Notes on W decay

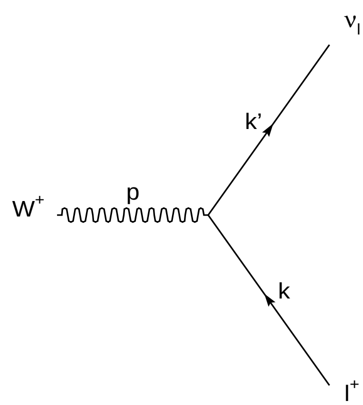
Revised: Roma, December 2001

1. Leptonic decay

We will consider leptonic decays of the W^+ boson, i.e. the processes

$$W^+ \to \ell^+ + \nu_\ell \ , \tag{1}$$

where ℓ^+ and ν_ℓ denote a positively charged lepton (e^+, μ^+ or τ^+) and the associated neutrino, respectively. The corresponding Feynman diagram is



with $p \equiv (\omega_p, \mathbf{p})$, $k \equiv (\omega_k, \mathbf{k})$ and $k' \equiv (\omega_{k'}, \mathbf{k}')$. The decay modes (1) provide $\sim 31.5\%$ of the observed W^+ width. The 2000 edition of the Particle Data Book reports the following branching ratios (recall: the branching ratio associated with the i-th decay mode is defined as $B(i) = \Gamma_i / \sum_i \Gamma_i$, Γ_i being the i-th mode decay rate):

$$B(W^+ \to e^+ + \nu_e) = 10.66 \pm 0.20\%$$

$$B(W^+ \to \mu^+ + \nu_\mu) = 10.49 \pm 0.29\%$$

$$B(W^+ \to \tau^+ + \nu_\tau) = 10.43 \pm 0.41\% \ .$$

Note that, within the experimental errors, the three leptonic decay channels have the same branching ratio.

The remaining $\sim 68.5\%$ of the decays ($68.5 \pm 0.6\%$ according to the particle data book) are of the type

$$W^+ \to \text{hadrons} \ .$$

Obviously, W^- decays are related to W^+ decays through charge conjugation.

As usual, we will start from the S-matrix element (V is the volume of the normalization box)

$$S_{if} = (2\pi)^4 \delta^{(4)}(p - k - k') \left(\frac{1}{2V\omega_p} \right)^{1/2} \left(\frac{m_\ell}{V\omega_k} \right)^{1/2} \left(\frac{m_{\nu_\ell}}{V\omega_{k'}} \right)^{1/2} \mathcal{M}_{if} \ . \tag{2}$$

The corresponding transition probability per unit time reads (T denotes the interaction time)

$$w_{if} = \frac{|S_{if}|^2}{T} = (2\pi)^4 \delta^{(4)}(p - k - k') \ \frac{1}{2\omega_p} \ \frac{m_\ell}{V\omega_k} \ \frac{m_{\nu_\ell}}{V\omega_{k'}} \ |\mathcal{M}_{if}|^2 \ , \tag{3}$$

and the differential decay rate can be written in terms of the above w_{if} in the form

$$\begin{aligned} d\Gamma &= w_{if} \ \frac{V}{(2\pi)^3} d^3k \frac{V}{(2\pi)^3} d^3k' \\ &= (2\pi)^4 \delta^{(4)}(p - k - k') \ \frac{2m_\ell 2m\nu_\ell}{2\omega_p} \ |\mathcal{M}_{if}|^2 \ \frac{d^3k}{(2\pi)^3 2\omega_k} \ \frac{d^3k'}{(2\pi)^3 2\omega_{k'}} \ . \end{aligned} \tag{4}$$

The explicit form of the invariant amplitude $\mathcal{M}_{if}$ will be specified at a later stage. First we want to rewrite $d\Gamma$ in a more compact form.

Using the momentum conserving δ-function to carry out the three-dimensional integration over the momentum of the produced neutrino ($\mathbf{k}'$) we find

$$\begin{aligned} &(2\pi)^4 \delta^{(4)}(p - k - k') \ \frac{d^3k}{(2\pi)^3 2\omega_k} \ \frac{d^3k'}{(2\pi)^3 2\omega_{k'}} \\ &= \frac{1}{(2\pi)^2} \delta(\omega_p - \omega_k - \omega_{k'}) \ \frac{1}{2\omega_k} \ \frac{1}{2\omega_{k'}} |\mathbf{k}|^2 d|\mathbf{k}| d\Omega_k \ . \end{aligned} \tag{5}$$

The right hand side of the above equation can be further simplified choosing to work in the rest frame of the W^+ boson, where

$$p \equiv (M_W, 0, 0, 0) \quad , \quad \mathbf{k} = -\mathbf{k}'$$

and

$$M_W = \omega_k + \omega_{k'} \ ,$$

implying

$$\frac{d|\mathbf{k}|}{dM_W} = \left(\frac{dM_W}{d|\mathbf{k}|}\right)^{-1} = \frac{1}{|\mathbf{k}|}\ \frac{\omega_k\omega_{k'}}{\omega_k + \omega_{k'}} = \frac{1}{|\mathbf{k}|}\ \frac{\omega_k\omega_{k'}}{M_W} \ .$$

Substitution into eq.(5) yields

$$\frac{1}{(2\pi)^2}\delta(M_W - \omega_k - \omega_{k'})\ \frac{1}{2\omega_k}\ \frac{1}{2\omega_{k'}}|\mathbf{k}|^2\ \frac{1}{|\mathbf{k}|}\ \frac{\omega_k\omega_{k'}}{M_W}\ dM_W \ ,$$

i.e. (use the δ-function to carry out the M_W integration)

$$\frac{1}{(2\pi)^2}\ \frac{1}{4}\ \frac{|\mathbf{k}|}{M_W}\ d\Omega_k \ .$$

Finally, the differential decay rate of eq.(4) can be rewritten using the above result as

$$d\Gamma = \frac{1}{32\pi^2}\ \frac{|\mathbf{k}|}{M_W^2}\ |\mathcal{M}_{if}|^2\ d\Omega_k \ . \tag{6}$$

Let us now go back to the invariant amplitude $\mathcal{M}_{if}$. Using Feynman rules appropriate to describe weak interactions we can write

$$\mathcal{M}_{if} = -ig_W\ \epsilon_r^\mu(\mathbf{p})\ \left[\bar{u}^{(s')}(\mathbf{k}')\gamma_\mu(1-\gamma_5)v^{(s)}(\mathbf{k})\right] \ . \tag{7}$$

In the W^+ rest frame, the leptons are produced with momenta of order $\sim M_W$ (recall: the W^+ mass is $M_W \sim 80$ GeV, to be compared to $m_e \sim .5{\times}10^{-3}$ GeV, $m_\mu \sim 105{\times}10^{-3}$ GeV, and $m_\tau \sim 1.8$ GeV). Hence, their rest masses can be safely neglected. We will use this approximation in the calculation of the squared invariant amplitude averaged over W polarization and summed over the spins of the produced particles:

$$\begin{aligned} |\overline{\mathcal{M}_{if}}|^2 = g_W^2\ &\left(\frac{1}{3}\sum_{r=1}^{3}\epsilon_r^\mu(\mathbf{p})\epsilon_r^\nu(\mathbf{p})\right) \\ \times &\left[\sum_{ss'}\bar{u}^{(s')}(\mathbf{k}')\gamma_\mu(1-\gamma_5)v^{(s)}(\mathbf{k})\bar{v}^{(s)}(\mathbf{k})\gamma_\nu(1-\gamma_5)u^{(s')}(\mathbf{k}')\right] \ . \end{aligned} \tag{8}$$

Consider first the polarizarion average, i.e. the quantity enclosed in round brackets. It can be readily evaluated writing the sum over r in the most general form compatible with the requirement of Lorentz invariance:

$$\sum_{r=1}^{3} \epsilon_r^\mu \epsilon_r^\nu = A g^{\mu\nu} + B p^\mu p^\nu \ .$$

From the above equation and the constraints ($r = 1, 2, 3$)

$$p_\mu \epsilon_r^\mu = 0 \quad , \ g_{\mu\nu} \epsilon_r^\mu \epsilon_r^\nu = \epsilon_r^\mu \epsilon_{r,\nu} = -1$$

it follows that

$$p_\mu p_\nu \sum_{r=1}^{3} \epsilon_r^\mu \epsilon_r^\nu = A p_\mu p_\nu g^{\mu\nu} + B p_\mu p_\nu p^\mu p^\nu = A M_W^2 + B M_W^4 = 0 \ ,$$

implying $B = -A/M_W^2$, and

$$g_{\mu\nu} \sum_{r=1}^{3} \epsilon_r^\mu \epsilon_r^\nu = A g_{\mu\nu} g^{\mu\nu} - \frac{A}{M_W^2} g_{\mu\nu} p^\mu p^\nu = 4A - \frac{A}{M_W^2} M_W^2 = -3 \ ,$$

implying $A = -1$. In conclusion

$$\frac{1}{3} \sum_{r=1}^{3} \epsilon_r^\mu(\mathbf{p}) \epsilon_r^\nu(\mathbf{p}) = \frac{1}{3} \left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{M_W^2} \right) \ . \tag{9}$$

Let us now turn to the calculation of the quantity enclosed in square brackets in eq.(8). As usual, the spin sum is performed using completeness of Dirac's spinors, with the result

$$\sum_{ss'} \bar{u}^{(s')}(\mathbf{k}') \gamma_\mu (1-\gamma_5) v^{(s)}(\mathbf{k}) \bar{v}^{(s)}(\mathbf{k}) \gamma_\nu (1-\gamma_5) u^{(s')}(\mathbf{k}') = \frac{Tr\left[\gamma_\mu (1-\gamma_5) \not{k} \gamma_\nu (1-\gamma_5) \not{k}'\right]}{2m_\ell 2m_{\nu_\ell}}, \tag{10}$$

and the trace appearing in the right hand side can be rewritten (see appendix of the notes on weak interactions)

$$2Tr\left(\gamma_\mu \not{k} \gamma_\nu \not{k}'\right) + 2Tr\left(\gamma_5 \gamma_\mu \not{k} \gamma_\nu \not{k}'\right) = 8\left[k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}(kk')\right] - 8i\epsilon_{\mu\rho\nu\sigma} k^\rho k'^\sigma \ . \tag{11}$$

Substitution of eqs.(9) and (11) into eq.(8) yields

$$|\overline{\mathcal{M}_{if}}|^2 = \frac{8}{3}\, g_W^2 \left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{M_W^2} \right) \left[k_\mu k'_\nu + k_\mu k'_\nu - g_{\mu\nu}(kk') - i\epsilon_{\mu\rho\nu\sigma} k^\rho k'^\sigma\right] \ . \tag{12}$$

Note that the quantity in round brakets is a symmetric tensor under the exchange $\mu \leftrightarrow \nu$. Hence, its contraction with the antisymmetric tensor $\epsilon_{\mu\rho\nu\sigma}$ vainshes and the above equation reduces to

$$|\overline{\mathcal{M}_{if}}|^2 = \frac{8}{3}\; g_W^2 \; \left[(kk') + 2\frac{(pk)(pk')}{M_W^2} \right] \; .$$

In the W rest frame, choosing the z axis along the direction of the emitted ℓ^+, we can write

$$k \equiv \left(\frac{M_W}{2}, 0, 0, \frac{M_W}{2} \right) \quad , \quad k' \equiv \left(\frac{M_W}{2}, 0, 0, -\frac{M_W}{2} \right) \; ,$$

implying

$$(kk') = (pk) = (pk') = \frac{M_W^2}{2} \; ,$$

and the squared invariant amplitude takes simple form

$$|\overline{\mathcal{M}_{if}}|^2 = \frac{8}{3}\; g_W^2 \; M_W^2 \; . \tag{13}$$

We are now in the position of carrying out the calculation of the differential decay rate, plugging the above result into eq.(6). The result is

$$d\Gamma = \frac{1}{24\pi^2}\; g_W^2 \; M_W \; d\Omega_k \; ,$$

leading to

$$\Gamma = \int d\Gamma = \frac{1}{6\pi}\; g_W^2 \; M_W \; .$$

Had we taken into account the nonvanishing ℓ^+ mass, m_ℓ, the result of our calculation would have been

$$\Gamma_y = \frac{1}{6\pi}\; g_W^2 \; M_W \; (1-y)^2 \left(1 + \frac{y}{2} \right)$$

with $y = (m_\ell/M_W)^2$. Even in the case of τ^+ production $y \sim 5 \times 10^{-4}$ and

$$\frac{\Gamma - \Gamma_y}{\Gamma} \sim 7.5 \times 10^{-4} \; ,$$

confirming that setting $m_\ell = 0$ is indeed a very accurate approximation, and the rate does not depend appreciably upon the specific decay channel. Note that this conclusion is consistent with the data, showing that the branching ratios of the three leptonic decay channels coincide within the experimental errors (see page 1).

The leptonic decay rate is given by the sum over the three possible channels, i.e.

$$\Gamma_\ell = 3\Gamma = \frac{1}{2\pi} g_W^2 \ M_W \ .$$

Denoting by Γ_{tot} the *total* decay rate, i.e. the rate including both leptonic and hadronic decays, we obviously have

$$\Gamma_{tot} > \Gamma_\ell \ ,$$

implying that the W lifetime $\tau = \Gamma_{tot}^{-1}$ satisfies the inequality

$$\tau < \frac{2\pi}{g_W^2 M_W} \ ,$$

whose right hand side can be estimated using the relation (see notes on weak interactions)

$$\frac{G}{\sqrt{2}} = \frac{g_W^2}{M_W^2}$$

and the value of the Fermi coupling constant obtained from muon decay

$$G \sim 1.02 \times 10^{-5} \frac{1}{m_p^2} \ ,$$

$m_p \sim .939$ GeV being the proton rest mass. The resulting upper limit to the W lifetime is

$$\tau < 1.7 \times 10^{-25} \mathrm{sec} \ .$$

The extremely small value of τ makes the direct detection of a free W boson impossible. Detection is feasible only through the analysis of the W decay products.