Chiral Symmetry in Nuclear Matter

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We use the chiral $\sigma$--$\omega$ model at zero temperature to study the role of chiral symmetry in nuclear matter. The central role played by the pion propagator in connection with the energy density of nuclear matter and the problem of chiral phase transitions is pointed out. Results obtained previously are reexamined from the standpoint of chiral symmetry. Consequences of baryon current conservation and the renormalization of the neutral vector boson field in connection with many-body problems are also treated in detail.

1. INTRODUCTION

Since the works of Lee and collaborators [1] on abnormal nuclear states there has been much interest in chiral models for nuclear matter and nuclei (see, for example, Refs. [2--9]). In particular, recent works [5--7] stressed that chiral models provide insights into the mechanism of saturation in nuclear matter. From the theoretical viewpoint, chiral symmetry provides important constraints on strong interaction processes not only in free space [10] but also, in a similar way, in the nuclear medium, and it is important to explore these constraints further and to study their consequences. Among chiral models, the linear $\sigma$ model [11, 12] is the simplest one, and therefore it has been widely used as a means to demonstrate chiral symmetry constraints both in free space [11--13] and in the nuclear medium [5, 6, 8, 28].

In this work, we will start from the $\sigma$ model lagrangian and add a neutral vector boson ($\omega$ meson) in order to account for the short range repulsion between nucleons. In such a model, which we will call the chiral $\sigma$--$\omega$ model, nucleons interact via exchange of scalar ($\sigma$), vector ($\omega$), and pseudoscalar ($\pi$) mesons. Besides its inclusion of pions, it differs from the more popular non-chiral $\sigma$--$\omega$ model [14, 15] (called QHD I in Ref. [16]) in nonlinear meson interaction terms (like $\sigma^3, \sigma^4$) which are required by chiral symmetry. The quantity we are mainly interested in is the energy density of nuclear matter. The aim of our investigation is threefold: First, we want to set up the general formalism consistent with symmetry constraints, which in principle enables one to calculate the energy density, including the effects of quantum fluctuations. We will use the results of Refs. [11, 12] to discuss the renormalization of the $\sigma$ model in nuclear matter. Second, we wish to
elucidate some general properties of the energy density which mainly follow from chiral invariance. In particular, we will generally demonstrate and explore the fact that the energy minimization condition is equivalent to the Goldstone theorem in nuclear matter. That is, if the energy density is minimized with respect to the nuclear matter expectation value of the scalar field, either the pion propagator has a pole at $q=0$ (Goldstone mode) or the nuclear matter expectation value of the scalar field vanishes (Wigner mode). The stability conditions for the two modes can also be formulated in terms of the pion propagator. These features will enable us to discuss the chiral phase transition to the abnormal state studied previously more transparently from the standpoint of chiral symmetry. Our third purpose is to study the form of the energy density in specific approximations. It is well known that the simple-minded Hartree approximation is, strictly speaking, not applicable in the present model due to the appearance of “tachyon poles” in the meson propagators. For this reason we attempt to discuss at least the formulation of the problem beyond the Hartree approximation. The numerical calculations based on an approximation to the Hartree–Fock scheme will be discussed in a different paper. In order to obtain numerical estimates in the framework of the Hartree approximation, it is common to add higher order loop terms to the Hartree result in a somewhat ad hoc manner. We will discuss results obtained by other authors and give some numerical estimates for the case of adding the pions to the scenario of Refs. We emphasize, however, that it is not the main purpose of this paper to present numerical results. Instead, it is to explain general and rather formal aspects of the chiral $\sigma-\omega$ model in nuclear matter and to illustrate these general considerations using simple examples. In this sense, the present paper should be considered to provide a basis for quantitative numerical works.

The paper is organized as follows: The general formalism is set up in Section 2, where special emphasis is placed on the renormalization procedure consistent with chiral symmetry and baryon current conservation. In Section 3 we discuss some general properties of the energy density, in particular its relation to the Goldstone theorem. There we also present some considerations on the Landau–Migdal force in the present model. Section 4 gives the form of the energy density when particular approximations are introduced. In Section 5 we use the Hartree approximation to illustrate some of our general results and to present some numerical estimates. A summary is given in Section 6.

## 2. Renormalization of the Chiral $\sigma-\omega$ Model in the Presence of Nuclear Matter

In this section we explain the formalism needed to discuss renormalizations in the presence of the nuclear medium. We emphasize the role of the hypothesis of the partially conserved axial vector current (PCAC) and of the conservation of the baryon current. As for the treatment in free space we follow Refs. [11, 12].
2.1. Lagrangian and Renormalization Constants

If we add the $\omega$ meson interacting with the nucleon to the chiral $\sigma$ model lagrangian [11], we have in terms of unrenormalized quantities (characterized by a subscript (0))

\[
\mathcal{L} = \bar{\Psi}_{(0)} \left[ i \nabla_{\mu} - g_{\omega} \phi_{(0)} + i \pi_{(0)} \cdot \tau \gamma_{5} - g_{\omega} \gamma_{\mu} V_{(0)}^{\mu} \right] \Psi_{(0)} + \frac{1}{2} \left[ (\partial_{\mu} \phi_{(0)})^{2} + (\partial_{\mu} \pi_{(0)})^{2} \right] - \frac{1}{4} G_{(0)\mu\nu} G_{(0)}^{\mu\nu} \\
- \frac{\mu_{d}^{2}}{2} \left( \phi_{(0)}^{2} + \pi_{(0)}^{2} \right) - \frac{\lambda^{2}}{4} \left( \phi_{(0)}^{2} + \pi_{(0)}^{2} \right)^{2} + \frac{m_{\omega}^{2}}{2} V_{(0)\mu} V_{(0)}^{\mu} + c_{0} \phi_{(0)}.
\] (2.1)

Here $\Psi, \phi, \pi,$ and $V_{\mu}$ are the fields of the nucleon, the $\sigma$ meson, the pion, and the $\omega$ meson, respectively, and $G_{\mu\nu} = \partial_{\mu} V_{\nu} - \partial_{\nu} V_{\mu}.$ The model contains the three coupling constants $g, g_{\omega},$ and $\lambda.$ The last term in (2.1) explicitly breaks the chiral symmetry, which leads to PCAC,

\[
\partial_{\mu} j_{A}^{\mu} = -c_{0} \pi_{(0)}
\] (2.2a)

with

\[
\vec{j}_{A}^{\mu} = \bar{\Psi}_{(0)} \gamma^{\mu} \gamma_{5} \frac{\tau}{2} \Psi_{(0)} - \partial_{\mu} \phi_{(0)} + \phi_{(0)} \partial_{\mu} \pi_{(0)}. \tag{2.2b}
\]

The invariance of the lagrangian under global phase transformations of the nucleon field leads to baryon current conservation

\[
\partial_{\mu} \vec{j}_{B}^{\mu} = 0
\] (2.3a)

with

\[
\vec{j}_{B}^{\mu} = \bar{\Psi}_{(0)} \gamma^{\mu} \Psi_{(0)}. \tag{2.3b}
\]

The assumption $\mu_{d}^{2} < 0$ leads to spontaneous symmetry breaking. The $\sigma$ and the $\omega$ meson fields have (constant) expectation values in nuclear matter, which we denote by $\bar{\sigma}$ and $\bar{\omega}_{\mu},$ respectively. (Here the tilde indicates that these quantities are density dependent. As the density goes to zero we have $\bar{\omega}_{\mu} \to 0$ (see Eq. (2.36) below) and $\bar{\sigma} \to \sigma.$) Accordingly we translate the fields as follows,

\[
\phi_{(0)} = \sigma_{(0)} + \bar{\sigma}
\] (2.4a)

\[
V_{(0)}^{\mu} = \omega_{(0)}^{\mu} + \bar{\omega}_{\mu}^{\mu}. \tag{2.4b}
\]
Furthermore, we introduce the renormalized fields, coupling constants, and mass parameters according to

\[
\Psi_{(0)} = \sqrt{Z_N} \Psi; \quad (\sigma_{(0)}, \bar{v}, \pi_{(0)}) = \sqrt{Z_M} (\sigma, \bar{v}, \pi); \quad (\omega_{(0)}, \tilde{\omega}) = \sqrt{Z_\omega} (\omega, \tilde{\omega})
\]

\[
\mu_0^2 = \frac{\mu^2 + \delta\mu^2}{Z_M}; \quad m_{\omega0}^2 = \frac{m_\omega^2 + \delta m_\omega^2}{Z_\omega}
\]

\[
\bar{g}_0 = \frac{Z_N}{Z_N \sqrt{Z_M}} \bar{g}, \quad \bar{g}_{\omega0} = \frac{Z_{\omega0}}{Z_N \sqrt{Z_\omega}} \bar{g}_\omega; \quad \bar{\lambda}_0^2 = \frac{Z_\lambda}{Z_M} \lambda^2, \quad c_0 = \frac{c}{\sqrt{Z_M}}.
\]

Here we introduced the wave function renormalization constants \(Z_N, Z_M,\) and \(Z_\omega\); the mass counterterms \(\delta\mu^2\) and \(\delta m_\omega^2\); and the vertex renormalization constants \(Z_\lambda, Z_{\omega0},\) and \(Z_\mu.\) Putting Eqs. (2.4) and (2.5) into (2.1) we obtain up to density independent \(c\)-number terms

\[
\mathcal{L} = \mathcal{L}_{MF} + \mathcal{L}_L + \mathcal{L}_{CT}
\]

with

\[
\mathcal{L}_{MF} = -\frac{1}{4} \lambda^2 Z_\lambda (\bar{v}^2 - v^2)^2 - \frac{m^2 + \delta m^2}{2} (\bar{v}^2 - v^2)
\]

\[
+ c (\bar{v} - v) + \frac{m_{\omega0}^2 + \delta m_{\omega0}^2}{2} \tilde{\omega} \tilde{\omega}
\]

\[
\mathcal{L}_L = \bar{\Psi} \left[ i Z_N \not{D} - \tilde{m}_N - \delta \tilde{m}_N - g Z_\lambda (\sigma + i \pi \cdot \gamma_5) 
\]

\[
- g_{\omega0} Z_{\omega0} \gamma_\mu (\omega^\mu + \tilde{\omega}) \right] \Psi
\]

\[
+ \frac{1}{2} \left[ Z_M (\partial_\mu \sigma)^2 - (\tilde{m}_\sigma^2 + \delta \tilde{m}_\sigma^2) \sigma^2 \right]
\]

\[
+ \frac{1}{2} \left[ Z_M (\partial_\mu \pi)^2 - (\tilde{m}_\pi^2 + \delta \tilde{m}_\pi^2) \pi^2 \right] - \frac{1}{4} Z_{\omega0} G_{\mu \nu} G^{\mu \nu}
\]

\[
+ \frac{m^2 + \delta m^2}{2} \omega_\mu \omega^\mu - \frac{1}{4} \lambda^2 Z_\lambda (\sigma^2 + \pi^2)^2 - \lambda^2 Z_\lambda \delta \sigma (\sigma^2 + \pi^2)
\]

\[
\mathcal{L}_{CT} = \sigma (c - \bar{v}(\tilde{m}_\sigma^2 + \delta \tilde{m}_\sigma^2)) + \omega_\mu \tilde{\omega}^\mu (m^2 + \delta m^2).
\]

In Eqs. (2.7) we defined the density dependent mass parameters and their counterterms as follows:
Equations (2.8), without tildes, define the masses \( m_N \), \( m_\sigma \), and \( m_\pi \) of the nucleon, the sigma meson, and the pion together with their counterterms for zero density. Following Ref. [12], we consider these masses as free parameters and define the coupling constants \( g \) and \( \lambda^2 \) by the relations

\[
m_N = g v
\]

\[
m_\sigma^2 - m_\pi^2 = 2 \lambda^2 v^2.
\]

Then Eqs. (2.8) for zero density determine the vertex renormalization constants \( Z_g \) and \( Z_\lambda \) in terms of the mass counterterms as

\[
Z_g = 1 + \frac{\delta m_N}{m_N}
\]

\[
Z_\lambda = 1 + \frac{\delta m_\pi^2 - \delta m_\sigma^2}{m_\pi^2 - m_\sigma^2}.
\]

Equation (2.8a) can be written in the form

\[
\tilde{m}_N + \delta \tilde{m}_N = m_N + \delta m_N + g Z_g (\tilde{v} - v).
\]

This relation, together with similar relations for the \( \sigma \) meson and the pion, can be used to derive separate relations for the mass parameters and the counterterms:

\[
\tilde{m}_N = g \tilde{v} = m_N \frac{\tilde{v}}{v}
\]

\[
\tilde{m}_\sigma^2 = m_\sigma^2 + 3 \lambda^2 (\tilde{v}^2 - v^2)
\]

\[
\tilde{m}_\pi^2 = m_\pi^2 + \lambda^2 (\tilde{v}^2 - v^2)
\]

and

\[
\delta \tilde{m}_N = g (Z_g - 1) \tilde{v} = \delta m_N \frac{\tilde{v}}{v}
\]

\[
\delta \tilde{m}_\sigma^2 = \delta m_\sigma^2 + 3 \lambda^2 (Z_\lambda - 1) (\tilde{v}^2 - v^2)
\]

\[
\delta \tilde{m}_\pi^2 = \delta m_\pi^2 + \lambda^2 (Z_\lambda - 1) (\tilde{v}^2 - v^2).
\]

Summarizing up to this point, if we impose renormalization conditions to determine the mass counterterms \( \delta m_N, \delta m_\sigma^2, \) and \( \delta m_\pi^2 \) in free space, the vertex renormalization constants and the density dependent mass parameters and counterterms are obtained from Eqs. (2.10) to (2.12).

In Eq. (2.6) we split the lagrangian into three parts: \( \mathcal{L}_{MF} \) is a constant depending
on the “mean fields” $\hat{\sigma}$ and $\hat{\pi}$. The lagrangian $\mathcal{L}_L$ consists of terms with at least two field operators. This part will generate the loop terms in the energy density. The last piece $\mathcal{L}_{\text{CT}}$ is linear in the fields $\sigma$ and $\omega^\mu$ and, as we will discuss below, generates counterterms in order to cancel contributions from loop terms to the nuclear matter expectation values of $\sigma$ and $\omega^\mu$.

Let us now discuss the renormalization conditions. For this we consider the self-energies $\Sigma_N, \Sigma_\sigma, \Sigma_\pi,$ and $\Sigma_\omega^{\mu\nu}$ of the nucleon, the $\sigma$ meson, the pion, and the $\omega$ meson in the nuclear medium as calculated from $\mathcal{L}_L$ of Eq. (2.7b). To exhibit the contributions due to the mass and wave function renormalization counterterms, we split them as

\begin{align}
\Sigma_N(k) &= \Sigma_N^f(k) + \delta m_N(k) - \bar{k}(Z_N - 1) \tag{2.13a} \\
\Sigma_\sigma(k) &= \Sigma_\sigma^f(k) + \delta m_\sigma^2(k^2) - k^2(Z_M - 1) \tag{2.13b} \\
\Sigma_\pi(k) &= \Sigma_\pi^f(k) + \delta m_\pi^2(k^2) - k^2(Z_M - 1) \tag{2.13c} \\
\Sigma_\omega^{\mu\nu}(k) &= \Sigma_\omega^{\mu\nu,f}(k) + g_\omega^{\mu\nu} \delta m_\omega^{\mu\nu} - (k^\mu k^\nu - k^2 g_\omega^{\mu\nu})(Z_\omega - 1), \tag{2.13d}
\end{align}

thereby introducing the “unrenormalized” self-energies $\Sigma$. In Eq. (2.13a) we introduced $\bar{k}^\mu = k^\mu - g_\omega Z_\omega / Z_N \bar{\omega}$. The renormalization conditions are imposed on the zero density values (denoted by a subscript $f$) of the self-energies as follows:

\begin{align}
\Sigma_N^f(k = \mu_N) &= 0, & \frac{\partial \Sigma_N^f}{\partial k}(k = \mu_N) &= 0 \tag{2.14a} \\
\Sigma_\sigma^f(k^2 = \mu_\sigma^2) &= 0 (\sigma = \pi), & \frac{\partial \Sigma_\sigma^f}{\partial k^2}(k^2 = \mu_\sigma^2) &= 0 \tag{2.14b} \\
\Sigma_\omega^{\mu\nu,f}(k^2 = \mu_\omega^2) &= 0, & \frac{\partial \Sigma_\omega^{\mu\nu,f}}{\partial k^2}(k^2 = \mu_\omega^2) &= 0. \tag{2.14c}
\end{align}

Here the renormalization points are denoted as $\mu_\sigma (\sigma = N, \sigma, \pi, \omega)$. In the case where the above expressions are complex (which happens, e.g., for $\mu_\sigma^2 > 2m_\sigma^2$), the real part must be taken. $\Sigma^{(t)}_{\omega}$ is the transverse part of the $\omega$ meson self-energy in free space:

\begin{align}
\Sigma^{(t)}_{\omega}(k) &= \left( - g_\omega^{\mu\nu} \frac{k^\mu k^\nu}{k^2} \right) \Sigma^{(t)}_{\omega}(k^2) + \frac{k^\mu k^\nu}{k^2} \Sigma^{(l)}_{\omega}.
\end{align}

The longitudinal part of the self-energy (2.15) is not an independent dynamical quantity: Baryon current conservation (2.3) requires that (see Ref. [19] or Appendix C of Ref. [20])

\begin{align}
k_\mu D^{\mu\nu}(k) &= \frac{k^\nu}{m_\omega^2 + \delta m_\omega^2} \tag{2.16a}
\end{align}
for the renormalized ω meson propagator. Due to the Dyson equation (2.20d) given below, Eq. (2.16a) is equivalent to

\[ k_\mu \Sigma_\omega^{\mu}(k) = -k^\nu \delta m_\omega^2, \quad k_\mu \Sigma_\omega^{\nu}(k) = 0, \quad (2.16b) \]

and hence \( \Sigma_\omega^{\mu}(k) = -\delta m_\omega^2 \), independent of \( k \). The quantity \( \delta m_\omega^2 \) is finite [19, 20], but the bare ω meson mass is infinite, as can be seen from the second equation in (2.5b). (Note that \( Z_\omega^{-1} \) diverges.)

The seven conditions (2.14) determine the mass counterterms and the wave function renormalization counterterms of Eqs. (2.13) in free space. Let \( M_\xi \) \(( \xi = \sigma, \omega, \pi, N) \) be the pole positions of the propagators in free space. Taking the pion as an example, we see that \( M_\pi^2 \) is related to the parameter \( m_\pi^2 \) in the lagrangian by

\[ m_\pi^2 = M_\pi^2 - \Sigma_\pi(M_\pi^2). \quad (2.17) \]

If we choose the renormalization points as \( \mu_\xi = m_\xi \), then the propagators in free space have poles at \( k^2 = m_\xi^2 = M_\xi^2 \) with residues equal to one (except for the σ meson, which can decay into two pions). In this case the \( m_\xi \) are the physical masses.

It is still necessary to determine the renormalization constant \( Z_{\omega NN} \) for the ωNN vertex. Here we follow Refs. [19, 20] and define the coupling constant \( g_\omega \) such that the ωNN source vertex \( \Gamma_\omega^{\mu}(q) \) in free space (f) satisfies

\[ \Gamma_\omega^{\mu}(q) \rightarrow -ig_\omega \gamma^\mu \quad \text{as} \quad q \rightarrow 0, \quad p \rightarrow m_N, \quad (2.18) \]

where \( q = p' - p \) is the momentum transferred by the ω meson to the nucleon. Then the Ward–Takahashi identity for baryon current conservation determines \( Z_{\omega NN} \) in terms of \( \delta m_\omega^2 \) and \( Z_N \):

\[ Z_{\omega NN} = Z_N \left( 1 + \frac{\delta m_\omega^2}{m_\omega^2} \right). \quad (2.19) \]

Equation (2.19) is derived in Appendix C of Ref. [20] and is equivalent to Eq. (3.11) in Ref. [19].

Now all renormalization constants are fixed, and in the next section we discuss the determination of the mean fields \( \bar{v} \) and \( \bar{w}^\mu \). Before that, however, we write the Dyson equations for the single particle Green functions for later use: In terms of the unrenormalized and renormalized self-energies, the single particle Green functions satisfy the Dyson equations

\[ S = \bar{S}_0^0 + \bar{S}_0 \Sigma_N S = S_0 + S_0 \Sigma_N S \quad (2.20a) \]

\[ \Delta_\pi = \bar{\Delta}_{\pi 0} + \bar{\Delta}_{\pi 0} \Sigma_\pi \Delta_\pi = \Delta_{\pi 0} + \Delta_{\pi 0} \Sigma_\pi \Delta_\pi \quad (2.20b) \]

\[ \Delta_\sigma = \bar{\Delta}_{\sigma 0} + \bar{\Delta}_{\sigma 0} \Sigma_\sigma \Delta_\sigma = \Delta_{\sigma 0} + \Delta_{\sigma 0} \Sigma_\sigma \Delta_\sigma \quad (2.20c) \]

\[ D^{\mu\nu} = \bar{D}_0^{\mu\nu} + \bar{D}_0^{\mu\nu} \bar{\Sigma}_\omega \Delta_\nu = D_0^{\mu\nu} + D_0^{\mu\nu} \Sigma_\omega \Delta_\nu. \quad (2.20d) \]
The lowest order unrenormalized propagators in these equations are

\[
\bar{S}_0(k) = \frac{1}{Z_N k - \delta m_N + i\delta_k}
\]

(2.21a)

\[
\bar{A}_{\pi 0} = \frac{1}{Z_M k^2 - \delta m^2 + i\delta^2}
\]

(2.21b)

\[
\bar{D}_0^{\mu\nu}(k) = \left( -g^{\mu\nu} + \frac{k^\mu k^\nu}{k^2} \right) \frac{1}{Z_\omega k^2 - m_\omega^2 - \delta m_\omega^2} + \frac{k^\mu k^\nu}{k^2} \frac{1}{m_\omega^2 + \delta m_\omega^2}.
\]

(2.21c)

In Eq. (2.21a) we introduced [16]

\[
\tilde{k}^\mu = k^\mu - g_\omega \frac{Z g_\omega}{Z_N} \tilde{w}^\mu = k^\mu - g_\omega \left( 1 + \frac{\delta m_\omega^2}{m_\omega^2} \right) \tilde{w}^\mu
\]

(2.22)

where \(n(k)\) is the Fermi distribution function. The lowest order renormalized propagators \(S_0, \ A_{\pi 0}, \ \text{and} \ D_0^{\mu\nu}\) are obtained from Eqs. (2.21) by setting the wave function renormalization constants equal to one and the mass counterterms equal to zero.

Since the unrenormalized \(\omega\) meson self-energy satisfies Eq. (2.16b), we see that the terms proportional to \(k^\mu k^\nu\) in \(D_0^{\mu\nu}\) do not contribute in Eq. (2.20d). In the actual calculation we can therefore replace

\[
\bar{D}_0^{\mu\nu} \to g^{\mu\nu} \bar{A}_{\omega 0} \equiv g^{\mu\nu} \left( -1 \right) \frac{Z_\omega k^2 - m_\omega^2 - \delta m_\omega^2}{Z_\omega k^2 - m_\omega^2 - \delta m_\omega^2}
\]

(2.23)

as long as we consider products with \(\Sigma_\omega\). Equation (2.20d) then becomes

\[
D^{\mu\nu} = \Delta^{\mu\nu}_{\omega} = \frac{k^\mu k^\nu}{m_\omega^2 + \delta m_\omega^2} Z_\omega \bar{A}_{\omega 0},
\]

(2.24)

where \(\Delta_{\omega}\) satisfies

\[
\Delta_{\omega}^{\mu\nu} = g^{\mu\nu} \bar{A}_{\omega 0} + \bar{A}_{\omega 0} \Sigma_\omega \Delta_{\omega}^{\mu\nu}.
\]

(2.20d')

The second term in (2.24), although necessary to make \(D^{\mu\nu}\) finite, will not contribute in our subsequent discussions, since eventually either it drops in the energy density after subtracting the vacuum expectation value, or the product with \(\Sigma_\omega\) is involved. In the nuclear medium there can occur mixing of the \(\sigma\) and \(\omega\) meson degrees of freedom; i.e., the self-energies \(\Sigma_\sigma\) or \(\Sigma_\omega\) contain intermediate states with one \(\omega\) or one \(\sigma\) meson, respectively. It is sometimes convenient to define a new self-energy [15] which is irreducible with respect to both \(\sigma\) and \(\omega\) meson lines, and to combine Eqs. (2.20c) and (2.20d') to

\[
\Delta^{ab} = \bar{A}_{0}^{ab} + \bar{A}_{0}^{bc} \Sigma_{cd} \Delta^{db}.
\]

(2.25)
Here, as in the following, latin indices take the values \( a = -1, 0, 1, 2, 3 \), while Greek indices run from 0 to 3. We define

\[
\begin{align*}
\Delta^{-1}_{-1} &= \Delta_{\sigma} \\
\Delta^{\mu\nu} &= \Delta^{\mu\nu}_{\omega_0}
\end{align*}
\]  

(2.26a)  

(2.26b)

and due to Eqs. (2.21b) and (2.23)

\[
\tilde{\Delta}_{ab}^{\mu\nu} = \begin{pmatrix}
\Delta^{-1}_{-1} = \Delta_{\sigma_0} & 0 \\
0 & \Delta^{\mu\nu}_{\omega_0} = g^{\mu\nu} \Delta_{\omega_0}
\end{pmatrix}
\]  

(2.27)

In the following developments we will make use of the field theoretical definition of \( \Sigma_x \) for the field \( x = (\sigma, \pi, \omega, N) \) in terms of the renormalized single particle Green function \( \Delta_x \)

\[
\langle 0 | T(\tilde{S}_x(x) z_x(x')) | 0 \rangle = \int d^4y \Sigma_x(x - y) \Delta_x(y - x'),
\]

(2.28)

where \( z_x \) is the operator for the field \( x \) and \((-i\tilde{S}_x)\) is equal to the “unrenormalized source” (see Section 2.2).

2.2. Determination of the Mean Fields

The quantities \( \tilde{v} \) and \( \tilde{\omega}^{\mu} \) are the nuclear matter expectation values of the scalar and vector fields \( \phi \) and \( V^\mu \), and therefore they are determined by the conditions (see Eqs. (2.4))

\[
\begin{align*}
\langle 0 | \sigma | 0 \rangle &= 0 \\
\langle 0 | \omega^{\mu} | 0 \rangle &= 0.
\end{align*}
\]  

(2.29a)  

(2.29b)

The equations of motion for \( \sigma \) and \( \omega \) derived from the lagrangian (2.6) are

\[
(Z_M \Box + m^2_\sigma + \delta m^2_\sigma) = j_\sigma = J_\sigma - i\tilde{S}_\sigma
\]

(2.30)

with

\[
J_\sigma = c - \tilde{v}(\tilde{m}^2_\sigma + \delta \tilde{m}^2_\sigma)
\]

(2.31a)

and

\[
Z_m \Box + m^2_\omega + \delta m^2_\omega) \omega^{\mu} = j_\omega^{\mu} = -J_\omega^{\mu} + i\tilde{S}_\omega^{\mu}
\]

(2.31b)

(2.32)

\footnote{The inverse of a matrix \( M \) is defined by \((M^{-1})^{ab} (M)_{bc} = g^{ac} \), with \( g_{ac} = (1 0) \). Indices \(-1\) may be written above or below.}
with
\[ J_{\omega}^\mu = \tilde{\nu}^\mu (m_\omega^2 + \delta m_\omega^2) \]  \hspace{1cm} (2.33a)
\[ i\tilde{S}_{\omega}^\mu = g_\omega Z_{\omega} \bar{\psi}_\omega \gamma^\mu \psi. \]  \hspace{1cm} (2.33b)

In Eqs. (2.30) and (2.32) we split the "unrenormalized sources" \( j_\sigma \) and \( j_\omega^\mu \) (which do not include the mass and wave function renormalization counterterms) into constant terms (from \( \mathcal{L}_{CT} \) of Eq. (2.7c)) and terms involving the quantized fields. The conditions (2.29) imply
\[ j_\sigma = J_\sigma - iS_\sigma = 0 \]  \hspace{1cm} (2.34a)
\[ -j_\omega^\mu = J_\omega^\mu - iS_\omega^\mu = 0, \]  \hspace{1cm} (2.34b)

where the quantities \( j_\sigma \) and \( S_\sigma \) without hats in these expressions are defined as the nuclear matter expectation values of the corresponding operators. Since the quantities \( S_\sigma \) and \( S_\omega \) themselves depend on the mean fields, Eqs. (2.34) can be considered as the eigenvalue equations [11] to determine \( \tilde{\nu} \) and \( \tilde{\nu}^\mu \). In terms of Feynman diagrams, \( S_\sigma \) and \( S_\omega^\mu \) are given by the sum of all loop diagrams with one external line (\( \sigma \) or \( \omega \), respectively; see Fig. 1 for \( S_\sigma \)). From (2.34) we see that the role of \( \mathcal{L}_{CT} \), Eq. (2.7c), is just to provide the necessary counterterms in order to cancel these loop terms. Let us first consider the simpler case of Eq. (2.34b). Using Eqs. (2.33) and (2.3b), we can write this equation as
\[ -j_\omega^\mu = \tilde{\nu}^\mu (m_\omega^2 + \delta m_\omega^2) - \frac{g_\omega Z_{\omega}^N}{Z_{\omega}} j_B^\mu = 0, \]  \hspace{1cm} (2.35)

where \( j_B^\mu = (\rho^\mu, j_B^\nu) \) is the baryon current in the nuclear matter ground state. Baryon current conservation implies Eq. (2.19), and therefore
\[ \tilde{\nu}^\mu = \frac{g_\omega}{m_\omega^2} j_B^\mu. \]  \hspace{1cm} (2.36)

Thus, the mean field produced by the \( \omega \) meson is completely determined by the baryon current, as expected. Now consider Eq. (2.34a) and compare it with the low energy theorem obtained from the axial Ward-Takahashi relation [11, 21] for the pionic decay vertex \( D^\mu(q) \) in the medium:
\[ q_{\mu} D^\mu(q) = i(c + \tilde{\nu}) A_\sigma^{-1}(q). \]  \hspace{1cm} (2.37)

---

**Fig. 1.** Graphical representation of Eq. (2.41). The dashed-dotted line denotes the \( \sigma \) meson, the dashed line the pion, and the full line the nucleon. The hatched circles represent the \( T \) matrices. The expressions obtained from the Feynman rules must be multiplied by the numbers shown in parentheses.
Here $\Delta_\pi^{-1}(q) = q^2 - m_\pi^2 - \Sigma_\pi(q)$ is the inverse renormalized pion propagator in the nuclear medium. Setting $q = 0$ in the above relation we obtain the low energy theorem

$$\tilde{v} \Delta_\pi^{-1}(0) = \tilde{v}(m_\pi^2 + \Sigma_\pi(0)) = c.$$  \hspace{1cm} (2.38)

Equation (2.38) expresses the Goldstone theorem in the nuclear medium [11, 21]: For exact chiral symmetry ($c = 0$) we have either $\Delta_\pi^{-1}(0) = 0$ (Goldstone mode; the pion is a Goldstone boson), or $\tilde{v} = 0$ (Wigner mode; the nucleon effective mass is zero). Instead of Eq. (2.34a) we could equally well use the Goldstone theorem (2.38) to determine the physical value $\tilde{v}_0$. (Note that $\Sigma_\pi$ is a function of $\tilde{v}$). In fact, we will show that the relation

$$j_\sigma = \tilde{v} \Delta_\pi^{-1}(0) + c$$  \hspace{1cm} (2.39)

holds for any value of $\tilde{v}$. If $\tilde{v} = \tilde{v}_0$, both sides of Eq. (2.39) vanish identically. Using the methods of Ref. [11], we derive Eq. (2.39) in Appendix A. In the following, however, we demonstrate Eq. (2.39) in a more explicit way, which will be more convenient for later use. Namely, from Eqs. (2.30) and (2.31a) we see that (2.39) implies the following relation between the source function $S_\sigma$ and the pion self-energy:

$$i \frac{S_\sigma}{\tilde{v}} = \Sigma_\pi(0).$$ \hspace{1cm} (2.40)

To demonstrate this relation explicitly, we take the expectation value of (2.31b) between the nuclear matter ground state to obtain

$$S_\sigma = -gZ_\pi \int \frac{d^4 q}{(2\pi)^4} \text{Tr} S(q) + 3\lambda^2 Z_\pi \tilde{v} \int \frac{d^4 q}{(2\pi)^4} (\Delta_\sigma(q) + \Delta_\pi(q))$$

$$- \lambda^2 Z_\pi \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} (\Delta_\sigma(q_1) \Delta_\sigma(q_2) \Delta_\sigma(q_1 + q_2) T(q_1, q_2, -q_1 - q_1; q_1, q_2, q_2)$$

$$+ 3\Delta_\pi(q_1) \Delta_\pi(q_2) \Delta_\pi(q_1 + q_2) T(-q_1 - q_2; q_1, q_2).$$ \hspace{1cm} (2.41)

This expression is shown graphically in Fig. 1. For the mesonic $T$ matrices we follow largely the conventions of Ref. [11]: $T(p_1, ..., p_n; q_1, q_2)$ is obtained from the connected part of the full $n$ sigma–$m$ pion Green function by amputating the external legs. The arguments of $T$ denote the incoming momenta, and the phase convention is such that $T$ equals the expression for the Feynman diagram calculated by using the Feynman rules given in Ref. [11]. (It is equal to $i$ times the $T$ matrix of Ref. [11].) For the three-meson vertices, which appear in Eq. (2.41), the $T$ matrix is equivalent to the $\sigma, \pi$ irreducible vertex. In Eq. (2.41) we used the fact that for the case $m = 2$ the isospin dependence is simply

$$T^{\mu\nu}(p_1, ..., p_n; q_1, q_2) - \delta^{\mu\nu} T(p_1, ..., p_n; q_1, q_2).$$  \hspace{1cm} (2.42)
On the other hand, the expression for the unrenormalized pion self-energy \( \Sigma^\mu_\pi = \delta^{\mu\nu} \Sigma^\nu_\pi \) is (see Fig. 2)

\[
\Sigma^\mu_\pi(k) = i g Z_\pi \int \frac{d^4q}{(2\pi)^4} \text{Tr}(\gamma_5 \sigma^\mu S(q) \Gamma^\mu_\pi(q, k + q) S(k + q))
\]

\[+ \delta^{\mu\nu} i \lambda^2 Z_\pi \int \frac{d^4q}{(2\pi)^4} (A_\sigma(q) + 5A_\pi(q))
\]

\[+ 2i\bar{\sigma} A_\pi(q) A_\pi(k + q) T(-q; k + q, -k))
\]

\[-i \lambda^2 Z_\pi \int \frac{d^4q_1}{(2\pi)^4} \int \frac{d^4q_2}{(2\pi)^4} \left( \delta^{\mu\nu} A_\sigma(q_1) A_\sigma(q_2) A_\pi(q_1 + q_2 - k)\right.
\]

\[\times T(q_1, q_2; -q_1 - q_2 + k, -k) + \frac{1}{3} A_\pi(q_1) A_\pi(q_2) A_\pi(q_1 + q_2 - k)
\]

\[\left. \times (\delta^{i\alpha} \delta^{j\beta} + \delta^{i\beta} \delta^{j\alpha} + \delta^{i\gamma} \delta^{j\lambda}) T^{\alpha\beta\gamma\lambda} (; q_1, q_2, -q_1 - q_2 + k, -k) \right).\]

For the NN-meson irreducible vertices \( \Gamma_\alpha (\alpha = \pi, \sigma, \omega) \) we follow the convention that \( \Gamma_\alpha(p', p) \) equals the expression for the amputated Feynman diagram with one outgoing (momentum \( p' \)) and one incoming (momentum \( p \)) nucleon line, and one external meson line. Equation (2.43) can be derived straightforwardly from the definition of the unrenormalized pion self-energy given in Eq. (2.28). The connection between (2.41) and (2.43) for \( k = 0 \) is provided by the low energy theorems derived from the axial Ward-Takahashi relations for the \( T \)-matrices, given by Eq. (6) in Section 5b of Ref. [11] for the purely mesonic case and reproduced in Appendix A, Eqs. (A.12). There we also demonstrate (see Eq. (A.13)) that by simply inserting these relations into (2.43) one is led directly to (2.40).

Before concluding this section, we add the following remarks. First, the symmetry breaking parameter \( c \) is expressed in terms of physical quantities by evaluating Eq. (2.37) in free space at the pole position of the pion propagator \( (q^2 = M_\pi^2) \) and using the definition \( D^\mu_\pi(q) = i q^\mu F_\pi(q^2) \) with \( F_\pi(M_\pi^2) = F_\pi \), the physical pion decay constant. One obtains

\[ c = F_\pi M_\pi^2. \]
Equation (2.38) for zero density determines the value of $v$. Using (2.44) and (2.17) we can write this equation as

$$v \left( 1 - \frac{\Sigma_{\pi l}(M^2_\pi)}{M^2_\pi} \right) = F_\pi. \quad (2.45)$$

Explicit one-loop calculations [12] of the pion self-energy, renormalizing at $\mu^2_\pi = m^2_\pi$, show that $\Sigma_{\pi l}(0) < M^2_\pi$, and therefore from Eq. (2.45), $v \approx F_\pi$. Note that for exact symmetry ($M^2_\pi = 0$) the relation $v = F_\pi$ holds exactly, since then the renormalization condition (2.14b) implies a vanishing derivative of $\Sigma_{\pi l}$ at $q^2 = 0$. The same conclusion holds with the renormalization prescription usually adopted in many-body calculations [1, 16], namely $\mu^2_\pi = 0$ and $m_\pi$ taken as the experimental pion mass. In this case, $v = F_\pi M^2_\pi/m^2_\pi \approx F_\pi$. Therefore, if we use the experimental values for the nucleon mass $m_N$ and the pion decay constant $F_\pi$, Eq. (2.9a) shows that the parameter $g$ has the value $g \approx 10$. The Goldberger–Treiman relation in the present framework is [12, 21]

$$M_N g_A = v g_{\pi NN}, \quad (2.46)$$

where $g_A$ and $g_{\pi NN}$ are defined via the irreducible NN axial vector and NN$\pi$ vertex, respectively, in the $q \to 0$ limit. If we renormalize the nucleon propagator at $\mu_N = m_N$, such that $m_N = M_N$, this relation can be written as

$$g = \frac{M_N}{v} = \frac{g_{\pi NN}}{g_A}. \quad (2.47)$$

It is possible to reproduce the experimental value $g_{\pi NN} \approx 13.5$ in the $\sigma$ model in the one-loop approximation using $g = 10$ as an input parameter [12]. On the other hand, in many works [4–6, 13], $g$ is considered as a free parameter and Eq. (2.9a) is used to determine the corresponding value of $v$. If then one uses Eq. (2.45) to assign a theoretical value to $F_\pi$, it is possible to carry out consistent calculations also in such a scheme.

Second, Eq. (2.38) implies the following behaviour of the quantity $(- A_{\pi}^{-1}(0))$, which can be used to define an “effective pion mass”: For $c \neq 0$, it is enhanced in the medium by a factor $v/\tilde{v} = m_N/m_\pi$. (We assume $m_N < m_\pi$.) For $c = 0$ (exact symmetry), it is zero as long as the system remains in the Goldstone mode but changes discontinuously to a non-zero (positive) value if at some density the system goes into the Wigner mode. (As we will discuss in the next section, the stability of the Wigner mode requires that $- A_{\pi}^{-1}(0) > 0$.) We also note that it is necessary to take into account the self-energies in the definition of “effective meson masses,” since the mass parameters defined in (2.11b) and (2.11c) are not positive definite (“Tachyon poles” in the Hartree approximation) [1, 3, 5, 17, 28]. We will come back to this point later.

Our third comment concerns the role of the counterterm lagrangian (2.7c) [11]. In actual calculations one can ignore it, treat $\tilde{v}$ as a free parameter, and leave out
all $\sigma$ tadpole diagrams in the calculation of Green functions. (Note that the role of $\mathcal{L}_{\text{CT}}$ is just to cancel these diagrams.) At the end of the calculation the physical value of $\bar{v}$ is determined from Eq. (2.34a) or (2.38), or by minimizing the energy density (see Section 3). Thus, unless the opposite is stated explicitly, all Green functions in this work are considered not to contain $\sigma$ tadpoles and to depend on $\bar{v}$ as a free parameter. Consider, for example, the quantity $S_\sigma$ itself: If we discard $\mathcal{L}_{\text{CT}}$, all propagator lines in the loops of Fig. 1 contain in principle also self-energy corrections due to $\sigma$ tadpoles. These $\sigma$-reducible contributions can be separated schematically as $S_\sigma = S_{\sigma,i} + B(S_{\sigma,i} + iJ_\sigma)$ with a $\sigma$-reducible block $B$, whose exact definition is not important for the argument. By definition, the quantity $S_{\sigma,i}$ is $\sigma$ irreducible and is thus given by the set of diagrams (Fig. 1) without $\sigma$ tadpoles on the internal propagator lines. From this it follows that the condition $J_\sigma - iS_\sigma = 0$ (Eq. (2.34a)) is equivalent to $J_\sigma - iS_{\sigma,i} = 0$. Thus, the eigenvalue equation formally does not change if we apply the prescription of omitting all $\sigma$ tadpoles, as discussed above. An analogous discussion holds for the treatment of $\omega$ tadpoles as well.

3. General Properties of the Energy Density

The hamiltonian density $\mathcal{H} = \sum_i p_i \dot{q}_i - \mathcal{L}$, where $q_i$ and $p_i$ are the fields and their conjugate momenta, can be split, analogously to Eq. (2.6), as $\mathcal{H} = \mathcal{H}_{\text{MF}} + \mathcal{H}_L + \mathcal{H}_{\text{CT}}$. We therefore write for the energy density $E = \langle 0 \mid \mathcal{H} \mid 0 \rangle$

$$E = E_{\text{MF}} + E_L$$

with the mean field dependent part

$$E_{\text{MF}} = \frac{1}{4} \lambda^2 Z_\lambda (\bar{v}^2 - v^2)^2 + \frac{m^2 + \delta m^2}{2} (\bar{v}^2 - v^2)$$

$$- c(\bar{v} - v) - \frac{m^2 + \delta m^2}{2} \tilde{\omega}_\mu \tilde{\omega}^\mu,$$

and the loop part $E_L$. It is convenient to separate the dependence of $E_{\text{MF}}$ on the renormalization points in the following way: Using Eqs. (2.10b), (2.9b) to express $Z_\lambda$ by the mass counterterms, and Eqs. (2.13) to replace the latter by the self-energies at zero momenta, we get

$$E_{\text{MF}} = U + \delta U$$

with

$$U = - \frac{\Delta_{\text{ct}}^{-1}(0) - \Delta_{\text{ct}}^{-1}(0)}{8v^2} (\bar{v}^2 - v^2) - \frac{\Delta_{\text{ct}}^{-1}(0)}{2} (\bar{v}^2 - v^2)$$

$$- c(\bar{v} - v) - \frac{m^2 + \delta m^2}{2} \tilde{\omega}_\mu \tilde{\omega}^\mu$$

$$\delta U = - \frac{\Sigma_{\text{ct}}(0) - \Sigma_{\text{ct}}(0)}{8v^2} (\bar{v}^2 - v^2)^2 - \frac{\Sigma_{\text{ct}}(0)}{2} (\bar{v}^2 - v^2).$$
The dependence on the renormalization points is contained in the finite "potential" $U$, which as a function of $\tilde{v}$ has the familiar double minimum shape. If we choose the renormalization points as $\mu^2 = 0 \ (\pi = \sigma, \pi, \omega)$, $U$ reduces to

$$U_0 = \frac{m^2 - m^2}{8\pi^2} (\tilde{v}^2 - v^2)^2 + \frac{m^2}{2} (\tilde{v}^2 - v^2) - c(\tilde{v} - v) - \frac{m^2}{2} \tilde{w}_\mu \tilde{w}^\mu. \quad (3.3b')$$

The divergencies, which eventually cancel against those in the loop part $E_L$, are contained in the term $\delta U$, which is independent of the choice of the renormalization points. Explicit forms for $E_L$ will be given in later sections. Here we wish to study some properties of $E = E(\tilde{v}, \tilde{w}^\mu, n_i)$ as a function of $\tilde{v}$, $\tilde{w}^\mu$, and $\{n_i\}$, where $n_i$ are the occupation numbers for nucleons.

3.1. Variations with Respect to the Mean Fields

The following relations hold:

$$\frac{\partial E}{\partial \tilde{v}} = -j_\sigma; \quad \frac{\partial E}{\partial \tilde{w}_\mu} = j^\mu_{\omega} \quad (3.4a)$$

$$\frac{\partial^2 E}{\partial \tilde{v}^2} = -A_{\omega}^{-1}(0); \quad \frac{\partial^2 E}{\partial \tilde{w}_\mu \partial \tilde{w}_\nu} = -(A_{\omega}^{-1}(0))^{\mu\nu}. \quad (3.4b)$$

Consider first the mean field term $E_{\text{MF}}$ of Eq. (3.2). Using Eqs. (2.8b) and (2.8c) we have

$$\frac{\partial E_{\text{MF}}}{\partial \tilde{v}} = -J_\sigma; \quad \frac{\partial E_{\text{MF}}}{\partial \tilde{w}_\mu} = -J^\mu_{\omega} \quad (3.5a)$$

$$\frac{\partial^2 E_{\text{MF}}}{\partial \tilde{v}^2} = \tilde{m}_\sigma^2 + \delta \tilde{m}_\sigma^2; \quad \frac{\partial^2 E_{\text{MF}}}{\partial \tilde{w}_\mu \partial \tilde{w}_\nu} = -g^{\mu\nu}(m^2 + \delta m^2), \quad (3.5b)$$

where $J_\sigma$ and $J^\mu_\omega$ are given by Eqs. (2.31a) and (2.33a). Thus, in order to prove Eqs. (3.4) we must show that for the loop terms

$$\frac{\partial E_L}{\partial \tilde{v}} = iS_\sigma; \quad \frac{\partial E_L}{\partial \tilde{w}_\mu} = iS^\mu_{\omega} \quad (3.6a)$$

$$\frac{\partial^2 E_L}{\partial \tilde{v}^2} = \Sigma_\sigma(0); \quad \frac{\partial^2 E_L}{\partial \tilde{w}_\mu \partial \tilde{w}_\nu} = \Sigma^{\mu\nu}_{\omega}(0). \quad (3.6b)$$

Thus, by differentiating the loop terms in the energy density we generate the amputated $n$-point meson Green functions with external momenta equal to zero.

---

2 In this work, the limit $q \to 0$ of any quantity $A(q)$ is defined as $A(0) = \lim_{q \to 0} \lim_{\omega \to 0} A(q)$. If the order of the limits is reversed, the corresponding quantities carry an index (st), i.e., $A^{\text{st}}(0) = \lim_{\omega \to 0} \lim_{q \to 0} A(q)$. 

(Here we treat the cases $n = 1, 2$.) Although this is intuitively clear and can be shown more elegantly using the concept of the effective action [22], we wish to indicate here a proof of relations (3.6) for completeness: One splits $\mathcal{L}_0$ of Eq. (2.7b) according to $\mathcal{L}_0 = \mathcal{L}_0 + \mathcal{L}_1$, where $\mathcal{L}_1$ contains all the terms with explicit dependence on $\tilde{v}$ and $\tilde{w}^\mu$. For example, one possible choice is to consider $\mathcal{L}_0$ as the free lagrangian depending on the free mass parameters. Use $\mathcal{L}_1$ to define an interaction picture (ip) and write for any operator product $\hat{P}(x_1, x_2, ...) = A(x_1) R(x_2) ...$

$$P(x_1, x_2, ...) \equiv \langle 0 | T \hat{P}(x_1, x_2, ...) | 0 \rangle = \frac{\langle 0 | T(\hat{P}^{ip}(x_1, x_2, ...) U) | 0 \rangle}{\langle 0 | U | 0 \rangle},$$

(3.7)

where $| 0 \rangle$ is is the ground state in the absence of $\mathcal{L}_1$ and

$$U = \exp \left\{ -i \int dt H_1^{ip}(t) \right\}.$$  

(3.8)

Using the following property of $U$,

$$\frac{\partial U}{\partial z} = -i \int dt T \left( \frac{\partial H_1^{ip}(t)}{\partial z} U \right) \quad (z = \tilde{v}, \tilde{w}^\mu),$$

and substituting $\frac{\partial H_1^{ip}}{\partial z} = -\int d^3 x \frac{\partial \mathcal{L}_1^{ip}}{\partial z}$, we obtain from Eq. (3.7)

$$\frac{\partial}{\partial z} P(x_1, x_2, ...) = \langle 0 | \frac{\partial}{\partial z} \hat{P}(x_1, x_2, ...) | 0 \rangle$$

$$+ i \int dx'_0 \int d^3 x' \left\{ \langle 0 | T \left( \hat{P}(x_1, x_2, ...) \frac{\partial \mathcal{L}_1(x')}{\partial z} \right) | 0 \rangle$$

$$- \langle 0 | \hat{P}(x_1, x_2, ...) | 0 \rangle \langle 0 | \frac{\partial \mathcal{L}_1(x')}{\partial z} | 0 \rangle \right\} \quad (z = \tilde{v}, \tilde{w}^\mu).$$

(3.9)

The order of integrations in (3.9) is important, since in a many-body system the operator $(\partial/\partial z) \mathcal{L}_1$ can give rise to particle–hole excitations at the Fermi surface which vanish when integrated over $x'$ first but survive when the order of integration is reversed [23]. The second term in $\{ \}$ of (3.9) subtracts the disconnected parts from the first term. For the case $P = \mathcal{H}^{\sigma}_L(x)$ the expression (3.9) is independent of $x$. Since $| 0 \rangle$ is an eigenstate of the hamiltonian and we can replace $\mathcal{H}^{\sigma}_L$ by the full hamiltonian density $\mathcal{H} + \mathcal{H}_{MF}$ in the last two terms, these cancel each other. Noting that $(\partial/\partial z) \mathcal{H}^{\sigma}_L = - (\partial/\partial z) \mathcal{L}_1$, one arrives at Eqs. (3.6a). To show the first equation of (3.6b), we take $\hat{P} = i \hat{S}_\sigma$ to obtain from (3.9) and (2.31b)

$$i \frac{\partial S_\sigma}{\partial \tilde{v}} = \lambda^2 Z Z \langle 0 \rangle 3 \sigma^+ + \pi^+ | 0 \rangle + i \int d^2 x' \langle 0 | T(\hat{S}_{\sigma}(x) \sigma(x')) | 0 \rangle_{\text{conn.}, \sigma \text{-irred}},$$

(3.10)

where the order of integrations is defined as in Eq (3.9). Remember that by $S_\sigma$ we
mean only the $\sigma$-irreducible pieces, although in the formal expression also the $\sigma$-reducible pieces enter (see the remarks at the end of Section 2). Since this irreducibility is not affected by the differentiation, we must take only the $\sigma$-irreducible pieces of the expression on the right-hand side of Eq. (3.10). If the equation of motion for $\sigma$ is used, the last term in (3.10) becomes

$$-i \int d^4x' \langle 0 | T(\mathcal{S}_\sigma(x)[Z_M \square_{\lambda} + \tilde{m}_\sigma^2 + \delta \tilde{m}_\sigma^2] \sigma(x')) | 0 \rangle$$

$$= -i^2 Z_\lambda \langle 0 | 3\sigma(x)^2 + \pi(x)^2 | 0 \rangle - \int d^4x'$$

$$\times \langle 0 | T(\mathcal{S}_\sigma(x) \mathcal{S}_\sigma(x')) | 0 \rangle. \quad (3.11)$$

In Eq. (3.11), first, we used the fact that according to our discussions at the end of Section 2 the source term of the quantized $\sigma$ field now does not include the term $J_\sigma$ of Eq. (2.31a) (since we discarded the counterterm $\mathcal{L}_{CT}$ and treat $\sigma$ as a free parameter); second, we used the canonical commutation relation between $\sigma$ and $\mathcal{T}$ to take the d'Alembert operator out of the $\mathcal{T}$ product. The first term on the right-hand side of (3.11) cancels the first one in (3.10), and for the second term we use the definition (2.28) of the unrenormalized self-energy. Taking the $\sigma$-irreducible part means that we should replace $A_\sigma \rightarrow \tilde{A}_{\sigma 0}$ in that expression, which leads to

$$i \frac{\partial S_\sigma}{\partial \tilde{v}} = \int d^4x' \mathcal{S}_\sigma(x-x') = \mathcal{S}_\sigma(q = 0). \quad (3.12)$$

The proof of the second equation in (3.6b) follows the same procedure and is not reproduced here.

The relations (3.4a) imply that the minimization conditions for the energy density with respect to the mean fields are equivalent to Eqs. (2.29).

### 3.2. Minimization of the Energy Density and the Goldstone Theorem

The considerations of Section 2 based on baryon current conservation and PCAC allow us to rewrite Eqs. (3.4a) as (see Eqs. (2.35), (2.19) and (2.39))

$$\frac{\partial E}{\partial \tilde{v}} = -\tilde{v} A_\sigma^{-1}(0) - c \quad (3.13)$$

$$\frac{\partial E}{\partial \tilde{w}_\mu} = (m^2_\omega + \delta m^2_\omega) \left( \tilde{w}^\mu + \frac{g_{\omega \mu}}{m^2_\omega} j^\mu \right). \quad (3.14)$$

From Eq. (3.13) it follows that the minimization condition of the energy density with respect to $\tilde{v}$ is equivalent to the Goldstone theorem (2.38). That is, for exact chiral symmetry ($c = 0$), the minimization of the energy density leads either to a
pole in the pion propagator at $q = 0$ (Goldstone mode) or to a vanishing expectation value $\tilde{\nu} = \langle 0 | \phi | 0 \rangle$ (Wigner mode):

$$\frac{\partial E}{\partial \tilde{\nu}} = -\tilde{\nu} \Delta_\pi^{-1}(0) = 0 \rightarrow \begin{cases} -\Delta_\pi^{-1}(0) = \tilde{m}_\pi^2 + \Sigma_\pi(0) = 0 & \text{(Goldstone mode)} \\ \tilde{\nu} = 0 & \text{(Wigner mode).} \end{cases}$$ (3.15a, 3.15b)

The stability criterion

$$\frac{\partial^2 E}{\partial \tilde{\nu}^2} = -\Delta_\sigma^{-1}(0) = \tilde{m}_\sigma^2 + \Sigma_\sigma(0) > 0$$ (3.16)

decides which of the two modes is actually realized. Due to the relation $-\Delta_\sigma^{-1}(0) = -\Delta_\pi^{-1}(0) - \tilde{\nu}(\partial / \partial \tilde{\nu}) \Delta_\pi^{-1}(0)$, which follows from Eqs. (3.15) and (3.16), or from the identities in Appendix A, the stability criterion can also be written in terms of the pion propagator as

$$\frac{\partial^2 E}{\partial \tilde{\nu}^2} = -\Delta_\sigma^{-1}(0) > 0$$

$$\begin{cases} \tilde{\nu} \frac{\partial}{\partial \tilde{\nu}} (-\Delta_\pi^{-1}(0)) = \tilde{\nu} \frac{\partial}{\partial \tilde{\nu}} (\tilde{m}_\pi^2 + \Sigma_\pi(0)) > 0 & \text{(Goldstone mode)} \\ -\Delta_\pi^{-1}(0) = \tilde{m}_\pi^2 + \Sigma_\pi(0) > 0 & \text{(Wigner mode).} \end{cases}$$ (3.17a, 3.17b)

These considerations are illustrated schematically in Figs. 3 and 4. Let us first consider the upper parts of these figures, which illustrate the self-consistency relation (3.15). The dashed line represents the quantity $-\tilde{m}_\pi^2$, which is equal to $m_\sigma^2/(1 - (\tilde{\nu}/v)^2)$ for exact symmetry (see Eq. (2.11c); we assume that the renormalization point is chosen such that $m_\sigma \to 0$ if $M_\sigma \to 0$, see Eq. (2.17)), and the full line represents $\Sigma_\pi(0)$. The situation is shown for three densities, increasing from left to right. The intersections of the dashed and the full line are solutions to Eq. (3.15a). Due to Eq. (3.17b), the Wigner mode solution ($\tilde{\nu} = 0$) is stable if the full line lies above the dashed one at $\tilde{\nu} = 0$, while the Goldstone mode solution ($\tilde{\nu} \neq 0$) is stable if for $\tilde{\nu} > 0$ ($\tilde{\nu} < 0$) the derivative of the full line is larger (smaller) than that of the dashed line at the intersection. (By “stable” we mean stability with respect to variations in $\tilde{\nu}$ at fixed density.) In the lower parts of the figures we show the corresponding curves for the binding energy per nucleon ($E_b/A - E/\rho - m_N$) as a function of $\tilde{\nu}$.

Figure 3 illustrates a continuous (second order) chiral phase transition: The physical value of $\tilde{\nu}$, which is a solution to $\Delta_\pi^{-1}(0) = 0$, decreases continuously to zero. At the critical density $\rho_c = \rho_2$, the value of the energy density at $\tilde{\nu} = 0$ turns from a maximum into a minimum. For densities higher than $\rho_c$, only the Wigner mode solutions exist.

Figure 4 represents an example of a discontinuous (first order) chiral phase transition: At the first density $\rho_1$, both the Goldstone and the Wigner mode
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Fig. 3. Schematic illustration of a continuous chiral phase transition. Exact chiral symmetry is assumed. The upper three diagrams show the quantity \( \frac{m_n^2 - m_p^2}{2} \) (dashed line) and the pion self-energy at \( q^2 = 0 \) (full line), such that \( \delta \rho_x (q^2) = \Sigma (0) - m_n^2 \). The lower three diagrams show the corresponding curves for the binding energy per nucleon. The plots refer to three different densities \( \rho_1, \rho_2, \rho_3 \) in ascending order. The horizontal axis in each diagram shows the ratio of the nuclear matter expectation value of the scalar field to its free value. The three diagrams in the same line have the same units on the vertical axis.
FIG. 4. Example of a discontinuous chiral phase transition. Exact chiral symmetry is assumed. The figure is organized in the same way as Fig. 3. The case shown here is actually the "quasiclassical approximation" discussed in Section 5.
solutions are stable and separated by an energy barrier. At the second density \( \rho_c' = \rho_2' \), the local minima and local maxima of the energy density for both \( \tilde{\nu} > 0 \) and \( \tilde{\nu} < 0 \) have coalesced. At this density the two equations

\[
\left. \begin{align*}
A_\pi^{-1}(0) &= A_\sigma^{-1}(0) = 0 \\
(\tilde{\nu} \neq 0)
\end{align*} \right) \quad (3.18)
\]

are satisfied simultaneously. These two equations determine the density \( \rho_c' \) and the corresponding value of \( \tilde{\nu} \). As we approach this density from below, the quantity \( (-A_\sigma(0)) \) and hence the attraction due to the \( \sigma \) meson exchange in the Landau–Migdal force (see Eq. (3.24) below) goes to infinity. For higher densities there is no longer a Goldstone mode solution. We use the notation \( \rho_c' \) to distinguish this "critical density" from \( \rho_c \), above which the energy density of the Wigner mode is an absolute minimum (\( \rho_c < \rho_c' \) in Fig. 4). The case shown in Fig. 4 is actually the "quasiclassical approximation" discussed in Section 5.

If the chiral symmetry is explicitly broken (c \( \neq 0 \)), a term \( (c/\tilde{\nu} - m_\pi^2) \) should be added to the dashed lines in Figs. 3 and 4 according to Eq. (3.13). In the example shown in Fig. 4 there then still exists a discontinuous transition to a very small value of \( \tilde{\nu} \). Equation (3.17a) is formally unchanged if \( c \neq 0 \). We remark that in the schematic plots of Figs. 3 and 4 we assumed that for any baryon density the energy density is real for all values of \( \tilde{\nu} \). In general this is not the case, since, for example, at low densities \( E \) may become complex at small \( \tilde{\nu} \) (see Ref. [3] or the examples in Section 5).

3.3. The Landau–Migdal Interaction

Let us now study some aspects of the dependence of \( E(\tilde{\nu}, \tilde{\nu}^\mu, n_i) \) on the occupation numbers \( \{n_i\} \). For this, let us assume that the conditions

\[
\frac{\partial E}{\partial \tilde{\nu}} = \frac{\partial E}{\partial \tilde{\nu}^\mu} = 0
\]

are already satisfied, and that accordingly \( \tilde{\nu} = \tilde{\nu}(n_i) \) and \( \tilde{\nu}^\mu = \tilde{\nu}(n_i) \). In the Landau–Migdal theory [24] the first and second variations of \( E \) with respect to the \( n_i \) determine the quasiparticle energy \( \varepsilon_i \) and the quasiparticle interaction \( f_{ij} \) at the Fermi surface:

\[
\varepsilon_i = \frac{dE}{dn_i} = \frac{\partial E}{\partial n_i} + \frac{\partial E}{\partial \tilde{\nu}} \frac{\partial \tilde{\nu}}{\partial n_i} + \frac{\partial E}{\partial \tilde{\nu}^\mu} \frac{\partial \tilde{\nu}^\mu}{\partial n_i} = \frac{\partial E}{\partial n_i}
\]

(3.20a)

\[
f_{ij} = \frac{d^2E}{dn_j dn_i} = \frac{d\varepsilon_i}{dn_j} = \frac{\partial \varepsilon_i}{\partial n_j} + \frac{\partial \varepsilon_i}{\partial \tilde{\nu}} \frac{\partial \tilde{\nu}}{\partial n_j} + \frac{\partial \varepsilon_i}{\partial \tilde{\nu}^\mu} \frac{\partial \tilde{\nu}^\mu}{\partial n_j}.
\]

(3.20b)

It does not seem possible to derive the general forms of \( \varepsilon_i \) and \( f_{ij} \) using only these prescriptions and the underlying field theoretical model. However, we can specify those terms in \( f_{ij} \) which arise from the density dependence of the mean fields, i.e., the last two terms in Eq. (3.20b), thereby generalizing some of the results in Ref. [25]. (Note that in the Hartree approximation used in Ref. [25] the first term on
the right-hand side of Eq. (3.20b) is zero; see Section 5.) To determine the quantities \( \partial E / \partial n_j \) and \( \partial \tilde{w}^\mu / \partial n_j \) we use the following conditions, which follow from (3.19),

\[
\frac{d}{dn_j} \frac{\partial E}{\partial \tilde{b}} + \frac{\partial^2 E}{\partial \tilde{b}^2} \frac{\partial \tilde{b}}{\partial n_j} + \frac{\partial^2 E}{\partial \tilde{w}^\mu \partial \tilde{b}} \frac{\partial \tilde{w}^\mu}{\partial n_j} = 0
\]

(3.21a)

\[
\frac{d}{dn_j} \frac{\partial E}{\partial \tilde{w}^\mu} = \frac{\partial^2 E}{\partial \tilde{w}^\nu \partial \tilde{w}^\mu} \frac{\partial \tilde{w}^\nu}{\partial n_j} + \frac{\partial^2 E}{\partial \tilde{b} \partial \tilde{w}^\mu} \frac{\partial \tilde{b}}{\partial n_j} = 0.
\]

(3.21b)

In the limit \( j_B \to 0 \) (the Landau–Migdal interaction is defined in this limit) the last terms in Eqs. (3.21) vanish: For \( \mu = i \) this is obvious since \( \partial^2 E / \partial \tilde{w}^i \partial \tilde{b} \) must be proportional to \( j_B \). For \( \mu = 0 \) we can use the form of \( \partial E / \partial \tilde{w}^0 \) as given by Eq. (3.14) with \( j_B = \sum_i n_i \). Noting that \( \tilde{n}^0 \) and \( \tilde{b} \) are independent variables, we get \( \partial^2 E / \partial \tilde{w}^0 \partial \tilde{b} = 0 \). Therefore, using Eqs. (3.4b) and (3.20a) we obtain from (3.21)

\[
\frac{\partial \tilde{b}}{\partial n_j} = A_\sigma(0) \frac{\partial \tilde{b}}{\partial \tilde{b}}
\]

(3.22a)

\[
\frac{\partial \tilde{w}^\mu}{\partial n_j} = A_\omega^\nu(0) \frac{\partial \tilde{w}^\mu}{\partial \tilde{w}^\nu}
\]

(3.22b)

In Appendix B we prove the relations

\[
\frac{\partial E}{\partial \tilde{b}} = i \tilde{\Gamma}(\mathbf{p}) \Gamma_\sigma(\mathbf{p}, \mathbf{p}) |_{p_0 = \epsilon, f(\mathbf{p}) \equiv i \gamma_\sigma(\mathbf{p})}
\]

(3.23a)

\[
\frac{\partial E}{\partial \tilde{w}^\mu} = i \tilde{\Gamma}(\mathbf{p}) \Gamma_\omega^\nu(\mathbf{p}, \mathbf{p}) |_{p_0 = \epsilon, f(\mathbf{p}) \equiv i \gamma_\omega^\nu(\mathbf{p})}
\]

(3.23b)

where the \( \Gamma_\alpha (\alpha = \sigma, \omega) \) are the irreducible \( \pi \)NN vertices and \( f(\mathbf{p}) \) is the correctly normalized quasiparticle spinor. Using Eqs. (3.22) and (3.23) in (3.20b) we obtain

\[
\tilde{f}_\gamma = \frac{\partial \tilde{b}}{\partial n_j} - \gamma_\sigma(\mathbf{p}_i) A_\sigma(0) \gamma_\sigma(\mathbf{p}_i) - \gamma_\omega^\nu(\mathbf{p}_i) A_\omega^\nu(0) \gamma_\omega(\mathbf{p}_i)
\]

(3.24)

with \( \gamma_\sigma \) and \( \gamma_\omega^\nu \) defined in Eqs. (3.23). We thus see that the \( \{n_i\} \) dependence of the mean fields gives rise to the direct interaction terms due to \( \sigma \) and \( \omega \) meson exchange, i.e., the last two terms in Eq. (3.24). Equations (3.21), (3.22), and (3.24) generalize Eqs. (8), (9), and (10) of Ref. [25], respectively. This correspondence will be made more explicit later when we discuss the Hartree approximation in Section 5.

Baryon current conservation can be applied to further specify the \( \omega \) meson exchange part, i.e., the last term in Eq. (3.24). The condition (2.16a) gives for the \( \omega \) meson propagator at \( k = 0 \) (see also Eq. (2.24))

\[
A_\omega^0(0) = \frac{1}{m_\omega^2 + \delta m_\omega^2}; \quad A_\omega^i(0) = 0; \quad A_\omega^{ii(\text{stat})}(0) = \frac{g_\omega^i}{m_\omega^2 + \delta m_\omega^2}.
\]

(3.25)
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(See the footnote in Section 3.1 concerning our definitions of the limit \( q \to 0 \). Note also that we set \( j_B = 0 \).) The last equation in (3.25) means that the \( \omega \) meson self-energy \( \Sigma_\omega^{\mu}(q) \) vanishes for \( q \to 0 \) if we first let \( q_0 \to 0 \) and then \( q \to 0 \): Setting \( \Sigma_\omega^{\mu}(0) = g^{\mu\nu}\Sigma_\omega(0) \) we have

\[
\Sigma_\omega^{(st)}(0) = 0, \tag{3.26}
\]

as follows also from Eq. (2.16b). In Eq. (3.24), however, we need the self-energy \( \Sigma_\omega(0) \), which is obtained by setting \( q \to 0 \) first. The difference between these two limits \([23]\) of the self-energy is due to particle-hole (ph) excitation processes at the Fermi surface, which contribute to \( \Sigma_\omega^{(st)}(0) \) but not to \( \Sigma_\omega(0) \). Denoting the contribution of these ph excitations by \( \Sigma_\omega^{(ph)} \), we have schematically \( \Sigma_\omega^{(st)}(0) = \Sigma_\omega(0) + \Sigma_\omega^{(ph)}(0) \), and due to Eq. (3.26), \( \Sigma_\omega(0) = -\Sigma_\omega^{(ph)}(0) \). Therefore the self-energy in the expression

\[
\Delta_\omega^{\mu}(0) = \frac{g^{\mu\nu}}{m_\omega^2 + \delta m_\omega^2 + \Sigma_\omega(0)} \tag{3.27}
\]

vanishes for zero density and, due to gauge invariance, can be related to ph excitation processes at the Fermi surface. (For the Hartree approximation, see Eq. (4.41d).) Next let us discuss the \( \omega \)NN vertices in Eq. (3.24). They are defined in Eq. (3.23b) as the matrix elements of the irreducible \( \omega \)NN vertex \( \gamma_\omega^{\mu} \). The relation to the \( \omega \)NN source vertex \( \Gamma_\omega^{\mu} \), on which we imposed the renormalization condition (2.18), is \([20]\) (using \( q = p' - p \))

\[
D^{\mu\nu}(q) \Gamma_\omega(p', p) = D^{\mu\nu}(q) \Gamma_\omega(p', p). \tag{3.28}
\]

The matrix element of the \( \omega \)NN source vertex

\[
\gamma_\omega^{\mu}(p) = \tilde{J}^{\mu}(p) \Gamma_\omega(p, p) \bigg|_{p_0 = c_p} f(p) \tag{3.29}
\]

at the Fermi surface has the same form as the isoscalar electromagnetic current (times a factor \(-2i\gamma_\omega\)), which was determined in Refs. \([23,26]\) from gauge invariance:

\[
\gamma_\omega^{\mu}(p) = -i\gamma_\omega \left( 1, \frac{p}{e_p} \right). \tag{3.30}
\]

Using Eq. (3.28) and the propagators as given in Eqs. (3.25) and (3.27), we obtain for the matrix elements of the irreducible \( \omega \)NN vertex

\[
\gamma_\omega^{\mu}(p) = -i\gamma_\omega \left( 1 + \frac{\delta m_\omega^2}{m_\omega^2} \right) \tag{3.31a}
\]

\[
\gamma_\omega(p) = -i\gamma_\omega \frac{p}{e_p} \left( 1 + \frac{\delta m_\omega^2 + \Sigma_\omega(0)}{m_\omega^2} \right). \tag{3.31b}
\]
Therefore we obtain finally for the $\omega$ meson exchange part $(f_{ij}^{(\omega)})$ of Eq. (3.24)

$$f_{ij}^{(\omega)} = -\gamma_{\omega\nu}(\mathbf{p}_i) A_{\omega\nu}(0) \gamma_{\omega\nu}(\mathbf{p}_j)$$

$$= \frac{g_{\omega}^2}{m_{\omega}^2} \left( 1 + \frac{\delta m_{\omega}}{m_{\omega}^2} \right) - \frac{g_{\omega}^2}{m_{\omega}^2} \mathbf{p}_i \cdot \mathbf{p}_j \left( 1 + \frac{\delta m_{\omega}^2 + \bar{\Sigma}_{\omega}(0)}{m_{\omega}^2} \right).$$

(3.32)

The $\sigma$ meson exchange part in (3.24) and the first term in (3.32) due to $\omega^0$ exchange contribute to the Landau–Migdal parameter $F_0$, while the second term in (3.32), which comes from $\omega$ exchange, contributes to $F_1$.

The first derivative of the binding energy per nucleon $E_B/A = E/\rho - m_N$ with respect to the density is given by

$$\frac{d(E_B/A)}{d\rho} = \frac{1}{\rho} \left( \frac{dE}{d\rho} - \frac{E}{\rho} \right) = \frac{1}{\rho} (E_F - E_B/A - m_N),$$

(3.33)

where $E_F$ is the Fermi energy, i.e., the value of $E_i$ at the Fermi surface. Hence at the saturation point the binding energy per nucleon is $E_B/A = E_F - m_N$. Due to our above considerations, the second derivative is given by

$$\frac{d^2(E_B/A)}{d\rho^2} = \frac{1}{\rho} \frac{dE_F}{d\rho} - \frac{2}{\rho} \frac{d(E_B/A)}{d\rho}$$

(3.34a)

with

$$\frac{dE_F}{d\rho} = \frac{1}{N_0} (1 + N_o f_0) = \frac{1}{N_0} (1 + F_0),$$

(3.34b)

$$\frac{1}{N_0} = \frac{\pi^2}{2 p_F^2} \frac{\partial E_F}{\partial p} \bigg|_{p = p_F}; \quad F_0 = N_o F_0,$$

(3.34c)

$$f_0 = \frac{\partial E_F}{\partial \rho} - \gamma_{\sigma}(p_F) A_{\sigma}(0) \gamma_{\sigma}(p_F) + \frac{g_{\omega}^2}{m_{\omega}^2} \left( 1 + \frac{\delta m_{\omega}^2}{m_{\omega}^2} \right).$$

(3.34d)

At the saturation point the second term on the right-hand side of (3.34a) is zero, and one obtains for the compressibility

$$K = 9 \rho^2 \frac{d^2(E_B/A)}{d\rho^2} = 3 p_F \frac{\partial E_F}{\partial p} \bigg|_{p = p_F} (1 + F_0).$$

(3.35)

4. FORM OF THE ENERGY DENSITY

In this section we derive the form of the loop part of the energy density (3.1). As usual [15, 16], we first express it generally in terms of single particle Green functions and then consider specific models for the latter. The pion self-energy has already been represented graphically by Fig. 2. In this section we will also make use
of the self-energies for the nucleon, the $\sigma$ meson, and the $\omega$ meson, which are shown in Fig. 5. It should be noted that some of the results derived in this section are essentially already known [1, 3–6]. However, as stated in the Introduction, we wish to present the general method of calculating the energy density beyond the Hartree approximation in accordance with the renormalization scheme discussed earlier, and to illustrate some of the more general considerations of Sections 2 and 3 in simple models. To achieve these aims in a self-contained manner, we rederive some results obtained previously.

4.1. General Form of the Energy Density

The loop part $E_L$ in Eq. (3.1) is the expectation value of

$$\mathcal{H}_L = \sum_i p_i \hat{q}_i - \mathcal{L}_L = \mathcal{H}_0 + \mathcal{H}_i,$$

where $q_i$ and $p_i$ are the fields and their conjugate momenta and $\mathcal{L}_L$ is given by Eq. (2.7b). The kinetic energy part $\mathcal{H}_0$ is understood to include the contributions due to the mass and wave function renormalization counterterms. In order to express the expectation value of the interaction part $\mathcal{H}_i$ in terms of single particle Green functions, we use the connection between the unrenormalized source $-i\tilde{S}_x$ and the unrenormalized self-energy $\Sigma_x$ for the field $x = \sigma, \pi, \omega, N$ as given in Eq. (2.28). In this way we obtain

$$E_L = \sum_x E_{0x} + E_i \quad (x = N, \sigma, \omega, \pi)$$

\[4.2\]

![Fig. 5. Graphical representation of the self-energies of the nucleon, the $\sigma$ meson, and the $\omega$ meson. The $\omega$ meson is denoted by a wavy line. For an explanation of the other symbols, see the caption to Fig. 1.](image)
with the "kinetic energies"

\[
E_{0N} = -i \int \frac{d^4k}{(2\pi)^4} \text{Tr} \, S \left( Z_N \gamma_0 k_0 - \not{S}_0^{-1} \right) \tag{4.3a}
\]

\[
E_{0\sigma} = i \int \frac{d^4k}{(2\pi)^4} \not{A}_{\sigma} \left( Z_M \not{k}^2 - \frac{1}{2} \not{A}_{\sigma0}^{-1} \right) \tag{4.3b}
\]

\[
E_{0\omega} = -i \int \frac{d^4k}{(2\pi)^4} \not{A}_{\omega\mu} \left( Z_\omega \not{k}^2 + \frac{1}{2} \not{A}_{\omega0}^{-1} \right) \tag{4.3c}
\]

\[
E_{0\pi} = 3E_{0\omega}(\sigma \rightarrow \pi), \tag{4.3d}
\]

and the "potential energy"

\[
E_1 = -i \int \frac{d^4k}{(2\pi)^4} \text{Tr} \left( \Sigma_N \right) + \frac{i}{3} \int \frac{d^4k}{(2\pi)^4} \left( \Sigma_{\sigma3} \not{A}_{\sigma} + 3\Sigma_{\pi3} \not{A}_{\pi} \right)
+ \frac{i}{4} \int \frac{d^4k}{(2\pi)^4} \left( \Sigma_{\sigma4} \not{A}_{\sigma} + 3\Sigma_{\pi4} \not{A}_{\pi} \right). \tag{4.4}
\]

Here we split the meson self-energies according to

\[
\Sigma_\pi = \Sigma_{\pi, \text{nuc}} + \Sigma_{\pi3} + \Sigma_{\pi4}, \tag{4.5}
\]

where the first and second terms refer to the first and second diagrams in Fig. 2, respectively, and \( \Sigma_{\pi4} \) stands for the remaining four diagrams. A similar notation is used for the \( \sigma \) meson; i.e., \( \Sigma_{\sigma, \text{nuc}} \) refers to the first diagram in Fig. 5b, \( \Sigma_{\sigma3} \) to the next two diagrams, and \( \Sigma_{\sigma4} \) to the rest. We note that due to the Dyson equation (2.20a) the term involving \( \not{S}_0^{-1} \) in (4.3a) could be cancelled against the first term in (4.4) [16].

The kinetic terms due to the mesons cancel parts of the potential energy, as can be seen as follows: Due to the Dyson equations (2.20b), (2.20c), and (2.20d'), we can replace

\[
\not{A}_{\alpha} \not{S}_0^{-1} \rightarrow \not{A}_{\alpha} \tag{4.6}
\]

in Eqs. (4.3b) to (4.3d). Therefore the terms involving \( \not{S}_0^{-1} \) in those equations give a contribution

\[
\frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \text{Tr} \left( \not{S}_N \right) - \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \left( \not{A}_{\sigma3} \not{A}_{\sigma} + 3\not{A}_{\pi3} \not{A}_{\pi} \right) - \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \left( \not{A}_{\sigma4} \not{A}_{\sigma} + 3\not{A}_{\pi4} \not{A}_{\pi} \right). \]
The following relation was used to derive Eq. (4.6),

\[ \int \frac{d^4k}{(2\pi)^4} \left( \Sigma_{\sigma,\text{nuc}} A_\sigma + 3\Sigma_{\pi,\text{nuc}} A_\pi + \text{Tr}(\Sigma_{\omega} A_\omega) \right) \]

\[ = -\int \frac{d^4k}{(2\pi)^4} \text{Tr}(\Sigma_N S), \quad (4.7) \]

which is verified by noting that the same types of vertex functions enter into the nucleon self-energy of Fig. 5a and the nucleon loop terms of the meson self-energies in Figs. 2, 5b, and 5c. Adding the contribution (4.6) to \( E_1 \) of Eq. (4.4) we see that the meson kinetic energies cancel half of the potential energy due to the nucleon loops. We obtain

\[ E_L = E_{0\text{N}} + i \int \frac{d^4k}{(2\pi)^4} k_0^2 \left[ Z_M (A_\sigma + 3A_\pi) - Z_\omega A_\omega \right] + E_1' \quad (4.8a) \]

with

\[ E_1' = \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \text{Tr}(\Sigma_N S) - \frac{i}{6} \int \frac{d^4k}{(2\pi)^4} (\Sigma_{\sigma3} A_\sigma + 3\Sigma_{\pi3} A_\pi) \]

\[ - \frac{i}{4} \int \frac{d^4k}{(2\pi)^4} (\Sigma_{\sigma4} A_\sigma + 3\Sigma_{\pi4} A_\pi). \quad (4.8b) \]

The terms involving \( k_0^2 \) in Eq. (4.8a), which are the remainders of the meson kinetic energies, are usually discarded in actual Hartree-Fock calculations done so far.

The method for expressing the energy density explained above does not directly make contact with the two-line irreducible graphical expansion [1, 3, 22, 27] of the energy density in terms of full propagators but bare vertices. In the latter method, the two-line reducible diagrams ("ring diagrams") are contained in a compact mathematical expression, which is sometimes more convenient in actual calculations. The connection to the expressions given above is provided by the relation

\[ \frac{\partial S}{\partial k_0} S^{-1} = -S \frac{\partial S^{-1}}{\partial k_0} = -S \left( Z_N \gamma_0 - \frac{\partial \Sigma_N}{\partial k_0} \right), \quad (4.9) \]

and analogous relations for the meson propagators. Using these equations, we can rewrite the kinetic energy terms (4.3a) to (4.3d) as

\[ E_{0\text{N}} = E_{N}^* - i \int \frac{d^4k}{(2\pi)^4} k_0 \text{Tr} \left( S \frac{\partial \Sigma_N}{\partial k_0} \right) \quad (4.10a) \]

\[ E_{0\sigma} + E_{0\omega} = E_{\sigma\omega}^* + \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} k_0 \text{Tr} \left( A_\sigma \frac{\partial \Sigma_N}{\partial k_0} \right) \quad (4.10b) \]

\[ E_{0\pi} = E_{\pi}^* + \frac{3i}{2} \int \frac{d^4k}{(2\pi)^4} k_0 A_\pi \frac{\partial \Sigma_\pi}{\partial k_0} \quad (4.10c) \]
with the "ring energies"

\[ E'_N = -i \int \frac{d^4k}{(2\pi)^4} \text{Tr}(\ln S - \tilde{S}_0^{-1} S) \]  
(4.11a)

\[ E'_{\sigma\omega} = \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \text{Tr}(\ln \Delta - \tilde{\Delta}_0^{-1} \Delta) \]  
(4.11b)

\[ E'_\pi = \frac{3i}{2} \int \frac{d^4k}{(2\pi)^4} (\ln \Delta_\pi - \tilde{\Delta}_0^{-1} \Delta_\pi) . \]  
(4.11c)

Here we combined the \( \sigma \) and \( \omega \) meson contributions by using the propagator (2.25). The total loop part then becomes

\[ E_L = E_t + E_i \]  
(4.12)

with

\[ E_t = \sum_{\alpha} E'_\alpha \quad (\alpha = N, \sigma\omega, \pi) \]  
(4.13)

and

\[ E_t = E_1 - i \int \frac{d^4k}{(2\pi)^4} k_0 \left\{ \text{Tr} \left( S \frac{\partial \tilde{\Sigma}_N}{\partial k_0} \right) - \frac{1}{2} \text{Tr} \left( \Delta \frac{\partial \tilde{\Sigma}_\pi}{\partial k_0} \right) - \frac{3}{2} \Delta_\pi \frac{\partial \tilde{\Sigma}_\pi}{\partial k_0} \right\} , \]  
(4.14)

where \( E_1 \) is given by Eq. (4.4). Although it is not obvious from the expression (4.14), \( E_t \) is equal to the sum of the two-line irreducible diagrams with two or more loops. We will demonstrate this explicitly in the next subsection for the Hartree–Fock approximation, which leads to the two-loop contribution to \( E_t \).

4.2. The Hartree–Fock (HF) Approximation

The HF approximation consists in taking only the two-loop part of \( E_t \) (Eq. (4.14)), or, equivalently, calculating the single particle propagators self-consistently in the one-loop approximation. The one-loop expressions for the unrenormalized self-energies are as follows (we use the same symbols as before for the HF self-energies and propagators):

\[ \Sigma_N(k) = -i \int \frac{d^4q}{(2\pi)^4} \gamma^a S(k + q) \gamma^b A^{ab}(q) \]  
\[ - 3i(gZ_g)^2 \int \frac{d^4q}{(2\pi)^4} \gamma_5 S(k + q) \gamma_5 \Delta_\pi(q); \]  
\[ \Sigma^{ab}(k) = \Sigma^{ab}_{\text{nuc}}(k) + \delta^{a-1} \delta^{b-1}(\Sigma_\sigma + \Sigma_\omega) \]  
(4.15)
with

\[ \Sigma_{\text{nuc}}^{ab}(k) = i \int \frac{d^4q}{(2\pi)^4} \text{Tr}(\Gamma^a S(k + q) \Gamma^b S(q)) \]  

(4.16b)

\[ \Sigma_{\sigma3}(k) = 6i(\lambda^2 Z_\sigma)^2 \delta^2 \int \frac{d^4q}{(2\pi)^4} (\Delta_\sigma(q) \Delta_\sigma(k + q) + 3 \Delta_\sigma(q) \Delta_\sigma(q)) \]  

(4.16c)

\[ \Sigma_{\sigma4}(k) = 3i\lambda^2 Z_\sigma \int \frac{d^4q}{(2\pi)^4} (\Delta_\sigma(q) + \Delta_\sigma(q)); \]  

(4.16d)

and

\[ \Sigma_{\pi}(k) = \Sigma_{\pi, \text{nuc}} + \Sigma_{\pi3} + \Sigma_{\pi4} \]  

(4.17a)

\[ \Sigma_{\pi, \text{nuc}}^{ab}(k) = i(gZ_\sigma)^2 \int \frac{d^4q}{(2\pi)^4} \text{Tr}(\gamma_5 S(k + q) \gamma_5 S(q)) \]  

(4.17b)

\[ \Sigma_{\pi3}(k) = 4i(\lambda^2 Z_\pi)^2 \delta^2 \int \frac{d^4q}{(2\pi)^4} \Delta_\pi(q) \Delta_\pi(k + q) \]  

(4.17c)

\[ \Sigma_{\pi4}(k) = i\lambda^2 Z_\pi \int \frac{d^4q}{(2\pi)^4} (\Delta_\pi(q) + 5 \Delta_\pi(q)). \]  

(4.17d)

In Eqs. (4.15) and (4.16b) we used the notation

\[ \Gamma^a = (-i g Z_\sigma, -i g_\omega Z_{\omega\nu} \gamma^\mu). \]  

(4.18)

Equations (4.15) to (4.17) are shown graphically in Fig. 6. Inserting the above forms of the self-energies into Eq. (4.14) we obtain (see Appendix C)

\[ -i \Sigma_{\pi} = \text{graphical representation} \]  

(a)

\[ -i \Sigma = \text{graphical representation} \]  

(b)

\[ -i \Sigma_{\omega} = \text{graphical representation} \]  

(c)

Fig. 6. Graphical representation of the HF self-energies (4.15) to (4.17). The wavy line with dots denotes the combined propagator for \( \sigma \) and \( \omega \). In the combined self-energy \( \Sigma \), mixing occurs only through the first diagram (the nucleon loop diagram). For the other symbols, see the caption to Fig. 1.
\[ E_i^{(HF)} = -\frac{i}{2} \int \frac{d^4 k}{(2\pi)^4} \text{Tr}(S \Sigma_N) + \frac{i}{6} \int \frac{d^4 k}{(2\pi)^4} (\Sigma_{\sigma^3} A_\sigma + 3 \Sigma_{\pi^3} A_\pi) + \frac{i}{4} \int \frac{d^4 k}{(2\pi)^4} (\Sigma_{\sigma^4} A_\sigma + 3 \Sigma_{\pi^4} A_\pi), \]  

(4.19)

where the first term can alternatively be written as

\[ -\frac{i}{2} \int \frac{d^4 k}{(2\pi)^4} \text{Tr}(S \Sigma_N) \]

\[ = \frac{i}{2} \int \frac{d^4 k}{(2\pi)^4} (\text{Tr}(A \Sigma_{\text{nuc}}) + 3 A_\pi \Sigma_{\pi,\text{nuc}}). \]  

(4.20)

\( E_i^{(HF)} \) is shown graphically in Fig. 7. The total HF energy density is then

\[ E^{(HF)} = E_{MF}^{(HF)} + E_L^{(HF)} = E_{MF}^{(HF)} + E_r^{(HF)} + E_i^{(HF)}. \]  

(4.21)

The three terms on the right-hand side of (4.21) are given by Eqs. (3.3), (4.13), (4.11), and (4.19), where now all single particle Green functions are solutions to Eqs. (4.15) to (4.17). Note that the ring energies (4.11) contain counterterm contributions through the unrenormalized propagators (see Eqs. (2.21)). As was noted in Ref. [15], those counterterm contributions in Eqs. (4.11b) and (4.11c) which are due to the nucleon loop self-energies can be combined with Eq. (4.20). The resulting expression then again has the form (4.20) with the unrenormalized meson self-energies replaced by the renormalized ones.

If we consider the HF energy density (4.21) as a functional of the free variables \( S, \Delta, \) and \( A_\pi, \) and require stability with respect to variations in them, we get back the

\[ \begin{align*}
\left( \begin{array}{c}
\begin{array}{c}
\Sigma
\end{array}
\end{array} \right)_{(1/2)} & \quad + \quad \left( \begin{array}{c}
\begin{array}{c}
\Sigma
\end{array}
\end{array} \right)_{(1/2)} \\
(a) & \\
\left( \begin{array}{c}
\begin{array}{c}
\Sigma
\end{array}
\end{array} \right)_{(1/4)} & + \quad \left( \begin{array}{c}
\begin{array}{c}
\Sigma
\end{array}
\end{array} \right)_{(1/4)} \\
(b) & \\
\left( \begin{array}{c}
\begin{array}{c}
\Sigma
\end{array}
\end{array} \right)_{(1/6)} & + \quad \left( \begin{array}{c}
\begin{array}{c}
\Sigma
\end{array}
\end{array} \right)_{(1/4)} & + \quad \left( \begin{array}{c}
\begin{array}{c}
\Sigma
\end{array}
\end{array} \right)_{(1/3)}
\end{align*} \]

Fig. 7. Graphical representation of the two-loop (HF) contribution to \( E_i, \) given by Eq. (4.19). The three terms in that equation correspond to (a), (b), and (c) in the figure.
Dyson equations (2.20a), (2.20b), and (2.25): From Eqs. (4.11) and 4.19) we have, using the form of the self-energies as given by Eqs. (4.15) to (4.17),

\[
\frac{\partial E^{(HF)}_y}{\partial S(k)} = -i(S^{-1} - \tilde{S}_y^{-1} + \Sigma_y) = 0
\]

(4.22a)

\[
\frac{\partial E^{(HF)}_y}{\partial A(k)} = \frac{i}{2}(A^{-1} - \tilde{A}_y^{-1} + \Sigma) = 0
\]

(4.22b)

\[
\frac{\partial E^{(HF)}_y}{\partial A_\pi(k)} = \frac{3i}{\Sigma}(A_\pi^{-1} - \tilde{A}_\pi^{-1} + \Sigma_\pi) = 0.
\]

(4.22c)

The self-energies in these equations arise from the variation of \( E^{(HF)}_y \) of Eq. (4.19), while the other terms come from the variation of the ring energies (4.11). Due to the relations (4.22) it is very simple to calculate the derivatives of the HF energy density with respect to the mean fields and thereby to derive Eqs. (3.6a) (or, as a consequence, Eqs. (3.4a)) in the present model: We need only take into account the mean field dependence of the various unrenormalized lowest order propagators in \( E^{(HF)}_y \) of Eq. (4.11) as well as the explicit \( \delta \) dependence due to \( \Sigma_{\sigma_3} \) and \( \Sigma_{\pi_3} \) in Eq. (4.19). Using the forms of the lowest order propagators given in Section 2 we have

\[
\frac{\partial \tilde{S}_0^{-1}}{\partial \delta} = -gZ \delta;
\]

\[
\frac{\partial \tilde{A}_0^{-1}}{\partial \delta} = -gZ \delta;
\]

\[
\frac{\partial (\tilde{A}_0^{-1})^{ab}}{\partial \delta} = -\delta^{a-1} \delta^{b-1} 6\delta^2 Z \delta;\]

\[
\frac{\partial \tilde{A}_\sigma^{-1}}{\partial \delta} = -2\delta^2 Z \delta.
\]

(4.23a)

(4.23b)

In deriving these relations we made use of Eqs. (2.8). We therefore obtain

\[
\frac{\partial E^{(HF)}_L}{\partial \delta} = -igZ \int \frac{d^4k}{(2\pi)^4} \text{Tr } S(k) + 3i\delta^2 Z \delta \int \frac{d^4k}{(2\pi)^4} (A_\sigma(k) + A_\pi(k))
\]

\[
-6(\lambda^2 Z \delta)^2 \delta \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} A_\sigma(k_1)
\]

\[
\times (A_\sigma(k_2) A_\sigma(k_1 + k_2) + A_\pi(k_2) A_\pi(k_1 + k_2))
\]

\[
- iS^{(HF)}_\sigma,
\]

(4.24)

which is the same as (2.41) with the full vertices replaced by the lowest order ones (\(-6i\lambda^2 Z \delta \delta \) for the \( \sigma^3 \) vertex and \(-2i\lambda^2 Z \delta \delta \) for the \( \sigma^\pi \) vertex). Variation with respect to \( \tilde{w}_\mu \) gives

\[
\frac{\partial E^{(HF)}_L}{\partial \tilde{w}_\mu} = -igZ \gamma_\mu \int \frac{d^4k}{(2\pi)^4} \text{Tr}(S(k) \gamma_\mu) = iS^{(HF)}_\mu,
\]

(4.25)
as is clear from the definition of $\tilde{S}_\sigma$, Eq. (2.33b). In Sections 2 and 3 we discussed the equivalence between the minimization condition for the energy density and the Goldstone theorem, which is expressed by Eq. (2.40). It is clear, however, that this relation does not hold if we simply replace $S_\sigma$ and $\Sigma_\sigma$ by their HF expressions given above, since $S_\sigma^{(HF)}$ contains two-loop terms (since it is derived from the two-loop approximation $E^{(HF)}$), while $\Sigma_\sigma$ of Eq. (4.17) does not. In general, if we calculate $E$ in the $l$-loop approximation, the pion self-energy satisfying (2.40) contains $l$ loops also and is different from the $(l - 1)$-loop expression used as an "input" to calculate $E$. Therefore, in the HF case, the relevant pion self-energy has the form (2.43), where the single particle Green functions are given by their HF expressions, while the vertex functions $\Gamma_\pi$ and $T$ are determined by the one-loop approximation and satisfy the low energy theorems (A.12) together with the HF propagators. As discussed earlier, this guarantees the validity of (2.40).

4.2. The Hartree Approximation

The HF approximation discussed in the previous subsection is a self-consistent two-loop approximation to the energy density. The Hartree approximation is obtained by neglecting all self-energies in the calculation of the loop part $E_L$, i.e., by neglecting $E_i$ of Eq. (4.14) as well as the counterterm contributions in Eqs. (4.11). It is a self-consistent one-loop approximation to the energy density. The variations (4.22) then give $S = S_0$, $A = A_0$, and $A_\pi = A_{\pi 0}$. The energy density thus becomes

$$E^{(H)} = E^{(H)}_{MF} + E^{(H)}_{t}.$$  \hspace{1cm} (4.26)

Here $E^{(H)}_{MF}$ is to be calculated from Eq. (3.3) with the propagators in free space determined (non-self-consistently) in the one-loop approximation, and $E^{(H)}_{t}$ is the sum of the three terms

$$E^{(H)}_{t, N} = -i \int \frac{d^4k}{(2\pi)^4} \text{Tr} \ln S_0$$ \hspace{1cm} (4.27a)

$$E^{(H)}_{t, \sigma 0} = i \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \ln \Delta_{\sigma 0}$$ \hspace{1cm} (4.27b)

$$E^{(H)}_{t, \pi} = \frac{3}{2} i \int \frac{d^4k}{(2\pi)^4} \ln \Delta_{\pi 0}.$$ \hspace{1cm} (4.27c)

Both terms in Eq. (4.26) individually are divergent, but their sum is finite. Let us first consider $E^{(H)}_{MF}$ of Eq. (3.3). In order to calculate the finite potential $U^{(H)}$ we need the free space self-energies for $\sigma$ and $\pi$. These are given in Appendix D for arbitrary choice of the renormalization points. If the mesonic renormalization points are taken to be zero, $U^{(H)}$ reduces to (3.3b'). The divergent term $\delta U$ is obtained from the unrenormalized $\sigma$ and $\pi$ self-energies in free space given in Appendix D. The result is
\[ 16\pi^2 \delta U = \frac{3}{2} \lambda^2 (m_n^2 + m_p^2)(v^2 - v'v^2) - 4m_n^2 g^2 (v^2 - v'v^2) \]

\[-\left( \Gamma \left( 2 - \frac{\gamma}{2} \right) - \ln m_n^2 \right) \left( 2 g^4 (v^2 - v'v^2)^2 + 4g^2 m_n^2 (v^2 - v'v^2) \right) \]

\[+ \left( \Gamma \left( 2 - \frac{\gamma}{2} \right) - \ln m_n^2 \right) \left( \frac{9}{4} \lambda^4 (v^2 - v'v^2)^2 + \frac{3}{2} \lambda^2 m_n^2 (v^2 - v'v^2) \right) \]

\[+ \left( \Gamma \left( 2 - \frac{\gamma}{2} \right) - \ln m_n^2 \right) \left( \frac{3}{4} \lambda^4 (v^2 - v'v^2)^2 + \frac{3}{2} \lambda^2 m_n^2 (v^2 - v'v^2) \right). \quad (4.28) \]

Here, \( \Gamma(x) \) is the gamma function and \( n \) is the number of dimensions. Next, we consider the loop terms (4.27). For the nucleonic contribution (4.27a) it is convenient to split the propagator \( S_0 \) according to (see Eqs. (2.21a) and (2.22))

\[ S_0(k) = \frac{1}{k - \hat{m} + i\delta} = S_{0F} + S_{0D} \quad (4.29a) \]

with

\[ S_{0F} = \frac{1}{k - \hat{m} + i\delta} \quad (4.29b) \]

\[ S_{0D} = (k + \hat{m}) \frac{n(k)}{\hat{E}_k} \delta(\hat{E}_0 - \hat{E}_k) \quad (4.29c) \]

with \( \hat{E}_k = \sqrt{k^2 + \hat{m}^2} \). By a partial integration we can separate the contribution due to \( S_{0D} \):

\[ E_{(H)}^{(N)} = -i \int \frac{d^4k}{(2\pi)^3} k^0 \text{Tr}(\gamma^0 S_{0D}) - i \int \frac{d^4k}{(2\pi)^4} \text{Tr} \ln S_{0F} \]

\[ = 4 \int \frac{d^3k}{(2\pi)^3} \hat{E}_k n(k) + g_\omega \frac{Z_{gN}}{Z_N} \tilde{w}^0 \rho + 4i \int \frac{d^4k}{(2\pi)^4} \ln(k^2 - \hat{m}^2 + i\delta). \quad (4.30) \]

Here, the first two terms on the right-hand side are due to the integral over \( S_{0D} \). Since \( Z_{N} = 1 \) in the Hartree approximation, we will omit it from now. With (4.30) used, the ring energy (4.27) becomes, after the vacuum expectation value is subtracted,

\[ E_{(H)} = 4 \int \frac{d^3k}{(2\pi)^3} \hat{E}_k n(k) + g_\omega \frac{Z_{gN}}{Z_N} \tilde{w}^0 \rho - \frac{m_N^4}{8\pi^2} F(y_N) \]

\[+ \frac{m_N^4}{64\pi^2} F(y_\sigma) + \frac{3m_N^4}{64\pi^2} F(y_{1/2}) + \delta E. \quad (4.31) \]
Here we introduced \( \alpha = N, \sigma, \pi \)

\[
y_{\alpha} = \left( \frac{m_{\alpha}}{m_{\alpha}^*} \right)^2,
\]

\[
\frac{m_{\alpha}^4}{32\pi^2} F(y_{\alpha}) = -i \int \frac{d^4k}{(2\pi)^4} \left\{ \ln \left( 1 - \frac{\delta_{\alpha}}{k^2 - m_{\alpha}^2} \right) + \frac{\delta_{\alpha}}{k^2 - m_{\alpha}^2} + \frac{1}{2} \left( \frac{\delta_{\alpha}}{k^2 - m_{\alpha}^2} \right)^2 \right\}
\]

(4.32)

with \( \delta_{\alpha} = m_{\alpha}^2 - m_{\alpha}^*, \) and

\[
\delta E = -\frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \left\{ 8 \left( \frac{\delta_{\alpha}}{k^2 - m_{\alpha}^2} + \frac{1}{2} \left( \frac{\delta_{\alpha}}{k^2 - m_{\alpha}^2} \right)^2 \right) - \left( \frac{\delta_{\alpha}}{k^2 - m_{\alpha}^2} + \frac{1}{2} \left( \frac{\delta_{\alpha}}{k^2 - m_{\alpha}^2} \right)^2 \right) - 3 \left( \frac{\delta_{\alpha}}{k^2 - m_{\alpha}^2} + \frac{1}{2} \left( \frac{\delta_{\alpha}}{k^2 - m_{\alpha}^2} \right)^2 \right) \right\}.
\]

(4.33)

In going from (4.27) to (4.31) we essentially only subtracted the term \( \delta E \) and added it again in order to separate the divergent term. Note that the function \( F(y) \) defined by Eq. (4.32) is finite. An elementary calculation of \( \delta E \) shows that

\[
\delta E = -\delta U,
\]

(4.34)

where \( \delta U \), the divergent part of \( E_{\text{MF}}^{(H)} \), has been given in Eq. (4.28). This shows that the energy density (4.26) is finite. The function \( F(y) \) is calculated to be

\[
F(y) = y^2 \ln y - \frac{3}{4} (y^2 - 1) + 2(y - 1).
\]

(4.35)

The energy density in the Hartree approximation is obtained by adding Eqs. (4.31) and (3.3),

\[
E^{(H)} = U^{(H)} + g_{\omega} Z_{\omega\rho} \rho \tilde{w}^0 + 4 \int \frac{d^3k}{(2\pi)^3} n(k) \tilde{E}_k
\]

\[
- \frac{m_N^4}{8\pi^2} F(y_N) + \frac{m_\pi^4}{64\pi^2} F(y_\pi) + \frac{3m_\pi^4}{64\pi^2} F(y_\pi),
\]

(4.36)

with \( U^{(H)} \) given by Eq. (3.3b) (see also Appendix D) or, for the choice \( \mu_\omega^2 = 0 \), by (3.3b'). Using Eqs. (2.36) and (2.19) with \( Z_N = 1 \), the terms due to the \( \omega \) meson in (4.36) can be combined to

\[
-\frac{m_\omega^2}{2} Z_{\omega\rho} \tilde{w}_\mu^\rho + g_{\omega} Z_{\omega\rho} \rho \tilde{w}^0 = \frac{1}{2} \frac{g_\omega^2 Z_{\omega\rho}}{m_\omega^2} (\rho^2 + \tilde{j}_\rho).
\]

(4.37)
If we set the mesonic renormalization points equal to zero and omit the pionic contributions, Eq. (4.36) reduces to the expression used in Refs. [5, 6]. Let us calculate the derivatives of the energy density (4.36) with respect to the mean fields. We obtain (without changing the notations for the Green functions)

\[
\frac{\partial E^{(1H)}}{\partial \bar{n}} = - \bar{v} \Lambda^{-1}(0) - c
\]  

(4.38a)

with

\[- \Lambda^{-1}(0) = \tilde{m}_\pi^2 + \Sigma_\pi(0),
\]

(4.38b)

\[\Sigma_\pi(0) = \Sigma_{af}(0) + \frac{\bar{v}^2 - v^2}{2v^2} (\Sigma_{af}(0) - \Sigma_{af}(0))
+ 4g^2 \int \frac{d^3k}{(2\pi)^3} \frac{n(k)}{E_k - \tilde{m}_\pi^2 g^2} G(y_\pi)
+ \frac{3m_\pi^2 \lambda^2}{16\pi^2} G(y_\sigma) + \frac{3m_\pi^2 \lambda^2}{16\pi^2} G(y_\pi)
\]

(4.38c)

\[G(y) = v \ln y - v + 1.
\]

(4.38d)

The term \(\tilde{m}_\pi^2\) in Eq. (4.38b) as well as the first two terms in (4.38c) involving the free space self-energies at \(q = 0\) has been derived by differentiating \(U^{(1H)}\) of Eq. (3.3b), while the last four terms in (4.36) give rise to the last four terms in (4.38c). It is not difficult to convince oneself that \(\Sigma_\pi(0)\) of Eq. (4.38c) is indeed the one-loop expression for the pion self-energy at \(q = 0\): If we split the renormalized free space self-energies in (4.38) according to \(\Sigma_\pi(0) = \Sigma_{af}(0) + \delta m_\pi^2 (\pm = \sigma, \pi)\), we see that the counterterm contribution contained in (4.38c) is just

\[\delta m_\pi^2 + (\bar{v}^2 - v^2) \lambda^2 (Z_\lambda - 1) = \delta \tilde{m}_\pi^2\]

due to Eqs. (2.10b) and (2.12c). Therefore Eq. (4.38c) takes the form

\[\Sigma_\pi(0) = \Sigma_\pi(0) + \delta \tilde{m}_\pi^2\]

where \(\Sigma_\pi(0)\) is obtained by replacing \(\Sigma_{af}(0) \to \Sigma_{af}(0)\) in (4.38c). A simple calculation, not reproduced here, then shows that this expression for \(\Sigma_\pi(0)\) is equivalent to the expression obtained from Eqs. (2.43a) and (2.43b) or the graphs in Fig. 6c calculated with the lowest order (Hartree) propagators and bare vertices. If we choose the renormalization points as \(\mu_\pi^2 = \mu_\sigma^2 = 0\), the first two terms in (4.38c) vanish. The next two terms are due to the nucleon loop diagram of Fig. 6c and the last two terms are due to the meson loop diagrams. Using Eqs. (4.29), the nucleon loop diagram in Fig. 6c can be split into a "density part," which involves at least one propagator \(S_{pd}\) and is given by the third term in (4.38c), and a "Feynman part," which is given by the fourth term in (4.38c). Since we are considering the limit \(k \to 0\), there are only particle–antinucleon (p\(\overline{N}\)) and no particle hole (ph) contributions to the nucleon loop term.
Equation (4.38a) is a special case of the Goldstone theorem (3.13). In particular we note that in the so-called “quasiclassical approximation,” which is obtained by neglecting the last three terms in (4.36) and choosing the form (3.3b’) for \( U \), only the third term in the pion self-energy (4.38c), i.e., the “density part” of the pN loop diagram, remains. It is repulsive. The attractive contribution due to the interaction of the pion with the condensed scalar field is contained in the mass parameter \( \tilde{m}_\pi^2 = m_\pi^2 + \lambda^2 (\bar{v}^2 - v^2) \). The cancellation between these two terms is analogous to the cancellation which leads to the small s-wave scattering length \( a_0^{(+)} \) in pion–nucleon scattering. In the framework of the quasiclassical approximation, the assumption of exact chiral symmetry then requires a complete cancellation of these two terms also in the nuclear medium (as long as we stay in the Goldstone mode), since then \( A^{-1}_0(0) = 0 \) becomes the energy minimization condition to determine the physical value of \( \bar{v} \). The Wigner mode \( (\bar{v} = 0) \) becomes stable in the quasiclassical approximation if the density is large enough that the third term in (4.38c) exceeds the value \( m_\pi^2/2 \) (which is equal to \( -\tilde{m}_\pi^2 \) at \( \bar{v} = 0 \); compare Eq. (3.17b) and Figs. 3, 4).

Let us briefly also discuss the forms of the \( \sigma \) and \( \omega \) meson self-energies as derived from the expression (4.36), since these determine the Landau–Migdal interaction (3.24) in the Hartree approximation. Differentiating (4.38a) once more we obtain

\[
\frac{\partial^2 E^{(H)}}{\partial \bar{v}^2} = -A^{-1}_0(0)
\]

with

\[
-A^{-1}_0(0) = \tilde{m}_\sigma^4 + \Sigma_\sigma(0), \quad (4.39b)
\]

\[
\Sigma_\sigma(0) = \Sigma_{\sigma\pi}(0) + \frac{3(\bar{v}^2 - v^2)}{2\bar{v}}(\Sigma_{\sigma\pi}(0) - \Sigma_{\pi\pi}(0))
\]

\[
+ 4g^2 \int \frac{d^3k}{(2\pi)^3} \frac{n(k) \tilde{G}_k^2}{\tilde{E}_k^4} - \frac{m_\pi^2 g^2}{2\pi^2} H_N(y_N)
\]

\[
+ \frac{3m_\sigma^2 \lambda^2}{16\pi^2} H_\sigma(y_\sigma) + \frac{3m_\pi^2 \lambda^2}{16\pi^2} H_\pi(y_\pi), \quad (4.39c)
\]

\[
H_N(y) = 3y \ln y - y + 1,
\]

\[
H_\sigma(y) = \left( y + \frac{6\lambda^2 \bar{v}^2}{m_\sigma^2} \right) \ln y - y + 1,
\]

\[
H_\pi(y) = \left( y + \frac{2\lambda^2 \bar{v}^2}{m_\pi^2} \right) \ln y - y + 1. \quad (4.39d)
\]

\( \Sigma_\sigma(0) \) of Eq. (4.39c) agrees with the expression obtained from Fig. 6b in the Hartree approximation. The third term in (4.39c) is the “density part” of the nucleon loop contribution in the limit \( |q| \rightarrow 0 \) first followed by \( q_0 \rightarrow 0 \). Only the pN contributions survive in this limit. The first two terms in (4.39c) vanish for the choice \( \mu_\sigma^2 = \mu_\pi^2 = 0 \). In the Wigner mode \( (\bar{v} = 0) \) the expression (4.39b) is equivalent to (4.38b).
Using (3.3b), differentiation of (4.36) with respect to \( \tilde{w}^\mu \) gives

\[
\frac{\partial E^{(H)}}{\partial \tilde{w}_\mu} = j_{\omega}^{\mu} = -m_\omega^2 Z_{\omega \omega} \tilde{w}^\mu + g_\omega Z_{\omega \omega} j_B^{\mu}
\]  

(4.40a)

with \( j_B^{\mu} = \rho \) and

\[
\begin{align*}
\rho &= 4 \int \frac{d^3k}{(2\pi)^3} n(k) \frac{k}{E_k}.
\end{align*}
\]

(4.40b)

Setting the right-hand side of Eq. (4.40a) equal to zero then gives Eq. (2.36).

Differentiating (4.40a) once more we obtain

\[
\frac{\partial^2 E^{(H)}}{\partial \tilde{w}_\mu \partial \tilde{w}_\nu} = -(\Delta_{\omega}^{-1}(0))^{\mu\nu} = -g^{\mu\nu} m_\omega^2 Z_{\omega \omega} + g_\omega Z_{\omega \omega} \frac{\partial j_B^{\nu}}{\partial \tilde{w}_\mu}.
\]

(4.41a)

Using (4.40b) together with (2.19) for \( Z_N = 1 \) this becomes

\[
\begin{align*}
(\Delta_{\omega}^{-1}(0))^{\mu} &= m_\omega^2 + \delta m_\omega^2, \\
(\Delta_{\omega}^{-1}(0))^{ij} &= g^{ij}(m_\omega^2 + \delta m_\omega^2) + \Sigma_{\omega}^{ij}(0)
\end{align*}
\]

(4.41b)

(4.41c)

with

\[
\Sigma_{\omega}^{ij}(0) = 4(g_\omega Z_{\omega \omega})^2 \int \frac{d^3k}{(2\pi)^3} n(k) \left( \frac{g^{ij}}{E_k} \right) \frac{k^i k^j}{E_k^3}
\]

\[
\xrightarrow{\rho \to 0} g^{ij}(g_\omega Z_{\omega \omega})^2 \frac{\rho}{\tilde{E}_F} \equiv g^{ij} \bar{\Sigma}_{\omega}(0),
\]

(4.41d)

where \( \tilde{E}_F = \sqrt{\tilde{p}_F^2 + \tilde{m}_F^2} \). Equation (4.41b) has been derived more generally in Section 3 (see Eq. (3.25)), and \( \bar{\Sigma}_{\omega}(0) \) of (4.41d) is the Hartree expression of the \( \omega \) meson self-energy at \( q = 0 \) (compare Eq. (3.27)). In accordance with the discussion in Section 3, \( \bar{\Sigma}_{\omega}(0) \) vanishes if the density goes to zero. The propagators given in (4.39) and (4.41) should be inserted into the general expression (3.24) to determine the Landau–Migdal interaction. The first term in (3.24) is zero in the Hartree approximation since the quasiparticle spectrum

\[
\epsilon_\rho = \tilde{E}_\rho + g_\omega Z_{\omega \omega} \tilde{w}^0
\]

(4.42)

does not depend explicitly on the distribution function. The vertex \( \gamma_\rho^\mu \) in the Hartree approximation can be derived from the definition (3.23b) or from the more explicit form (3.31b) using (4.42) and the expression (4.41d) for \( \bar{\Sigma}_{\omega}(0) \). One obtains

\[
\gamma_\omega^{\mu}(p) = -ig_\omega Z_{\omega \omega} \frac{p^\mu}{\tilde{E}_\rho} \quad (p^\mu = (\tilde{E}_\rho, p)).
\]

(4.43a)

As expected, this is just the matrix element of the free \( \gamma NN \) vertex, since in the Hartree approximation the nucleon self-energy \( \Sigma_N \) and therefore, due to gauge
invariance, also the correction to the irreducible $\omega$NN vertex is zero. (This is in contrast to the source vertex which is renormalized by the RPA-type vertex corrections [20, 26].) Equation (3.23a) gives for $\gamma_{\sigma}$

$$\gamma_{\sigma}(p) = -ig \frac{\hat{\eta}_\sigma}{E_p}.$$  

(4.43b)

We therefore obtain for the Landau–Migdal interaction in the Hartree approximation from (3.24)

$$f_{\sigma} = -g^2 \frac{\hat{m}_\sigma^2}{\hat{E}_p \hat{E}_p} \frac{1}{\hat{m}_\sigma^2 + \Sigma_{\sigma}(0)} + \frac{g^2 Z_{\omega \sigma}}{m_\omega^2} \frac{\hat{p}_i \cdot \hat{p}_j}{\hat{E}_p \hat{E}_p} \frac{g^2 Z_{\omega \sigma}^2}{\hat{m}_\sigma^2 + \Sigma_\omega(0)}$$

(4.44)

with $\Sigma_{\sigma}(0)$ given by (4.39c) and $\Sigma_{\omega}(0) = \Sigma_{\omega}(0) + \delta m_\omega^2$ with $\Sigma_{\omega}(0)$ given by (4.41d). In the quasiclassical approximation, where $Z_{\omega \sigma} = 1$ and only the third term in (4.39c) contributes, Eq. (4.44) differs from the result derived in Ref. [25] only by the appearance of $\hat{m}_\sigma^2$ instead of the free mass parameter $m_\sigma^2$.

5. Simple Examples

In this section we use the form of the energy density in the Hartree approximation, Eq. (4.36), to discuss various simple cases, including those studied previously by other authors [1, 4–6, 9]. Our aim is to illustrate the connection to the more general results derived in Sections 2 and 3. In the numerical calculations discussed below, we will fix the coupling constant $g = 10$ due to reasons discussed in Section 2. (Note, however, that for the cases discussed in Sections 5.1 and 5.2 only the ratio $g^2/m_\sigma^2$ enters.) Thus, fixing also $m_\omega = 783$ MeV, the free parameters are $m_\sigma$ and $g_{\omega \sigma}$. Concerning $m_\pi$, we will refer either to the case $m_\pi = 0$ or to the case $m_\pi = 140$ MeV.

5.1. The Quasiclassical Approximation

The quasiclassical approximation consists in neglecting all loop terms in (4.36) except the loop term which gives the energy of the filled Fermi sea. In the frame where the total three-momentum of the system is zero, we have

$$E^{(1)} = U_0 + \frac{1}{2} \frac{g^2 Z_{\omega \sigma}}{m_\omega^2} \rho^2 + 4 \int_{\rho F} d^3 k \frac{d^3 k}{(2\pi)^3} E(k),$$

(5.1)

where $U_0$ is given by (3.3b’), and $Z_{\omega \sigma} = 1$. By differentiating this expression with respect to $\hat{v}$ we obtain the corresponding expression for the pion self-energy (see Eq. (4.38c))

$$\Sigma^{(1)}_{\pi} = 4g^2 \int_{\rho F} d^3 k \frac{d^3 k}{(2\pi)^3} \frac{1}{E_A},$$

(5.2)
As discussed earlier, this is the density part of the pN loop diagram in the pion self-energy. For simplicity, let us discuss the case of exact chiral symmetry ($c = m_\pi = 0$). The solution of the self-consistency equation

\[-A_\pi^{-1}(0) = \tilde{m}_\pi^2 + \Sigma_\pi^{(1)}(0) = m_\pi^* = 0\]  \hspace{1cm} (5.3)

has already been illustrated graphically in Fig. 4. Here, for convenience of comparison, we use parameters identical to those in Ref. [9], namely $m_\sigma = 0.8$ GeV and $g_\omega = 9.1$. (The choice of parameters has no influence on the qualitative features discussed below.) The stability condition is

\[-A_\sigma^{-1}(0) = \tilde{m}_\sigma^2 + \Sigma_\sigma^{(1)}(0) = m_\sigma^* > 0\]  \hspace{1cm} (5.4a)

with (see Eq. (4.39c))

\[\Sigma_\sigma^{(1)}(0) = 4g^2 \int^{\rho_F} d^3k \frac{k^2}{(2\pi)^3 \tilde{E}_k^3}\]  \hspace{1cm} (5.4b)

which is the density part of the pN loop contribution to the $\sigma$ self energy. As is clear from Fig. 4, due to the large curvature of the pion self-energy at $\tilde{v} = 0$ there exist both the Goldstone mode and the Wigner mode solutions up to a density $\rho_c = \rho_2$. For $\rho > \rho_c$ only the solution $\tilde{v} = 0$ remains. In Fig. 8 we show the effective nucleon mass $m_N = \tilde{v}$ and the binding energy per nucleon. As noted by other authors [9], independent of the choice of parameters, the normal state (Goldstone mode) does not show saturation with respect to variations in the density. On the contrary, for $g_\omega \neq 0$, the abnormal state saturates. (Note that since in the abnormal state the attractive $\sigma$ meson exchange part of the Landau–Migdal force (4.44) is ineffective, an extremum of $E_B/A(\rho)$ is necessarily a minimum.)

**Fig. 8.** The nucleon effective mass (a) and the binding energy per nucleon (b) in the quasiclassical approximation. Beyond a certain density $\rho_c$ the normal state disappears and only the abnormal state, where $\tilde{m}_N = 0$, exists. At the critical density $\rho_c < \rho_2$ the system goes into the Wigner mode and the effective mass becomes zero, as indicated by the dashed line in (a). The parameters used here are $g = 10$, $m_\sigma = 0.8$ GeV, $m_\pi = 0$, and $g_\omega = 9.1$. The phase transition shown here is analyzed in more detail in Fig. 4.
Let us look at these results more closely: We have seen that the discontinuous phase transition is caused by the large curvature of \( \Sigma^{(1)}_\pi(0) \) at \( \tilde{v} = 0 \). In fact,

\[
T^{(1)}(0, 0; 0, 0) = \frac{\partial^2}{\partial \tilde{v}^2} \Sigma^{(1)}_\pi(0) \propto \ln \tilde{v} \to -\infty \quad \text{as} \quad \tilde{v} \to 0.
\]

(5.5)

The reason for this singularity is the following: The quantity (5.5) is that part of the nucleon loop contribution to the \( \sigma^2 \pi^2 \) vertex (with all external momenta equal to zero) which depends explicitly on the density. The full nucleon loop contribution is graphically represented by Fig. 9. For \( \tilde{v} \to 0 \) the loop integral behaves like \( \int d^4k/k^4 \). However, since due to the time ordering at least one of the internal nucleon lines must be a hole line, the Pauli principle forbids \(|k| < p_F\) and thus excludes the infrared singularity. However, since the quantity (5.5) is only one part (the Pauli correction term) of the full nucleon loop contribution, the infrared singularity is present. In other words, once we include also the “Feynman part” \( \Sigma^{(2)}_\pi \) in the pion self-energy, its curvature at \( \tilde{v} = 0 \) will be finite. Thus, one is naturally led to include the nucleon vacuum fluctuation term, which will be discussed in the next subsection.

At the density where \( \Sigma^{(1)}_\pi(0) = m_\pi^2/2 \) at \( \tilde{v} = 0 \), the abnormal state becomes stable with respect to variations in \( \tilde{v} \). This occurs at

\[
p_F = \frac{1}{\sqrt{2}} \frac{m_\sigma \pi}{g}.
\]

(5.6)

which, with the present parameters, corresponds to \( 0.3 \rho_0 \) (\( \rho_0 = 0.17 \text{ fm}^{-3} \)). The “critical density” \( \rho'_c \) (Fermi momentum \( p_{Fc} \)) and the corresponding value \( \chi_c = \tilde{v}_{c}/\nu \) are determined by the two equations (3.18); i.e., in the present case

\[
m_\pi^* = m_\sigma^* = 0,
\]

(5.7)
which can be brought into the form

\[
C - z^3 + z = Cz \left( z - (z^2 - 1) \ln \frac{z + 1}{z - 1} \right)
\]

(5.8a)

\[
x_c = \frac{\rho_F g}{m_N} \sqrt{z^2 - 1}
\]

(5.8b)

with

\[
C = 2 \left( \frac{m_N \bar{g}}{\pi m_\sigma} \right)^2, \quad z = 2 \left( \frac{\rho_F g}{\pi m_\sigma} \right)^2.
\]

(5.8c)

(The trivial solution \( z = 1, x_c = 0 \) corresponds to the case (5.6), where the abnormal state turns from an energy maximum into an energy minimum.) With the present parameters we obtain \( \rho_c \approx 0.9 \rho_0 \) and \( x_c \approx 0.55 \) (see Fig. 4). The critical density where the Wigner mode solution becomes an absolute minimum lies slightly below \( \rho_c \approx 0.8 \rho_0 \). The instability of the normal state can therefore also be viewed in the following way: In the Goldstone phase we can use the equation \( m_{\pi}^* \equiv 0 \) to bring \( m_\sigma^* \) into the form

\[
\frac{m_\sigma^*}{m_\sigma^2} = 1 - \frac{C \rho^2}{m_N^2 E_F} \quad \text{(Goldstone phase)}
\]

(5.9)

with \( E_F = \sqrt{p_F^2 + \tilde{m}_N^2} \). The quantity (5.9) determines the \( \sigma \) meson exchange part of the Landau–Migdal parameter \( f_0 \) (obtained by setting \(-\Delta_{-1}(0) = m_\sigma^* \) in Eq. (3.34d)). The attraction increases monotonically as the density increases, until a critical density is reached where \( m_\sigma^* \) vanishes. This behavior is due to the attractive contributions from the nonlinear terms, contained in the mass parameter \( \tilde{m}_N^2 \). Note that if we had replaced \( \tilde{m}_N^2 \rightarrow m_\sigma^2 \), the quantity \(-\Delta_{-1}(0)) \) would be positive definite.

5.2. The Role of the Nucleon Vacuum Fluctuation Term

Including the nucleon vacuum fluctuation term in (4.36), we obtain the energy density

\[
E^{(2)} = E^{(1)} - \frac{m_N^4}{8\pi^2} F(y_N),
\]

(5.10)

where \( E^{(1)} \) is given by Eq. (5.1). For simplicity we assume the renormalization points \( \mu_\sigma = \mu_\omega = \mu_\pi = 0 \). This is the model investigated in Ref. [4]. In the calculations below we use \( m_\pi = 0.6 \) GeV and \( g_\sigma = 4.8 \), which are very close to the values used in Ref. [4]. (Note that we fixed \( g = 10 \).) From Eq. (4.38c) we obtain the corresponding expression for the pion self-energy

\[
\Sigma_\pi^{(2)}(0) = \Sigma_\pi^{(1)}(0) - \frac{m_N^2 g^2}{2\pi^2} (y_N \ln y_N - y_N + 1).
\]

(5.11)
\( \Sigma_{\pi}^{(2)} \) is the full pN loop contribution. For simplicity we again refer to the case of exact chiral symmetry \((c - m_\pi = 0)\). The second term in (5.11), which is the vacuum fluctuation term, is negative definite. It vanishes for \( \tilde{v} = v \) and takes the value \(-m_N^2 g^2/(2\pi^2)\) at \( \tilde{v} = 0 \). The solution of the self-consistency equation

\[
-\Delta_{\pi}^{-1}(0) = \tilde{m}_\pi^2 + \Sigma_{\pi}^{(2)}(0) = 0
\]

is illustrated in Fig. 10 for \( \rho = 0.2 \text{ fm}^{-3} \). The vacuum fluctuation term is small for \( \tilde{v} = v \), but it changes the form of the pion self-energy drastically for smaller values of \( \tilde{v} \). The curvature at \( \tilde{v} = 0 \) is now finite. The stability criterion (3.17a) is obviously satisfied for the Goldstone mode solution. As the density increases, the full line in Fig. 10 moves upwards without changing its shape appreciably. If the full line touches the dashed one at \( \tilde{v} = 0 \), the Wigner mode becomes stable (see Eq. (3.17b)) and beyond this density the “effective pion mass” \(-\Delta_{\pi}^{-1}(0)\) is positive; i.e., Eq. (5.12) no longer has a solution. The Fermi momentum corresponding to this critical density is, using \( \Sigma_{\pi}(0) = g^2/\pi^2(p_F^2 - m_N^2/2) \),

\[
p_{Fc}^2 = \frac{m_N^2}{2} + \frac{\pi^2 m_N^2}{2g^2} \approx \frac{m_N^2}{2},
\]

and therefore the critical density is roughly 15 times the normal nuclear matter density independent of the choice of parameters. There is certainly little meaning in using the present schematic model at such high densities, but the conclusion we can draw is that due to the large attractive contributions of the vacuum fluctuations to the pion self-energy the chiral phase transition becomes a continuous (second order) one and the critical density is shifted to very high values. This change of the

![Fig. 10. Illustration of the self-consistency equation (5.12) for \( \rho = 0.2 \text{ fm}^{-3} \). The dashed line shows \(-m_N^2 = \lambda^2(v^2 - \tilde{v}^2)\) and the full line shows the pion self-energy (5.11). As the density increases, the full line moves upwards and, at the critical density (5.13), touches the dashed line at \( \tilde{v}/v = 0 \). Compare this with the fictitious case shown in Fig. 3.](image)
Fig. 11. The nucleon effective mass (a) and the binding energy per nucleon (b) when the nucleon vacuum fluctuation term is included. The parameters are $g = 10$, $m_a = 0.6 \text{ MeV}$, $m_a = 0$, and $g_{\sigma} = 4.86$.

Phase transition from a discontinuous to a continuous one due to the inclusion of the nucleon vacuum fluctuations has also been observed in Ref. [28].

In Fig. 11 we show the effective nucleon mass and the binding energy per nucleon. We see that for densities larger than the critical density (5.13) there is no minimum in the binding energy curve. One can easily show from Eq. (3.33) that for $\bar{n} = 0$ the binding energy per nucleon actually has a minimum at a density which is, however, slightly below the critical density, and hence the abnormal state is still unstable there with respect to variations in $\bar{n}$. The results are repeated, on a smaller scale, by the dashed lines in Fig. 12.

Fig. 12. The full lines show the $\sigma$ meson mass parameter of Eq. (5.4) and the nucleon effective mass (a), and the binding energy per nucleon (b) when the vacuum fluctuation term due to the sigma is included in addition to that due to the nucleon. The parameters are the same as those used in Fig. 11. The dashed lines reproduce the results shown in Fig. 11 for comparison.
The $\sigma$ meson self-energy in the present model is given by the sum of the third and fourth terms in Eq. (4.39c). Once the self-consistency equation (5.12) is satisfied we have

$$\frac{-\Delta_{\sigma}^{-1}(0)}{m_{\sigma}^2} = 1 - \frac{C}{m_N^2} \frac{p_F^2}{E_F} + \frac{g_s^2 m_N^2}{\pi^2 m_{\sigma}^2} \left( 1 - \frac{E}{V} \right)^2$$

(Goldstone mode),

(5.14)

where $C$ was introduced in Eq. (5.8c). The second term on the right-hand side of (5.14) is due to (but not identical with) the repulsive contribution due to the nucleon vacuum fluctuation term. This sizable term excludes the pole in the $\sigma$ meson propagator at $q = 0$ and thereby also the discontinuous phase transition observed in the quasiclassical approximation.

Some results are shown in more detail in Table I in the column "QCL + N." There we split the binding energy per nucleon at the saturation point into the four contributions $E_1$, $E_2$, $E_\sigma$, and $E_\pi$, which are due to the first three terms, the fourth, fifth, and sixth terms in Eq. (4.36), respectively. Further we give the numerical value of (5.14), which, however, seems extremely large. The next three rows give the three contributions to the Landau–Migdal parameter $F_0$ shown in Eq. (3.34d), i.e., $F_0^{(c)}$, $F_0^{(s)}$, and $F_0^{(w)}$ corresponding to the first, second, and third terms in (3.34d), respectively. Finally we show the total $F_0$ and the resulting compressibility.

The above discussion has shown the necessity of including the nucleon vacuum fluctuation term in order to exclude the discontinuous phase transition at normal densities. The actual size of its contribution to various physical quantities is, however, extremely large, as can be seen from the third and sixth rows of Table I or from Fig. 10.

| TABLE I |
| Nuclear Matter Properties in the Hartree Approximation |

<table>
<thead>
<tr>
<th>Case</th>
<th>QCL + N</th>
<th>QCL + N + $\sigma$</th>
<th>QCL + N + $\sigma + \pi$</th>
</tr>
</thead>
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<tr>
<td>$\rho_0$, $E_\sigma/A$</td>
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<td>(0.17, -17.4)</td>
<td>(0.18, -15.8)</td>
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<tr>
<td>$E_1$ (MeV)</td>
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<td>-45.8</td>
<td>-39.5</td>
</tr>
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<td>$E_2$ (MeV)</td>
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<td>30.0</td>
<td>36.1</td>
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<td>-16.8</td>
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<td>$E_\pi$ (MeV)</td>
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<td>0</td>
<td>4.4</td>
</tr>
<tr>
<td>$-\Delta_{\pi}^{-1}(0)/m_\pi^2$</td>
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<td>5.0</td>
<td>2.2</td>
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<td>$F_0^{(c)}$</td>
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</tr>
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<td>0.62</td>
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<tr>
<td>$K$ (MeV)</td>
<td>115</td>
<td>131</td>
<td>482</td>
</tr>
</tbody>
</table>

* The results shown in the three columns are based on the expressions (5.10), (5.17), and (5.18) of the energy density. With $y = 10$ and $m_\omega = 0.783$ GeV in all three cases, the other parameters are $m_\pi = 0.6$ GeV, $m_\sigma = 0$, and $g_\omega = 4.86$ in the first two cases, and $m_\pi = 1$ GeV, $m_\sigma = 0.14$ GeV, and $g_\omega = 2.81$ in the third case. The first row shows the resulting saturation density $\rho_0$ in fm$^{-3}$ and the corresponding binding energy per nucleon $E_\sigma/A$ in MeV. For further explanation, see the text.
5.3. The Role of the Meson Loop Terms

We now discuss the role of the last two terms in the Hartree energy density (4.36), which represent the "zero point energy" contributions of the boson fields. As we discussed before, these terms are ill defined since $y_\sigma$ and $y_\pi$ are not positive definite, which is especially serious for the pionic contribution. The sigma meson contribution has been treated in many works [1, 5, 6] by replacing the Hartree mass parameter $m_\sigma^2$ by the quantity $m_\sigma^{*2}$ defined in (5.4), which is the self-consistent $\sigma$ meson mass parameter in the quasiclassical approximation. In a similar way one can replace $m_\pi^2$ by $m_\pi^{*2}$ of Eq. (5.3), which is the self-consistent pion mass parameter in the quasiclassical approximation. (By "self-consistent mass parameters" we mean the quantities $(-d_\sigma'(0))$ and $(-d_\pi'(0))$ satisfying Eq. (3.16) and the Goldstone theorem (3.13) for a given model energy density E.) Thus, in order to obtain some insight into the role of the meson loop terms, we replace

$$y_\sigma \to \hat{y}_\sigma = \frac{m_\sigma^{*2}}{m_\sigma^2}; \quad y_\pi \to \hat{y}_\pi = \frac{m_\pi^{*2}}{m_\pi^2}$$

(5.15)

in the last two terms of the energy density (4.36). As discussed earlier, these mass parameters $m_\sigma^{*2}$ and $m_\pi^{*2}$ include the density dependent part of the pN polarization term at $q = 0$. We must note, however, that there are many problems associated with such an ad hoc replacement: First, although it seems quite natural, it is not unique. This is clear from the fact that the replacement (5.15) cannot be done from the start, since then the cancellation of the divergences, expressed by Eq. (4.34), would no longer be valid. If one includes the "density parts" of the polarizations consistently in the ring energies (4.11b) and (4.11c) from the beginning, the finiteness of the result is due to the $1/k^2$ behaviour of the polarizations for large meson momenta [15]. Therefore it is not possible to include the polarizations uniquely by replacing them by their values at $k = 0$. Second, the prescription (5.15) destroys chiral symmetry. In particular, the Goldstone theorem (4.38a) no longer holds: If we differentiate the energy density $\hat{E}^{(H)}$ modified due to (5.15) with respect to $\hat{v}$ and write the result in the form (4.38a), thereby defining a modified quantity $-\hat{A}_\pi^{-1}(0) = \hat{m}_\pi^2 + \hat{\Sigma}_\pi(0)$, the quantity $\hat{\Sigma}_\pi(0)$ is obtained from (4.38c) by replacing

$$G(y_\alpha) \to (1 + c_\alpha)G(\hat{y}_\alpha) \quad (\alpha = \sigma, \pi)$$

$$c_\sigma = \frac{1}{6\lambda^2\hat{v}} \frac{\partial}{\partial \hat{v}} \Sigma_\sigma^{(1)}(0)$$

$$c_\pi = \frac{1}{2\lambda^2\hat{v}} \frac{\partial}{\partial \hat{v}} \Sigma_\pi^{(1)}(0)$$

(5.16)

with $\Sigma_\sigma^{(1)}$ and $\Sigma_\pi^{(1)}$ given by Eqs. (5.4b) and (5.2). The presence of the factors $c_\alpha$ makes it impossible to interpret the result of the differentiation as a pion self-energy, and, moreover, gives rise to the following further problem: For $\hat{v} \to 0$ these factors behave like $\ln \hat{v}$ (compare the discussion around Eq. (5.5)). This has the
consequence that the second derivative of the energy density with respect to \( \bar{\varphi} \) (i.e., the quantity \(-\tilde{\mathcal{J}}^{-1}(0)\)) goes to minus infinity as \( \bar{\varphi} \to 0 \). (The first derivative is finite due to the factor \( \bar{\varphi} \) in (4.38a).) In other words, due to the ad hoc replacements (5.15) the point \( \bar{\varphi} = 0 \) is always a maximum of the energy density, and the transition to the abnormal state cannot take place no matter how high the density is. On the other hand, for the cases discussed below, the factors \( c_\varphi \) are quite small compared to 1 for densities around the normal nuclear matter density and \( \bar{\varphi}/\bar{\varphi} \gtrsim 0.7 \). Then the nucleonic term (5.11) is dominant in \( \tilde{\mathcal{J}}_\pi(0) \) and we get a behaviour similar to that shown in Fig. 10. For values \( \bar{\varphi}/\bar{\varphi} \lesssim 0.5 \), however, the \( c_\varphi \) become large and the "depth" of \( \tilde{\mathcal{J}}_\pi(0) \) at \( \bar{\varphi} = 0 \) becomes infinite, which prevents the Wigner mode from becoming stable. Thus, while the method (5.15) of including the effect of the polarizations might be reasonable for normal densities, where we expect \( \bar{\varphi} \ll \bar{\varphi} \), it certainly gives incorrect results for high densities. A more consistent treatment of the self-energy corrections to the meson propagators by starting from the HF framework discussed in Section 4 is now in progress [18]. One can show that if the polarizations \( \Sigma_\pi^{(1)}(q) \) and \( \Sigma_\sigma^{(1)}(q) \) are taken into account consistently from the start, the three diseases discussed above are cured. Such an attempt, using approximate forms for the polarizations, has been made in Ref. [3].

With these limitations in mind, let us discuss the role of the \( \sigma \) meson loop term; i.e., we consider

\[
E^{(3)} = E^{(2)} + \frac{m_\sigma^4}{64\pi^2} F(\hat{y}_\sigma), \tag{5.17}
\]

where \( E^{(2)} \) is given by Eq. (5.10). As noted in Refs. [5, 6], the size of the \( \sigma \) meson loop term depends sensitively on the assumed value for \( m_\sigma \). If we use parameters identical to those in the previous subsection (\( g = 10, \ m_\sigma = 0.6 \) GeV), its contribution is negligibly small, even for higher densities. This can be seen from Fig. 12 by comparing the results for the binding energy per nucleon and the effective nucleon mass including the \( \sigma \) meson loop term (full lines) with those including only the nucleon loop term (dashed lines). Figure 12 also shows the mass parameter \( m_\sigma^* \) which is used in the calculation of the meson loop.

Comparison of the values in the second and third columns of Table I shows again the small influence of the \( \sigma \) meson loop term. Concerning the parameter \( F_0 \), we must note that due to the replacements (5.15) in the energy density the quasiparticle spectrum also changes due to Eq. (3.20a) and in fact now depends explicitly on the density. Therefore also the first term in Eq. (3.34d) is non-zero, and the \( \sigma NN \) vertex (3.23a) is changed from its Hartree value (4.42b). For the present case, however, these modifications are negligibly small, as can be seen from Table I.

Finally we consider

\[
E^{(4)} = E^{(H)} = E^{(3)} + \frac{3m_\pi^2}{64\pi^2} F(\hat{y}_\pi), \tag{5.18}
\]
where $E^{(3)}$ is given by Eq. (5.17) and the last term represents the effect of the pion loop. Use of the same parameters ($g = 10$, $m_\sigma = 0.6$ GeV) leads, however, to an unreasonably high value for $m^{*2}_\pi$ and therefore also for $\tilde{\gamma}_\pi$ even at low densities. This can be foreseen, since using simply the values of $\tilde{v}$ and $\rho$ at the saturation point shown by the full lines in Fig. 12, we obtain $m^{*2}_\pi/m_\pi = 2.6$. This somewhat artificial problem, which has also been discussed in Ref. [17], arises because the mass parameter $m^{*2}_\pi$ of Eq. (5.3) is a "self-consistent" mass parameter (in the sense defined above) only in the quasiclassical approximation. Only in that case does it agree with the inverse pion propagator ($-A^{-1}_\pi(0)$) which satisfies the Goldstone theorem, Eq. (4.38a), and vanishes for exact symmetry once the energy density is minimized. When we go beyond the quasiclassical approximation, no such constraint on $m^{*2}_\pi$ exists. If one insists on using $m^{*2}_\pi$, one must raise the value of $m_\sigma$ up to 1 GeV or higher, while keeping $g = 10$ fixed, in order to counterbalance the strong repulsion due to $\Sigma^{(1)}_\pi$ by the attractive contributions due to the nonlinear meson interaction terms contained in the parameter $\tilde{m}^{2}_\pi$. For $g = 10$, $m_\sigma = 1$ GeV, $m_\pi = 0.14$ GeV, and $g_\omega = 2.8$, we get the results shown in Fig. 13. Even in this case the repulsion due to the pion loop is substantial for higher densities, as is clear from the increase of $m^{*2}_\pi$ with increasing density, and accordingly the value of $g_\omega$ must be reduced in order to get the right saturation point. The results using this parameter set are collected in the last column of Table I. Due to the larger value of $m_\sigma$, the $\sigma$ ring energy now cancels about half of the nucleonic ring energy. The repulsive pionic contribution is still moderate at the normal nuclear matter density. The value of ($-A^{-1}_\pi(0)$) is changed little, compared to its value in the two cases discussed above, with the largest contribution still due to the nucleon vacuum fluctuation term. The modifications of the quasiparticle spectrum due to the replacement of $\tilde{m}^{2}_\pi$ by $m^{*2}_\pi$ in the energy density lead to enhancement of the $\sigma$NN vertex $\gamma_\sigma$ of Eq. (3.23a), thus increasing the contribution to $F_0$ due to the second

![Fig. 13. The meson mass parameters of Eqs. (5.3) and (5.4) and the nucleon effective mass (a), and the binding energy per nucleon (b) when the vacuum fluctuation term due to the pion is included in addition to those due to the sigma and the nucleon. The parameters are $g = 10$, $m_\sigma = 1$ GeV, $m_\pi = 0.14$ GeV, and $g_\omega = 2.8$.](image-url)
term in (3.34d). This additional attraction, however, is cancelled largely by the contribution due to the first term in (3.34d), and the result is a positive total value \( F_0 = 0.82 \) and a compressibility of \( K = 482 \text{ MeV} \). We also wish to mention that the behaviour of the effective meson masses with increasing density shown in Fig. 13 is qualitatively consistent with recent results obtained in effective quark theories [34].

Altogether, the assessment of the vacuum fluctuation energies due to the meson fields poses many problems, especially concerning the pionic contribution. Based on the present approximate treatment we cannot draw firm conclusions. In our estimate, the repulsive quantum fluctuation contribution due to the nucleon dominates over the mesonic contributions at normal densities. It is partially cancelled by the attractive \( \sigma \) meson loop term, the degree of cancellation depending on the assumed value for \( m_\sigma \). The pionic contribution is repulsive and increases with increasing density. A more reliable treatment of the meson loop contributions based on the formalism of Section 4 will be discussed in a future paper [18].

6. CONCLUSION

In this paper we used the chiral \( \sigma-\omega \) model to study the role of chiral invariance in the nuclear medium. We discussed the formalism needed to carry out calculations in nuclear matter, particularly emphasizing a renormalization procedure consistent with symmetry requirements. Our main object was to elucidate certain constraints imposed by chiral symmetry and baryon current conservation. In particular, we focussed our attention on the Goldstone theorem in nuclear matter and its connection to the energy density and the problem of chiral phase transitions. These general considerations have been illustrated in the Hartree approximation, where we have shown how one can conveniently discuss the phase transition in terms of the pion propagator. A reliable treatment of the meson vacuum fluctuation terms requires at least the inclusion of particle–antinucleon and particle–hole polarization insertions in the meson propagators, and for this reason we also discussed some formal aspects of the energy density in the Hartree–Fock approximation. Details of the procedure for taking into account these polarizations consistently as well as the numerical work required will be discussed in a separate paper [18].

At present there is much interest in the possibility of chiral phase transitions on the quark level in connection with the existence of the quark-gluon plasma at finite temperature [29, 30] and finite temperature and density [31–34]. The developments of the present paper, utilizing the equivalence of the self-consistency condition and the Goldstone theorem in order to obtain information on the nature of the phase transition, might turn out to be useful in discussing these topics, too. For example, it is interesting that in the framework of the Nambu–Jona–Lasinio model in the mean field approximation a relation analogous to (3.15) has been derived in Ref. [33] for finite density and temperature.

Apart from chiral symmetry, some developments contained in the present paper might be of general interest for recently developed and widely used many-body
theories. For example, the constraints imposed by the baryon current conservation on the omega-nucleon interaction in the medium and the renormalization of the omega meson field have been investigated systematically. The results of Ref. [25] on the part of the Landau–Migdal interaction which arises from the density dependence of the mean fields produced by the $\sigma$ and $\omega$ mesons have been generalized. For the case of the Hartree–Fock approximation we have demonstrated explicitly the connection between more conventional approaches, in which the energy density is viewed as the sum of kinetic and potential energy terms, and the diagramatic expansion method, thereby putting some of the formal developments in Ref. [15] on a somewhat broader basis.

In conclusion, the chiral $\sigma$–$\omega$ model provides us with a useful tool for obtaining insight into the way in which chiral symmetry governs strong interaction processes in the nuclear medium. We attempted to clarify some of the consequences of these constraints and to illustrate them by simple examples. Further attempts in this direction are necessary in order to decide whether the model is sufficiently powerful to study many-body problems for purposes going beyond a formal exploration of symmetry constraints.

APPENDIX A: LOW-ENERGY THEOREMS IN THE $\sigma$ MODEL

1. Proof of Eq. (2.39)

In the following we will discuss two possible ways to derive Eq. (2.39).

(a) The axial Ward–Takahashi identities in the limit $q \to 0$, where $q$ is the momentum of the external axial vector field, impose so-called “consistency relations” or “low-energy theorems” on the strong interaction vertices. These identities are given in Section 5b of Ref. [11], Eq. (6) (referred to as Eq. (L5b.6)), for the mesonic $T$ matrices ($T$), and in Eq. (11) for the $\sigma$, $\pi$-irreducible parts ($\Gamma$). The latter quantities are related to the $T$ matrices by

$$T(p_1, \ldots, p_n; q_1, \ldots, q_m) = \Gamma_{n,m}(p_1, \ldots, p_n; q_1, \ldots, q_m) + \sigma, \pi\text{-irreducible part.}$$

Here the arguments denote the incoming meson momenta. The identities for the $\Gamma$'s are

$$\delta \Gamma_{n,m+1}^{i_1 \ldots i_m}(p_1, \ldots, p_n; q_1, \ldots, q_m, 0)$$

$$= \sum \delta^{i_j} \Gamma_n^{i_1 \ldots i_j \ldots i_m}(p_1, \ldots, p_n, q_1, \ldots, q_j, \ldots, q_m)$$

$$- \sum \Gamma_n^{i_1 \ldots i_{m-1} j}(p_1, \ldots, p_n, q_1, \ldots, q_{m-1}, p_1) - i\epsilon \delta_{n,0} \delta_{m,1} \delta^{i_j}. \quad (A.1)$$

The notation $\hat{q}_s$ means that the variable $q_s$ must be omitted. The isospin indices $i_x$ and $j$ are written in the same order as the pion momentum variables. For $n = 0$ the second term on the right-hand side of (A.1) is zero by definition. We have

$$\Gamma_{2,0}(p, -p) = i \Delta^{-1}_\sigma(p); \quad \Gamma_{ij}^{\sigma}(p, -p) - i \Delta^{-1}_\pi(p) \delta^{ij}; \quad \Gamma_{1,0}(0) = \hat{j}_\sigma, \quad (A.2)$$
where \( j_\sigma \) is the \( \sigma \)-irreducible part of the \( \sigma \) source as used in the main text. At first sight it therefore seems that Eq. (2.39) is simply the \( n = 0, m = 1 \) case of Eq. (A.1). One must note, however, that in Ref. [11, Eq. (L5b.3)], \( \Lambda_{1,0} \) is assumed to vanish identically, in which case (A.1) for \( n = 0, m = 1 \) reduces to the Goldstone theorem (2.38) which holds only at the physical point \( \vec{v}_0 \). One can, however, show quite easily that Eq. (2.39) is a necessary consequence of the relations (A.1) for \( (n, m) \neq (0, 1) \). (A direct proof of Eq. (2.39) will be given below.) For this, define a function (we leave out isospin indices)

\[
F_{0,1}(\vec{v}) = \vec{v}\Gamma_{0,2}(p, p) \Gamma_{1,0}(0) + ic.
\]

We know, due to Eqs. (2.34a) and (2.38), that \( F_{0,1}(\vec{v}_0) = 0 \). One can easily show that also all derivatives with respect to \( \vec{v} \) vanish at the physical point: \( F_{0,1}^{(k)}(\vec{v}_0) = 0 \) for \( k = 0, 1, 2, \ldots \). It then follows that \( F_{0,1}(\vec{v}) = 0 \) for all \( \vec{v} \). To show that all derivatives vanish at \( \vec{v}_0 \), we note that by generalizing the method discussed in Section 3.1 one can show that

\[
\Gamma_{n,m}^{(k)}(p_1, \ldots, p_n; q_1, \ldots, q_m) = \Gamma_{n+k,m}(p_1, \ldots, p_n, q_1, \ldots, q_m, 0) \quad \text{for general integers } n, m.
\]

i.e., taking the \( k \)th derivative with respect to \( \vec{v} \) generates a Green function with \( k \) additional (amputated) \( \sigma \) meson legs with zero momenta, indicated by the symbol \( k^{(0)} \). Using (A.4) one finds that the \( k \)th derivative of Eq. (A.3) is given by

\[
F_{0,1}^{(k)}(\vec{v}) = F_{0+k^{(0)},1}(\vec{v}),
\]

where for general integers \( n, m \) the function \( F_{n,m} \) is defined by

\[
F_{n,m}(\vec{v}) = \vec{v}\Gamma_{n,m+1}(p_1, \ldots, p_n; q_1, \ldots, q_m, 0)
- \sum_s \Gamma_{n+1,m-1}(p_1, \ldots, p_n, q_s; q_1, \ldots, q_s, q_m)
+ \sum_t \Gamma_{n-1,m+1}(p_1, \ldots, q_t, \ldots, p_n; q_1, \ldots, q_m, p_t) + ic \delta_{n0} \delta_{m1}.
\]

Due to Eqs. (A.1) and (A.5), all derivatives of \( F_{0,1} \) vanish at \( \vec{v}_0 \), and therefore \( F_{0,1} \) vanishes identically. This establishes Eq. (2.39) as the generalization of the Goldstone theorem (2.38). Moreover, from the above it is clear that we have generally

\[
F_{n,m}^{(k)}(\vec{v}) = F_{n+k^{(0)},m}(\vec{v}),
\]

which confirms that, if the identities (A.1) hold at the physical point, they also hold for any value of \( \vec{v} \).

(b) Let us now discuss a direct derivation of Eq. (2.39). For this, we note that, according to our discussions at the end of Section 2, once we regard \( \vec{v} \) as a free parameter, Eq. (2.29a) is not satisfied and therefore, a priori, all Green functions
calculated from the lagrangian (2.7b) contain also $\sigma$ tadpole insertions. Let us characterize those Green functions by a hat; e.g., $\hat{f}_{0,2}$ contains also the Feynman graph shown in Fig. 14. The Goldstone theorem was derived in Ref. [11] under the assumption $\langle \sigma \rangle = 0$. It is given in Eq. (L5b.4), which is equivalent to Eq. (2.38). If the restriction $\langle \sigma \rangle = 0$ is lifted, it is changed to

$$i(\tilde{v} + \langle \sigma \rangle) \hat{\Gamma}_{0,2}(0, 0) = c; \quad (A.7)$$

i.e., instead of $\tilde{v}$ we have the expectation value $\langle \phi \rangle$ of the original scalar field on the left-hand side, and the inverse pion propagator includes also the $\sigma$ tadpole insertions. Since the quantity $\Gamma_{1,0}$ is by definition the $\sigma$-irreducible part of the one-point Green function $\langle \sigma \rangle$, we have

$$\Gamma_{1,0} = -\hat{\Gamma}_{2,0}(0, 0) \langle \sigma \rangle. \quad (A.8)$$

The $\sigma$-reducible part of $\hat{f}_{0,2}$ (i.e., the part involving $\sigma$ tadpole insertions) can be separated as follows,

$$\hat{f}_{0,2}(0, 0) = \Gamma_{0,2}(0, 0) + \langle \sigma \rangle \hat{\Gamma}_{1,2}(0, 0, 0), \quad (A.9)$$

where the $\sigma\pi^2$ vertex $\hat{\Gamma}_{1,2}$ satisfies (see Eq. (A.1))

$$i\tilde{v} \hat{\Gamma}_{1,2}(0, 0, 0) = \hat{\Gamma}_{2,0}(0, 0) - \hat{\Gamma}_{0,2}(0, 0, 0). \quad (A.10)$$

The second term on the right-hand side of (A.9) is graphically represented by Fig. 14. Note that since the vertex $\hat{\Gamma}_{1,2}$ itself contains all possible $\sigma$ tadpole insertions in the $\sigma\pi^2$ vertex, the connected part of the second term in (A.9) includes all $\sigma$ tadpole insertions in the pion self-energy. Using (A.9) and (A.10) in (A.7) we obtain

$$c = i\tilde{v} \Gamma_{0,2}(0, 0) + i \langle \sigma \rangle \hat{\Gamma}_{2,0}(0, 0)$$

$$= i\tilde{v} \Gamma_{0,2}(0, 0) - i\Gamma_{1,0}(0; 0)$$

$$= -\tilde{v} A^{-1}_\pi(0) + j_\sigma, \quad (A.11)$$

where $A^{-1}_\pi(0)$ is the $\pi$ self-energy. Fig. 14. Graphical representation of the second term on the right-hand side of Eq. (A.9). The hatched rectangular box stands for the vertex $\hat{\Gamma}_{1,2}$. 
where in the second step we used (A.8) and in the third one (A.2). Equation (A.11) is the same as Eq. (2.39).

2. Demonstration of the Identity (2.40)

The low-energy theorems for the mesonic \( T \) matrices can be derived from the identities for the irreducible vertices, Eq. (A.1), and are given in Eq. (L5b.6) of Ref. [11]. Eq. (2.43) we need the special cases

\[
\bar{v} T(-q; q, 0) = i(\Delta_{\sigma}^{-1}(q) - \Delta_{\pi}^{-1}(q)) \tag{A.12a}
\]

\[
\bar{v} T(q_1, q_2; -q_1 - q_2, 0) = \Delta_{\sigma}(q_1 + q_2) \Delta_{\pi}^{-1}(q_1 + q_2) T(q_1, q_2, -q_1 - q_2; -q_1 - q_2)
- \Delta_{\pi}(q_1) \Delta_{\sigma}^{-1}(q_1) T(q_2; q_1, -q_1 - q_2)
- \Delta_{\pi}(q_2) \Delta_{\sigma}^{-1}(q_2) T(q_1; q_2, -q_1 - q_2) \tag{A.12b}
\]

\[
\bar{v} T^{\alpha\beta\gamma}(q_1, q_2, -q_1 - q_2, 0) = \delta^{\alpha\beta} \Delta_{\sigma}(q_1) \Delta_{\pi}^{-1}(q_1) T^{\beta\gamma}(q_1; q_2, -q_1 - q_2)
+ \delta^{\beta\gamma} \Delta_{\sigma}(q_1 + q_2) \Delta_{\pi}^{-1}(q_1 + q_2) T^{\gamma\alpha}(-q_1 - q_2; q_1, q_2)
+ \delta^{\gamma\alpha} \Delta_{\sigma}(q_2) \Delta_{\pi}^{-1}(q_2) T^{\alpha\beta}(q_2; q_1, -q_1 - q_2). \tag{A.12c}
\]

If we insert these relations together with the identity for the \( \pi NN \) vertex [35],

\[
\bar{v} \Gamma^j_\pi(q, q) = -\frac{\tau^j}{2} \{ S^{-1}(q), \gamma_5 \}, \tag{A.12d}
\]

into Eq. (2.43) we obtain

\[
\Sigma_\pi(0) = -i \frac{g Z_{\pi}}{v} \int \frac{d^4 q}{(2\pi)^4} \text{Tr} S(q) \tag{A.13a}
\]

\[
+ 3i \alpha^2 Z_{\pi} \int \frac{d^4 q}{(2\pi)^4} (\Delta_{\sigma}(q) + \Delta_{\pi}(q)) \tag{A.13b}
\]

\[
- i \frac{\lambda^2 Z_{\pi}}{v} \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} (\Delta_{\sigma}(q_1) \Delta_{\pi}(q_2) \Delta_{\sigma}(q_1 + q_2) T(q_1, q_2, -q_1 - q_2; -q_1 - q_2)
+ 3 \Delta_{\sigma}(q_2) \Delta_{\sigma}(q_2) T(q_1, q_2, -q_1 - q_2; q_1, q_2)), \tag{A.13c}
\]

where we also used (2.42). The terms (A.13a), (A.13b), and (A.13c) correspond to (2.43a), (2.43b), and (2.43c), respectively. Comparing (A.13) with (2.41) we arrive at (2.40).

APPENDIX B: PROOF OF RELATIONS (3.23)

The spinor \( f(p) \) obeys the Dirac equation

\[
(S^{-1}(p)|_{p_0 = \epsilon_p}) f(p) = 0 \tag{B.1}
\]
and the normalization condition

\[ \tilde{f}(p) \left( \frac{\partial S^{-1}(p)}{\partial \rho_0} \right)_{\rho_0 = \epsilon_p} f(p) = 1. \]  

(B.2)

Equation (B.2) is derived in Appendix B of Ref. [26], and we will comment on it at the end of this appendix. We differentiate (B.1) with respect to the mean fields and then multiply from the left by \( \tilde{f}(p) \):

\[ \frac{\partial \rho_0}{\partial z} \left( \frac{\partial S^{-1}(p)}{\partial \rho_0} \right)_{\rho_0 = \epsilon_p} f(p) + \tilde{f}(p) \frac{\partial S^{-1}(p)}{\partial \rho_0} \left. \right|_{\rho_0 = \epsilon_p} f(p) = 0 \quad (z = \tilde{v}, \tilde{w}^\mu). \]  

(B.3)

Due to (B.2) we obtain

\[ \frac{\partial \rho_0}{\partial z} \left( \frac{\partial S^{-1}(p)}{\partial \rho_0} \right)_{\rho_0 = \epsilon_p} f(p) = -\tilde{f}(p) \frac{\partial S^{-1}(p)}{\partial \rho_0} \left. \right|_{\rho_0 = \epsilon_p} f(p) \quad (z = \tilde{v}, \tilde{w}^\mu). \]  

(B.4)

Let us calculate the derivative of the nucleon propagator using the formula (3.9). For \( z = 0 \) we obtain

\[ \frac{\partial}{\partial \rho_0} S(x' - x) = -i \int d^4 y ' \langle 0 | T(\psi(x')\bar{\psi}(x) \hat{S}_\sigma(y)) | 0 \rangle_{\text{conn. } \sigma - \text{irred.}}. \]  

(B.5)

Here, as in Eq. (3.10), we must take only the \( \sigma \)-irreducible part, since we understand that the propagator \( S \) does not include \( \sigma \) tadpoles, i.e., is \( \sigma \) irreducible. Further evaluation of (B.5) follows closely the steps which led from Eq. (3.10) to (3.12): We use the equation of motion for \( \sigma \) to replace \( -i \hat{S}_\sigma \) in (B.5) by \( (Z_M \Box + \tilde{m}^2 + \delta \tilde{m}^2) \sigma \). Since \( \delta \) commutes with the nucleon field at equal times, the d’Alembert operator can be taken out of the \( T \) product, and we can then use the definition of the irreducible \( \sigma \)NN vertex,

\[ \langle 0 | T(\psi(x')\bar{\psi}(x)\sigma(y')) | 0 \rangle \]

To obtain the \( \sigma \)-irreducible piece of (B.5), we again must replace \( \Delta_\sigma \to \tilde{\Delta}_\sigma \) effectively (see Eq. (2.21b)), and we obtain

\[ \frac{\partial S(x' - x)}{\partial \rho_0} = i \int d^4 y ' \int d^4 u ' \int d^4 u S(x' - u') \Gamma_\sigma(u' - y, u - y) S(u - x). \]  

(B.6)

In momentum space this becomes

\[ \frac{\partial S(p)}{\partial \rho_0} = i S(p) \Gamma_\sigma(p, p) S(p). \]  

(B.8)
or

$$\frac{\delta S^{-1}(p)}{\delta \bar{v}} = -i\Gamma_\sigma(p, p). \quad (B.9a)$$

A completely equivalent treatment for the case \( z = \tilde{w}^\mu \) gives

$$\frac{\delta S^{-1}(p)}{\delta \tilde{w}_\mu} = -i\Gamma_\mu(p, p). \quad (B.9b)$$

Using Eqs. (B.9) in (B.4) we arrive at Eqs. (3.23).

Let us add a few comments concerning Eq. (B.2): It is equivalent to the condition [26] that the quantity \( \Sigma_\sigma f_\sigma(p) \) be the residue of \( S(p) \) at the quasiparticle pole \( \rho_0 = \epsilon_\rho \). For zero density \( \epsilon_\rho \to E_\rho = \sqrt{p^2 + M_\rho^2} \). If we choose the renormalization point \( \mu_N = m_N \) in Eq. (2.14a), where \( m_N \) is the nucleon mass parameter in the lagrangian, then \( M_\rho = m_N \), and for zero density \( f(p) \to \sqrt{M_\rho/E_\rho} u(p) \) with \( u\bar{u} = 1 \), as follows from Eq. (B.2). For other choices of \( \mu_N \) the spinors in free space, normalized as \( Uu = 1 \), must be multiplied by a (finite) renormalization factor, which depends on the self-energy and is determined from Eq. (B.2).

APPENDIX C: PROOF OF EQ. (4.19)

Let us start from the expression for \( E_I \), Eq. (4.14), in the HF approximation. We split it into two parts,

$$E_I = E_{I,\text{nuc}} + E_{I,\text{mes}},$$

with

$$E_{I,\text{nuc}} = i \int \frac{d^4k}{(2\pi)^4} k^0 \left\{ \text{Tr} \left( \frac{\delta S}{\delta k_0} \Sigma_N \right) + \frac{1}{2} \text{Tr} \left( A \frac{\delta \Sigma_{\text{nuc}}}{\delta k_0} \right) + \frac{3}{2} A_\sigma \frac{\delta \Sigma_{n,\text{nuc}}}{\delta k_0} \right\}$$

$$+ \frac{1}{2} k^0 A_\sigma \frac{\delta \Sigma_{\text{mes}}}{\delta k_0} + \frac{3}{2} k^0 A_\sigma \frac{\delta \Sigma_{n,\text{mes}}}{\delta k_0} \quad \left( C.1 \right)$$

$$E_{I,\text{mes}} = i \int \frac{d^4k}{(2\pi)^4} \left\{ \frac{1}{4} (\Sigma_{\sigma4} A_\sigma + 3 \Sigma_{\sigma4} A_\rho) + \frac{1}{3} (\Sigma_{\sigma3} A_\sigma + 3 \Sigma_{\pi3} A_\rho) \right\}$$

$$+ \frac{1}{2} k^0 A_\sigma \frac{\delta \Sigma_{\text{mes}}}{\delta k_0} + \frac{3}{2} k^0 A_\sigma \frac{\delta \Sigma_{n,\text{mes}}}{\delta k_0} \quad \left( C.2 \right),$$

where we used the notations introduced in (4.16a) and (4.17a) for the meson self-energies. Let us also split the nucleon self-energy as \( \Sigma_N = \Sigma_{N1} + \Sigma_{N2} \) according to
the two terms in (4.15). Consider first the second term in (C.1), which has the form

\[ \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} k^0 \Delta^a_{ab} \frac{\partial \Sigma_{\text{nuc}}^b}{\partial k^0} \]

\[ = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} k^0 \Delta^a_{ab}(q) \text{Tr} \left( \Gamma^b \frac{\partial S(k+q)}{\partial q^a} \Gamma^a S(q) \right) \]

\[ = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} q^0 \Delta^a_{ab}(q) \text{Tr} \left( \Gamma^b S(k+q) \Gamma^a \frac{\partial S(k)}{\partial k^0} \right), \quad (C.3) \]

where the second equality follows from a partial integration and interchanging \( k \) and \( q \). We combine (C.3) with the part involving \( \Sigma_{N1} \) in the first term of (C.1):

\[ A_{\sigma\omega} = \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} k^0 \left\{ \text{Tr} \left( \frac{\partial S}{\partial k^0} \Sigma_{N1} \right) + \frac{1}{2} \text{Tr} \left( \Delta \frac{\partial \Sigma_{\text{nuc}}}{\partial k^0} \right) \right\} \]

\[ = \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \Delta^a_{ab}(q) \text{Tr} \left( \Gamma^b S(k) \Gamma^a \frac{\partial S(k+q)}{\partial k^0} \right) \left( k^0 + q^0 \right) \]

\[ = \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \Delta^a_{ab}(q) \text{Tr} \left( S(k) \Gamma^b S(k+q) \Gamma^a \right) \]

\[ = \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \Delta^a_{ab}(q) \text{Tr} \left( S(k) \Gamma^b \frac{\partial S(k+q)}{\partial k^0} \Gamma^a \right) \left( k^0 + q^0 \right), \quad (C.4) \]

If we replace \( k \to k - q \) and then \( q \to -q \) in the last term and note that \( \Delta^a_{ab}(-q) = \Delta^a_{ba}(q) \) (this relation follows from the form of the self-energy (4.16b) and the Dyson equation (2.25)), it is seen [15] that the last term is equal to \(-A_{\sigma\omega}\), and hence

\[ A_{\sigma\omega} = -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \Delta^a_{ab}(q) \text{Tr} \left( S(k) \Gamma^b S(k+q) \Gamma^a \right). \]

Proceeding in the same way for the pionic part \( A_\pi \), we obtain

\[ E_{l,\text{nuc}} = \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \left( \text{Tr}(A \Sigma_{\text{nuc}} + 3A \Sigma_{\text{nuc}}) \right) \]

\[ = -\frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \text{Tr}(S \Sigma_{N1}), \quad (C.5) \]

as in Eqs. (4.19) and (4.20). This contribution is shown graphically in Fig. 7a. Consider now the purely mesonic terms. The first term in (C.2) appears, in the same
form, in (4.19) and is shown graphically in Fig. 7c. The sum of the last two terms in (C.2) is

\[
B = \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} k^0 \left\{ \Delta_\sigma \frac{\partial \Sigma_{\alpha 3}}{\partial k_0} + 3 \delta_\pi \frac{\partial \Sigma_{\pi 3}}{\partial k_0} \right\} 
\]

\[= -9(\lambda^2Z_\sigma)^2 \tilde{v}^2 \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} k^0 \Delta_\sigma(k) \Delta_\sigma(q) \frac{\partial \Delta_\sigma(k + q)}{\partial k_0} \]

\[-6(\lambda^2Z_\pi)^2 \tilde{v}^2 \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \Delta_\pi(k) \Delta_\pi(q) \frac{\partial \Delta_\pi(k + q)}{\partial k_0} \left(k_0 + \frac{q_0}{2}\right) \]

\[= B_1 + B_2. \quad (C.6)\]

We now consider the terms \(B_1\) and \(B_2\) separately and treat them in the same way as \(A_{\alpha 00}\) in Eq. (B.4); i.e., first we make a partial integration in \(k_0\), which produces two terms, one involving simply a product of propagators and the other involving one derivative. In the latter term we replace \(k \rightarrow k - q\) and then \(q \rightarrow -q\) to obtain

\[
B_1 = 3(\lambda^2Z_\sigma)^2 \tilde{v}^2 \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \Delta_\sigma(k) \Delta_\sigma(q) \Delta_\sigma(k + q) 
\]

\[= \frac{i}{6} \int \frac{d^4k}{(2\pi)^4} (\Sigma_{\alpha 3} \Delta_\sigma + 3 \Sigma_{\pi 3} \Delta_\sigma). \quad (C.7)\]

\[
B_2 = 3(\lambda^2Z_\pi)^2 \tilde{v}^2 \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \Delta_\pi(k) \Delta_\pi(q) \Delta_\pi(k + q). \]

Therefore \(B\) can be written as

\[
B - B_1 + B_2 = \frac{i}{6} \int \frac{d^4k}{(2\pi)^4} (\Sigma_{\alpha 3} \Delta_\sigma + 3 \Sigma_{\pi 3} \Delta_\sigma). \]

Using this in (C.2) we obtain

\[
E_{i,mes} = \frac{i}{4} \int \frac{d^4k}{(2\pi)^4} (\Sigma_{\alpha 3} \Delta_\alpha + 3 \Sigma_{\pi 3} \Delta_\pi) + \frac{i}{6} \int \frac{d^4k}{(2\pi)^4} (\Sigma_{\sigma 3} \Delta_\sigma + 3 \Sigma_{\pi 3} \Delta_\pi), \quad (C.8)\]

as in Eq. (4.19). The second term in Eq. (C.8) is shown graphically in Fig. 7b.

**APPENDIX D: LOWEST ORDER \(\sigma\) AND \(\pi\) SELF-ENERGIES IN FREE SPACE**

In the potential (3.3b) there enter the inverse propagators in free space

\[
A_{\alpha}^{-1}(k^2) = k^2 - m_\alpha^2 - \Sigma_\alpha(k^2) \quad (\alpha = \sigma, \pi) \quad (D.1a)
\]

with

\[
\Sigma_\alpha(k^2) = \Sigma_\alpha(k^2) + \delta m_\alpha^2 - (Z_M - 1) k^2, \quad (D.1b)
\]
where the unrenormalized self-energy $\Sigma_\pi$ is calculated from the Feynman diagrams in Figs. 6b and 6c in lowest order, and the renormalization constants are determined from Eq. (2.14b) as

$$
(Z_M - 1) = \frac{\partial \Sigma_\pi}{\partial k^2}_{k^2 = \mu^2} \Bigg|_{k^2 = \mu^2},
$$

$$
\delta m^2_\pi - \mu^2_\pi (Z_M - 1) - \Sigma_\pi(\mu^2_\pi).
$$

If $\mu^2_\pi > 2m^2_\pi$, the real part of $\Sigma_\pi(\mu^2_\pi)$ should be taken in Eq. (D.2b). The unrenormalized self-energies are (see Figs. 6b, 6c and Eqs. (4.16), (4.17))

$$
\Sigma_\sigma(k^2) = \frac{\lambda^2}{2} (3F_1(m^2_\sigma) + 3F_1(m^2_\sigma) + 6\lambda^2v^2F_2(k^2, m^2_\sigma, m^2_\sigma) + 18\lambda^2v^2F_2(k^2, m^2_\sigma, m^2_\sigma)) - 8g^2F_1(m^2_\sigma) - 4g^2(k^2 - 4m^2_\pi)F_2(k^2, m^2_\pi, m^2_\pi),
$$

$$
\Sigma_\pi(k^2) = \frac{\lambda^2}{2} (5F_1(m^2_\pi) + F_1(m^2_\sigma) + 4\lambda^2v^2F_2(k^2, m^2_\pi, m^2_\pi)) - 8g^2F_1(m^2_\pi) + 4g^2k^2F_2(k^2, m^2_\pi, m^2_\pi).
$$

Here we introduced the functions [12]

$$
F_1(m^2) = i \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 - m^2},
$$

$$
F_2(k^2, m^1_\pi, m^2_\pi) = i \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2 - m^1_\pi)((k - q)^2 - m^2_\pi)},
$$

which are calculated by using dimensional regularization,

$$
F_1(m^2) = - \frac{m^2}{16\pi^2} \left( \Gamma \left(2 - \frac{n}{2}\right) - \ln m^2 + 1 \right),
$$

$$
F_2(k^2, m^1_\pi, m^2_\pi) = - \frac{1}{16\pi^2} \left( \Gamma \left(2 - \frac{n}{2}\right) - f_2(k^2, m^1_\pi, m^2_\pi) \right),
$$

with

$$
f_2(k^2, m^1_\pi, m^2_\pi) = \int_0^1 dx \ln(m^1_\pi + (m^2_\pi - m^1_\pi)x - k^2x(1 - x)).
$$

From the expression (D.3) and (D.4) the counterterms (D.2) are calculated, and the results are inserted into (D.1b). For $k^2 = 0$, which is the case required in Eq. (3.3b), we obtain
\[
16\pi^2 \Sigma_{\sigma}(0) = 6\lambda^4 v^2 (f_2(0, m^2_\sigma, m^2_\pi) - f_2(\mu^2_\sigma, m^2_\sigma, m^2_\pi)) \\
+ 18\lambda^4 v^2 (f_2(0, m^2_\sigma, m^2_\pi) - f_2(\mu^2_\sigma, m^2_\sigma, m^2_\pi)) \\
- 16\lambda^2 m^2_N (f_2(0, m^2_N, m^2_N) - f_2(\mu^2_N, m^2_N, m^2_N)) \\
+ 4\lambda^4 \mu^2_N f_2(\mu^2_\sigma, m^2_\sigma, m^2_\pi) + 4\lambda^2 \mu^2_N f_2(\mu^2_\sigma, m^2_\sigma, m^2_\pi) \\
+ 4\lambda^2 \mu^2_N (f_2(\mu^2_\sigma, m^2_N, m^2_N) - f_2(\mu^2_\sigma, m^2_N, m^2_N)), \quad (D.7)
\]

\[
16\pi^2 \Sigma_{\pi}(0) = 4\lambda^2 v^2 (f_2(0, m^2_\sigma, m^2_\pi) - f_2(\mu^2_\sigma, m^2_\sigma, m^2_\pi)) \\
+ 4\lambda^4 \mu^2_N f_2(\mu^2_\sigma, m^2_\pi, m^2_\pi) + 4\lambda^2 \mu^2_N f_2(\mu^2_\sigma, m^2_\pi, m^2_\pi). \quad (D.8)
\]

Using the form (D.6c) one can calculate these expressions analytically. In particular, for \(m_1 = m_2\) the function \(f_2\) takes the simple form

\[
f_2(\mu^2, m^2, m^2) = \ln m^2 - A_{\mu m},
\]

where

\[
A_{\mu m} = \begin{cases} 
2 - 2y_{\mu m} \arctg(y_{\mu m}^{-1}) & \text{if } \mu < 2m \\
2 - y_{\mu m} \ln|1 + y_{\mu m}|(1 - y_{\mu m}) & \text{if } \mu > 2m
\end{cases}
\]

with \(y_{\mu m} = \sqrt{4m^2/\mu^2 - 1}\). The self-energies at finite density for arbitrary choice of the renormalization points are obtained by inserting (D.7) and (D.8) into (4.38c) and (4.39c).

In the Hartree approximation, the dependence of the energy density on the renormalization point \(\mu^2\) can be absorbed into a redefinition of the coupling constant \(g_\omega\), as is clear from the formulae in the main text. (For the explicit form of \(\delta m^2_\omega\), see Ref. [20].)

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**References**