Notes on covariant quantization of the electromagnetic field

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1. Introduction

In Dirac's first formulation of the quantum theory of radiation, the electromagnetic field was separated into a radiation field and a static Coulomb interaction. While the radiation field was then treated using a quantum mechanical approach, the Coulomb interaction was regarded as a classical instantaneous potential. Obviously, in this approach relativistic invariance was lost.

Although Dirac's formalism is very convenient for many applications, a formulation in which the four potential $A^\mu(x)$ transforms in a manifestly covariant fashion is needed to describe fully relativistic situations, as well as to establish that the theory is renormalizable.

One would tend to believe that, being observable classically, the electromagnetic field should be easily quantizable using the canonical procedure. Unfortunately, the problem turns out to be far from trivial. We will see that, in order to maintain covariance in the quantization procedure, one has to abandon the requirement of positive definite metric in Hilbert space and work in a space of states with indefinite norm. This difficulty is a consequence of the fact that in covariant quantization the number of variables exceeds the number of independent dynamical degrees of freedom, as the four components of $A^\mu(x)$ cannot all be treated as independent variables. Obvioulsy, the additional degrees of freedom give vanishing contributions to the amplitudes describing physical processes, and physical states belong to the subspace of positive norm states.

2. Fermi's Lagrangian

The electromagnetic field is described by the celebrated Maxwell's equations

$$\begin{aligned} \nabla \cdot \mathbf{E} = 0 \quad &, \quad \nabla \times \mathbf{B} = \frac{\partial \boldsymbol{E}}{\partial t} \ , \\ \nabla \cdot \mathbf{B} = 0 \quad &, \quad \nabla \times \mathbf{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \ , \end{aligned} \tag{1}$$

that can be rewritten in terms of the components of the four potential $A^{\mu} \equiv (\phi, \mathbf{A})$

$$\mathbf{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t} \quad , \quad \mathbf{B} = \nabla \times \mathbf{A} \ . \tag{2}$$

Definig the antisymmetric field tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} , \tag{3}$$

it can be easily seen that Maxwell's equations are equivalent to

$$\partial_\nu F^{\mu\nu} = 0 \ . \tag{4}$$

Moreover, replacing the fields with their expressions in terms of the potentials, the tensor defined in eq.(3) can be cast in the form

$$F^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\nu \ . \tag{5}$$

Combining eqs.(4) and (5) leads to

$$\partial_\nu F^{\mu\nu} = \partial_\nu \partial^\nu A^\mu - \partial_\nu \partial^\mu A^\nu = 0, \tag{6}$$

i.e. to the equation

$$\Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = 0 \ , \tag{7}$$

which is of course also equivalent to Mawxell's equations. Note that, besides being written in a manifestly covariant form, the above equation has the important property of being invariant under the *gauge transformation*

$$A^\mu(x) \to A'^\mu(x) = A^\mu(x) + \partial^\mu \Lambda(x) \ . \tag{8}$$

As a first step towards canonical quantization, let us try to derive eq.(4) from a Lagrangian through Hamilton's principle. Introducing infinitesimal variations of the four potential $\delta A^\mu(\mathbf{x}, t)$ satisfying $\delta A^\mu(\mathbf{x}, t_1) = \delta A^\mu(\mathbf{x}, t_2) = 0$ we can write

$$\int_{t_1}^{t_2} d^4x \ \partial_\nu F^{\mu\nu} \ \delta A_\mu = - \int_{t_1}^{t_2} d^4x \ F^{\mu\nu} \ \delta(\partial_\nu A_\mu) = 0 \ . \tag{9}$$

Substitution of eq.(5) in the above equation yields

$$-\frac{1}{2} \int_{t_1}^{t_2} d^4x \ F^{\mu\nu} \ \delta F_{\mu\nu} = 0 \tag{10}$$

i.e.

$$\delta \int_{t_1}^{t_2} d^4x \ \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right) = 0 \ , \tag{11}$$

implying that the lagrangian we are looking for is

$$\mathcal{L}_0 = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \ . \tag{12}$$

Unfortunately, the lagrangian of eq.(12) is not suitable to carry out canonical quantization, since from the definition of conjugate field

$$\pi^\mu(x) = \frac{\partial \mathcal{L}_0}{\partial \dot{A}_\mu} \tag{13}$$

it follows that

$$\pi^\mu(x) = -F^{\mu 0}(x) \ , \tag{14}$$

implying in turn (see eq.(3)) that the field conjugate to $A^0(x)$ vanishes identically:

$$\pi^0(x) = -F^{00}(x) \equiv 0 \ . \tag{15}$$

The above equation is clearly incompatible with canonical commutation rules.

In 1932, Fermi proposed to use, instead of $\mathcal{L}_0$, the lagrangian

$$\mathcal{L} = \mathcal{L}_0 - \frac{1}{2}(\partial^\mu A_\mu)(\partial^\nu A_\nu) = -\frac{1}{2}(\partial_\nu A_\mu)(\partial^\nu A^\mu) \ . \tag{16}$$

Note that, unlike $\mathcal{L}_0$, Fermi's lagrangian is *not* gauge invariant. However, it allows for canonical quantization, since

$$\pi^\mu(x) = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = -\dot{A}_\mu(x) \ , \tag{17}$$

and leads to the field equation

$$\Box A^\mu(x) = 0 \ , \tag{18}$$

which is compatible with eq.(7) (i.e. with Maxwell's equations) provided the condition

$$\partial_\mu A^\mu = 0 \ , \tag{19}$$

known as *Lorentz condition*, is satisfied. It has to be pointed out that Lorentz condition does not specify the vector potential in a unique way, since the function $\Lambda(x)$ appearing in eq.(8) can always be chosen in such a way as to satisfy eq.(19). In fact, if $A_\mu(x)$ fulfills eq.(19), the same is true for $A'_\mu(x) = A(x) + \partial_\mu \Lambda(x)$ provided

$$\Box \Lambda(x) = 0 \ .$$

Any gauge in which eq.(19) is fulfilled is called *Lorentz gauge*. It is apparent that, unlike *Coulomb gauge*, $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$, Lorentz gauge is Lorentz covariant. We will first carry out canonical quantization using Fermi's lagrangian. Fulfillment of the Lorentz condition will be required at a later stage.

As usual, the field is expanded in Fourier components and split into positive and negative energy parts:

$$A^\mu(x) = A^{\mu+}(x) + A^{\mu-}(x) \ , \tag{20}$$

with

$$A^{\mu+}(x) = \sum_{r\mathbf{k}} \left(\frac{1}{2V\omega_k}\right)^{1/2} \epsilon_r^\mu(\mathbf{k}) a_r(\mathbf{k}) \mathrm{e}^{-ikx} \tag{21}$$

$$A^{\mu-}(x) = \sum_{r\mathbf{k}} \left(\frac{1}{2V\omega_k}\right)^{1/2} \epsilon_r^\mu(\mathbf{k}) a_r^\dagger(\mathbf{k}) \mathrm{e}^{ikx} \ . \tag{22}$$

In the above equations, the sum over the vawe vectors is extended to all $\mathbf{k}$ satisfying periodic boundary conditions

$$k_\alpha = n_\alpha \ \frac{\pi}{L} \quad , \quad (\alpha = 1,2,3) \ ,$$

where n_α is an integer number and $L = V^{1/3}$, while $\omega_k = |\mathbf{k}|$.

The polarization states are described by the vectors $\epsilon_r^\mu(\mathbf{k})$, satisfying the orthonormality and completeness relations

$$\epsilon_r(\mathbf{k})\epsilon_s(\mathbf{k}) = \epsilon_{r\mu}(\mathbf{k})\epsilon_s^\mu(\mathbf{k}) = -\zeta_r \ \delta_{rs} \tag{23}$$

$$\sum_r \zeta_r \ \epsilon_r^\mu(\mathbf{k})\epsilon_r^\nu(\mathbf{k}) = -g^{\mu\nu} \ , \tag{24}$$

with $\zeta \equiv (-1,1,1,1)$.

To make the interpretation of our results easier, we will use a specific choice of the polarization vectors:

$$\epsilon_0^\mu \equiv n^\mu \equiv (1,0,0,0) \ , \tag{25}$$

$$\epsilon_r^\mu \equiv (0, \boldsymbol{\epsilon}_r) \ , \quad r = 1,2,3 \ , \tag{26}$$

$\boldsymbol{\epsilon}_1(\mathbf{k})$ and $\boldsymbol{\epsilon}_2(\mathbf{k})$ being mutually orthogonal unit vectors belonging to the plane perpendicular to the wave vector $\mathbf{k}$, i.e. satisfying

$$\boldsymbol{\epsilon}_r \cdot \boldsymbol{\epsilon}_s = \delta_{rs} \ , \quad r,s = 1,2 \ ,$$

$$\mathbf{k} \cdot \boldsymbol{\epsilon}_r \ = 0 \ , \quad r = 1,2 \ ,$$

while $\boldsymbol{\epsilon}_3(\mathbf{k})$ is the unit vector along the direction of $\mathbf{k}$.

$$\boldsymbol{\epsilon}_3 = \frac{\mathbf{k}}{|\mathbf{k}|} \ .$$

Note that the components of $\epsilon_3(\mathbf{k})$ can be written in the manifestly covariant form

$$\epsilon_3^\mu = \frac{k^\mu - (kn)n^\mu}{[(kn)^2 - k^2]^{1/2}} \ , \tag{27}$$

where we have not explicitely required that $k^2 = 0$, as it would be appropriate in the case of real photons.

Using eq.(17), giving the conjugate fields corresponding to Fermi's hamiltonian, we can immediately write the equal time canonical commutation rules

$$[A^\mu(\mathbf{x},t), A^\nu(\mathbf{x}',t)] = \left[\dot{A}^\mu(\mathbf{x},t), \dot{A}^\nu(\mathbf{x}',t)\right] = 0 \ , \tag{28}$$

$$\left[A^\mu(\mathbf{x},t), \dot{A}^\nu(\mathbf{x}',t)\right] = -ig^{\mu\nu}\delta^{(3)}(\mathbf{x} - \mathbf{x}') \ . \tag{29}$$

Substitution of the field expansion (eqs.(20)-(22)) then leads to the commutation rules for the coefficients $a_r(\mathbf{k})$ and $a_r^\dagger(\mathbf{k})$

$$[a_r(\mathbf{k}), a_s(\mathbf{k}')] = \left[a_r^\dagger(\mathbf{k}), a_s^\dagger(\mathbf{k}')\right] = 0 \tag{30}$$

$$\left[a_r(\mathbf{k}), a_s^\dagger(\mathbf{k}')\right] = \zeta_r \delta_{rs} \delta_{\mathbf{k}\mathbf{k}'} \ . \tag{31}$$

Eqs.(30) and (31) show that, while for $r = 1, 2$ and 3 $a_r(\mathbf{k})^\dagger$ and $a_r(\mathbf{k})$ can be naturally interpreted as the usual bosonic creation and annihilation operators, respectively, extending the same intepretation to the case of $r = 0$ is made impossible by the presence of the factor $\zeta_0 = -1$ in the right hand side of eq.(31). In order to preserve the familiar commutation rules, in this case one would have to interchange the roles of $a_r(\mathbf{k})^\dagger$ and $a_r(\mathbf{k})$. However, with this interpretation the hamiltonian (N denotes normal product)

$$H = \int d^3x N \left[\pi^\mu(x)\dot{A}_\mu(x) - \mathcal{L}(x)\right] \ , \tag{32}$$

does not have a lower bound. For example, it can be easily shown that the state containing one scalar (i.e. $r = 0$) photon has negative energy.

3. Gupta-Bleuler formalism

A way to circumvent the difficulty represented by eq.(31), without running into the problem of having a hamiltonian not bounded from below, was proposed independently by Gupta and Bleuler in 1950.

The formalism developed by Gupta and Bleuler maintains the familiar intepretation of $a_r^\dagger(\mathbf{k})$ and $a_r(\mathbf{k})$ as creation and annihilation operators of scalar ($r = 0$), tranverse ($r = 1$ and 2) and longitudinal ($r = 3$). photons, respectively. This amounts to assume the existence of a vacuum state satisfying

$$a_r(\mathbf{k})|0\rangle = 0 \tag{33}$$

for all r and $\mathbf{k}$, or, equivalently,

$$A_\mu^+(x)|0\rangle = 0 \tag{34}$$

for all μ and x.

The state containing one photon of type r carrying momentum $\mathbf{k}$ can then be readily written as

$$|\mathbf{k}r\rangle = a_r^\dagger(\mathbf{k})|0\rangle \ , \tag{35}$$

and its energy is positive and independent of r, as it follows from

$$H|\mathbf{k}r\rangle = \sum_{\mathbf{k}',s} \omega_{k'} \zeta_s a_s^\dagger(\mathbf{k}') a_s(\mathbf{k}') a_r^\dagger(\mathbf{k})|0\rangle = \omega_k |\mathbf{k}r\rangle \ ,$$

H being the hamiltonian defined in eq.(32).

A very unpleasant feature of this approach emerges when considering the normalization of the state. In fact, one finds

$$\langle \mathbf{k}r|\mathbf{k}r\rangle = \langle 0|a_r(\mathbf{k})a_r^\dagger(\mathbf{k})|0\rangle = \zeta_r\langle 0|0\rangle \ ,$$

implying that the state containing one scalar ($r = 0$) photon has negative normalization. The same pathology afflicts all states containing an odd number of scalar photons.

Although one may be tempted to dismiss this problem on the ground that scalar (as well as longitudinal) photons are not observed, the occurrence of states of negative norm poses a nontrivial conceptual problem, challanging the whole probabilistic interpretation of quantum mechanics. On the other hand, we have to keep in mind that the theory based on the Fermi's lagrangian (16), that we have considered so far, is not equivalent to Maxwell's theory unless we impose the subsidiary condition expressed by eq.(19).

Classically, fulfillment of the Lorentz condition guarantees that the total energy is positive definite. In the quantized theory, however, eq.(19) cannot be satisfied as an operator identity, since this would lead to contraddict the result (see section 3 on photon propagator)

$$[\partial_\mu A^\mu(x), A^\nu(x')] = i\partial_\mu D_F^{\mu\nu}(x - x') \neq 0 \ .$$

The above argument can be rephrased saying that there is no physical state $|\psi\rangle$ satisfying

$$\partial_\mu A^\mu(x)|\psi\rangle = 0 \tag{36}$$

for all x. Gupta and Bleuler then proposed to replace eq.(36) with a weaker condition, holding for the part of the field expansion involving annihilation operators only, i.e.

$$\partial_\mu A^{\mu +}(x)|\psi\rangle = 0 \ . \tag{37}$$

Using eq.(37) and the adjoint relation

$$\langle\psi|\partial_\mu A^{\mu -}(x) \tag{38}$$

we can write

$$\langle\psi|\partial_\mu A^{\mu +}(x) + \partial_\mu A^{\mu -}(x)|\psi\rangle = 0 \ , \tag{39}$$

showing that Lorentz condition is fulfilled for expectation values, i.e. that Maxwell theory is recovered in the classical limit.

The physical meaning of the Gupta Bleuler implementation of Lorentz condition can be best understood in momentum space. Substitution of the $A^{\mu+}(x)$ expansion (eq.(21)) into eq.(37) leads to

$$[a_3(\mathbf{k}) - a_0(\mathbf{k})] \, |\psi\rangle = 0 \tag{40}$$

for all $\mathbf{k}$, showing that the numbers of longitudinal and scalar photons allowed in a given state are *not* independent. On the other hand, note that the number of transverse (i.e. physical) photons is not constrained. Restricting Hilbert space to the subspace of states satisfying eq.(40) amounts to eliminate the states with negative norm. For example, it can be easily seen that the state with one scalar photon does not satisfy the condition of eq.(40), and therefore does not belong to the Hilbert space of Gupta Bleuler theory.

From eq.(40) and its adjoint it follows that

$$\langle\psi| \left[a_3^\dagger(\mathbf{k}) - a_0^\dagger(\mathbf{k})\right] [a_3(\mathbf{k}) - a_0(\mathbf{k})] \, |\psi\rangle = \langle\psi| \left[a_3^\dagger(\mathbf{k})a_3(\mathbf{k}) - a_0^\dagger(\mathbf{k})a_0(\mathbf{k})\right] |\psi\rangle = 0 \ , \tag{41}$$

implying that the expectation value of the hamiltonian, i.e. the sum of the energies carried by the photons, in the state $|\psi\rangle$ is

$$\begin{aligned}
\langle\psi|H|\psi\rangle &= \sum_{\mathbf{k}} \omega_k \langle\psi| \sum_{r=0}^{3} \zeta_r a_r^\dagger(\mathbf{k}) a_r(\mathbf{k}) |\psi\rangle \\
&= \sum_{\mathbf{k}} \omega_k \langle\psi| - a_0^\dagger(\mathbf{k})a_0(\mathbf{k}) + a_1^\dagger(\mathbf{k})a_1(\mathbf{k}) + a_2^\dagger(\mathbf{k})a_2(\mathbf{k}) + a_3^\dagger(\mathbf{k})a_3(\mathbf{k}) |\psi\rangle \\
&= \langle\psi| \sum_{r=1}^{2} a_r^\dagger(\mathbf{k}) a_r(\mathbf{k}) |\psi\rangle \ .
\end{aligned} \tag{42}$$

The above equation shows that, as expected, longitudinal and scalar photons do not contribute to the energy of the state. The same result holds true for the expectation value of any physical observable.

In conclusion, requiring that Lorentz condition be satisfied in the form given in eq.(37) makes the covariant formulation of the quantized theory fully equivalent to the standard formulation in terms of tranverse degrees of freedom (i.e. real photons) only.

4. The photon propagator

In order to obtain the photon propagator, we have to generalize the equal time commutation relations, eqs.(28) and (29) to the case in which the constraint $t = t'$ is released.

We start splitting $A^\mu(x)$ into positive and negative energy parts, to rewrite the commutator $[A^\mu(x), A^\nu(x')]$ (use the field expansion and the commutation rules of eq.(30))

$$\begin{aligned}[A^\mu(x), A^\nu(x')] &= \left[A^{\mu+}(x), A^{\nu+}(x')\right] + \left[A^{\mu+}(x), A^{\nu-}(x')\right] \\ &+ \left[A^{\mu-}(x), A^{\nu+}(x')\right] + \left[A^{\mu-}(x), A^{\nu-}(x')\right] \\ &= \left[A^{\mu+}(x), A^{\nu-}(x')\right] + \left[A^{\mu-}(x), A^{\nu+}(x')\right] \ . \end{aligned} \tag{43}$$

Substitution of the expansions of eqs.(21) and (22) into the first term in the last line of the above equation then yields (use eqs.(31) and (24) and replace $\sum_k \to [V/(2\pi)^3] \int d^3k$)

$$\begin{aligned}\left[A^{\mu+}(x), A^{\nu-}(x')\right] &= \frac{1}{2V}\sum_{\mathbf{k},r}\sum_{\mathbf{k}',r'}\frac{1}{\sqrt{\omega_k\omega_{k'}}}\left[a_r(\mathbf{k}), a^\dagger_{r'}(\mathbf{k}')\right]\epsilon^\mu_r(\mathbf{k})\epsilon^\nu_{r'}(\mathbf{k}')\mathrm{e}^{-i(kx-k'x')} \\ &= -g^{\mu\nu}\frac{1}{(2\pi)^3}\int\frac{d^3k}{2\omega_k}\mathrm{e}^{ik(x-x')} \ . \end{aligned} \tag{44}$$

Defining now

$$D_F^{\mu\nu\pm}(x-x') = \mp i(-g^{\mu\nu})\frac{1}{(2\pi)^3}\int\frac{d^3k}{2\omega_k}\,\mathrm{e}^{\mp ik(x-x')} \ , \tag{45}$$

we can finally write

$$\left[A^{\mu+}(x), A^{\nu-}(x')\right] = iD_F^{\mu\nu+}(x-x') \tag{46}$$

and the analog relation

$$\left[A^{\mu-}(x), A^{\nu+}(x')\right] = iD_F^{\mu\nu-}(x-x') \ . \tag{47}$$

Let us now consider the vacuum expectation value of the *time ordered product*

$$T\left\{A^\mu(x)A^\nu(x')\right\} = i\theta(t-t')A^\mu(x)A^\nu(x') + \theta(t'-t)A^\nu(x')A^\mu(x) \ , \tag{48}$$

where $\theta(t)$ is the usual step function ($\theta(t) = 1$ at $t > 0$, $\theta(t) = 0$ at $t < 0$). From

$$\langle 0| \left[A^{\mu+}(x), A^{\nu-}(x')\right] |0\rangle = \langle 0|A^{\mu+}(x)A^{\nu-}(x')|0\rangle$$
$$= \langle 0|A^{\mu}(x)A^{\nu}(x')|0\rangle = iD_F^{\mu\nu+}(x-x') \tag{49}$$

and

$$\langle 0| \left[A^{\mu-}(x), A^{\nu+}(x')\right] |0\rangle = -\langle 0|A^{\nu+}(x')A^{\mu-}(x)|0\rangle$$
$$= -\langle 0|A^{\nu}(x')A^{\mu}(x)|0\rangle = iD_F^{\mu\nu-}(x-x') \ , \tag{50}$$

it clearly follows that

$$\langle 0|T\left\{A^{\mu}(x)A^{\nu}(x')\right\}|0\rangle = i\left[\theta(t-t')D_F^{\mu\nu+}(x-x') - \theta(t'-t)D_F^{\mu\nu-}(x-x')\right] \ . \tag{51}$$

The above equation defines the Feynman propagator of the photon, $D_F^{\mu\nu}$, to be identified with the quantity in square brackets appearing in the right hand side, i.e.

$$D_F^{\mu\nu}(x) = \theta(t)D_F^{\mu\nu+}(x) - \theta(-t)D_F^{\mu\nu-}(x) \ . \tag{52}$$

We will now obtain the expression of the photon propagator in momentum space, $D_F^{\mu\nu}(k)$, obviously related to $D_F^{\mu\nu}(x)$ through

$$D_F^{\mu\nu}(x) = \int \frac{d^4k}{(2\pi)^4} \ D_F^{\mu\nu}(k) \ \mathrm{e}^{-ikx} \ . \tag{53}$$

Substitution of eq.(45) into eq.(52) leads to

$$D_F^{\mu\nu}(x) = \theta(t)\left(-\frac{i}{(2\pi)^3}\right)(-g^{\mu\nu})\int d^3k \frac{\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{x}-\omega_k t)}}{2\omega_k}$$
$$- \theta(-t)\left(\frac{i}{(2\pi)^3}\right)(-g^{\mu\nu})\int d^3k \frac{\mathrm{e}^{-i(\mathbf{k}\cdot\mathbf{x}-\omega_k t)}}{2\omega_k} \ . \tag{54}$$

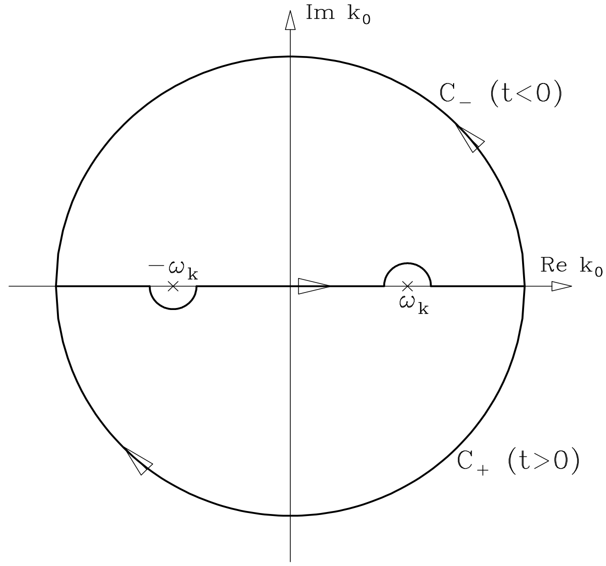
The three dimensional momentum integrations appearing in the above equation can be turned into four dimensional integrations noting that at $t > 0$

$$\int_{C_+} dk_0 \frac{\mathrm{e}^{-ik_0 t}}{k^2 + i\epsilon} = -2\pi i \ \frac{\mathrm{e}^{-i\omega_k t}}{2\omega_k} \tag{55}$$

while at $t < 0$

$$\int_{C_-} dk_0 \frac{\mathrm{e}^{-ik_0 t}}{k^2 + i\epsilon} = -2\pi i \ \frac{\mathrm{e}^{i\omega_k t}}{2\omega_k} \ . \tag{56}$$

where $C_\pm$ denote the integration contours shown in the figure, and the limit $\epsilon \to 0$ must be taken at the end of the integration.



Substitution of eqs.(55) and (56) into eq.(54) shows that the momentum-space propagator can be written in the compact form

$$D_F^{\mu\nu}(k) = \frac{-g^{\mu\nu}}{k^2 + i\epsilon} \tag{57}$$

or, equivalently, (use the completeness relation of the polarization vectors given by eq.(24))

$$D_F^{\mu\nu}(k) = \frac{1}{k^2 + i\epsilon} \sum_r \zeta_r \epsilon_r^\mu(\mathbf{k}) \epsilon_r^\nu(\mathbf{k}) \ . \tag{58}$$

To further clarify that only the physical degrees of freedom contribute to the amplitudes describing observed processes, we will now use eq.(58) and the definition of the polarization vectors, eqs.(25)-(27), to rewrite the photon propagator in the form

$$\begin{aligned} D_F^{\mu\nu}(k) &= \frac{1}{k^2 + i\epsilon} \left\{ \sum_{r=1}^{2} \epsilon_r^\mu(\mathbf{k}) \epsilon_r^\nu(\mathbf{k}) - n^\mu n^\nu \right. \\ &+ \left. \frac{(k^\mu k^\nu - (kn)\,[k^\mu n^\nu + k^\nu n^\mu) + (kn)^2 n^\mu n^\nu]}{[(kn)^2 - k^2]} \right\} \\ &= D_{F,T}^{\mu\nu}(k) + D_{F,C}^{\mu\nu}(k) + D_{F,R}^{\mu\nu}(k). \end{aligned} \tag{59}$$

The first term in the last line of the above equation represents the contribution of tranverse photons, whereas the remaining two terms can be obtained rearranging the scalar and longitudinal photon contributions, with the result (in the $\epsilon \to 0$ limit)

$$D^{\mu\nu}_{F,C}(k) = \frac{n^\mu n^\nu}{|\mathbf{k}|^2} \tag{60}$$

$$D^{\mu\nu}_{F,R}(k) = \frac{1}{k^2} \; \frac{k^\mu k^\nu - (kn)\,(k^\mu n^\nu + k^\nu n^\mu)}{|\mathbf{k}|^2} \; . \tag{61}$$

Fourier transforming the only nonvanishing component of $D^{\mu\nu}_{F,C}(k)$ (i.e. the one having $\mu = \nu = 0$) back to coordinate space we find

$$D^{00}_{F,C}(x) = \int \frac{d^4k}{(2\pi)^4} \; \frac{\mathrm{e}^{-ikx}}{|\mathbf{k}|^2} = \delta(x_0) \; \frac{1}{4\pi} \; \frac{1}{|\mathbf{x}|} \; , \tag{62}$$

showing that the quantity defined by eq.(60) is colosely related to the familiar Coulomb potential.

Keeping in mind that in Dirac's theory the electromagnetic field is described in terms of transverse photons and electrostatic interaction only, and that in our formalism these contributions correspond to $D^{\mu\nu}_{F,T}$ and $D^{\mu\nu}_{F,C}$, we can anticipate that the remaining term, $D^{\mu\nu}_{F,R}$, does not contribute to the amplitudes describing physical processes. The mechanism driving the disappearance of the contribution coming from $D^{\mu\nu}_{F,R}$ can be easily understood from the following example.

Consider the electromagnetic current associated with an electron:

$$j^\mu(x) = \overline{\psi}(x)\gamma^\mu(x) \; .$$

The continuity equation

$$\partial_\mu j^\mu(x) = 0$$

implies that in momentum space

$$k_\mu j^\mu(k) = 0 \; , \tag{63}$$

with

$$j^\mu(k) = \int d^4x \; \mathrm{e}^{ikx} \; j^\mu(x).$$

From eq.(61) (showing that all terms in the definition of $D^{\mu\nu}_{F,R}(k)$ are proportional to k^μ and/or k^ν) and eq.(63) it follows immediately that the electron current *does not couple* to $D^{\mu\nu}_{F,R}(k)$, i.e. that

$$D^{\mu\nu}_{F,R}(k) j_\nu(k) = j_\mu(k) D^{\mu\nu}_{F,R}(k) = 0 \ .$$

As a consequence, $D^{\mu\nu}_{F,R}$ does not contribute to the amplitude describing, e.g., electron scattering.