Notes on electron-neutrino scattering

Roma, January 2001

1. Introduction

The development of a gauge theory of weak and electromagnetic interactions leads to the following SU(2)×U(1) invariant lagrangian, describing the interactions between leptons and gauge bosons.

$$\mathcal{L}_I^{\ell B} = -j^\mu(x)A_\mu(x) - \frac{g}{2\sqrt{2}}\left[J^{\mu\dagger}(x)W_\mu(x) + J^\mu(x)W_\mu^\dagger(x)\right]$$
$$- \frac{g}{\cos\theta_w}\left[J_3^\mu(x) - \frac{1}{e}j_\mu(x)\sin^2\theta_w\right]Z_\mu(x) \ . \qquad (1)$$

The first term in the right hand side of the above equation is the standard QED interaction, while the second term describes the coupling of the charge-lowering and charge-rising currents (the sum includes all leptons),

$$J_\mu(x) = \sum_\ell \overline{\psi}_\ell(x)\gamma_\mu(1-\gamma_5)\psi_{\nu_\ell}(x)$$

$$J_\mu^\dagger(x) = \sum_\ell \overline{\psi}_{\nu_\ell}(x)\gamma_\mu(1-\gamma_5)\psi_\ell(x) \ ,$$

to the complex vector field $W_\mu(x)$. It clearly appears that this term describes the weak interactions that we have discussed when studying muon decay, provided we identify the charged W bosons with the quanta of the $W_\mu(x)$ field and require (see notes on weak interactions)

$$\frac{g}{2\sqrt{2}} = g_w = M_W\left(\frac{G}{\sqrt{2}}\right)^{1/2} \ , \qquad (2)$$

where M_W and g_w are the W boson mass and coupling constant, respectively, and G is the Fermi coupling constant.

Finally, the second line of eq.(1) describes the neutral current

$$J_3^\mu(x) - \frac{1}{e}j_\mu(x)\sin^2\theta_w = \frac{1}{4}\overline{\psi}_{\nu_\ell}(x)\gamma_\mu(1-\gamma_5)\psi_{\nu_\ell}(x)$$
$$- \frac{1}{4}\overline{\psi}_\ell\gamma_\mu\left[(1-4\sin^2\theta_w) - \gamma_5\right]\psi_\ell(x) \qquad (3)$$

coupled to the real vector field $Z_\mu(x)$, whose quantum is the neutral vector boson Z_0.

Invariance under SU(2)×U(1) gauge transformation requires the masses of the gauge bosons, as well as all lepton masses, to be zero. Nonvanisging masses are generated through spontaneous symmetry breaking induced by a spin zero scalar field having nonvanishing vacuum expectation value, called Higgs field. The Lagrangian describing the interaction between lettons and the Higgs field reads

$$\mathcal{L}_I^{\ell H} = -\frac{1}{v} m_\ell \overline{\psi}_\ell(x)\psi_\ell(x)\sigma(x) - \frac{1}{v} m_{\nu_\ell}\overline{\psi}_{\nu_\ell}(x)\psi_{\nu_\ell}(x)\sigma(x) \ . \tag{4}$$

Eqs.(1), (3) and (4) show that electron neutrino scattering processes

$$\begin{array}{ll} \nu_\mu + e^- \to \nu_\mu + e^- & \overline{\nu}_\mu + e^- \to \overline{\nu}_\mu + e^- \\ \nu_e + e^- \to \nu_e + e^- & \overline{\nu}_e + e^- \to \overline{\nu}_e + e^- \ , \end{array} \tag{5}$$

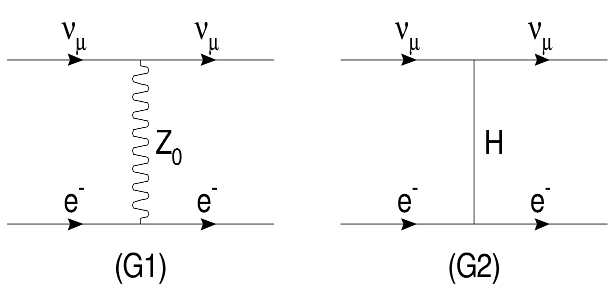
can occur through exchange of the neutral bosons Z_0 and H (H denotes the quantum of the Higgs field).

2. Calculation of the electron-neutrino scattering cross section

We will consider $\nu_\mu + e^- \to \nu_\mu + e^-$ first. The corresponding amplitude consists of two contributions

$$\mathcal{M}_{if} = \mathcal{M}_{if}(Z_0) + \mathcal{M}_{if}(H) \ , \tag{6}$$

described by diagrams G1 and G2.



Using the appropriate Feynman rules, we can readily write the amplitude associated with diagram G1 as (see eq.(3))

$$\begin{aligned}\mathcal{M}_{if}(Z_0) &= \frac{g^2}{\cos^2\theta_w}\,\frac{1}{4}\left[\overline{u}(\nu_\mu')\gamma_\mu(1-\gamma_5)u(\nu_\mu)\right] iD_F^{\mu\nu}(k_Z,M_Z)\\ &\qquad\times\frac{1}{4}\left\{\overline{u}(e')\gamma_\nu\left[(1-4\sin^2\theta_w)-\gamma_5\right]u(e)\right\}\\ &= -\frac{g^2}{\cos^2\theta_w}\,\frac{1}{4}\left[\overline{u}(\nu_\mu')\gamma_\mu(1-\gamma_5)u(\nu_\mu)\right] iD_F^{\mu\nu}(k_Z,M_Z)\left[\overline{u}(e')\gamma^\nu(g_V-g_A\gamma_5)u(e)\right]\ ,\end{aligned} \tag{7}$$

where we have defined

$$g_V = 2\sin^2\theta_w - \frac{1}{2} \quad , \quad g_A = -\frac{1}{2}\ ,$$

and the Z_0 propagator has the form suitable for a massive spin one particle

$$D_F^{\mu\nu}(k_Z,M_Z) = i\,\frac{-g^{\mu\nu}+k_Z^\mu k_Z^\nu/M_Z^2}{k_Z^2-M_Z^2+i\epsilon}\ . \tag{8}$$

The amplitude associated with H exchange reads

$$\mathcal{M}_{if}(H) = -\frac{1}{v^2}m_e m_{\nu_\mu}\overline{u}(\nu_\mu')u(\nu_\mu)i\Delta(k_H,M_H)\overline{u}(e')u(e)\ , \tag{9}$$

the scalar boson propagator being given by

$$i\Delta(k_H,M_H) = i\frac{1}{k_H^2-M_H^2+i\epsilon}\ . \tag{10}$$

The measured Z_0 mass is $M_Z \approx 94$ GeV, while the latest guess of the H mass is $M_H \sim 114$ GeV (from the analysis of candidate $e^+e^- \to Z_0 \to HZ_0$ events at CERN e^+e^- collider LEP). As both M_Z and M_H are large, we can safely assume

$$k_Z^2 << M_Z^2 \quad , \quad k_H^2 << M_H^2\ , \tag{11}$$

and disregard the terms containing k_Z and k_H in eqs.(8) and (10). Using this approximation, eq.(2) and the relation between the W and Z_0 masses ,

$$M_W = M_Z\cos\theta_w\ ,$$

we can rewrite eq.(7) in the simplified form

$$\mathcal{M}_{if}(Z_0) = -i\frac{G}{\sqrt{2}}\left[\overline{u}(\nu'_\mu)\gamma_\mu(1-\gamma_5)u(\nu_\mu)\right]\left[\overline{u}(e')\gamma^\mu(g_V - g_A\gamma_5)u(e)\right] \ . \tag{12}$$

The amplitude $\mathcal{M}_{if}(H)$ can also be rewritten in terms of the Fermi coupling constant using (11) and

$$M_W = \frac{1}{2} = vg \ ,$$

implying

$$\frac{1}{v^2} = \sqrt{2}G \ .$$

The resulting expression is

$$\mathcal{M}_{if}(H) = i\sqrt{2}G\frac{m_e m_{\nu_\mu}}{M_H^2}\left[\overline{u}(\nu'_\mu)u(\nu_\mu)\right]\left[\overline{u}(e')u(e)\right] \ . \tag{13}$$

Comparison between the above equation and eq.(12) shows that

$$\frac{\mathcal{M}_{if}(H)}{\mathcal{M}_{if}(Z_0)} \sim \frac{m_e m_{\nu_\mu}}{M_H^2} \sim 0 \ ,$$

i.e. that the contribution of $\mathcal{M}_{if}(H)$ can be disregarded in eq.(6).

Let us now calculate the cross section of the process depicted by diagram G1. We will denote p, p', k and k' the initial and final four momenta carried by the neutrino and the electron, respectively.

The differential cross section is defined as

$$d\sigma = \frac{1}{\text{flux}} \ w_{if} \ \frac{V}{(2\pi)^3}d^3p'\frac{V}{(2\pi)^3}d^3k' \ , \tag{14}$$

with

$$w_{if} = \frac{|S_{if}|^2}{T} \ ,$$

where V and T denote the normalization volume and interaction time and

$$S_{if} = (2\pi)^4\delta^{(4)}(p'+k'-p-k)\left(\frac{1}{2V\omega_k}\right)\left(\frac{1}{2V\omega_p}\right)\left(\frac{1}{2V\omega_{k'}}\right)\left(\frac{1}{2V\omega_{p'}}\right)|\mathcal{M}_{if}|^2$$

Squaring the above S_{if} and substituting into eq.(14) we obtain (after average or sum over the spins)

$$d\sigma = \frac{1}{F}\,\overline{|\mathcal{M}_{if}|}^2\, dQ\ , \tag{15}$$

where the quantity F, obtained combining the incident flux and the normalization of the initial states, reads

$$F = |\mathbf{v}_k - \mathbf{v}_p| 4\omega_k\omega_p = \left|\frac{\mathbf{k}}{\omega_k} - \frac{\mathbf{p}}{\omega_p}\right| 4\omega_k\omega_p = 4|\mathbf{k}\omega_p - \mathbf{p}\omega_k|\ . \tag{16}$$

Note that F defined by the above equation is manifestly Lorentz invariant, as it can be rewritten in the form $F = 4[(kp)^2 - m_e^2 m_{\mu_\nu}^2]^{1/2}$.

The phase-space factor dQ is also Lorentz invariant, as follows from its definition

$$dQ = \frac{1}{(2\pi)^2}\delta^{(4)}(p' + k' - p - k)\ \frac{d^3p'}{2\omega_{p'}}\ \frac{d^3k'}{2\omega_{k'}}\ . \tag{17}$$

The δ-function appearing in the above equation can be exploited to perform the integration over the final electron momentum $\mathbf{k}'$, with the result

$$dQ = \frac{1}{(2\pi)^2}\delta(\omega_{p'} + \omega_{k'} - \omega_p - \omega_k)\ \frac{|\mathbf{p}'|^2 d|\mathbf{p}'|}{2\omega_{p'}2\omega_{k'}} d\Omega_{p'}\ . \tag{18}$$

The cross section calcualtion can be most easily carried out working in the center of mass (CM) frame, defined by the requirement

$$\mathbf{p} + \mathbf{k} = \mathbf{p}' + \mathbf{k}' \ = 0,$$

and assuming that all lepton masses can be neglected with respect to the corresponding momenta. Under these conditions we can write

$$p \equiv (|\mathbf{p}|, \mathbf{p}) \qquad k \equiv (|\mathbf{p}|, -\mathbf{p})$$
$$p' \equiv (|\mathbf{p}'|, \mathbf{p}') \qquad k' \equiv (|\mathbf{p}'|, -\mathbf{p}')\ ,$$

with

$$|\mathbf{p}| = |\mathbf{p}'|\ ,$$

as required by energy conservation. It follows that

$$s = (k+p)^2 = (k'+p')^2 = 2(kp) = 2(k'p') = 4|\mathbf{p}|^2$$

and, defining the total CM energy $W = \omega_k + \omega_p = \sqrt{s}$,

$$d|\mathbf{p}| = \frac{1}{2} dW \ .$$

Substituting into eq.(18) and using the δ-function to eliminate the W integration we obtain

$$dQ = \frac{1}{32\pi^2}\ d\Omega_{p'} \ ,$$

and combining the above result with eqs.(15) and (16) (note that in the CM frame $F = 8|\mathbf{p}|^2 = 2s$) finally leads to

$$d\sigma = \frac{1}{64\pi^2}\frac{1}{s}\overline{|\mathcal{M}_{if}|}^2 d\Omega_{p'} \tag{19}$$

We are left wit the familiar problem of calculating the spin summed squared invariant amplitude, i.e. (note that in the average over the initial spin states, we have to take into account the fact that, while electrons can assume two spin states, neutrinos are always in the state corresponding to negative helicity)

$$\begin{aligned}\overline{|\mathcal{M}_{if}|}^2 &= \frac{G^2}{2}\left[\sum_{rr'} \overline{u}^{(r')}(p')\gamma_\mu(1-\gamma_5)u^{(r)}(p)\overline{u}^{(r)}(p)\gamma_\nu(1-\gamma_5)u^{(r')}(p')\right] \\ &\times \left[\frac{1}{2}\sum_{qq'} \overline{u}^{(q')}(k')\gamma^\mu(g_V - g_A\gamma_5)u^{(q)}(k)\overline{u}^{(q)}(k)\gamma^\nu(g_V - g_A\gamma_5)u^{(q')}(k') \ .\right] \\ &= \frac{G^2}{4}\ Tr\left[\gamma_\mu(1-\gamma_5)\not{p}\gamma_\nu(1-\gamma_5)\not{p}'\right]\ Tr\left[\gamma^\mu(g_V - g_A\gamma_5)\not{k}\gamma^\nu(g_V - g_A\gamma_5)\not{k}'\right] \ .\end{aligned} \tag{20}$$

Using the standard theorems on traces of products of γ-matrices (see appendix of the notes on weak interactions) the above equations can be rewritten

$$\begin{aligned}\overline{|\mathcal{M}_{if}|}^2 = \frac{G^2}{4}\ &\left\{8\left[p_\mu {p'}_\nu + p_\nu {p'}_\mu - g^{\mu\nu}(pp')\right] + 8i\epsilon_{\mu\rho\nu\sigma}p^\rho {p'}^\sigma\right\} \\ &\times \left\{4(g_V^2 + g_A^2)\left[k^\mu k'^\nu + k^\nu k'^\mu - g^{\mu\nu}(kk')\right] + 8ig_V g_A \epsilon^{\mu\rho'\nu\sigma'}k_{\rho'}{k'}_{\sigma'}\right\} \ .\end{aligned} \tag{21}$$

Contractions involving symmetric and antisymmetric tensors vanish. The remaining contributions yield

$$\overline{|\mathcal{M}_{if}|}^2 = \frac{G^2}{4}\left\{64(g_V^2+g_A^2)\left[(pk)(p'k')+(pp')(kk')\right] - 64 i g_V g_A \epsilon_{\mu\rho\nu\sigma}\epsilon^{\mu\rho'\nu\sigma'} p^\rho p'^\sigma k_{\rho'} k'_{\sigma'}\right\} \ , \quad (22)$$

leading to (use $\epsilon_{\mu\rho\nu\sigma}\epsilon^{\mu\rho'\nu\sigma'} = -2(g^\rho_{\rho'} g^\sigma_{\sigma'} - g^\rho_{\sigma'} g^\sigma_{\rho'})$)

$$\overline{|\mathcal{M}_{if}|}^2 = \frac{G^2}{4}\left\{64(g_V^2+g_A^2)\left[(pk)(p'k')+(pp')(kk')\right]\right.$$
$$\left. +128 g_V g_A\left[(pk)(p'k')-(pp')(kk')\right]\right\} \ . \quad (23)$$

In the CM frame, and neglecting all lepton masses

$$(pk) = (p'k') = 2|\mathbf{p}|^2 = \frac{s}{2} \ ,$$

$$(pk') = (p'k) = |\mathbf{p}|^2(1-\cos\theta) = \frac{s}{4}(1-\cos\theta)$$

and eq.(23) becomes

$$\overline{|\mathcal{M}_{if}|}^2 = 4G^2 s^2\left\{(g_V^2+g_A^2)\left[1+\frac{1}{4}(1-\cos\theta)^2\right] + 2g_V g_A\left[1-\frac{1}{4}(1-\cos\theta)^2\right]\right\} \ . \quad (24)$$

Substitution into eq.(19) and angular integration finally lead to the following *total* cross section for the process $\nu_\mu + e^- \to \nu_\mu + e^-$

$$\sigma(\nu_\mu e) = \frac{G^2}{3\pi}\ s(g_V^2 + g_A^2 + g_V g_A). \quad (25)$$

Obvioulsly, had we considered the similar process involving antineutrinos, i.e. $\overline{\nu}_\mu + e^- \to \overline{\nu}_\mu + e^-$, the result would have been

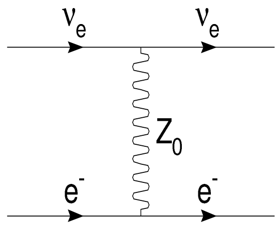
$$\sigma(\overline{\nu}_\mu e) = \frac{G^2}{3\pi}\ s(g_V^2 + g_A^2 - g_V g_A), \quad (26)$$

since the replacement $\nu_\mu \to \overline{\nu}_\mu$ amounts to change $g_A \to -g_A$ in the Z_0 exchange amplitude.

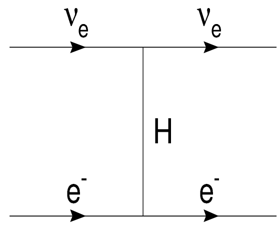
Besides neutral current Z_0 and H exchange (represented by diagrams G3 and G4), the process $\nu_\mu + e^- \to \nu_\mu + e^-$ can proceed through charged W boson exchange (diagram G5).

As in the previous discussion we will disregard the contribution of the amplitude involving H exchange, and write the amplitude associated with W exchange in the limit $k_W << M_W$, so that the momentum dependence of the W propagator disappears. The result is

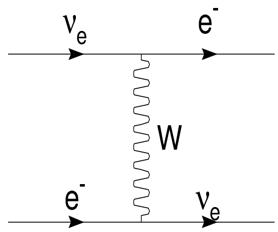
$$\mathcal{M}_{if}(W) = i\frac{G}{\sqrt{2}} \; [\overline{u}(e')\gamma_\mu(1-\gamma_5)u(\nu_e)]\,[\overline{u}(\nu_e')\gamma^\mu(1-\gamma_5)u(e)] \tag{27}$$



(G3)



(G4)



(G5)

Eq.(27) can be cast in a more convenient form using Fierz identity†, stating that for any four Dirac spinors u_i (i=1,2,3,4)

$$[\overline{u}_1\gamma_\mu(1-\gamma_5)u_2]\,[\overline{u}_3\gamma^\mu(1-\gamma_5)u_4] = -\,[\overline{u}_1\gamma_\mu(1-\gamma_5)u_4]\,[\overline{u}_3\gamma^\mu(1-\gamma_5)u_2] \ .$$

Use of the above identity in eq.(27) yields

$$\mathcal{M}_{if}(W) = -i\frac{G}{\sqrt{2}} \; [\overline{u}(e')\gamma_\mu(1-\gamma_5)u(e)]\,[\overline{u}(\nu_e')\gamma^\mu(1-\gamma_5)u(\nu_e)] \ . \tag{28}$$

Comparison between eq.(28) and eq.(12) shows that the two amplitudes have the same structure, except for the fact that in eq.(28) $g_V = g_A = 1$. Hence, the Z_0 and W exchange contributions to $\nu_e + e^- \to \nu_e + e^-$ can be both included in the calculation using eq.(12) with the replacement

$$g_V \to g_V + 1 \quad , \quad g_A \to g_A + 1.$$

The same procedure can obviously be employed to obtain the amplitude for $\overline{\nu}_e + e^- \to \overline{\nu}_e + e^-$ scattering from the amplitude of the corresponding process involving $\overline{\nu}_\mu$.

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†A clear proof of Fierz identity can be found in: L.B. Okun *Leptons and quarks* (Elsevier Science Ltd., 1985)