Roma, Dec. 17, 2017

## Spin of the photon

Consider the third component of the angular-momentum

$$J^3 = \int d^3x \Big\{ \sum_{i=1}^{3} \dot{A}^i \left( x^1 \frac{\partial}{\partial x_2} - x^2 \frac{\partial}{\partial x_1} \right) A^i \qquad (1)$$
$$- \left( \dot{A}^1 A^2 - \dot{A}^2 A^1 \right) \Big\}$$

where

$$A^i(x) = \sum_k \frac{1}{\sqrt{2V\omega_k}} \sum_{r=1}^{2} \epsilon^i_{\vec{k}r} \left( a_{\vec{k}r}\, e^{-ikx} + a^\dagger_{\vec{k}r}\, e^{ikx} \right) \qquad kx \to \omega_k x_0 - \vec{k}x$$

and $\omega_k = |\vec{k}|$.

We want to obtain the commutator $[J^3, a^\dagger_{k\lambda}]$, which receives non vanishing contributions from the terms in the second line of Eq. (1), representing the photon spin.

As a first step, we rewrite the commutator in the form

$$[J^3, a^\dagger_{kr}] = \int d^3x \frac{1}{\sqrt{2V\omega_k}} \Big[ \epsilon^1_{kr}\, e^{-ikx} \overleftrightarrow{\partial_0} A^2(x) \qquad (2)$$
$$- \epsilon^2_{kr}\, e^{-ikx} \overleftrightarrow{\partial_0} A^1(x) \Big]$$

\# Proof of eq. (2)

$$[J^3, a^\dagger_{kr}] = -\int d^3x \, [\dot{A}^1 A^2 - \dot{A}^2 A^1 , a^\dagger_{kr}]$$

$$= -\int d^3x \Big\{ \dot{A}^1 [A^2, a^\dagger_{kr}] + [\dot{A}^1, a^\dagger_{kr}] A^2$$

$$- \text{same with } 1 \rightleftarrows 2 \Big\}$$

From

$$\int d^3x \, (\partial_0 A^1) \, [A^2, a^\dagger_{kr}]$$

$$= \int d^3x \, (\partial_0 A^1) \sum_{k'r'} \frac{1}{\sqrt{2V\omega_{k'}}} \, \epsilon^2_{k'r'} \, [a_{k'r'}, a^\dagger_{kr}] \, e^{-ikx}$$

and

$$\int d^3x \, [(\partial_0 A^1), a^\dagger_{kr}] \, A^2$$

$$= \int d^3x \sum_{k'r'} \frac{1}{\sqrt{2V\omega_{k'}}} \, \epsilon^1_{k'r'} \, [a_{k'r'}, a^\dagger_{kr}] \, (\partial_0 e^{-ikx}) \, A^2$$

it follows that

$$[J^3, a^\dagger_{kr}] = -\int d^3x \frac{1}{\sqrt{2V\omega_k}} \Big[ \epsilon^2_{kr} (\partial_0 A^1) e^{-ikx}$$

$$+ \epsilon^1_{kr} (\partial_0 e^{-ikx}) A^2 - \epsilon^1_{kr} (\partial_0 A^2) e^{-ikx}$$

$$- \epsilon^2_{kr} (\partial_0 e^{-ikx}) A^1 \Big] \text{, i.e. Eq. (2)} \quad \#$$

Consider now the first contribution to the right hand side of Eq. (2)

$$\int d^3x \, \frac{1}{\sqrt{2V\omega_k}} \, \epsilon^1_{kr} \left( e^{-ikx} \overleftrightarrow{\partial_0} A^2 \right)$$

$$= \int d^3x \, \frac{1}{\sqrt{2V\omega_k}} \, \epsilon^1_{kr} \Big\{ e^{-ikx} \sum_{k'r'} \frac{(-i\omega_{k'})}{\sqrt{2V\omega_{k'}}} \, \epsilon^2_{k'r'} \times \left[ a_{k'r'} e^{-ikx} - a^+_{k'r'} e^{ik'x} \right]$$

$$- (-i\omega_k) e^{-ikx} \sum_{k'r'} \frac{\epsilon^2_{k'r'}}{\sqrt{2V\omega_{k'}}} \left[ a_{k'r'} e^{-ik'x} + a^+_{k'r'} e^{ik'x} \right] \Big\}$$

$$= \frac{1}{2\omega_k} \, \epsilon^1_{kr} (-i\omega_k) \sum_{r'} \Big\{ \cancel{\epsilon^2_{-kr'} \, a_{-kr'} \, e^{-2i\omega_k x_0}}$$

$$- \epsilon^2_{kr'} \, a^+_{kr'} - \cancel{\epsilon^2_{-kr'} \, a_{-kr'} \, e^{-2i\omega_k x_0}}$$

$$- \epsilon^2_{kr'} \, a^+_{kr'} \Big\}$$

$$= i \, \epsilon^1_{kr} \sum_{r'} \epsilon^2_{kr'} \, a^+_{kr'}$$

Collecting together the two contributions we find

$$\left[ J^3, a^+_{k\lambda} \right] = i \Big[ \epsilon^1_{kr} \sum_{r'} \epsilon^2_{kr'} \, a^+_{kr'} - \epsilon^2_{kr} \times \sum_{r'} \epsilon^1_{kr'} \, a^+_{kr'}$$

Finally, using $\epsilon_{k1} = (1,0,0)$, $\epsilon_{k2} \equiv (0,1,0)$ we obtain

$$[J^3, a^\dagger_{kr}] = i\,\epsilon^1_{kr}\, a^\dagger_{k2} - i\,\epsilon^2_{kr}\, a^\dagger_{k1}$$

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