^{-ipx} + d_p^\dagger v_r(p)e^{ipx}\right) \tag{13}$$

[1]Note that since A^i is quantized, $\langle 0|\,A^i\,|0\rangle = 0$ indeed, A^i can either annihilate or create a photon leading to an initial and final state with a different number of photons.

where the N_p factors denote the normalization of the spinors. $N_p = (2VE_p)^{\frac{1}{2}}$. The matrix element $\langle f| S^{(1)} |i\rangle$ reads

$$\langle f| S^{(1)} |i\rangle = \frac{ie^2}{N_{p_1} N_{p_2} N_{p_3} N_{p_4}} \int d^4x \int \frac{d^3x'}{4\pi|\vec{x} - \vec{x}'|}$$
$$\langle f| \sum_{p_1,p_2,p_3,p_4} \left(c^\dagger_{p_1} \bar{u}_{p_1} e^{ip_1x} + d_{p_1} \bar{v}_{p_1} e^{-ip_1x}\right) \gamma^0 \left(c^\dagger_{p_2} \bar{u}_{p_2} e^{ip_2x'} + d_{p_2} \bar{v}_{p_2} e^{-ip_2x'}\right) \gamma^0$$
$$\left(c_{p_3} u_{p_3} e^{-ip_3x'} + d^\dagger_{p_3} v_{p_3} e^{ip_3x'}\right) \left(c_{p_4} u_{p_4} e^{-ip_4x} + d^\dagger_{p_4} v_{p_4} e^{ip_4x}\right) |i\rangle \quad (14)$$

We are considering the scattering channel, which means that there is an electron e^- in position x and an positron e^+ in position x', hence we have to consider for $\bar{\psi}(x)$ the operator $c^\dagger$, for $\psi(x)$ the operator c, for $\bar{\psi}(x')$ the operator d and for $\psi(x')$, $d^\dagger$.

Hence,

$$\langle f| S^{(1)} |i\rangle =$$
$$\frac{ie^2}{N_p N_{p'} N_k N_{k'}} \int d^4d^3x' \, \frac{e^{i(p'-p)x} e^{i(k'-k)x'}}{4\pi|\vec{x} - \vec{x}'|} \left(\bar{u}_{p'} \gamma^0 u_p\right) \left(\bar{v}_k \gamma^0 v_{k'}\right) \quad (15)$$

The time integration in eq. (15) can be carried out right away, since

$$\int dx^0 \, e^{i\left(E_{p'} - E_p + E_{k'} - E_k\right)} = 2\pi \, \delta(E_{p'} - E_p + E_{k'} - E_k) \quad (16)$$

The integration over d^3x, d^3x' can be carried out using the new variables ξ, λ

$$\begin{cases} \vec{\xi} = \frac{\vec{x} + \vec{x}'}{2} \\ \vec{\lambda} = \vec{x} - \vec{x}' \end{cases} \Rightarrow \begin{cases} \vec{x} = \vec{\xi} + \frac{\vec{\lambda}}{2} \\ \vec{\lambda} = \vec{\xi} - \frac{\vec{\lambda}}{2} \end{cases} \quad (17)$$

the Jacobian of the transformation in eq. (17) is equal to 1 thus, we find

$$\int d^3x d^3x' \, \frac{e^{-i\left(\vec{p}' - \vec{p}\right)\cdot\vec{x}} e^{-i\left(\vec{k}' - \vec{k}\right)\cdot\vec{x}'}}{4\pi|\vec{x} - \vec{x}'|} = \int d^3\xi e^{-i\left(\vec{p}' - \vec{p} + \vec{k}' - \vec{k}\right)\cdot\vec{\xi}} \int d^3\lambda \, \frac{e^{-i\left(\vec{p}' - \vec{p} + \vec{k} - \vec{k}'\right)\cdot\frac{\vec{\lambda}}{2}}}{4\pi|\vec{\lambda}|}$$
$$= (2\pi)^3 \, \delta^3 \left(\vec{p}' + \vec{k}' - \vec{p} - \vec{k}\right) \int d^3\lambda \frac{e^{-i\vec{q}\cdot\vec{\lambda}}}{4\pi|\vec{\lambda}|} \quad (18)$$

where, $\vec{q} = \vec{p}' - \vec{p} = \vec{k}' - \vec{k}$.

Using, (the proof of eq. (19) is given in the appendix C)

$$\int \frac{d^3\lambda}{4\pi} \frac{e^{-i\vec{q}\cdot\vec{\lambda}}}{|\vec{\lambda}|} = \frac{1}{\vec{q}^2} \quad (19)$$

we can write,

$$\langle f| S^{(1)} |i\rangle = (2\pi)^4 \, \delta^4 \left(p' - p + k' - k\right) \frac{1}{N_p N_{p'} N_k N_{k'}} M^{(1)}_{if} \quad (20)$$

where we have defined the element $M_{if}^{(1)}$,

$$M_{if}^{(1)} = ie^2 \frac{1}{\vec{q}^2} \left(\bar{u}_{p'} \gamma^0 u_p\right) \left(\bar{v}_{k'} \gamma^0 v_k\right) \tag{21}$$

Now consider the second term $S^{(2)}$. There are two different contributions to the matrix element between the states $|i\rangle$ and $|f\rangle$. We call these two contributions $S_A^{(2)}$, $S_B^{(2)}$ (in appendix B we write more explicitly how to write $S_A^{(2)}$, $S_B^{(2)}$).

$$\langle f| S^{(2)} |i\rangle = \langle f| S_A^{(2)} |i\rangle \; + \; \langle f| S_B^{(2)} |i\rangle \tag{22}$$

$$\begin{aligned} \langle f| S_A^{(2)} |i\rangle = -\frac{e^2}{2} \int d^4x d^4x' T \left[A^\mu(x) A^\nu(x')\right] \frac{1}{N_p N_{p'} N_k N_{k'}} \\ \{-e^{i(p'-p)x} e^{i(k'-k)x'} \left(\bar{u}_{p'} \gamma_\mu u_p\right) \left(\bar{v}_k \gamma_\nu v_{k'}\right) \\ -e^{i(p'-p)x'} e^{i(k'-k)x} \left(\bar{u}_{p'} \gamma_\nu u_p\right) \left(\bar{v}_k \gamma_\mu v_{k'}\right)\} \end{aligned} \tag{23}$$

and,

$$\begin{aligned} \langle f| S_B^2 |i\rangle = \frac{e^2}{2} \int d^4x d^4x' T \left[A^\mu(x) A^\nu(x')\right] \frac{1}{N_p N_{p'} N_k N_{k'}} \\ \{e^{i(p'+k')x} e^{-i(p+k)x'} \left(\bar{u}_{p'} \gamma_\mu v_{k'}\right) \left(\bar{v}_k \gamma_\nu u_p\right) + \\ e^{i(p'+k')x'} e^{-i(p+k)x} \left(\bar{u}_{p'} \gamma_\nu v_{k'}\right) \left(\bar{v}_k \gamma_\mu u_p\right)\} \end{aligned} \tag{24}$$

The matrix element $\langle f| S_A^{(2)} |i\rangle$ describes electron-positron scattering, while $\langle f| S_B^{(2)} |i\rangle$ is associated with the process in which the initial state e^+e^- annihilates at x' and the final state e^+e^- pair is created in x.

We will discuss e^+e^- scattering only.

Using,

$$T\left[A(x)B(x')\right] \; = \; N\left[A(x)B(x')\right] \; + \; \langle 0| T\left[A(x)B(x')\right] |0\rangle \tag{25}$$

and defining,

$$iD_F^{\mu\nu}(x - x') = \langle 0| T\left[A(x)B(x')\right] |0\rangle \tag{26}$$

we can rewrite $\langle f| S_A^{(2)} |i\rangle$ as,

$$\begin{aligned} \langle f| S_A^{(2)} |i\rangle = e^2 \int d^4 d^4x' \frac{1}{N_p N_{p'} N_k N_{k'}} iD_F^{\mu\nu}(x - x') \\ e^{i(p'-p)x} e^{(k'-k)x'} \left(\bar{u}_{p'} \gamma_\mu u_p\right) \left(\bar{v}_k \gamma_\nu v_{k'}\right) \end{aligned} \tag{27}$$

integration over $\xi = (x + x')/2$ leads to

$$\langle f| S_A^{(2)} |i\rangle = (2\pi)^4 \delta(p' - p + k' - k) \frac{1}{N_p N_{p'} N_k N_{k'}} M_{if}^{2A} \tag{28}$$

with $q = p - p' = k' - k$ and M_{if}^{2A},

$$M_{if}^{2A} = ie^2 \int d^4\lambda \, D_F^{\mu\nu}(\lambda) \, e^{iq\lambda} \; (\bar{u}_{p'}\gamma_\mu u_p) \, (\bar{v}_k \gamma_\nu v_{k'}) \tag{29}$$

The photon propagator $D_F^{\mu\nu}(x-x')$ can be obtained using the field expansion for $A^\mu(x)$.

$$A^\mu(x) = \sum_{\vec{k},r=1}^{2} \frac{1}{(2V\omega_k)^{1/2}} \epsilon_r^\mu(\vec{k}) \left(a_r(\vec{k}) e^{-ikx} \; + \; a_r^\dagger(\vec{k}) e^{ikx)} \right) \tag{30}$$

leading to

$$\begin{aligned} iD_F^{\mu\nu}(x-x') &= \langle 0| \, A^\mu(x) A^\nu(x') \, |0\rangle \, \theta(t-t') \\ &+ \langle 0| \, A^\nu(x') A^\mu(x) \, |0\rangle \, \theta(t'-t) \qquad (31) \\ &= \theta(t-t') \sum_{\vec{k}} \frac{1}{2V\omega_k} e^{-ik(x-x')} \sum_{r=1}^{2} \epsilon_r^\mu(\vec{k}) \epsilon_r^\nu(\vec{k}) \\ &+ \theta(t'-t) \sum_{\vec{k}} \frac{1}{2V\omega_k} e^{-ik(x'-x)} \sum_{r=1}^{2} \epsilon_r^\mu(\vec{k}) \epsilon_r^\nu(\vec{k}) \qquad (32) \end{aligned}$$

The polarization vectors in the transverse gauge have no time component. They can be written as

$$\epsilon_r^\mu(\vec{k}) = (0, \hat{\epsilon}_r(\vec{k})), \qquad r = 1, 2 \tag{33}$$

wit the unit vector $\hat{\epsilon}_r(\vec{k})$ satisfying,

$$\hat{\epsilon}_r(\vec{k}) \cdot \hat{\epsilon}_{r'}(\vec{k}) = \delta_{r,r'} \tag{34}$$

$$\hat{\epsilon}_r(\vec{k}) \cdot \hat{k} = 0 \tag{35}$$

$$\sum_{r=1}^{2} \epsilon_r^i(\vec{k}) \epsilon_{r'}^j(\vec{k}) = \delta^{ij} - \hat{k}^i \hat{k}^j \tag{36}$$

Introducing time-like unit vector $\eta^\mu = (1, 0, 0, 0)$ (in the frame in which the electromagnetic field has been quantized) we can construct a set of four independent orthogonal vectors, $\epsilon_r^\mu(\vec{k})$, $\epsilon_3^\mu(\vec{k})$, η^μ

where,

$$\epsilon_r^\mu(\vec{k}) = (0, \hat{\epsilon}_r(\vec{k})), \qquad r = 1, 2 \tag{37}$$

$$\epsilon_3^\mu(\vec{k}) = \frac{k^\mu - (k\eta)\eta^\mu}{\left[(k\eta)^2 - k^2\right]^{1/2}} \tag{38}$$

$$\eta^\mu = (1, 0, 0, 0) \tag{39}$$

This set of vectors satisfy,

$$\eta^{\mu\nu} - \sum_{r=1}^{2} \epsilon_r^\mu(\vec{k}) \epsilon_r^\nu(\vec{k}) \; - \; \epsilon_3^\mu(\vec{k}) \epsilon_3^\nu(\vec{k}) = g^{\mu\nu} \tag{40}$$

To write the photon propagator in a more compact function we use

$$\theta(t-t')\sum_{\vec{k}}\frac{e^{-ik(x-x')}}{2V\omega_k} = \theta(t-t')\int\frac{d^3k}{(2\pi)^3}\frac{e^{-ik(x-x')}}{2\omega_k}$$
$$= \theta(t-t')\int\frac{d^3k}{(2\pi)^3}\frac{e^{-i\omega_k(t-t')}}{2\omega_k}e^{i\vec{k}\cdot(\vec{x}-\vec{x})} \tag{41}$$

whe $t > t'$ we have

$$\frac{e^{-i\omega_k(t-t')}}{2\omega_k} = i\int\frac{dk_0}{2\pi}\frac{e^{-ik_0(t-t')}}{k^2+i\epsilon} \tag{42}$$

with $\epsilon = 0^+$, while at $t' > t$,

$$\frac{e^{-i\omega_k(t'-t)}}{2\omega_k} = i\int\frac{dk_0}{2\pi}\frac{e^{-ik_0(t'-t)}}{k^2+i\epsilon} \tag{43}$$

Collecting all things together we obtain,

$$iD_F^{\mu\nu}(x-x') = i\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-x')}}{k^2+i\epsilon}\left[-g^{\mu\nu}-\epsilon_3^{\mu}(\vec{k})\epsilon_3^{\nu}(\vec{k})+\eta^{\mu}\eta^{\nu}\right]$$
$$= i\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-x')}}{k^2+i\epsilon}\left[-g^{\mu\nu}-\frac{k^{\mu}k^{\nu}}{\vec{k}^2}+(k^{\mu}\eta^{\nu}+k^{\nu}\eta^{\mu})\frac{k_0}{\vec{k}^2}-\eta^{\mu}\eta^{\nu}(\frac{k_0^2}{\vec{k}^2}-1)\right] \tag{44}$$

Subsituting the expression of $iD_F^{\mu\nu}(x-x')$ into eq. (29) we find (an integration over λ gives a $\delta^4(k-q)$) and $\frac{k_0^2}{\vec{k}^2}-1=\frac{q^2}{\vec{q}^2}$,

$$M_{if}^{2A} = \frac{ie^2}{q^2+i\epsilon}\left(-g^{\mu\nu}-\frac{q^{\mu}q^{\nu}}{\vec{q}^2}+(q^{\mu}\eta^{\nu}+q^{\nu}\eta^{\mu})\frac{q^0}{\vec{q}^2}-\eta^{\mu}\eta^{\nu}\frac{q^2}{\vec{q}^2}\right)(\bar{u}_{p'}\gamma_{\mu}u_p)(\bar{v}_{k'}\gamma_{\nu}v_k) \tag{45}$$

Note first that the last contribution in the $\epsilon \to 0$ limit is

$$-ie^2\frac{1}{\vec{q}^2}(\bar{u}_{p'}\gamma_0 u_p)(\bar{v}_k\gamma_0 v_{k'}) \tag{46}$$

and cancels exactly $M_{if}^{(1)}$ defined in eq. (21). Moreover, the contributions containing q^{μ} vanish since,

$$q^{\mu}\bar{u}_{p'}\gamma_{\mu}u_p = (p'-p)^{\mu}\bar{u}_{p'}\gamma_{\mu}u_p = \bar{u}_{p'}(\not{p}-\not{p}')u_p = (m-m)\bar{u}_{p'}u(p) = 0 \tag{47}$$

In conclusion we can write,

$$M_{if}^{2A} = ie^2 D_F^{\mu\nu}(p-p')(\bar{u}_{p'}\gamma_{\mu}u_p)(\bar{v}_{k'}\gamma_{\nu}v_k) \tag{48}$$

where the photon propagator $iD_F^{\mu\nu}(q)$ in the momentum space is,

$$iD_F^{\mu\nu}(q) = \frac{-ig^{\mu\nu}}{q^2+i\epsilon} \tag{49}$$

A Time-orderd product

Show that

$$T\left[A(x)B(x')\right] = N\left[A(x)B(x')\right] + \langle 0|\, T\left[A(x)B(x')\right] |0\rangle \tag{50}$$

Using the definition of normal product and time order product we obtain

$$T\left[A(x)B(x')\right] = \theta(t)A(x)B(x') \pm \theta(-t)B(x')A(x) \tag{51}$$

where the $+/-$ sign corresponds to boson/fermions and using

$$N(A\,B) = A\,B - \langle 0|\, A\,B\, |0\rangle \tag{52}$$

and $N(A\,B) = \pm N(B\,A)$ we find,

$$\begin{aligned} T[A\,B] &= \theta(t)\left[N(A\,B) + \langle 0|\, AB\,|0\rangle\right] \pm \theta(-t)\left[N(B\,A) + \langle 0|\, BA\,|0\rangle\right] && (53)\\ &= [\theta(t) + \theta(-t)]N(AB) \;+\; \langle 0|\, \theta(t)AB \;+\; \theta(-t)BA\,|0\rangle && (54)\\ &= N(AB) + \langle 0|\, T(AB)\,|0\rangle && (55)\end{aligned}$$

B Contribution to scattering and annihilation channel

We show which contributions correspond to e^+e^- scattering and e^+e^- annihilation.

$$S^{(2)} = -\frac{e^2}{2}\int d^4x d^4x' N\left[\bar{\psi}(x)\gamma^\mu A_\mu(x)\psi(x)\bar{\psi}(x')\gamma^\mu A_\mu(x')\psi(x')\right] \tag{56}$$

Taking,

$$\begin{aligned}\psi &\backsim \sum_p \left(c_p\, u\, e^{-ipx} \;+\; d_p^\dagger\, v\, e^{ipx}\right)\\ \bar{\psi} &\backsim \sum_{p'} \left(c_{p'}^\dagger\, \bar{u}\, e^{ip'x} \;+\; d_{p'}\, \bar{v}\, e^{-ipx}\right)\end{aligned}$$

The terms contribuiting to e^+e^- channel are

$$\sum_{p,p',k,k'} \left[c_{p'}^\dagger c_p\, \bar{u}_{p'}\gamma^\mu u_p\, e^{i(p'-p)x} \;-\; d_p^\dagger d_{p'}\bar{v}_{p'}\gamma^\mu v_p\, e^{-i(p'-p)x}\right]\cdot \left[c_{k'}^\dagger c_k\, \bar{u}_{k'}\gamma^\nu u_k\, e^{i(k'-k)x'} \;-\; d_k^\dagger d_{k'}\bar{v}_{k'}\gamma^\nu v_k\, e^{-i(k'-k)x'}\right] \tag{57}$$

In eq. (57) we can either take the combination $c_{p'}^\dagger c_p d_{k'}^\dagger d_k$ or $d_p^\dagger d_{p'} c_k^\dagger c_{k'}$. The terms contribuiting to e^+e^- annihilation are

$$\sum_{p,p',k,k'} \left[c^\dagger_{p'} d^\dagger_p \bar{u}_p \gamma^\mu v_p \, e^{i(p'+p)x} \, + \, d_{p'} d_p \bar{v}_{p'} \gamma^\mu u_p \, e^{-i(p+p')x}\right] \cdot$$

$$\left[c^\dagger_{k'} d^\dagger_k \bar{u}_k \gamma^\mu v_k \, e^{i(k'+k)x} \, + \, d_{k'} d_k \bar{v}_{k'} \gamma^\mu u_k \, e^{-i(k+k')x}\right] \quad (58)$$

In eq. (58) we can take either $c^\dagger_{p'} d^\dagger_p c^\dagger_{k'} d^\dagger_k$ or $d_{p'} d_p c^\dagger_{k'} d^\dagger_k$.

Note that scattering channel and annihilitation channel have different sign, since the scattering channel has an overall minus sign.

C *Yukawa* integral and Green function of ∇^2

Let us consider this *Yukawa* integral and Green function of ∇^2

$$\int d^3x \frac{e^{-\mu x}}{x} \, e^{i\vec{q}\cdot\vec{x}} \quad (59)$$

Let $q = |\vec{q}|$, $x = |\vec{x}|$

$$\int d^3x \frac{e^{-\mu x}}{x} \, e^{i\vec{q}\cdot\vec{x}} = 2\pi \int_0^\infty dx \, x^2 \int_0^\pi d\theta \sin(\theta) \frac{e^{-\mu x}}{x} \, e^{iqx\cos(\theta)}$$
$$= \frac{4\pi}{q} \int_0^\infty dx \, \frac{e^{-\mu x}}{x} \, \sin(qx) = \frac{4\pi}{q^2} \int_0^\infty dy \, e^{-\tilde{\mu} y} \, \sin(y) = \frac{4\pi}{q^2} \int_0^\infty dy \, I(\tilde{\mu})$$

where, $\tilde{\mu} = \frac{\mu}{q}$ and $y = qx$.

$$I(\tilde{\mu}) = \int_0^\infty dy \, e^{-\tilde{\mu} y} \, \sin(y) = -\frac{1}{\tilde{\mu}} \left(e^{-\tilde{\mu} y} \, \sin(y) \mid_0^\infty - \int_0^\infty dy \, e^{-\tilde{\mu} y \cos(y)} \right)$$
$$= -\frac{1}{\tilde{\mu}} \left(e^{-\tilde{\mu} y} \, \cos(y) + \int_0^\infty dy \, e^{-\tilde{\mu} y} \, \sin(y) \right) = -\frac{1}{\tilde{\mu}} \left[-1 \, + \, I(\tilde{\mu}) \right]$$

Which implies that,

$$I(\tilde{\mu}) = \frac{1}{1 + \tilde{\mu}^2} \quad (60)$$

and finally,

$$\int d^3x \frac{e^{-\mu x}}{x} \, e^{i\vec{q}\cdot\vec{x}} = \frac{4\pi}{q^2 + \mu^2} \quad (61)$$

taking the limit $\mu \to 0$, we find

$$\int d^3x \, \frac{e^{i\vec{q}\cdot\vec{x}}}{x} = \frac{4\pi}{q^2} \quad (62)$$

The latter integral can be also evaluated by the Green function of the Laplacian operator.

$$\nabla^2_x \frac{1}{4\pi |\vec{x} - \vec{x}'|} = -\delta^3(\vec{x} - \vec{x}') \quad (63)$$

Let $G(\vec{x}-\vec{x}')$ be the Green function of ∇^2.

$$G(\vec{x}-\vec{x}') = \int \frac{d^3q}{(2\pi)^3}\,\hat{G}(q)\,e^{i\vec{q}\cdot\vec{x}} \tag{64}$$

acting with ∇_x^2 on $G(\vec{x}-\vec{x}')$,

$$\nabla_x^2 G(\vec{x}-\vec{x}') = \int \frac{d^3q}{(2\pi)^3}\,\hat{G}(q)(-|\vec{q}|^2)\,e^{i\vec{q}\cdot\vec{x}} \tag{65}$$

Taking eq. (63) and eq.(65) together, we find

$$\frac{1}{4\pi|\vec{x}-\vec{x}'|} = \int \frac{d^3q}{(2\pi)^3}\,\frac{e^{i\vec{q}\cdot\vec{x}}}{|\vec{q}|^2} \tag{66}$$

which leads to,

$$\int d^3x \frac{e^{i\vec{q}\cdot\vec{x}}}{4\pi|\vec{x}-\vec{x}'|} = \int d^3x\,\frac{d^3p}{(2\pi)^3}\,\frac{e^{i(\vec{p}+\vec{q})\cdot\vec{x}}}{|\vec{p}|^2} = \frac{1}{|\vec{q}|^2} \tag{67}$$