Notes on Weak Interactions

Roma, November 2000

1. Fermi's theory of β-decay

Nuclear β-decay was the first observed manifestation of weak interactions. It involves the transformation of a proton into a neutron, or viceversa. For example, the reactions

$$^{10}C \to \ ^{10}B^* + e^+ + \nu_e \tag{1}$$

$$^{14}O \to \ ^{14}N^* + e^+ + \nu_e \tag{2}$$

are driven by the elementary process

$$p \to n + e^+ + \nu_e \ . \tag{3}$$

A free proton cannot decay into a neutron, since $m_n > m_p$ ($m_n - m_p$ = 939.6 - 938.3 = 1.3 MeV). In fact, proton lifetime is larger than 10^{30} years. On the other hand, a neutron can decay through

$$n \to p + e^- + \overline{\nu}_e \ , \tag{4}$$

its measured lifetime being $\tau \sim 900$ sec.

Fermi's theory of β-decay, developed in 1934, is based on the analogy with nuclear γ-ray emission, i.e. the process in which a nucleus makes a transition, e.g. from an excited state to the ground state, emitting a photon. According to Fermi, if one thinks of the proton and the neutron as different states of the same particle belonging to charge eigenvalues 1 and 0, respectively†, β-decay can be interpreted as a transition between these two states associated with emission of an electron and an antineutrino (or a positron and a neutrino).

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†This idea was first proposed by W. Heisenberg

In Fermi's original paper, both heavy particles (initial and final nucleus) are described using nonrelativistic wave functions. This is expected to be a reasonable approximation since the typical momenta carried by the particles produced in β-decay are of the order of the neutron-proton mass difference, i.e. ~ 1 MeV. Hence, even in the case of β-decay of a single neutron, the final state proton has a momentum to mass ratio $(|\mathbf{k}_p|/m_p) \sim 10^{-3}$. We will first describe Fermi's original derivation of the β-decay tansition amplitude. A fully relativistic generalization, needed to treat weak interaction processes involving light particles only, will be discussed at a later stage.

The β-decay hamiltonian must be able to connect the two charge states corresponding to proton and neutron and create a pair of light particles (either e^- and $\overline{\nu}_e$ or e^+ and ν_e). To be specific, let us discuss neutron β-decay, i.e. process (4). If we denote by τ^+ the charge rising operator that transforms a neutron into a proton and by $a_r^\dagger$ and $b_s^\dagger$ the operators that create an e^- and a $\overline{\nu}_e$ in the states specified by the indices s and r, respectively, we can write Fermi's hamiltonian in the form

$$H = \frac{G}{\sqrt{2}} \tau^+ \sum_{rs} c^*_{rs} \, a^\dagger_r b^\dagger_s \ , \tag{5}$$

where G is the weak coupling constant and the presence of the factor $(1/\sqrt{2})$ has a historical origin. The quantity c^*_{rs} is in general a function of the coordinates and momenta of the participating particles. Note that the above hamiltonian must be complemented with its hermitean conjugate, describing the inverse process.

Neglecting spin and relativity altogether, and *assuming that the interaction have zero range*, the simplest choice for c^*_{rs} is

$$c^*_{rs} = \phi^*_r(x)\xi^*_s(x) \ , \tag{6}$$

where $\phi_r(x)$ and $\xi_s(x)$ are wave functions describing the electron and the antineutrino, respectively, evaluated at the same position occupied by the heavy particles.

To take into account relativity and the fact that both light particles carry spin 1/2, $\phi_r(x)$ and $\xi_s(x)$ have first to be identified with four-dimensional spinors, solutions of the Dirac e-quation. In addition, these spinors have to be combined in such a way as to construct quantities having simple transformation properties under Lorentz transformations. Motivated by the analogy with photon emission, Fermi proposed to replace c^*_{rs} of eq.(6) with a quantity exhibiting the same tranformation properties of the electromagnetic potential A^μ, that transforms as a four-vector. This would amount to substituting c^*_{rs} with

$$\overline{\psi}_{e,r}(x)\gamma^\mu\psi_{\nu,s}(x) \ , \tag{7}$$

where $\overline{\psi}_{e,r}(x) = \psi^\dagger_{e,r}(x)\gamma^0$ and $\psi^\dagger_{e,r}(x)$ is a positive energy solution of the Dirac equation. Note that to obtain eq.(7) we have also made use of the fact that, according to Dirac's theory, creation of a positive energy antineutrino is equivalent to absorption of a negative energy neutrino carrying opposite momentum. Hence, $\psi_{\nu,s}(x)$ is a negative energy solution of the Dirac equation.

The quantity defined by eq.(7) has the structure of a four-current. However, in the nonrelativistic approximation the effect of the heavy particles reduces to the appearance of a scalar potential, that couples to the $\mu = 0$ component of the four-current only †. Therefore, substituting eq.(7) into into eq.(5) and keeping only the component relevant in the nonrelativistic approximation we can write

$$H = \frac{G}{\sqrt{2}}\,\tau^+ \sum_{rs} \overline{\psi}_{e,r}(x)\gamma^0\psi_{\nu,s}(x)a^\dagger_r b_s \ , \tag{8}$$

where b_s is the operator that annihilates a negative energy neutrino in the state s.

The transition amplitude for neutron β-decay is defined in terms of the matrix element of the above hamiltonian between the initial state

$$|i\rangle = \Psi_n(x)|0\rangle \tag{9}$$

and the final state

$$|f\rangle = \Psi_p(x)|e^-(r)\ \overline{\nu}_e(s)\ \rangle. \tag{10}$$

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†This feature will be discussed in detail in the next section

Note that $|0\rangle$ is the vacuum state of Dirac's theory, i.e. a state in which all negative energy levels are filled and all positive energy levels are empty. The heavy particles are described by

$$\Psi_n(x) = \varphi_n(x)|n\rangle \tag{11}$$

and

$$\Psi_p(x) = \varphi_p(x)|p\rangle \ , \tag{12}$$

where $\varphi_n(x)$ and $\varphi_p(x)$ are nonrelativistic wave functions and $|p\rangle$ and $|n\rangle$ satisfy

$$\langle p|\tau^+|n\rangle = 1 \ . \tag{13}$$

Using

$$\langle \ e^-(r) \ \overline{\nu}_e(s)| \sum_{r's'} \ a^\dagger_{r'} b^\dagger_{s'} \ | \ 0 \ \rangle = \delta_{rr'}\delta_{ss'} \tag{14}$$

we can write the transition amplitude (to simplify the notation, the indices specifying the states of light particles are omitted)

$$T_{fi} = (2\pi) \ \langle f|H|i\rangle \ \delta(E_f - E_i) = \frac{G}{\sqrt{2}} \int d^4x[\varphi^*_p(x)\varphi_n(x)][\overline{\psi}_e(x)\gamma^0\psi_\nu(x)] \ . \tag{15}$$

We will get back to the details of the calculation of the neutron β-decay transition amplitude in the next section.

The most immediate relativistic generalization of eq.(15) involves the replacement of the scalar potential associated with the heavy particles with the four-current

$$\overline{\psi}_p(x)\gamma_\mu\psi_n(x) \ , \tag{16}$$

where $\psi_p(x)$ and $\psi_n(x)$ are now four-dimensional Dirac's spinors. The resulting transition matrix elements reads (compare to eq.(15))

$$T_{fi} = \frac{G}{\sqrt{2}} \int d^4x[\overline{\psi}_p(x)\gamma_\mu\psi_n(x)][\overline{\psi}_e(x)\gamma^\mu\psi_\nu(x)] \ . \tag{17}$$

Fermi theory was successfully employed for over twenty years, until Lee and Yang argued, in 1956, that parity may not be conserved in weak interactions.

The first clearcut experimental evidence of parity non conservation came from the classic ^{60}Co decay experiment, in 1957. The nucleus ^{60}Co has angular momentum J=5. In the experiment, performed by Wu *et al*, it was alligned in the state with J_z=5 and was observed to decay to the J_z=4 state of ^{60}Ni, the difference J_z=1 being carried by the spins of the e^- and $\bar{\nu}_e$. As a consequence, both the produced e and $\bar{\nu}_e$ had J_z=1/2.

The measurements showed that most e^- were emitted antiparallel to the spin of ^{60}Co, implying in turn that the corresponding $\overline{\nu}_e$ were emitted in the opposite direction. Thus, the ^{60}Co experiment provided evidence that the required J_z=1 came from a right handed $\overline{\nu}_e$ and a left handed e^-. The absence of the "mirror" particles (i.e. left handed $\overline{\nu}_e$ and right handed e^-) was a clear violation of parity.

The $\overline{\psi}_e(x)\gamma^\mu\psi_\nu(x)$ current appearing in eq.(17), being of vector nature, cannot explain the observed behavior, since it leads to predict equal number of e^- and $\overline{\nu}_e$ emitted parallel and antiparallel to the nuclear spin. However, it can be easily modified in such a way as to allow production of right handed $\overline{\nu}_e$ and left handed e^- only. The new current can be written as the difference between a vector (V) part, involving γ^μ only, and an axial-vector (A) part, involving the product $\gamma^\mu\gamma_5$:

$$\overline{\psi}_e(x)\gamma^\mu(1-\gamma_5)\psi_\nu(x) \ . \tag{18}$$

Using the definition of γ_5 and spinors normalized in a way appropriate to describe massless particles, it can be readily verified that a current with the above V-A structure creates only right handed $\overline{\nu}_e$ and (in the limit of negligible electron mass) left handed e^- (see Appendix again).

A generalization of the Fermi's hamiltonian of eq.(17) that properly describes the observed parity violation, originally proposed by Lee and Yang, can be written in the form

$$H = \frac{1}{\sqrt{2}}\left[\overline{\psi}_p(x)\gamma_\mu(G_V - G_A\gamma_5)\psi_n(x)\right]\left[\overline{\psi}_e(x)\gamma_\mu(1-\gamma_5)\psi_\nu(x)\right] \ , \tag{19}$$

where G_V and G_A denote the coupling constants associated with the vector and axial vector part of the np current, respectively.

If there were only weak interactions one would have $G_V = G_A = G$ (this is indeed the case for the current associated with the light particles). The np current, however, is also affected by strong interactions. Strong interactions do not change the weak charge, i.e. the vector current $\overline{\psi}_p \gamma_\mu \psi_n$ is conserved, but can flip the neutron spin, leading to $G_A \neq G_V$. Neutron β-decay data yield $\widetilde{G} = G_A/G_V \sim 1.2$.

2. Neutron β-decay

Using the results of the previous section, we can write the transition amplitude for neutron β-decay, defined as

$$T_{fi} = (2\pi) \, \langle f|H|i\rangle \; \delta(E_f - E_i), \tag{20}$$

in the form

$$\begin{aligned} T_{fi} &= \frac{G}{\sqrt{2}} \int d^4x \left[\overline{\psi}_p(x)\gamma^\mu(1-\widetilde{G}\gamma_5)\psi_n(x)\right] \left[\overline{\psi}_e(x)\gamma_\mu(1-\gamma_5)\psi_\nu(x)\right] \\ &= N_p N_n N_e N_\nu \int d^4x \mathcal{M}_{fi} \; \mathrm{e}^{i(k_n - k_p - k_e - k_\nu)x} \qquad (21) \\ &= N_p N_n N_e N_\nu \; \mathcal{M}_{fi} \; (2\pi)^4 \delta^{(4)}(k_n - k_p - k_e - k_\nu) \; , \qquad (22) \end{aligned}$$

where N_i $(i \equiv n, p, e, \nu)$ denotes the normalization factor of the Dirac spinor associated with particle i (see Appendix) and we have set $G = G_V$ and $\widetilde{G} = G_A/G_V$. The invariant amplitude $\mathcal{M}_{fi}$ reads

$$\mathcal{M}_{fi} = \frac{G}{\sqrt{2}} \; \overline{u}^{(s)}(k_p)\gamma^\mu(1-\widetilde{G}\gamma_5)u^{(r)}(k_n)\bar{u}^{(s')}(k_e)\gamma_\mu(1-\gamma_5)v^{(r')}(k_\nu). \tag{23}$$

and the transition probability per unit time and unit volume is defined as

$$W_{fi} = \frac{\overline{|T_{fi}|}^2}{T\Omega} \; , \tag{24}$$

where T is the interaction time, $\Omega = L^3$ is the volume of the normalization box and $\overline{|T_{fi}|}^2$ denotes the squared transition amplitude *averaged over the spin states of the neutron and* *summed over all possible spins of the final state particles.* The calculation of W_{fi} requires the evaluation of the square of the δ-function. Using the relation, valid at large L,

$$2\pi\delta(k_\alpha) \sim \int_{-L/2}^{L/2} dx_\alpha e^{ik_\alpha x_\alpha} = \frac{2}{x_\alpha}\sin\left(\frac{L}{2}x_\alpha\right) , \tag{25}$$

(α= 1, 2, 3) implying

$$[2\pi\delta(k_\alpha)]^2 = 2\pi\delta(k_\alpha)\int_{-L/2}^{L/2} dx_\alpha e^{ik_\alpha x_\alpha} = 2\pi\delta(x_\alpha)\ L \tag{26}$$

and the analog relation involving the energy conserving δ-function we can rewrite

$$[(2\pi)^4\delta^{(4)}(k_n - k_p - k_e - k_\nu)]^2 = T\Omega\ (2\pi)^4\delta^{(4)}(k_n - k_p - k_e - k_\nu) . \tag{27}$$

With the above result and the normalization factors given in the Appendix, eq.(24) can be rewritten (we denote $k_p^0 = E_p$, $k_n^0 = E_n$, $k_e^0 = E_e$ and $k_\nu^0 = E_\nu$)

$$W_{fi} = \frac{1}{\Omega^4}\ \frac{1}{2E_n}\frac{1}{2E_p}\frac{1}{2E_e}\frac{1}{2E_\nu}(2\pi)^4\delta^{(4)}(k_n - k_p - k_e - k_\nu)\overline{|\mathcal{M}_{fi}|}^2 . \tag{28}$$

The differential transition rate is related to W_{fi} through

$$d\Gamma = \frac{1}{\overline{\rho}_n}W_{fi}\ \rho_p\rho_e\rho_\nu , \tag{29}$$

where $\overline{\rho}_n = \Omega^{-1}$ is the neutron density, whereas ρ_i gives the number of final states with momentum between $\mathbf{k}_i$ and $\mathbf{k}_i + d\mathbf{k}_i$ available to particle i ($i \equiv p, e, \nu$). For particles enclosed in a cubic box and described by plane waves satisfying periodic boundary conditions

$$\rho_i = \frac{\Omega}{(2\pi)^3}\ d^3k_i . \tag{30}$$

Collecting together all the above results we can finally write the differential decay rate as

$$d\Gamma = \frac{1}{2E_n}\ \overline{|\mathcal{M}_{fi}|}^2\ \frac{d^3k_p}{2E_p(2\pi)^3}\frac{d^3k_e}{2E_e(2\pi)^3}\frac{d^3k_\nu}{2E_\nu(2\pi)^3}\ (2\pi)^4\delta^{(4)}(k_n - k_p - k_e - k_\nu) . \tag{31}$$

We will now proceed and use eq.(31) to carry out an explicit calculation of the neutron β-decay rate. As in Fermi's original paper on nuclear β-decay, we will use the nonrelativistic approximation to describe the neutron and the proton.

The nonrelativistic limit of the Dirac spinors, normalized as discussed in the Appendix, reads

$$\overline{u}^{(s)}(k_p) \sim \sqrt{2m_p}\begin{pmatrix}\chi_s^\dagger & 0\end{pmatrix} = \overline{u}_p^{(s)} \ , \tag{32}$$

and

$$u^{(s)}(k_n) \sim \sqrt{2m_n}\begin{pmatrix}\chi_s \\ 0\end{pmatrix} = u_n^{(s)} \ . \tag{33}$$

It follows that

$$\overline{u}_p^{(s)}\gamma_0 u_n^{(s')} = \sqrt{2m_p}\sqrt{2m_n}\begin{pmatrix}\chi_s^\dagger & 0\end{pmatrix}\begin{pmatrix}\mathrm{I} & 0 \\ 0 & -\mathrm{I}\end{pmatrix}\begin{pmatrix}\chi_{s'} \\ 0\end{pmatrix} = \sqrt{2m_p}\sqrt{2m_n}\delta_{ss'} \tag{34}$$

and ($\alpha = 1,\ 2,\ 3$)

$$\overline{u}_p^{(s)}\gamma_\alpha u_n^{(s')} = \sqrt{2m_p}\sqrt{2m_n}\begin{pmatrix}\chi_s^\dagger & 0\end{pmatrix}\begin{pmatrix}0 & \sigma_\alpha \\ -\sigma_\alpha & 0\end{pmatrix}\begin{pmatrix}\chi_{s'} \\ 0\end{pmatrix} = 0. \tag{35}$$

A similar calculation leads to

$$\overline{u}_p^{(s)}\gamma_0\gamma_5 u_n^{(s')} = 0 \tag{36}$$

and

$$\overline{u}_p^{(s)}\gamma_\alpha\gamma_5 u_n^{(s')} = \sqrt{2m_p}\sqrt{2m_n}\ \chi_s^\dagger\sigma_\alpha\chi_{s'}, \tag{37}$$

showing that the axial vector part of the current induces transitions in which the neutron spin can be flipped over. Here we are interested in an order of magnitude estimate of the decay rate and will disregard the contribution of these transitions, setting $\widetilde{G} = 0$ in eq.(22).

We will also make the rather crude assumption that the final state proton carry vanishing momentum, and will neglect the three-dimensional delta function resulting from momentum conservation altogether.

Average over the neutron spin and sum over the spin of the proton gives

$$\frac{1}{2}\sum_{ss'} |\overline{u}_p^{(s)}\gamma_0 u_n^{(s')}|^2 = 2m_p 2m_n \ , \tag{38}$$

while for the contribution of light particles we find (summation over α, β, δ and ϵ is implicit)

$$\begin{aligned}
&\sum_{ss'} |\overline{u}^{(s)}(k_e)\gamma_0(1-\gamma_5)v^{(s')}(k_\nu)|^2 \\
&= \sum_{ss'} \overline{u}^{(s)}(k_e)\gamma_0(1-\gamma_5)v^{(s')}(k_\nu)\overline{v}^{(s')}(k_\nu)\gamma_0(1-\gamma_5)u^{(s)}(k_e) \\
&= \sum_{ss'} \overline{u}_\alpha^{(s)}(k_e)[\gamma_0(1-\gamma_5)]_{\alpha\beta}v_\beta^{(s')}(k_\nu)\overline{v}_\delta^{(s')}(k_\nu)[\gamma_0(1-\gamma_5)]_{\delta\epsilon}u_\epsilon^{(s)}(k_e) \ .
\end{aligned} \tag{39}$$

Using the completeness relations of the Dirac spinors (see Appendix) and neglecting the electron mass the above sum can be rewritten

$$\begin{aligned}
&(\not{k}_e)_{\epsilon\alpha}[\gamma_0(1-\gamma_5)]_{\alpha\beta}(\not{k}_\nu)_{\beta\delta}[\gamma_0(1-\gamma_5)]_{\delta\epsilon} \\
&= Tr\,[\not{k}_e\gamma_0(1-\gamma_5)\not{k}_\nu\gamma_0(1-\gamma_5)] \\
&= Tr\left[\not{k}_e\gamma_0\not{k}_\nu(1+\gamma_5)^2\gamma_0\right] && (40) \\
&= 2\ Tr\,[\not{k}_e\gamma_0\not{k}_\nu(1+\gamma_5)\gamma_0] \ . && (41)
\end{aligned}$$

Using the traces of combinations of γ matrices given in the Appendix we finally get

$$\overline{|\mathcal{M}_{fi}|}^2 = \sum_{ss'} |\overline{u}^{(s)}(k_e)\gamma_0(1-\gamma_5)v^{(s')}(k_\nu)|^2 = 8E_eE_\nu(1+\cos\theta) \ , \tag{42}$$

where θ is the angle between the momenta of the two light particles. Note that the above equation implies that the two light particles are preferably emitted in the same direction, while back to back emission, corresponding to $\theta = \pi$, is forbidden. This is a consequence of the fact that we are only considering the contribution of the vector part of the np current, that does not flip the neutron spin. Hence, the total spin of the produced e^- and $\overline{\nu}_e$ must be zero.

Collecting all the above results and substituting into the definition of the differential decay rate we find, within our set of approximations

$$\begin{aligned}
d\Gamma &= \frac{1}{2m_n}\frac{1}{2m_p}\frac{d^3k_e}{2E_e(2\pi)^3}\frac{d^3k_\nu}{2E_\nu(2\pi)^3}\ \frac{G^2}{2}2m_p\ 2m_n\ 8E_eE_\nu \\
&\times (1+\cos\theta)(2\pi)\delta(E_e+E_\nu+m_p-m_n) \\
&= \frac{G^2}{2}\frac{d^3k_e}{(2\pi)^3}\frac{d^3k_\nu}{(2\pi)^3}\ 2(1+\cos\theta)(2\pi)\delta(E_e+E_\nu+m_p-m_n) \ .
\end{aligned} \tag{43}$$

To carry out the integration and obtain the decay rate Γ we use (neglect electron mass again)

$$d^3k_e\, d^3k_\nu = (2\pi)^2 2d\cos\theta E_e^2 E_\nu^2 dE_e dE_\nu, \tag{44}$$

with the result

$$\begin{aligned}\Gamma &= 2\frac{G^2}{(2\pi)^3}\int E_e^2 dE_e \int E_\nu^2 dE_\nu \delta(E_e + E_\nu + m_p - m_n) \int_{-1}^{1} d\cos\theta\ (1+\cos\theta) \\ &= 4\frac{G^2}{(2\pi)^3}\int_0^{m_n - m_p} E_e^2 dE_e (E_e + m_p - m_n)^2 \\ &= 4\frac{G^2}{(2\pi)^3}\frac{(m_n - m_p)^5}{30}\end{aligned} \tag{45}$$

We will now substitute the measured neutron lifetime, $\tau = \Gamma^{-1} \sim 900$ sec, and the neutron-proton mass difference, $m_n - m_p = 1.3$ MeV, and invert eq.(45) to obtain an order of magnitude estimate of the weak coupling constant G. Knowing that $c = 3\times 10^{10}$ cm sec^{-1} $= 3\times 10^{23}$ fm sec^{-1} (1 fm $= 10^{-13}$ cm) and $\hbar c = 197$ MeV fm, in our system of units in which $\hbar = c = 1$, we can write G^2 in units of MeV^{-4} substituting 1 sec $= 3\times 10^{23}\times 197^{-1}$ MeV^{-1}. We find

$$\begin{aligned}G^2 &= \frac{(2\pi)^3}{4}\ \frac{30}{(m_n - m_p)^5\tau} = \frac{(2\pi)^3}{4}\ \frac{30}{1.3^5\times 900}\text{MeV}^{-5}\ \text{sec}^{-1} \\ &= \frac{(2\pi)^3}{4}\ \frac{30\times 197}{1.3^5\times 900\times 3\times 10^{23}}\ \text{MeV}^{-4} \\ &\sim \frac{3.7}{10^{22}}\ \text{MeV}^{-4}\ ,\end{aligned} \tag{46}$$

leading to (remember that the proton mass is $m_p = 938$ MeV)

$$G \sim \frac{\sqrt{3.7}}{10^5}\left(\frac{.938}{938}\right)^2\ \text{MeV}^{-4} \sim 1.8\ \frac{10^{-5}}{m_p^2}\ . \tag{47}$$

A proper calculation, taking into account the contribution of the axial vector current proportional to $\tilde{G}$ and the nonvanishing electron mass yields $G \sim 10^{-5}/m_p^2$.

3. Muon decay

In this section we will calculate the rate of the process in which a muon of four momentum p decays into an electron carrying four momentum p', a ν_μ with four momentum k and a $\overline{\nu}_e$ with four momentum k':

$$\mu^-(p) \rightarrow e^-(p') + \overline{\nu}_e(k') + \nu_\mu(k) \ . \tag{48}$$

The associated invariant amplitude reads:

$$\mathcal{M}_{fi} = \frac{G}{\sqrt{2}} \left[\bar{u}^{(s)}(k)\gamma^\mu(1-\gamma_5)u^{(r)}(p)\right] \left[\bar{u}^{(s')}(p')\gamma_\mu(1-\gamma_5)v^{(r')}(k')\right] \ . \tag{49}$$

Denoting $k_0 = \omega$, $k'_0 = \omega'$, $p_0 = E$ and $p'_0 = E'$, we can write the differential decay rate in the form

$$d\Gamma = \frac{1}{2E}\overline{|\mathcal{M}_{fi}|}^2 dQ \tag{50}$$

with

$$dQ = \frac{d^3p'}{2E'(2\pi)^3}\frac{d^3k'}{2\omega'(2\pi)^3}\frac{d^3k}{2\omega(2\pi)^3}(2\pi)^4\delta^{(4)}(p-p'-k-k') \ . \tag{51}$$

We will use the relation

$$\int \frac{d^3k}{2\omega} = \int d^4k\theta(\omega)\delta(k^2) \tag{52}$$

to carry out the k integration in the right hand side of eq.(51). The results is

$$\begin{aligned} dQ &= \frac{d^3p'}{2E'(2\pi)^3}\frac{d^3k'}{2\omega'(2\pi)^3}\int \frac{d^3k}{2\omega(2\pi)^3}(2\pi)^4\delta^{(4)}(p-p'-k-k') \\ &= \frac{d^3p'}{2E'(2\pi)^3}\frac{d^3k'}{2\omega'(2\pi)^3}\int \frac{d^3k}{(2\pi)^3}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}'-\mathbf{k}-\mathbf{p}') \\ &\times \int d\omega\theta(\omega)(2\pi)\delta(E-E'-\omega-\omega')\delta(k^2) \ , \end{aligned} \tag{53}$$

leading to

$$dQ = \frac{d^3p'}{2E'(2\pi)^3}\frac{d^3k'}{2\omega'(2\pi)^3}\theta(E-E'-\omega')(2\pi)\delta\left((p-p'-k')^2\right) \ . \tag{54}$$

Now we want to compute the squared invariant transition amplitude, averaged over the spin of the muon and summed over all possible spins of the final state particles, i.e.

$$\overline{|\mathcal{M}_{fi}|}^2 = \frac{1}{2}\sum_{r,r',s,s'} \left[\bar{u}^{(s)}(k)\gamma^\mu(1-\gamma_5)u^{(r)}(p)\bar{u}^{(r)}(p)\gamma^\nu(1-\gamma_5)u^{(s)}(k)\right] \tag{55}$$

$$\times\left[\bar{u}^{(s')}(p')\gamma_\mu(1-\gamma_5)v^{(r')}(k')\bar{v}^{(r')}(k')\gamma_\nu(1-\gamma_5)u^{(s')}(p')\right] \ . \tag{56}$$

We will neglect the electron mass, but not the muon mass, denoted by m. Consider first the contribution of the square bracket involving the μ and ν_μ spinors. Using the completeness of Dirac's spinors (see Appendix) we find (summation over α, β, δ and ϵ is implicit):

$$\begin{aligned}\sum_{r,s} \quad & \bar{u}^{(s)}_\alpha(k)[\gamma^\mu(1-\gamma_5)]_{\alpha\beta}u^{(r)}_\beta(p)\bar{u}^{(r)}_\delta(p)[\gamma^\nu(1-\gamma_5)]_{\delta\epsilon}u^{(s)}_\epsilon(k)\\ &= [\gamma^\mu(1-\gamma_5)]_{\alpha\beta}(\not{p}+m)_{\beta\delta}[\gamma^\nu(1-\gamma_5)]_{\delta\epsilon}\not{k}_{\epsilon\alpha}\\ &= Tr\left[\gamma^\mu(1-\gamma_5)(\not{p}+m)\gamma^\nu(1-\gamma_5)\not{k}\right]\end{aligned} \tag{57}$$

The calculation of the above trace can be readily carried out using the following result, that can be easily obtained from the traces of products of γ matrices given in the Appendix:

$$Tr\,(\gamma_\mu\not{p}\gamma_\nu\not{k}) = 4[p_\mu k_\nu + p_\nu k_\mu - g_{\mu\nu}(pk)] \ , \tag{58}$$

leading to

$$Tr\,[\gamma_\mu(1-\gamma_5)\not{p}\gamma_\nu(1-\gamma_5)\not{k}] = 2Tr\,(\gamma_\mu\not{p}\gamma_\nu\not{k}) + 8i\epsilon_{\mu\rho\nu\sigma}p^\rho k^\sigma \ . \tag{59}$$

Moreover, it can be shown that (see Appendix)

$$Tr\,(\gamma_\mu\not{p}\gamma_\nu\not{k})\,Tr\,(\gamma^\mu\not{k}'\gamma^\nu\not{p}') = 32[(pk')(kp') + (pp')(kk')] \ , \tag{60}$$

$$Tr\,(\gamma_\mu\not{p}\gamma_\nu\gamma_5\not{k})\,Tr\,(\gamma^\mu\not{k}'\gamma^\nu\gamma_5\not{p}') = 32[(pk')(kp') - (pp')(kk')] \ , \tag{61}$$

and

$$Tr\,[\gamma^\mu(1-\gamma_5)\not{p}\gamma^\nu(1-\gamma_5)\not{k}]\,Tr\,[\gamma_\mu(1-\gamma_5)\not{k}'\gamma_\nu(1-\gamma_5)\not{p}'] = 256(pk')(kp') \tag{62}$$

The final result is (note that all terms linear in the muon mass vanish, since they involve traces of an odd number of γ matrices):

$$\overline{|\mathcal{M}_{fi}|}^2 = \frac{G^2}{2}\,\frac{1}{2}Tr\,[\gamma^\mu(1-\gamma_5)(\not p + m)\gamma^\nu(1-\gamma_5)\not k]\,Tr\,[\gamma_\mu(1-\gamma_5)\not k'\gamma_\nu(1-\gamma_5)\not p']$$
$$= 64G^2(pk')(kp')\ . \qquad (63)$$

The right hand side of eq.(63) can be most easily evaluated in the reference frame in which the muon is at rest, where $p \equiv (m,0,0,0)$, the result being (use $p'-k=p-k'$)

$$(pk')(kp') = \frac{1}{2}m\omega'(k+p')^2 = \frac{1}{2}m\omega'(p-k')^2\ , \qquad (64)$$

i.e.

$$2(pk')(kp') = m\omega'(m^2-2m\omega')\ . \qquad (65)$$

Collecting together the results of eqs.(50), (54), (63) and (65) we can finally write the differential muon decay rate in the form (note that the θ-function in eq.(54) can be omitted since the condition $E-E'-\omega'>0$ is always fulfilled)

$$d\Gamma = \frac{32G^2}{2m}m\omega'(m^2-2m\omega')\frac{d^3p'}{2E'(2\pi)^3}\frac{d^3k'}{2\omega'(2\pi)^3}\delta\left[(p-p'-k')^2\right]\ . \qquad (66)$$

Using (we are treating the electron as a massless particle, implying $E'=|\mathbf{k}'|$)

$$(p-p'-k')^2 = m^2-2mE'-2m\omega'+2E'\omega'(1-\cos\theta)\ , \qquad (67)$$

θ being the angle between the e^- and $\overline{\nu}_e$ momenta, and

$$d^3p'd^3k' = (4\pi)E'^2dE'(2\pi)d\cos\theta\omega'^2d\omega' \qquad (68)$$

we obtain

$$d\Gamma = \frac{G^2}{2\pi^3}dE'd\omega'm\omega'(m-2\omega')\delta\left(\frac{m^2-2mE'-2m\omega'}{2E'\omega'}+1-\cos\theta\right)d\cos\theta\ . \qquad (69)$$

The $\cos\theta$ integration can be readily carried out using the δ-function. The result is

$$d\Gamma = \frac{G^2}{2\pi^3}dE'd\omega'm\omega'(m-2\omega') \qquad (70)$$

with the constraints, coming from the requirement $-1\leq\cos\theta\leq 1$

$$\frac{m}{2} - E' \leq \omega' \leq \frac{m}{2} \ , \tag{71}$$

$$0 \leq E' \leq \frac{m}{2} \ . \tag{72}$$

Note that the above inequalities have a very straightforward physical interpretation. For example, the first inequality simply states that if the electron is produced with zero momentum, all the available energy, i.e. the muon mass, is equally shared between the ν_μ and the $\overline{\nu}_e$.

From eq.(70) we can also calculate the energy spectrum of the emitted electrons. Integration over ω' yields

$$\frac{d\Gamma}{dE'} = \frac{mG^2}{2\pi^3} \int_{m/2-E'}^{m/2} d\omega'\omega'(m - 2\omega') = \frac{mG^2}{2\pi^3}\frac{m^2 E'^2}{6}\left(3 - \frac{4E'}{\omega'}\right) \ . \tag{73}$$

The above prediction turns out to be in remarkably good agreement with experimental data.

Finally, the muon decay rate can be obtained carrying out the integration of the spectrum of eq.(73) over the electron energy:

$$\Gamma = \int_0^{m/2} dE' \left(\frac{d\Gamma}{dE'}\right) = \frac{G^2}{192\pi^3} m^5 \ . \tag{74}$$

We will now substitute the measured muon lifetime, $\tau = 2.2 \times 10^{-6}$, sec and the muon mass, $m = 105$ MeV, and invert eq.(74) to obtain the value of the weak coupling constant G. Following the procedure described in the previous section we can write

$$G^2 = \frac{192\pi^3}{\tau m^5} = \frac{192\pi^3 \times 197}{105^5 \times 2.2 \times 10^{-6} \times 3 \times 10^{23}} \ \mathrm{MeV}^{-4} \sim 2.5 \times 10^{-22} \mathrm{MeV}^{-4} \ , \tag{75}$$

leading to (remember that the proton mass is $m_p = 938$ MeV)

$$G \sim \sqrt{2.5} \times 10^{-5} \left(\frac{.938}{938}\right)^2 \mathrm{MeV}^{-4} \sim 1.4 \ \frac{10^{-5}}{m_p^2} \ . \tag{76}$$

Comparison with the value of G obtained from the rate of neutron β-decay suggests that the weak coupling constant is the same for leptons and nucleons, i.e. that β-decay and muon decay are different manifestations of the same underlying physical mechanism. In fact, proper calculations show that the coupling constants are equal within a few percent.

3. Interpretation of the Fermi's coupling constant

The weak interaction invariant amplitude (e.g. eq.(49)) can be rewritten in the more compact form

$$\mathcal{M}_{fi} = \frac{G}{\sqrt{2}} J^\mu J_\mu \ , \tag{77}$$

where $J_\mu = \bar{u}\gamma_\mu(1-\gamma_5)u$. In lowest order perturbation theory, the amplitude associated with the electromagnetic interaction of, e.g., an electron and a muon reads

$$\mathcal{M}^{em}_{fi} = j^\mu \left(-\frac{e^2}{q^2} \right) j_\mu \ , \tag{78}$$

where $j_\mu = \bar{u}\gamma_\mu u$ and q denotes the four momentum carried by the exchanged photon. Comparison between eqs.(77) and (78) shows that the two amplitudes have a similar structure, reflecting the current-current nature of the interaction. In the weak amplitude, however, there is no propagator, and the weak coupling constant $G/\sqrt{2}$ replaces the quantity e^2/q^2. This feature is consequence of the fact that weak interactions have extremely short range (in fact, in Fermi's hamiltonian the range is assumed to be zero), and implies in turn that G is not just a dimensionless coupling, but has dimensions GeV^{-2}. To pursue the analogy between electromagnetic and weak interactions, one must therefore assume that weak interactions are due to emission and absorption of a charged particle, playing the role of the photon, whose mass M is so large that the corresponding propagator can be approximated according to

$$\frac{g^2}{M^2 - q^2} \sim \frac{g^2}{M^2} \ , \tag{79}$$

g being a dimensionless coupling constant. Within this picture, the Fermi coupling G can be immediately related to M through

$$\frac{G}{\sqrt{2}} = \frac{g^2}{M^2} \ . \tag{80}$$

The existence of the particle exchanged in the weak interaction responsible for β-decay, the W boson, has been experimentally confirmed, and its measured mass is $M_W \sim 80$ GeV.

A1. Dirac's spinors

We will use a normalization suitable to describe massless particles. A plane wave solution of the Dirac's equation will be written:

$$\psi(x) = \frac{1}{\sqrt{2\omega_k}}\, u^{(s)}(\mathbf{k})\, \frac{\mathrm{e}^{-ikx}}{\sqrt{\Omega}}\ , \tag{81}$$

where $\Omega = L^3$ is the volume of the normalization box and $\omega_k = +\sqrt{|\mathbf{k}|^2 + m^2}$, m being the particle mass.

The $u^{(s)}$ corresponding to positive energy solutions (s=1,2) read

$$u^{(s)}(\mathbf{k}) = N \begin{pmatrix} \chi_s \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{\omega_k+m}\ \chi_s \end{pmatrix}\ , \tag{82}$$

while for negative energy we can write

$$u^{(s+2)}(\mathbf{k}) = N \begin{pmatrix} -\frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{\omega_k+m}\ \chi_s \\ \chi_s \end{pmatrix}\ . \tag{83}$$

In eqs.(82) and (83) χ_s are the Pauli spinors

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\ ,\ \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\ , \tag{84}$$

and $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ with

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ ,\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\ ,\ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ . \tag{85}$$

Normalization of $\psi(x)$ according to

$$\int d^3x\ \psi^{\dagger}(x)\psi(x)\ = 1 \tag{86}$$

requires

$$u^{(s)^\dagger}(\mathbf{k}) u^{(s)}(\mathbf{k}) = 2\omega_k \ , \tag{87}$$

implying in turn

$$N = \sqrt{\omega_k + m} \ . \tag{88}$$

Defining the adjoint spinor $\bar{u}^{(s)}(\mathbf{k}) = u^{(s)^\dagger}\gamma_0$ we finally find

$$\bar{u}^{(r)} u^{(s)} = \delta_{rs} \ 2m \tag{89}$$

and

$$\sum_s u_\alpha^{(s)}(\mathbf{k}) \bar{u}_\beta^{(s)}(\mathbf{k}) = (\gamma_\mu k^\mu + m)_{\alpha\beta} = (\not{k} + m)_{\alpha\beta} \tag{90}$$

- *Proof*: using (82) and

$$\bar{u}^{(s)}(\mathbf{k}) = N \begin{pmatrix} \chi_s^\dagger & -\dfrac{\boldsymbol{\sigma} \cdot \mathbf{k}}{\omega_k + m} \chi_s^\dagger \end{pmatrix} \tag{91}$$

  the matrix of elements $u_\alpha^{(s)}(\mathbf{k}) \bar{u}_\beta^{(s)}(\mathbf{k})$ can be readily constructed, with the result:

$$N^2 \begin{pmatrix} \mathrm{I} & -\frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{\omega_k+m} \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{\omega_k+m} & -\left(\frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{\omega_k+m}\right)^2 \end{pmatrix} = \begin{pmatrix} \omega_k + m & -\boldsymbol{\sigma} \cdot \mathbf{k} \\ \boldsymbol{\sigma} \cdot \mathbf{k} & \omega_k + m \end{pmatrix} \tag{92}$$

$$= \gamma_0 \omega_k - \boldsymbol{\gamma} \cdot \mathbf{k} + \mathrm{I}m = \gamma_\mu k^\mu + m = \not{k} + m \ . \tag{93}$$

According to Dirac's theory, the negative energy solutions (83) can be associated with antiparticles carrying positive energy and momentum -**k**. Hence, antiparticles are described by the spinors

$$v^{(1)}(\mathbf{k}) = u^{(4)}(-\mathbf{k}) \ , \quad v^{(2)}(\mathbf{k}) = u^{(3)}(-\mathbf{k}) \ , \tag{94}$$

yielding

$$v^{(s)^\dagger}(\mathbf{k}) v^{(s)}(\mathbf{k}) = 2\omega_k \ , \tag{95}$$

$$\bar{v}^{(s)}(\mathbf{k})v^{(s)}(\mathbf{k}) = -\delta rs\ 2m\ , \tag{96}$$

and

$$\sum_s v_\alpha^{(s)}(\mathbf{k})\bar{v}_\beta^{(s)}(\mathbf{k}) = (\not{k} - m)_{\alpha\beta}\ . \tag{97}$$

In the limit of nonrelativistic particles, having $(k/m) <<1$, eqs.(81) and (94) reduce to

$$u^{(s)}(\mathbf{k}) \rightarrow \sqrt{2m}\begin{pmatrix}\chi_s \\ 0\end{pmatrix} \tag{98}$$

$$u^{(s)}(\mathbf{k}) \rightarrow \sqrt{2m}\begin{pmatrix}0 \\ \chi_s\end{pmatrix}\ . \tag{99}$$

A massless particle has $\omega_k = |\mathbf{k}|$ and travels at the speed of light. Its velocity cannot be changed by a Lorentz transformation. As a consequence, for massless particles helicity (given by $\sigma \cdot \mathbf{k}/2|\mathbf{k}|$) is an intrinsic property, independent of the reference frame.

The zero mass limits of positive and negative energy spinors read

$$\begin{aligned} u^{(1)} &= \sqrt{\omega_k}\begin{pmatrix}\chi_1 \\ \chi_1\end{pmatrix}\ ,\ u^{(2)} = \sqrt{\omega_k}\begin{pmatrix}\chi_2 \\ -\chi_2\end{pmatrix} \\ u^{(3)} &= \sqrt{\omega_k}\begin{pmatrix}-\chi_1 \\ \chi_1\end{pmatrix}\ ,\ u^{(4)} = \sqrt{\omega_k}\begin{pmatrix}\chi_2 \\ \chi_2\end{pmatrix}\ . \end{aligned} \tag{100}$$

Particles with positive and negative helicity are called *right handed* and *left handed*, respectively. Hence, the spinors $u^{(1)}$ and $u^{(4)}$ describe right handed particles, while $u^{(2)}$ and $u^{(3)}$ describe left handed particles.

Using the Pauli-Dirac representation of the γ matrices:

$$\gamma_0 = \begin{pmatrix}\mathrm{I} & 0 \\ 0 & -\mathrm{I}\end{pmatrix}\ ,\ \boldsymbol{\gamma} = \begin{pmatrix}0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0\end{pmatrix}\ , \tag{101}$$

the matrix

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 \tag{102}$$

can be written

$$\gamma_5 = \begin{pmatrix} 0 & \mathrm{I} \\ \mathrm{I} & 0 \end{pmatrix} . \tag{103}$$

It follows that

$$\frac{1}{2}(1-\gamma_5)\, u^{(2)} = u^{(2)} \ , \ \frac{1}{2}(1-\gamma_5)\, u^{(3)} = u^{(3)} \tag{104}$$

and

$$\frac{1}{2}(1-\gamma_5)\, u^{(1)} = \frac{1}{2}(1-\gamma_5)\, u^{(4)} = 0 \ . \tag{105}$$

Right and left handed neutrinos are described by $u^{(1)}$ and $u^{(2)}$, respectively, whereas $u^{(3)}$ and $u^{(4)}$ describe right and left handed antineutrinos. From eqs.(104) and (105) it follows that a correvt of the V-A form

$$\bar{\psi}\gamma_\mu (1-\gamma_5)\,\psi \tag{106}$$

couples to right handed antineutrinos and left handed neutrinos only.

A2. Traces of combinations of γ matrices

We will make use of the following properties of the γ matrices:

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} \ ,$$

$$\gamma_0^2 = \mathrm{I} \ , \quad \gamma_\alpha^2 = -\mathrm{I} \qquad (\alpha = 1,2,3)$$

and ($\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$)

$$\{\gamma_5, \gamma_\mu\} = 0 \ ,$$

$$\gamma_5^2 = \mathrm{I} \ .$$

Moreover, remember that for any A and B (commuting or noncommuting)

$$Tr(AB) = Tr(AB) \ .$$

[T1] $Tr\left(\gamma_\mu \gamma_\nu\right) = 4 g_{\mu\nu}$

- Proof:

$$\begin{aligned} Tr\left(\gamma_\mu \gamma_\nu\right) &= \frac{1}{2}\left[Tr\left(\gamma_\mu \gamma_\nu\right) + Tr\left(\gamma_\nu \gamma_\mu\right)\right] \\ &= \frac{1}{2} Tr\left(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu\right) \\ &= \frac{1}{2}\left(2 g_{\mu\nu}\right) = g_{\mu\nu} Tr(\mathrm{I}) = 4 g_{\mu\nu} \ . \end{aligned}$$

[T2] The trace of the product of any odd number of γ matrices vanishes

- Proof:

$$Tr\left(\gamma_{\mu_1} ... \gamma_{\mu_n}\right) = Tr\left(\gamma_{\mu_1} ... \gamma_{\mu_n} \gamma_5 \gamma_5\right) \ .$$

Use $\{\gamma_\mu, \gamma_5\} = 0$ to move γ_5 to the left in n steps. We obtain

$$Tr\left(\gamma_{\mu_1} ... \gamma_{\mu_n}\right) = (-)^n Tr\left(\gamma_5 \gamma_{\mu_1} ... \gamma_{\mu_n} \gamma_5\right) \ .$$

Using now $Tr(AB) = Tr(BA)$ and $\gamma_5^2 = 1$ leads to

$$Tr\left(\gamma_{\mu_1} ... \gamma_{\mu_n}\right) = (-)^n Tr\left(\gamma_{\mu_1} ... \gamma_{\mu_n}\right) \ ,$$

which for n odd implies $Tr\left(\gamma_{\mu_1} ... \gamma_{\mu_n}\right) = 0$.

[T3] $Tr\left(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\right) = 4\left(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}\right)$

- Proof:

$$\begin{aligned} Tr(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) &= 2g_{\rho\sigma}Tr(\gamma_\mu\gamma_\nu) - Tr(\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\rho) \\ &= 8g_{\rho\sigma}g_{\mu\nu} - 2g_{\nu\sigma}Tr(\gamma_\mu\gamma_\rho) + Tr(\gamma_\mu\gamma_\sigma\gamma_\nu\gamma_\rho) \\ &= 8g_{\rho\sigma}g_{\mu\nu} - 8g_{\nu\sigma}g_{\mu\rho} + 2g_{\mu\sigma}Tr(\gamma_\nu\gamma_\rho) - Tr(\gamma_\sigma\gamma_\mu\gamma_\nu\gamma_\rho) \\ &= 8\left(g_{\rho\sigma}g_{\mu\nu} - g_{\nu\sigma}g_{\mu\rho} + g_{\mu\sigma}g_{\nu\rho}\right) - Tr\left(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\right) \ . \end{aligned}$$

Note that [T3] implies that the trace of the product of four γ matrices vanishes unless the product involves two pairs of γ matrices with the same indices.

[T4] $Tr\left(\gamma_5\gamma_{\mu_1}...\gamma_{\mu_n}\right) = 0$ for any odd n

- Proof: The result simply follows from [T2] and the definition of γ_5, involving four γ matrices

[T5] $Tr\left(\gamma_5\gamma_\mu\gamma_\nu\right) = 0$

- Proof:

$$\begin{aligned} Tr(\gamma_5\gamma_\mu\gamma_\nu) &= iTr(\gamma_0\gamma_1\gamma_2\gamma_3\gamma_\mu\gamma_\nu) \\ &= 2ig_{\mu\nu}Tr(\gamma_0\gamma_1\gamma_2\gamma_3) - iTr(\gamma_0\gamma_1\gamma_2\gamma_3\gamma_\nu\gamma_\mu) \ . \end{aligned}$$

The first term inthe second line vanishes due to [T3]. The second can be rewritten

$$-iTr(\gamma_0\gamma_1\gamma_2\gamma_3\gamma_\nu\gamma_\mu) = 2ig_{\nu 3}Tr(\gamma_0\gamma_1\gamma_2\gamma_\mu) + Tr(\gamma_0\gamma_1\gamma_2\gamma_\nu\gamma_3\gamma_\mu)$$

and again, due to [T3] $Tr(\gamma_0\gamma_1\gamma_2\gamma_\mu) = 0$. This procedure can be repeated over and over, with the final result

$$Tr\left(\gamma_5\gamma_\mu\gamma_\nu\right) = -Tr\left(\gamma_\nu\gamma_5\gamma_\mu\right) = -Tr\left(\gamma_5\gamma_\mu\gamma_\nu\right) \ .$$

[T6] $Tr\left(\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\right) = -4i\epsilon_{\mu\nu\rho\sigma}$ where

$$\epsilon_{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } (\mu\nu\sigma\rho) \text{ is an even permutation of } (0123) \\ -1 & \text{if it is an odd permutation} \\ 0 & \text{otherwise} \end{cases}$$

- Proof: Show first that if any two indices are the same

$$Tr\left(\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\right) = 0 \ .$$

Taking $\mu = \nu$, $\mu \neq \rho$, $\mu \neq \sigma$ we find (use $\gamma_0^2 = \mathrm{I}$, $\gamma_\alpha^2 = -\mathrm{I}$ (α=1,2,3) and [T5])

$$Tr\left(\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\right) = -Tr\left(\gamma_5(\gamma_\mu)^2\gamma_\nu\gamma_\sigma\right) = \pm Tr\left(\gamma_5\gamma_\nu\gamma_\sigma\right) = 0 \ .$$

Obvioulsy, the trace $Tr\left(\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\right)$ also vanishes if three or four indices among μ, ν, ρ and σ are the same, since $Tr(\gamma_5\gamma_\mu) = Tr(\gamma_5) = 0$.

In order to have a nonzero trace the indices $\mu\nu\rho\sigma$ must be any permutation of 0123. Consider first the identical permutation. In this case

$$\begin{aligned} \gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma &= i\gamma_0\gamma_1\gamma_2\gamma_3\gamma_0\gamma_1\gamma_2\gamma_3 \\ &= -i\gamma_0\gamma_1\gamma_2(\gamma_3)^2\gamma_0\gamma_1\gamma_2 = -i\gamma_0\gamma_1(\gamma_2)^2(\gamma_3)^2\gamma_0\gamma_1 \\ &= i\gamma_0(\gamma_1)^2(\gamma_2)^2(\gamma_3)^2\gamma_0 = i(\gamma_0)^2(\gamma_1)^2(\gamma_2)^2(\gamma_3)^2 = -i\mathrm{I} \ . \end{aligned}$$

Hence,

$$Tr(\gamma_5\gamma_0\gamma_1\gamma_2\gamma_3) = -iTr(\mathrm{I}) = -4i \ .$$

Using the anticommutation rules of the γ matrices it can be easily verified that any even permutation of the indices leads to the same result, while for odd permutations one gets $4i$.

[T7] $Tr\,(\gamma_\mu \not{p} \gamma_\nu \not{k}) = 4\,[p_\mu k_\nu + p_\nu k_\mu - g_{\mu\nu}(pk)]$

- Proof:

$$\begin{aligned} Tr\,(\gamma_\mu \not{p} \gamma_\nu \not{k}) &= p^\rho k^\sigma Tr(\gamma_\mu \gamma_\rho \gamma_\nu \gamma_\sigma) \\ &= 4p^\rho k^\sigma \left(g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho} - g_{\mu\nu} g_{\rho\sigma}\right) \ . \end{aligned}$$

[T8] $Tr\,(\gamma_\mu (1-\gamma_5) \not{p} \gamma_\nu (1-\gamma_5) \not{k}) = 2Tr\,(\gamma_\mu \not{p} \gamma_\nu \not{k}) + 8i\epsilon_{\mu\rho\nu\sigma} p^\rho k^\sigma$

- Proof:

$$\begin{aligned} \gamma_\mu (1-\gamma_5) \not{p} \gamma_\nu (1-\gamma_5) \not{k} &= \gamma_\mu \not{p} \gamma_\nu \not{k} - \gamma_\mu \gamma_5 \not{p} \gamma_\nu \not{k} - \gamma_\mu \not{p} \gamma_\nu \gamma_5 \not{k} + \gamma_\mu \gamma_5 \not{p} \gamma_\nu \gamma_5 \not{k} \\ &= \gamma_\mu \not{p} \gamma_\nu \not{k} + \gamma_\mu (\gamma_5)^2 \not{p} \gamma_\nu \not{k} - \gamma_\mu \gamma_5 \not{p} \gamma_\nu \not{k} - \gamma_\mu \gamma_5 \not{p} \gamma_\nu \not{k} \\ &= 2(\gamma_\mu \not{p} \gamma_\nu \not{k}) + 2(\gamma_5 \gamma_\mu \not{p} \gamma_\nu \not{k}) \ . \end{aligned}$$

Use [T6] to get the result.

[T9] $Tr\,(\gamma_\mu \not{p} \gamma_\nu \not{k})\, Tr\,(\gamma^\mu \not{k}' \gamma^\nu \not{p}') = 32\,[(pk')(kp') + (pp')(kk')]$

• Proof:

$$
\begin{aligned}
Tr\,(\gamma_\mu \not{p} \gamma_\nu \not{k})\, Tr\,(\gamma^\mu \not{k}' \gamma^\nu \not{p}') &= 16\,[p_\mu k_\nu + p_\nu k_\mu - g_{\mu\nu}(pk)]\,[k'^\mu p'^\nu + k'^\nu p'^\nu - g^{\mu\nu}(k'p')] \\
&= 16\,[(pk')(kp') + (pp')(kk') - (kp)(k'p') \\
&+ (pp')(kk') + (pk')(kp') - (pk)(p'k') \\
&- \ (k'p')(pk) - (k'p')(pk) + 4(pk)(k'p')] \ .
\end{aligned}
$$

[T10] $\quad Tr\,(\gamma_\mu \not{p} \gamma_\nu \gamma_5 \not{k})\, Tr\,(\gamma^\mu \not{k}' \gamma^\nu \gamma_5 \not{p}') = 32\,[(pk')(kp') - (pp')(kk')]$

• Proof: Using [T8] we can write

$$
\begin{aligned}
Tr\,(\gamma_\mu \not{p} \gamma_\nu \gamma_5 \not{k})\, Tr\,(\gamma^\mu \not{k}' \gamma^\nu \gamma_5 \not{p}') &= Tr\,(\gamma_5 \gamma_\mu \not{p} \gamma_\nu \not{k})\, Tr\,(\gamma_5 \gamma^\mu \not{k}' \gamma^\nu \not{p}') \\
&= p^\rho k^\sigma Tr\,(\gamma_5 \gamma_\mu \gamma_\rho \gamma_\nu \gamma_\sigma)\, k'_{\rho'} p'_{\sigma'} Tr\left(\gamma_5 \gamma^\mu \gamma^{\rho'} \gamma^\nu \gamma^{\sigma'}\right) \\
&= p^\rho k^\sigma k'_{\rho'} p'_{\sigma'} (-4i\epsilon_{\mu\rho\nu\sigma})(-4i\epsilon^{\mu\rho'\nu\sigma'})
\end{aligned}
$$

Use now the contraction property of the antisymmetric tensor $\epsilon_{\mu\rho\nu\sigma}$:

$$
\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\rho'\sigma'} = -2\left(g^\rho_{\rho'} g^\sigma_{\sigma'} - g^\rho_{\sigma'} g^\sigma_{\rho'}\right)
$$

to rewrite

$$
\begin{aligned}
& Tr\,(\gamma_\mu \not{p} \gamma_\nu \gamma_5 \not{k})\, Tr\,(\gamma^\mu \not{k}' \gamma^\nu \gamma_5 \not{p}') \\
&= -16 p^\rho k^\sigma k'_{\rho'} p'_{\sigma'} \epsilon_{\mu\rho\nu\sigma} \epsilon^{\mu\rho'\nu\sigma'} \\
&= -16 p^\rho k^\sigma k'_{\rho'} p'_{\sigma'} \left[-2\left(g^\rho_{\rho'} g^\sigma_{\sigma'} - g^\rho_{\sigma'} g^\sigma_{\rho'}\right)\right] \\
&= 32\,[(pk')(kp') - (pp')(kk')] \ .
\end{aligned}
$$

[T11] $\quad Tr\,[\gamma^\mu (1-\gamma_5) \not{p} \gamma^\nu (1-\gamma_5) \not{k}]\, Tr\,[\gamma_\mu (1-\gamma_5) \not{k}' \gamma_\nu (1-\gamma_5) \not{p}'] = 256 (pk')(kp')$

- Proof: Use [T8] to obtain

$$\begin{aligned}
& Tr\left[\gamma^\mu(1-\gamma_5)\not{p}\gamma^\nu(1-\gamma_5)\not{k}\right] Tr\left[\gamma_\mu(1-\gamma_5)\not{k}'\gamma_\nu(1-\gamma_5)\not{p}'\right] \\
&= \left[2Tr(\gamma^\mu\not{p}\gamma^\nu\not{k}) + 8i\epsilon^{\mu\rho\nu\sigma}p_\rho k_\sigma\right]\left[2Tr(\gamma_\mu\not{k}'\gamma_\nu\not{p}') + 8i\epsilon_{\mu\rho'\nu\sigma'}(k')^{\rho'}(p')^{\sigma'}\right] \\
&= 4Tr(\gamma^\mu\not{k}'\gamma^\nu\not{p}')Tr(\gamma_\mu\not{k}'\gamma_\nu\not{p}') - 64\epsilon^{\mu\rho\nu\sigma}\epsilon_{\mu\rho'\nu\sigma'}p_\rho k_\sigma(k')^{\rho'}(p')^{\sigma'} \\
&+ 16i\left[Tr(\gamma^\mu\not{p}\gamma^\nu\not{k})\epsilon_{\mu\rho'\nu\sigma'}(k')^{\rho'}(p')^{\sigma'} + Tr(\gamma_\mu\not{k}'\gamma_\nu\not{p}')\epsilon^{\mu\rho\nu\sigma}p_\rho k_\sigma\right] \ .
\end{aligned}$$

Using [T7], [T8] and the contraction property of $\epsilon_{\mu\nu\rho\sigma}$ already exploited to obtain [T10], the above expression can be rewritten:

$$\begin{aligned}
& 64\left[p_\mu k_\nu + p_\nu k_\mu - g_{\mu\nu}(pk)\right]\left[(k')^\mu(p')^\nu + (k')^\nu(p')^\mu - g^{\mu\nu}(k'p')\right] \\
&+ 64i\left[p^\mu k^\nu + p^\nu k^\mu - g^{\mu\nu}(pk)\right]\epsilon_{\mu\rho'\nu\sigma'}(k')^{\rho'}(p')^{\sigma'} \\
&+ \left[(k')_\mu(p')_\nu + (k')_\nu(p')_\mu - g_{\mu\nu}(k'p')\right]\epsilon^{\mu\rho\nu\sigma}p_\rho k_\sigma \\
&+ 128\left[(pk')(kp') - (pp')(kk')\right] \ .
\end{aligned}$$

The contributions of the second and third line vanish (they both involve contraction of a symmetric tensor with an antisymmetric tensor), while the sum of the first and fourth line yields

$$128\left[(pk')(kp') + (pp')(kk')\right] + 128\left[(pk')(kp') - (pp')(kk')\right] \ .$$