NON-PERTURBATIVE ASPECTS OF GAUGE THEORIES

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Solutions to problems
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Problem 1

Write the resolution of identity that defines the Nicolai map as a Hitchin system in terms of the non-hermitian connection $B_\lambda = (A_z + \lambda D_u)dz + (A_{\bar{z}} + \lambda^{-1} D_{\bar{u}})d\bar{z}$.

Solution

First of all we want to write equations

$$\begin{cases} F^-_{01} = \mu^-_{01} \\ F^-_{02} = \mu^-_{02} \\ F^-_{03} = \mu^-_{03} \end{cases} \tag{1}$$

in terms of complex coordinates $z = x_0 + ix_1$, $\bar{z} = x_0 - ix_1$, $u = x_2 + ix_3$ and $\bar{u} = x_2 - ix_3$ on $R^2 \times R^2_\theta$, where θ is the non-commutative parameter satisfying $[\partial_u, \partial_{\bar{u}}] = \theta^{-1}1$.
By definition

$$F^-_{\mu\nu} = F_{\mu\nu} - \tilde{F}_{\mu\nu}, \quad \tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F_{\rho\sigma} \tag{2}$$

therefore

$$\begin{aligned} F^-_{01} &= F_{01} + F_{32} \\ F^-_{02} &= F_{02} + F_{13} \\ F^-_{03} &= F_{03} + F_{21} \end{aligned} \tag{3}$$

Introducing $y^\alpha = (z, \bar{z}, u, \bar{u})$ one has

$$F_{01} = \frac{\partial y^\alpha}{\partial x^0}\frac{\partial y^\beta}{\partial x^1}F_{\alpha\beta} = \frac{\partial z}{\partial x^0}\frac{\partial \bar{z}}{\partial x^1}F_{z\bar{z}} + \frac{\partial \bar{z}}{\partial x^0}\frac{\partial z}{\partial x^1}F_{\bar{z}z} = -2iF_{z\bar{z}} \tag{4}$$

$$F_{32} = \frac{\partial y^\alpha}{\partial x^3}\frac{\partial y^\beta}{\partial x^2}F_{\alpha\beta} = \frac{\partial u}{\partial x^3}\frac{\partial \bar{u}}{\partial x^2}F_{u\bar{u}} + \frac{\partial \bar{u}}{\partial x^0}\frac{\partial u}{\partial x^2}F_{\bar{u}u} = 2iF_{u\bar{u}} \tag{5}$$

$$F_{02} = F_{zu} + F_{z\bar{u}} + F_{\bar{z}u} + F_{\bar{z}\bar{u}} \tag{6}$$

$$F_{13} = -F_{zu} + F_{z\bar{u}} + F_{\bar{z}u} - F_{\bar{z}\bar{u}} \tag{7}$$

$$F_{03} = i(F_{zu} - F_{z\bar{u}} + F_{\bar{z}u} - F_{\bar{z}\bar{u}}) \tag{8}$$

$$F_{21} = i(F_{uz} - F_{u\bar{z}} + F_{\bar{u}z} - F_{\bar{u}\bar{z}}) \tag{9}$$

thus

$$F^-_{01} = \mu^-_{01} \longrightarrow i(F_{u\bar{u}} - F_{z\bar{z}}) = \mu_0, \quad \mu_0 = \frac{1}{2}\mu^-_{01} \tag{10}$$

Now defining $D_u = \partial_u + iA_u$ and $D_{\bar{u}} = \partial_{\bar{u}} + iA_{\bar{u}}$ one has

$$\begin{aligned} [D_u, D_{\bar{u}}] &= [\partial_u, \partial_{\bar{u}}] + i([\partial_u, A_{\bar{u}}] + [A_u, \partial_{\bar{u}}]) - [A_u, A_{\bar{u}}] \\ &= \theta^{-1}1 + i(\partial_u A_{\bar{u}} - \partial_{\bar{u}} A_u) - [A_u, A_{\bar{u}}] \end{aligned} \tag{11}$$

so

$$F_{u\bar{u}} = \partial_u A_{\bar{u}} - \partial_{\bar{u}} A_u + i[A_u, A_{\bar{u}}] = i(\theta^{-1}1 - [D_u, D_{\bar{u}}]) \tag{12}$$

and (10) becomes

$$-iF_{z\bar{z}} + [D_u, D_{\bar{u}}] - \theta^{-1}1 = \mu_0 \tag{13}$$

Moreover

$$F^-_{02} = 2(F_{z\bar{u}} + F_{\bar{z}u}) = \mu^-_{02} \tag{14}$$
$$F^-_{03} = 2i(F_{\bar{z}u} - F_{z\bar{u}}) = \mu^-_{03} \tag{15}$$

can be combined in

$$F_{z\bar{u}} = \frac{1}{4}(\mu^-_{02} + i\mu^-_{03}) = n \tag{16}$$
$$F_{\bar{z}u} = \frac{1}{4}(\mu^-_{02} - i\mu^-_{03}) = \bar{n} \tag{17}$$

Defining $D_z = \partial_z + i[A_z, \ldots]$ and $D_{\bar{z}} = \partial_{\bar{z}} + i[A_{\bar{z}}, \ldots]$ then

$$-iD_z D_{\bar{u}} = -i\partial_z(\partial_{\bar{u}} + iA_{\bar{u}}) + [A_z, \partial_{\bar{u}} + iA_{\bar{u}}] = \partial_z A_{\bar{u}} - \partial_{\bar{u}} A_z + i[A_z, A_{\bar{u}}] = F_{z\bar{u}} \tag{18}$$

and $-iD_{\bar{z}} D_u = F_{\bar{z}u}$. Actually eq. (1) becomes

$$\begin{cases} -iF_{z\bar{z}} + [D_u, D_{\bar{u}}] - \theta^{-1}1 = \mu_0 \\ -iD_z D_{\bar{u}} = n \\ -iD_{\bar{z}} D_u = \bar{n} \end{cases} \tag{19}$$

Now defining the non-hermitian connection $B_\lambda = (A_z + \lambda D_u)dz + (A_{\bar{z}} + \lambda^{-1} D_{\bar{u}})d\bar{z}$, then

$$\begin{aligned} &F_{z\bar{z}}(B_\lambda) = \partial_z B_{\bar{z}} - \partial_{\bar{z}} B_z + i[B_z, B_{\bar{z}}] = \partial_z A_{\bar{z}} + \lambda^{-1}\partial_z D_{\bar{u}} - \partial_{\bar{z}} A_z \\ &-\lambda\partial_{\bar{z}} D_u + i([A_z, A_{\bar{z}}] + \lambda^{-1}[A_z, D_{\bar{u}}] + \lambda[D_u, A_{\bar{z}}] + [D_u, D_{\bar{u}}]) \\ &= F_{z\bar{z}} + i[D_u, D_{\bar{u}}] + \lambda^{-1} D_z D_{\bar{u}} - \lambda D_{\bar{z}} D_u \\ &= i(\mu_0 + \theta^{-1}1 + \lambda^{-1} n - \lambda\bar{n}) \end{aligned} \tag{20}$$

therefore the first of eq. (19) can be replaced by

$$-iF_{z\bar{z}}(B_\lambda) = \mu_0 + \theta^{-1}1 + \lambda^{-1} n - \lambda\bar{n} \tag{21}$$

Problem 2

Consider the second order contributions to $\langle \mathrm{Tr}\Psi_\lambda \rangle$, where

$$\Psi_\lambda = P \exp i \oint (A_z + \lambda D_u)dz + (A_{\bar{z}} + \lambda^{-1} D_{\bar{u}})d\bar{z} \tag{22}$$

is the twistor Wilson loop.

1. Find the terms that are obviously zero.
2. On the basis of dimensional considerations show how terms that are not obviously zero vanish for $\theta \to \infty$.
3. Explain why dimensional considerations could be insufficient to demostrate that some of these terms vanish for $\theta \to \infty$.
4. Show if all these terms vanish for finite θ.

Solution

The lowest non-trivial order in perturbation theory is

$$\langle \oint_s [(A_z + \lambda D_u)dz + (A_{\bar{z}} + \lambda^{-1} D_{\bar{u}})d\bar{z}](s) \oint_{s'<s} [(A_z + \lambda D_u)dz + (A_{\bar{z}} + \lambda^{-1} D_{\bar{u}})d\bar{z}](s') \rangle \tag{23}$$

where $D_u = \partial_u + iA_u$ and $D_{\bar{u}} = \partial_{\bar{u}} + iA_{\bar{u}}$.

1) Non-commutativity breaks the Euclidian $O(4)$ symmetry leaving a residual $O(2) \times O(2)$ symmetry in the planes $(z, \bar{z})$ and $(u, \bar{u})$. Therefore the only vevs that respect $O(2)$ invariance of the vacuum involve the products $\partial_u \partial_{\bar{u}}$ and $\partial_{\bar{u}} \partial_u$ and fields contractions.

The trace in (23) is over the tensor product of the $U(N)$ Lie algebra and the infinite dimensional Fock space that defines the Hilbert space representation of the non-commutative plane $(u, \bar{u})$. In fact the commutation relation $[u, \bar{u}] = \theta 1$ implies $u = \sqrt{\theta}\ a$ and $\bar{u} = \sqrt{\theta}\ a^\dagger$, where a and $a^\dagger$ are the annihilation and creation operators respectively. Moreover imposing $\partial_u u = \partial_{\bar{u}} \bar{u} = 1$ one gets $\partial_u = -[\theta^{-1}\bar{u},\ .\ ] = -[\theta^{-1/2} a^\dagger,\ .\ ]$ and $\partial_{\bar{u}} = [\theta^{-1} u,\ .\ ] = [\theta^{-1/2} a,\ .\ ]$ with $[\partial_u, \partial_{\bar{u}}] = \theta^{-1} 1$.

Finally, since the Euclidean metric in complex coordinates $(z, \bar{z}, u, \bar{u})$ is off-diagonal, the only non vanishing contractions are $\langle A_z A_{\bar{z}} \rangle$ and $\langle A_u A_{\bar{u}} \rangle$.

2) In the large-θ limit in the Feynman gauge $\langle A_z A_{\bar{z}} \rangle = \langle A_u A_{\bar{u}} \rangle$ so that in (23) they cancel because of the i in the definition of D_u and $D_{\bar{u}}$. Thus one is left with

$$\oint ds \oint_{s'<s} ds' \langle \partial_u(s) \partial_{\bar{u}}(s') \dot{z}(s) \dot{\bar{z}}(s') + \partial_{\bar{u}}(s) \partial_u(s') \dot{\bar{z}}(s) \dot{z}(s') \rangle \tag{24}$$

Let's consider the first term in the sum. This can be written as

$$\partial_u(s)\partial_{\bar{u}}(s') = \frac{[\partial_u(s), \partial_{\bar{u}}(s')]}{2} + \frac{\{\partial_u(s), \partial_{\bar{u}}(s')\}}{2} \tag{25}$$

The term with the commutator is $\sim \theta^{-1}1$ and comes out of the integral. The one with the anticommutator, being symmetric in s and s', comes out of the integral as well (the dependence on s and s' was a sort of fake dependence that has been introduced originally to remember the order between the two operators, but once this has been taken into account the dependence disappears). Before taking the vev one has to trace this expression over the Fock space. The term with the commutator gives $\theta^{-1}\mathrm{Tr}1$ while the one with the anticommutator, since $\{\partial_u, \partial_{\bar{u}}\} \sim \theta^{-1}\mathcal{N}$ where $\mathcal{N} = a^\dagger a$ is the number operator, gives $\theta^{-1}\sum_n \langle n|\mathcal{N}|n\rangle = \theta^{-1}\sum_n n$. Once divided by the normalization factor Tr1 the first contribution is finite while the second is infinite. But both the results must be multiplied by the integral of a constant over the loop that is zero. The same argument applies to the second term in (24), therefore these terms are zero even for finite θ.

3) We have seen that the contribution involving the anticommutator is infinite, thus, even if by dimensional analysis this goes like θ^{-1}, it is not guaranteed to vanish in the $\theta \to \infty$ limit. It is anyway zero since it is multiplied by the integral of a constant over the loop that is zero.

4) For finite theta not all the terms vanish. In fact, for instance, the terms $\langle A_z A_{\bar{z}}\rangle$ and $\langle A_u A_{\bar{u}}\rangle$ are only equal (and thus cancel) in the large-θ limit.

Problem 3

Find whether in Nekrasov formula for the $\mathcal{N} = 2$ SYM localization 'commutes' with Pauli-Villars regularization of the zero modes of the determinants.

Solution

In Nekrasov computation of the partition function of $\mathcal{N} = 2$ SYM theory there are two localizations. In the first localization, where susy is used, one reduces the partition function to a sum of finite dimensional integrals over the moduli space of instantons. Afterwards, by means of a second localization, one can reduce the finite dimensional integrals to a sum of contributions of (noncommutative) Abelian instantons that are the fixed points of a torus action. In order to perform this second localization, *i.e.* in order to have a vanishing coboundary integral, one has to compactify the moduli space of the instantons performing a noncommutative deformation of the gauge theory, *i.e.* defining it on a noncommutative space-time. In this case in fact the scale size parameter ρ of the instanton is bounded below at $\sqrt{\theta}$, where θ is the noncommutative parameter, so that the UV non-compactness is cured. The torus action on the noncommutative deformation of the moduli space that allows one to perform the second localization is the diagonal Cartan subgroup of $SU(N) \times SO(4)$. The latter is the group acting on the instantons moduli space in a $SU(N)$ gauge theory on ordinary commutative space-time.
The number of bosonic collective coordinates, *i.e.* the dimension of the muduli space, for an instanton of charge $k = 1$ in ordinary $SU(2)$ is eight. These are the position and the size ρ of the instantons which are coordinates on the manifold $SO(5,1)/SO(5)$ of the Euclidean conformal group divided the subgroup $SO(5)$ (consisting of $SO(4)$ rotations and a combinations of conformal boosts and traslations) that leaves the instanton invariant up to gauge transformations. In addition there are three gauge orientation collective coordinates corresponding to a rigid $SU(2)$ symmetry.
Instantons in $SU(N)$ can be obtained by embedding $SU(2)$ instantons in $SU(N)$. If one considers the embedding in which the instanton resides in the 2×2 block on the lower right of a $N \times N$ matrix, it is easy to see that the transformations generating new solutions belong to $SU(N)/SU(N-2) \times U(1)$ where $SU(N-2) \times U(1)$ is the stability group leaving the $SU(N)$ instanton solution invariant. The number of collective coordinates, *i.e.* the dimension of the previous coset space, is $4N - 5$ that together with the position and the scale of the $SU(2)$ solution yields a total of $4N$ ($4Nk$ for an instanton of charge k) collective coordinates. On the top of that there are also fermionic collective coordinates. A k-instanton solution in $SU(N)$ has $2Nk$ fermionic collective coordinates for fermions in the adjoint representation and k for fermion in the fundamental representation.

Once the second localization is performed, namely a noncommutative space-time is introduced, one has to consider the fixed points of the aforementioned torus action. These are $U(N)$ (noncommutative) instantons which split as a sum of $U(1)$ noncommutative instantons corresponding to N commuting $U(1)$ subgroups of $U(N)$. But these instantons have no moduli and thus no zero modes, therefore one has to regularize first and then localize at the fixed points otherwise one would miss the correct powers of the Pauli-Villars regulator.

Problem 4

Compute the 1-loop β function of pure YM by the background field method.

Solution

The partition function of pure YM is

$$Z = \int \mathcal{D}A \; e^{-S_{YM}} \tag{26}$$

where

$$S_{YM} = \frac{1}{2g^2} \int d^4x \; \mathrm{tr} F_{\mu\nu} F_{\mu\nu} = \frac{1}{4g^2} \int d^4x \; F^a_{\mu\nu} F^a_{\mu\nu} \tag{27}$$

In the previous formula, we used the following normalization for the trace in the fundamental representation

$$\mathrm{tr}(t^a t^b) = \frac{1}{2} \, \delta^{ab} \tag{28}$$

The YM field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \tag{29}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f^{abc} A^b_\mu A^c_\nu \tag{30}$$

Now split the gauge field into a classical background field and a fluctuating quantum field $A_\mu = \bar{A}_\mu + \delta A_\mu$. Then the YM field strength decomposes as follows

$$F_{\mu\nu}(\bar{A} + \delta A) = F_{\mu\nu}(\bar{A}) + D_\mu(\bar{A})\delta A_\nu - D_\nu(\bar{A})\delta A_\mu + i[\delta A_\mu, \delta A_\nu] \tag{31}$$

where $D_\mu(\bar{A}) = \partial_\mu + i\bar{A}_\mu$ is the covariant derivative in the adjoint represention, *i.e.*

$$D_\mu(\bar{A})\delta A_\nu = \partial_\mu \delta A_\nu + i[\bar{A}_\mu, \delta A_\nu] \tag{32}$$

$$(D_\mu(\bar{A})\delta A_\nu)^a = \partial_\mu \delta A^a_\nu - f^{abc} \bar{A}^b_\mu \delta A^c_\nu = D^{ac}_\mu(\bar{A}) \delta A^c_\nu \tag{33}$$

with

$$D^{ac}_\mu(\bar{A}) = \partial_\mu \delta^{ac} - f^{abc} A^b_\mu = \partial_\mu \delta^{ac} + i A^{ac}_\mu, \quad A^{ac}_\mu = A^b_\mu (T^b)^{ac} = i f^{abc} A^b_\mu \tag{34}$$

To define the functional integral we must gauge-fix using the Faddeev-Popov procedure. We choose a gauge fixing condition which is covariant with respect to the background gauge field

$$G(A) = D_\mu \delta A_\mu - c \tag{35}$$

The Faddeev-Popov determinant involves the variation of this operator with respect to the gauge transformation

$$\delta A_\mu \to \delta A_\mu - D_\mu \omega \tag{36}$$

namely, one has

$$\det\left(\frac{\delta G}{\delta\omega}\right) = \det(-\Delta_{\bar A}) = \int D\eta D\bar\eta \ \exp\left[\int d^4x \ \bar\eta(-\Delta_{\bar A})\eta\right] \tag{37}$$

where

$$\Delta_{\bar A} = D_\mu(\bar A) D_\mu(\bar A) = \partial^2 + i\partial_\mu A_\mu + 2iA_\mu\partial_\mu - A_\mu A_\mu \tag{38}$$

and η and $\bar\eta$ are anticommuting fields (FP ghosts) belonging to the adjoint representation. As usual we can promote the gauge-fixing term to the exponent, to quantize the theory in the background field analogue of Feynman-'t Hooft gauge. Namely, we can insert into the functional integral

$$\int Dc \exp\left[-\frac{1}{\xi g^2}\int d^4x \ \mathrm{tr}(c^2)\right]\delta(D_\mu\delta A_\mu - c) \tag{39}$$

that corresponds to adding a new term $\mathrm{tr}(D_\mu\delta A_\mu)^2/\xi g^2$ to the Lagrangian. Then the gauge-fixed Lagrangian is

$$\mathcal{L}_{FP} = \frac{1}{2g^2}\ \mathrm{tr}F^2_{\mu\nu}(\bar A + \delta A) + \frac{1}{\xi g^2}\ \mathrm{tr}(D_\mu\delta A_\mu)^2 + \ \bar\eta\ (-\Delta_{\bar A})\eta \tag{40}$$

Keeping only the quadratic terms in δA_μ (since we are doing a 1-loop computation)

$$\begin{aligned} &F^2_{\mu\nu}(\bar A + \delta A) = F^2_{\mu\nu}(\bar A) + (D_\mu(\bar A)\delta A_\nu - D_\nu(\bar A)\delta A_\mu)^2 + 2iF_{\mu\nu}(\bar A)[\delta A_\mu, \delta A_\nu] \\ &= F^2_{\mu\nu}(\bar A) + 2(D_\mu(\bar A)\delta A_\nu)^2 - 2D_\mu(\bar A)\delta A_\nu D_\nu(\bar A)\delta A_\mu + 2iF_{\mu\nu}(\bar A)[\delta A_\mu, \delta A_\nu] \end{aligned} \tag{41}$$

where we used that $F_{\mu\nu}(\bar A)D_\mu(\bar A)\delta A_\nu = 0$, since integrating by parts this gives the equation of motion $D_\mu(\bar A)F_{\mu\nu}(\bar A) = 0$. Again integrating by parts and using[1]

$$D_\mu D_\nu = D_\nu D_\mu + iF_{\mu\nu} \tag{42}$$

$$\mathrm{tr}(\delta A_\nu[F_{\mu\nu}, \delta A_\mu]) = \mathrm{tr}(F_{\mu\nu}[\delta A_\mu, \delta A_\nu]) = -\mathrm{tr}(\delta A_\mu[F_{\mu\nu}, \delta A_\nu]) \tag{43}$$

then the quadratic form becomes

$$\begin{aligned} &\mathrm{tr}\{(D_\mu(\bar A)\delta A_\nu - D_\nu(\bar A)\delta A_\mu)^2 + 2iF_{\mu\nu}(\bar A)[\delta A_\mu, \delta A_\nu]\} \\ &= \mathrm{tr}\{-2\delta A_\mu\Delta_{\bar A}\delta_{\mu\nu}\delta A_\nu + 2\delta A_\nu D_\mu D_\nu\delta A_\mu + 2iF_{\mu\nu}[\delta A_\mu, \delta A_\nu]\} \\ &= \mathrm{tr}\{-2\delta A_\mu\Delta_{\bar A}\delta_{\mu\nu}\delta A_\nu + 2\delta A_\nu D_\nu D_\mu\delta A_\mu + 2i\delta A_\nu F_{\mu\nu}\delta A_\mu + 2iF_{\mu\nu}[\delta A_\mu, \delta A_\nu]\} \\ &= \mathrm{tr}\{-2\delta A_\mu\Delta_{\bar A}\delta_{\mu\nu}\delta A_\nu - 2(D_\mu\delta A_\mu)^2 - 4i\delta A_\mu[F_{\mu\nu}, \delta A_\mu]\} \end{aligned} \tag{44}$$

[1]Henceforth we will sometimes forget the bar on A since no confusion can arise.

Moreover, we can choose $\xi = 1$ (Feynmann gauge) in (40) in such a way to cancel the second term in (44).
In the end the quadratic form becomes

$$\begin{aligned}&\mathrm{tr}\,\{\delta A_\mu(-2\Delta_{\bar{A}}\delta_{\mu\nu} - 4i\ adF_{\mu\nu})\delta A_\nu\} = \mathrm{tr}\{\delta A^a_\mu t^a[(-2\Delta_{\bar{A}}\delta_{\mu\nu} - 4i\ adF_{\mu\nu})\delta A_\nu]^b t^b\}\\&= \delta A^a_\mu[(-\Delta_{\bar{A}}\delta_{\mu\nu} - 2i\ adF_{\mu\nu})\delta A_\nu]^a = \delta A^a_\mu(-\Delta_{\bar{A}}\delta_{\mu\nu} - 2i\ adF_{\mu\nu})^{ac}\delta A^c_\nu\\&= \delta A^a_\mu[-(\Delta_{\bar{A}})^{ac}\delta_{\mu\nu} + 2f^{abc}F^b_{\mu\nu}]\delta A^c_\nu\end{aligned} \tag{45}$$

where $adF_{\mu\nu} = [F_{\mu\nu}, .]$, $(adF_{\mu\nu})^{ac} = F^b_{\mu\nu}(T^b)^{ac} = if^{abc}F^b_{\mu\nu}$, and

$$(\Delta_{\bar{A}})^{ac} = (D_\mu)^{ad}(D_\mu)^{dc} = \partial^2\delta^{ac} + i\partial_\mu A^{ac}_\mu + 2iA^{ac}_\mu\partial_\mu - A^{ad}_\mu A^{dc}_\mu \tag{46}$$

Plugging in (26) and integrating over the fluctuations and the ghost fields, one gets[2]

$$Z = e^{-S_{YM}(\bar{A})}det^{-1/2}(-\Delta_{\bar{A}}\delta_{\mu\nu} - 2i\ adF_{\mu\nu})det(-\Delta_{\bar{A}}) = e^{-\Gamma_{1-loop}(\bar{A})} \tag{47}$$

where $\Gamma_{1-loop}(\bar{A})$ is the effective action for the classical fields $\bar{A}_\mu$ to 1-loop order.
The first term in the determinant $\Delta_{\bar{A}}\delta_{\mu\nu}$ is called the orbital term, while $2i\ adF_{\mu\nu}$ is called the spin term.
Moreover

$$det^{-1/2}(-\Delta_{\bar{A}}\delta_{\mu\nu} - 2i\ adF_{\mu\nu})) = det^{-1/2}(-\Delta_{\bar{A}}\delta_{\mu\nu})det^{-1/2}(1 - 2i(-\Delta_{\bar{A}})^{-1}\ adF_{\mu\nu})) \tag{48}$$

The first term gives

$$det^{-1/2}(-\Delta_{\bar{A}}\delta_{\mu\nu}) = det^{-2}(-\Delta_{\bar{A}}) \tag{49}$$

Therefore

$$e^{-\Gamma_{1-loop}(\bar{A})} = e^{-S_{YM}(\bar{A})}det^{-1/2}(1 - 2i(-\Delta_{\bar{A}})^{-1}\ adF_{\mu\nu}))det^{-1}(-\Delta_{\bar{A}}) \tag{50}$$

Let's start with the second determinant, namely

$$\begin{aligned}&det^{-1}(-\Delta_{\bar{A}}) = det^{-1}(-\partial^2 - i\partial_\mu A_\mu - 2iA_\mu\partial_\mu + A_\mu A_\mu)\\&= det^{-1}(-\partial^2)det^{-1}\left[1 + (-\partial^2)^{-1}(-i\partial_\mu A_\mu - 2iA_\mu\partial_\mu + A_\mu A_\mu)\right]\end{aligned} \tag{51}$$

Now using

$$det(1 + M) = e^{\mathrm{tr}\log(1+M)} = e^{\mathrm{tr}M - \mathrm{tr}(M)^2/2 + \dots} \tag{52}$$

then

$$\begin{aligned}&det^{-1}(-\Delta_{\bar{A}}) = det^{-1}(-\partial^2)\exp\left\{-\mathrm{tr}[(-\partial^2)^{-1}(-i\partial_\mu A_\mu - 2iA_\mu\partial_\mu + A_\mu A_\mu)]\right\}\\&\times\exp\left\{\mathrm{tr}[(-\partial^2)^{-1}(-i\partial_\mu A_\mu - 2iA_\mu\partial_\mu + A_\mu A_\mu)(-\partial^2)^{-1}(-i\partial_\mu A_\mu - 2iA_\mu\partial_\mu + A_\mu A_\mu)]/2\right\}\end{aligned}$$

[2]We can factorize an irrelevant 2 overall in the quadratic form.

The term $det^{-1}(-\partial^2)$ is an irrelevant constant while terms linear in A_μ vanish since $tr(t^a) = 0$. Moreover the term $\mathrm{tr}[(-\partial^2)^{-1} A_\mu A_\mu]$ is quadratically divergent and we can neglect since we are only interested in log divergencies. Thus we are left with

$$det^{-1}(-\Delta_{\bar{A}}) \sim \exp\left\{\mathrm{tr}[(-\partial^2)^{-1}(i\partial_\mu A_\mu + 2iA_\mu\partial_\mu)(-\partial^2)^{-1}(i\partial_\mu A_\mu + 2iA_\mu\partial_\mu)]/2\right\} \tag{53}$$

Let's call $V = i\partial_\mu A_\mu + 2iA_\mu\partial_\mu$, then

$$\mathrm{tr}[(-\partial^2)^{-1}V(-\partial^2)^{-1}V)] = \mathrm{tr}\int d^4x \int d^4y G(x-y)V(y)G(y-x)V(x) \tag{54}$$

where

$$G(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x-y)}}{p^2}, \quad G(y-x) = \int \frac{d^4q}{(2\pi)^4}\frac{e^{-iq(y-x)}}{q^2} \tag{55}$$

and the trace is now simply a trace over gauge and space-time indices. Moreover one has

$$V(y)e^{-iq(y-x)} = [2q_\mu A_\mu(y) + i\partial^y_\mu A_\mu(y)]e^{-iq(y-x)} \tag{56}$$

that, substituting

$$A_\mu(y) = \int \frac{d^4k}{(2\pi)^4}\ e^{-iky}A_\mu(k) \tag{57}$$

gives

$$V(y)e^{-iq(y-x)} = \int \frac{d^4k}{(2\pi)^4}\ e^{-iky}A_\mu(k)[2q_\mu + k_\mu]e^{-iq(y-x)} \tag{58}$$

With the same steps

$$V(x)e^{-ip(x-y)} = \int \frac{d^4\ell}{(2\pi)^4}\ e^{-i\ell x}A_\mu(\ell)[2p_\mu + \ell_\mu]e^{-ip(x-y)} \tag{59}$$

Plugging all back in (54) and integrating over d^4x (it gives $(2\pi)^4\delta^{(4)}(q-p-\ell)$) and d^4y (it produces $(2\pi)^4\delta^{(4)}(p-k-q)$) one obtains

$$\begin{aligned}\mathrm{tr}[(-\partial^2)^{-1}V(-\partial^2)^{-1}V)] &= \mathrm{tr}\int \frac{d^4p}{(2\pi)^4}\int \frac{d^4k}{(2\pi)^4}\int d^4\ell \int d^4q\ A_\mu(\ell)A_\nu(k) \\ &\times \frac{(2p_\mu + \ell_\mu)(2q_\nu + k_\nu)}{p^2q^2}\ \delta^{(4)}(q-p-\ell)\delta^{(4)}(p-k-q)\end{aligned} \tag{60}$$

Finally, integrating over d^4q ($q \to p-k$) and over $d^4\ell$ ($\ell \to -k$) this becomes

$$\mathrm{tr}[(-\partial^2)^{-1}V(-\partial^2)^{-1}V)] = \mathrm{tr}\int \frac{d^4k}{(2\pi)^4}\int \frac{d^4p}{(2\pi)^4}A_\mu(-k)A_\nu(k)\frac{(2p_\mu - k_\mu)(2p_\nu - k_\nu)}{p^2(p-k)^2} \tag{61}$$

The log divergent part of the integral over d^4p must be gauge invariant, so

$$\int \frac{d^4p}{(2\pi)^4}\frac{(2p_\mu - k_\mu)(2p_\nu - k_\nu)}{p^2(p-k)^2} = \Pi(k^2)(k^2\delta_{\mu\nu} - k_\mu k_\nu) + \ldots \tag{62}$$

where the dots stand for the quadratically divergent part.
Taking the trace over space-time indices one gets

$$\int \frac{d^4p}{(2\pi)^4}\frac{4p^2+k^2-4pk}{p^2(p-k)^2} = 3k^2\Pi(k^2) \tag{63}$$

We are interested in log divergencies, *i.e.* $\int d^4p/p^4$, therefore, expanding in power of k/p

$$\int \frac{d^4p}{(2\pi)^4}\frac{4p^2+k^2-4pk}{p^2(p-k)^2} = \int \frac{d^4p}{(2\pi)^4}\frac{4p^2+k^2-4pk}{p^4[1+(k^2-2kp)/p^2]} \xrightarrow{(1+\varepsilon)^{-1}\sim 1-\varepsilon+\varepsilon^2}$$
$$\sim \int \frac{d^4p}{(2\pi)^4}\frac{-4k^2+k^2+8(pk)^2/p^2}{p^4} = -\int \frac{d^4p}{(2\pi)^4}\frac{k^2}{p^4} = -\frac{2k^2}{(4\pi)^2}\log\frac{\Lambda}{\mu} = 3k^2\Pi(k^2)$$

In the previous formula we used a standard trick in field theory. Namely, symmetry allows us to replace

$$p_\mu p_\nu \to A p^2 \delta_{\mu\nu} \tag{64}$$

into the integral. Tracing both sides, one gets $A = 1/4$. Similarly $(pk)^2 \to p^2k^2/4$. Moreover, we regularized the following integral

$$\int \frac{d^4p}{p^4} = 2\pi^2 \int_0^\infty \frac{dp}{p} \to 2\pi^2 \log\frac{\Lambda}{\mu} \tag{65}$$

So one gets

$$\Pi(k^2) = -\frac{2}{3(4\pi)^2}\ \log\frac{\Lambda}{\mu} \tag{66}$$

Lets check now that the log divergent part of (62) is really transverse, *i.e.* it vanishes when contracted with k_μ. Again expanding the denominator in (62) and contracting with k_μ one gets

$$\int \frac{d^4p}{(2\pi)^4}\frac{(2p_\mu-k_\mu)(2p_\nu-k_\nu)}{p^2(p-k)^2} \sim \int \frac{d^4p}{(2\pi)^4}\frac{1}{p^4}\left[\frac{-8(pk)k^2p_\nu}{p^2}+\frac{16(pk)^3p_\nu}{p^4}-\frac{4(pk)^2k_\nu}{p^2}+k^2k_\nu\right]$$

Using $(pk)^2 \to p^2k^2/4$ and[3] $p_\mu p_\nu p_\rho p_\sigma \to p^4(\delta_{\mu\nu}\delta_{\rho\sigma}+\delta_{\mu\rho}\delta_{\nu\sigma}+\delta_{\mu\sigma}\delta_{\nu\rho})/24$ then

$$\int \frac{d^4p}{(2\pi)^4}\frac{(2p_\mu-k_\mu)(2p_\nu-k_\nu)}{p^2(p-k)^2} \sim \left(-\frac{8}{4}+\frac{16}{24}\times 3\right)k^2k_\nu \int \frac{d^4p}{(2\pi)^4}\frac{1}{p^4} = 0 \tag{67}$$

Plugging back in (53) then

$$\begin{aligned}
det^{-1}(-\Delta_{\bar{A}}) &\sim \exp\left[-\frac{1}{3(4\pi)^2}\mathrm{tr}\int \frac{d^4k}{(2\pi)^4}A_\mu(k^2\delta_{\mu\nu}-k_\mu k_\nu)A_\nu \log\frac{\Lambda}{\mu}\right] \\
&= \exp\left[-\frac{1}{3(4\pi)^2}\int \frac{d^4k}{(2\pi)^4}A^{ac}_\mu(k^2\delta_{\mu\nu}-k_\mu k_\nu)A^{ca}_\nu \log\frac{\Lambda}{\mu}\right] \\
&= \exp\left[-\frac{N}{3(4\pi)^2}\left(\frac{1}{4}\int d^4x\ F^a_{\mu\nu}F^a_{\mu\nu}\right)\log\left(\frac{\Lambda}{\mu}\right)^2\right]
\end{aligned} \tag{68}$$

[3]This must be a completely symmetric tensor, thus $p_\mu p_\nu p_\rho p_\sigma \to Bp^4(\delta_{\mu\nu}\delta_{\rho\sigma}+\delta_{\mu\rho}\delta_{\nu\sigma}+\delta_{\mu\sigma}\delta_{\nu\rho})$. Tracing both sides twice, one gets $B = 1/24$.

where in the last step we used

$$f^{abc}f^{abd} = N\delta^{cd} \tag{69}$$

and that, for the term quadratic in A^a_μ,

$$\frac{1}{4}\int d^4x F^a_{\mu\nu}F^a_{\mu\nu} \sim \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} A^a_\mu(-k)A^a_\nu(k)(k^2\delta_{\mu\nu} - k_\mu k_\nu) \tag{70}$$

Now we have to compute the first determinant in (50). Since $\mathrm{tr}F_{\mu\nu} = 0$ then

$$det^{-1/2}(1 - 2i(-\Delta_{\bar{A}})^{-1}\ adF_{\mu\nu})) = \exp\left\{-\mathrm{tr}[(-\Delta_{\bar{A}})^{-1}adF_{\mu\nu}(-\Delta_{\bar{A}})^{-1}adF_{\nu\mu}]\right\} \tag{71}$$

Considering the lowest order $(-\Delta_{\bar{A}}) \sim (-\partial^2)$, then

$$\begin{aligned}\mathrm{tr}[(-\partial)^{-1}adF_{\mu\nu}(-\partial)^{-1}adF_{\nu\mu}] &= \mathrm{tr}\int d^4x\int d^4y G(x-y)adF_{\mu\nu}(y)G(y-x)adF_{\nu\mu}(x) \\ &= -N\int d^4x\int d^4y[G(x-y)]^2F^a_{\mu\nu}(y)F^a_{\mu\nu}(x)\end{aligned} \tag{72}$$

where we used

$$\mathrm{tr}(adF_{\mu\nu}adF_{\nu\mu}) = (adF_{\mu\nu})^{ac}(adF_{\nu\mu})^{ca} = if^{abc}F^b_{\mu\nu}if^{cda}F^d_{\nu\mu} = -NF^c_{\mu\nu}F^c_{\mu\nu} \tag{73}$$

In the coordinates space

$$G(x-y) = \frac{1}{4\pi^2(x-y)^2} \tag{74}$$

Since we are interested in log divergencies, we can expand $F_{\mu\nu}(y) = F_{\mu\nu}(x) + \ldots$ and keep only the first term. Thus, defining $z = x - y$

$$\begin{aligned}\mathrm{tr}[(-\partial)^{-1}adF_{\mu\nu}(-\partial)^{-1}adF_{\nu\mu}] &= -\frac{N}{(4\pi^2)^2}\int d^4xF^a_{\mu\nu}(x)F^a_{\mu\nu}(x)\int\frac{d^4z}{z^4} \\ &= -\frac{2\pi^2N}{(4\pi^2)^2}\int d^4xF^a_{\mu\nu}(x)F^a_{\mu\nu}(x)\log\frac{\Lambda}{\mu} \\ &= -\frac{4N}{(4\pi)^2}\left(\frac{1}{4}\int d^4xF^a_{\mu\nu}F^a_{\mu\nu}\right)\log\left(\frac{\Lambda}{\mu}\right)^2\end{aligned} \tag{75}$$

Therefore

$$det^{-1/2}(1 - 2i(-\Delta_{\bar{A}})^{-1}\ adF_{\mu\nu})) = \exp\left[\frac{4N}{(4\pi)^2}\left(\frac{1}{4}\int d^4xF^a_{\mu\nu}F^a_{\mu\nu}\right)\log\left(\frac{\Lambda}{\mu}\right)^2\right] \tag{76}$$

In the end, plugging all back in (50), one gets for the 1-loop effective action

$$\begin{aligned}\Gamma_{1-loop} &= S_{YM} + \left[\frac{N}{3(4\pi)^2} - \frac{4N}{(4\pi)^2}\right]\left(\frac{1}{4}\int d^4xF^a_{\mu\nu}F^a_{\mu\nu}\right)\log\left(\frac{\Lambda}{\mu}\right)^2 \\ &= \left[\frac{1}{g^2} - \frac{11N}{3(4\pi)^2}\log\left(\frac{\Lambda}{\mu}\right)^2\right]\left(\frac{1}{4}\int d^4xF^a_{\mu\nu}F^a_{\mu\nu}\right)\end{aligned} \tag{77}$$

Therefore the original fixed coupling constant in the effective action is replaced by a running coupling constant

$$\frac{1}{2g^2(\Lambda)} = \frac{1}{2g^2(\mu)} + \frac{11N}{3}\frac{1}{(4\pi)^2}\log\left(\frac{\Lambda}{\mu}\right) \tag{78}$$

or

$$g^2(\Lambda) = \frac{g^2(\mu)}{1 + 2\frac{11N}{3(4\pi)^2}\ g^2(\mu)\log\left(\frac{\Lambda}{\mu}\right)} \tag{79}$$

that is the solution at one-loop of the renormalization group equation for the β function

$$\beta(g) = \frac{\partial g}{\partial \log\Lambda} = -\beta_0 g^3, \quad \beta_0 = \frac{11N}{3(4\pi)^2} g^3 \tag{80}$$

Eq. (78) can be also written as

$$\Lambda\ e^{-\frac{1}{2\beta_0 g^2(\Lambda)}} = \mu\ e^{-\frac{1}{2\beta_0 g^2(\mu)}} \tag{81}$$

Thus, the combination

$$\Lambda_{YM} = \mu_0\ e^{-\frac{1}{2\beta_0 g^2(\mu_0)}} \tag{82}$$

where μ_0 is any fixed scale, is independent of the choice of scale μ_0, *i.e.* it is a renormalization group invariant (at one-loop). It is called the strong coupling scale of YM.