Notes on quantization of the electromagnetic field

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1. Summary of Maxwell's theory

The celebrated Maxwell's equations, summarizing the basic laws of electricity and magnetism, are a set of coupled, first-order partial differential equations linking the electric and magnetic fields, $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$, to the charge density and current $\rho(\mathbf{x},t)$ and $\mathbf{j}(\mathbf{x},t)$. Using a system of units in which $c=1$ they can be written in the form

$$\boldsymbol{\nabla}\cdot\mathbf{E}=\rho\ , \tag{1}$$

$$\boldsymbol{\nabla}\times\mathbf{B}=\mathbf{j}+\frac{\partial \boldsymbol{E}}{\partial t}\ , \tag{2}$$

$$\boldsymbol{\nabla}\cdot\mathbf{B}=0\ , \tag{3}$$

$$\boldsymbol{\nabla}\times\mathbf{E}=-\frac{\partial \boldsymbol{B}}{\partial t}\ . \tag{4}$$

From eqs.(1) and (2) it follows that charge density and current satisfy the *continuity equation*

$$\boldsymbol{\nabla}\cdot\mathbf{j}+\frac{\partial \rho}{\partial t}=0\ , \tag{5}$$

implying conservation of the electric charge

$$Q=\int d^3x\ \rho(\mathbf{x},t)\ . \tag{6}$$

To obtain eq.(6) we have used the identity stating that, given any vector $\mathbf{v}$, $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{v})\equiv 0$. From the same identity and from eq.(3) it follows that the magnetic field $\mathbf{B}$ can be written as the curl of a new vector field $\mathbf{A}(\mathbf{x},t)$ according to:

$$\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}. \tag{7}$$

The above equation defines the *vector potential* $\mathbf{A}$. Substitution of eq.(7) into eq.(4) yields

$$\boldsymbol{\nabla}\times\left(\mathbf{E}+\frac{\partial\mathbf{A}}{\partial t}\right)=0\ , \tag{8}$$

implying in turn that the quantity enclosed in round brackets can be written as the gradient of a scalar function: (minus) the *scalar potential* $\phi(\mathbf{x},t)$ (recall that, for any scalar function ψ, $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\psi)\equiv 0$). The electric field can then be rewritten

$$\mathbf{E}=-\boldsymbol{\nabla}\phi-\frac{\partial\mathbf{A}}{\partial t}\ . \tag{9}$$

Obviously, the electric and magnetic fields defined by eqs.(7) and (9) satisfy Maxwell's equations (3) and (4) by construction. Substitution of eqs.(7) and (9) into the remainig pair of equations, i.e. eqs.(1) and (2), yields

$$\boldsymbol{\nabla}\cdot\left(\boldsymbol{\nabla}\phi+\frac{\partial\mathbf{A}}{\partial t}\right)=\nabla^2\phi+\frac{\partial}{\partial t}\boldsymbol{\nabla}\cdot\mathbf{A}=-\rho \tag{10}$$

and

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A})+\frac{\partial}{\partial t}\left(\boldsymbol{\nabla}\phi+\frac{\partial\mathbf{A}}{\partial t}\right)=\mathbf{j}\ . \tag{11}$$

Finally, using the vector identity $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{v})=\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\cdot\mathbf{v})-\nabla^2\mathbf{v})$ the above equation can be rewritten

$$\nabla^2\mathbf{A}-\frac{\partial^2\mathbf{A}}{\partial t^2}-\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{A}-\frac{\partial\phi}{\partial t}\right)=\mathbf{j}\ . \tag{12}$$

Eqs.(7), (9), (10) and (12) provide an alternate formulation of Maxwell's theory, in which the physical fields $\mathbf{B}$ and $\mathbf{E}$ are replaced by the potentials ϕ and $\mathbf{A}$. Eqs.(7) and (9) give $\mathbf{B}$ and $\mathbf{E}$ in terms of ϕ and $\mathbf{A}$, while eqs.(10) and (12) are the equations of motion describing the evolution of the scalar and vector potentials. It has to be noted, however, that eqs.(10) and (12) are coupled second order partial differential equations, and no longer first order equations as eqs.(1) and (2). In the new formulation the state of the fields at time t is determined by the values of both $\mathbf{A}$ and $\partial \mathbf{A}/\partial t$ at the initial time t_0.

From eqs.(7) and (9) it follows that the physical fields *are not* uniquely specified. In fact, replacing

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} - \boldsymbol{\nabla}\Lambda \tag{13}$$

and

$$\phi \to \phi' = \phi + \frac{\partial \Lambda}{\partial t} \ , \tag{14}$$

Λ being any scalar function, obvioulsy leaves the magnetic field $\mathbf{B}$ unaffected, since $\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\Lambda) \equiv 0$. In addition, substitution of $\mathbf{A}'$ and ϕ' given by the above equations into the definition of the electric field, eq.(9), leads to

$$\begin{aligned} \mathbf{E} \to \mathbf{E}' &= -\boldsymbol{\nabla}\left(\phi + \frac{\partial \Lambda}{\partial t}\right) - \frac{\partial}{\partial t}\left(\mathbf{A} - \boldsymbol{\nabla}\Lambda\right) \\ &= -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t} = \mathbf{E} \ . \end{aligned}$$

The combination of transformations (13) and (14) goes under the name of *gauge transformation.*

The ambiguity in the definition of the fields implied by the invariance of Maxwell's theory under gauge transformations can be reduced choosing a specific *gauge*, i.e. requiring that the potentials satisfy a specific constraint, or *gauge condition.* The two most commonly used gauges are the *Lorentz gauge*, defined by the condition

$$\boldsymbol{\nabla}\mathbf{A} + \frac{\partial \phi}{\partial t} = 0 \ , \tag{15}$$

and the *Coulomb gauge*, defined by

$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0 \ . \tag{16}$$

Let us consider the *Lorentz gauge.* First, it has to noticed that eq.(15) still does not specify the vector and scalar potentials in a *unique* fashion. In fact, if $\mathbf{A}$ and ϕ satisfy eq.(15), so do $\mathbf{A}'$ and ϕ' given by

$$\mathbf{A}' = \mathbf{A} - \boldsymbol{\nabla}\Lambda' \tag{17}$$

and

$$\phi' = \phi + \frac{\partial \Lambda'}{\partial t} \ , \tag{18}$$

provided the scalar function Λ' is a solution of

$$\nabla^2 \Lambda' - \frac{\partial^2 \Lambda'}{\partial t^2} = 0 \ . \tag{19}$$

Use of the Lorentz gauge condition allows one to decouple eqs.(10) and (12), that can be rewritten in the more symmetric form

$$\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = \Box \phi = \rho \tag{20}$$

and

$$\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \Box \mathbf{A} = \mathbf{j} \ . \tag{21}$$

On the ohter hand, the equations obtained using the *Coulomb gauge* condition are still coupled. Eq.(10) reduces to

$$\nabla^2 \phi = -\rho \ , \tag{22}$$

i.e. to Poisson's equation for the scalar potential generated by the charge distribution ρ, while from eq.(12) one obtains

$$\Box \mathbf{A} = \mathbf{j} - \boldsymbol{\nabla} \frac{\partial \phi}{\partial t} \ . \tag{23}$$

2. Fourier decomposition of the classical radiation field

If there are no charges, i.e. for $\rho(\mathbf{x}, t) \equiv 0$, the unique solution of Poisson's equation (22),

$$\phi = \int d^3x \ \frac{\rho(\mathbf{x}, t)}{|\mathbf{x} - \mathbf{x}'|} \ , \tag{24}$$

is

$$\phi \equiv 0 \ . \tag{25}$$

Hence, in the Coulomb gauge, and in absence of charges and currents, the scalar potential vanishes and the vector potential satisfies the equations

$$\Box \mathbf{A} = 0 \tag{26}$$

$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0 \ . \tag{27}$$

Note that eq.(26) represents a set of three equations, satisfied by the components of $\mathbf{A}$, while eq.(27) is a relation *between* the components of the vector potential, implying that they are not linearly independent.

A solution of eq.(26) satisfying periodic boundary conditions in each of the three space dimensions of a cubic box of side L can be written

$$\mathbf{u_k}(\mathbf{x}, t) = \frac{1}{V} \ \boldsymbol{\epsilon}_\mathbf{k} \ \mathrm{e}^{i(\mathbf{k}\cdot\mathbf{x} - \omega_k t)} \ , \tag{28}$$

where $V = L^3$, $\omega_k = |\mathbf{k}|$ and the components of the wave vector $\mathbf{k}$ fulfill

$$k_x = n_x \ \frac{2\pi}{L} \quad , \quad k_y = n_y \ \frac{2\pi}{L} \quad , \quad k_z = n_z \ \frac{2\pi}{L} \ , \tag{29}$$

with $n_x, n_y, n_z = 0, \pm 1 \pm 2 \ldots$.

Requiring that $\mathbf{u_k}(\mathbf{x}, t)$ of eq.(28) also satisfy the Coulomb gauge condition leads to the *transversality* condition for the *polarization vector*

$$\mathbf{k} \cdot \boldsymbol{\epsilon}_\mathbf{k} = 0 \ , \tag{30}$$

implying that, for any given wave vector $\mathbf{k}$, $\boldsymbol{\epsilon}_\mathbf{k}$ lies in the plane perpendicular to $\mathbf{k}$. Hence, we can define two real unit vectors, $\boldsymbol{\epsilon}_{\mathbf{k}1}$ and $\boldsymbol{\epsilon}_{\mathbf{k}2}$, such that $\boldsymbol{\epsilon}_{\mathbf{k}1}$, $\boldsymbol{\epsilon}_{\mathbf{k}2}$ and $\mathbf{k}/|\mathbf{k}|$ form a set of mutually orthogonal unit vectors satisfying

$$\boldsymbol{\epsilon}_{\mathbf{k}r} \cdot \boldsymbol{\epsilon}_{\mathbf{k}r'} = \delta_{rr'} \tag{31}$$

$$\boldsymbol{\epsilon}_{\mathbf{k}r} \cdot \mathbf{k} = 0 \ . \tag{32}$$

In conclusion, the general solution of the equation of motion (26) satisfying the Coulomb gauge condition (27) can be written as a linear combination of the complete set of functions defined by eq.(28) according to (remember that the $\mathbf{A}$ is a real function)

$$\begin{aligned} \mathbf{A}(\mathbf{x}, t) &= \sum_\mathbf{k} \sum_{r=1}^{2} [c_{\mathbf{k}r} \mathbf{u}_{\mathbf{k}r}(\mathbf{x}, t) + c^*_{\mathbf{k}r} \mathbf{u}^*_{\mathbf{k}r}(\mathbf{x}, t)] \\ &= \sum_\mathbf{k} \sum_{r=1}^{2} [c_{\mathbf{k}r}(t) \mathbf{u}_{\mathbf{k}r}(\mathbf{x}) + c^*_{\mathbf{k}r}(t) \mathbf{u}^*_{\mathbf{k}r}(\mathbf{x})] \ , \end{aligned} \tag{33}$$

where

$$c_{\mathbf{k}r}(t) = c_{\mathbf{k}r} e^{-i\omega_k t} \tag{34}$$

and the functions

$$\mathbf{u}_{\mathbf{k}r}(\mathbf{x}) = \frac{1}{\sqrt{V}} \, \boldsymbol{\epsilon}_{\mathbf{k}r} \, e^{i\mathbf{k}\cdot\mathbf{x}} \; . \tag{35}$$

satisfy the orthogonality and normalization condition

$$\int d^3x \; \mathbf{u}^*_{\mathbf{k}r}(\mathbf{x}) \mathbf{u}_{\mathbf{k}'r'}(\mathbf{x}) = \frac{\boldsymbol{\epsilon}_{\mathbf{k}r} \cdot \boldsymbol{\epsilon}_{\mathbf{k}r'}}{V} \int d^3x \; e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} = \delta_{rr'} \delta_{\mathbf{k}\mathbf{k}'} \; . \tag{36}$$

3. Energy of the classical radiation field

The classical energy of the electromagnetic field, given by

$$H = \frac{1}{2} \int d^3x \; \left(|\mathbf{B}|^2 + |\mathbf{E}|^2 \right) = \frac{1}{2} \int d^3x \; \left(|\boldsymbol{\nabla} \times \mathbf{A}|^2 + \left| \frac{\partial \mathbf{A}}{\partial t} \right|^2 \right) \; , \tag{37}$$

can be rewritten using the Fourier decomposition of the vector potential $\mathbf{A}$, eq.(33), wih the result

$$\begin{aligned} H = \frac{1}{2} \sum_{\mathbf{k}r} \sum_{\mathbf{k}'r'} \int d^3x \; \Big\{ & [c_{\mathbf{k}r}(t) \; \boldsymbol{\nabla} \times \mathbf{u}_{\mathbf{k}r}(\mathbf{x}) + c.c.] \cdot [c^*_{\mathbf{k}'r'}(t) \; \boldsymbol{\nabla} \times \mathbf{u}^*_{\mathbf{k}'r'}(\mathbf{x}) + c.c.] \\ & + \left[\frac{\partial c_{\mathbf{k}r}}{\partial t} \; \mathbf{u}_{\mathbf{k}r}(\mathbf{x}) + c.c. \right] \cdot \left[\frac{\partial c^*_{\mathbf{k}'r'}}{\partial t} \; \mathbf{u}^*_{\mathbf{k}'r'}(\mathbf{x}) + c.c. \right] \Big\} \; . \end{aligned} \tag{38}$$

The calculation of the magnetic contribution to H involves integrations of the type

$$\begin{aligned} I_B &= \int d^3x \; (\boldsymbol{\nabla} \times \mathbf{u}_{\mathbf{k}r}(\mathbf{x})) \cdot (\boldsymbol{\nabla} \times \mathbf{u}^*_{\mathbf{k}'r'}(\mathbf{x})) \\ &= \int d^3x \; \boldsymbol{\nabla} \cdot [\mathbf{u}_{\mathbf{k}r}(\mathbf{x}) \times (\boldsymbol{\nabla} \times \mathbf{u}^*_{\mathbf{k}'r'}(\mathbf{x}))] + \int d^3x \; \mathbf{u}_{\mathbf{k}r}(\mathbf{x}) \cdot [\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{u}^*_{\mathbf{k}'r'}(\mathbf{x}))] \\ &= - \int d^3x \; \mathbf{u}_{\mathbf{k}r}(\mathbf{x}) \nabla^2 \mathbf{u}^*_{\mathbf{k}'r'}(\mathbf{x}) \\ &= \frac{|\mathbf{k}'|^2}{V} \; \boldsymbol{\epsilon}_r \cdot \boldsymbol{\epsilon}_{r'} \int d^3x \; e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} = \omega^2_{k'} \; \delta_{rr'} \delta_{\mathbf{k}\mathbf{k}'} \; . \end{aligned}$$

The above result can be easily obtained using the periodic boundary conditions and the identities $(\boldsymbol{\nabla} \times \mathbf{u}) \cdot (\boldsymbol{\nabla} \times \mathbf{v}) = \boldsymbol{\nabla} \cdot [\mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{v})] + \mathbf{u} \cdot [\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{v})]$ and $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{u}) = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \cdot \mathbf{u}) - \nabla^2 \mathbf{u}$.

The electric contribution involves integrals of the type

$$I_E = \int d^3x \; \frac{\partial c_{\mathbf{k}r}}{\partial t} \mathbf{u}_{\mathbf{k}r}(\mathbf{x}) \cdot \frac{\partial c^*_{\mathbf{k}'r'}}{\partial t} \mathbf{u}^*_{\mathbf{k}'r'}(\mathbf{x}) = \omega_k \omega_{k'} \; \delta_{rr'} \delta_{\mathbf{k}\mathbf{k}'} \; c_{\mathbf{k}r}(t) c^*_{\mathbf{k}'r'}(t) \; . \tag{39}$$

Collecting all terms together we finally find

$$H = \sum_{\mathbf{k}r} \omega^2_k \; [c_{\mathbf{k}r}(t) c^*_{\mathbf{k}r}(t) + c^*_{\mathbf{k}r}(t) c_{\mathbf{k}r}(t)] \; , \tag{40}$$

where the coefficients $c_{\mathbf{k}r}(t)$ satisfy the differential equation

$$\ddot{c}_{\mathbf{k}r}(t) \;\; = \;\; -\omega^2_k \; c_{\mathbf{k}r}(t) \; , \tag{41}$$

i.e. the equation of motion of a classical harmonic oscillator of angular frequency ω_k and unit mass. Note that, the coefficients $c_{\mathbf{k}r}(t)$ being complex numbers, $[c_{\mathbf{k}r}(t) c^*_{\mathbf{k}r}(t) + c^*_{\mathbf{k}r}(t) c_{\mathbf{k}r}(t)] = 2 c_{\mathbf{k}r}(t) c^*_{\mathbf{k}r}(t)$. The reason why H has been written as in eq.(40) will become apparent in the next section.

The fact that the energy of the radiation field can be cast in the form (40), with $c_{\mathbf{k}r}(t)$ satisfying eq.(41), suggests that H can be seen as the classical energy of a collection of independent harmonic oscillators of unit mass, oscillating in the directions specified by $\boldsymbol{\epsilon}_{\mathbf{k}r}$ with angular frequencies ω_k. Comparison between eq.(40) and the classical harmonic oscillator energy

$$H = \sum_{\mathbf{k}r} \frac{1}{2} \left(p_{\mathbf{k}r}^2 + \omega_k^2 x_{\mathbf{k}r}^2 \right) , \tag{42}$$

where $x_{\mathbf{k}r}$ and $p_{\mathbf{k}r}$ are classical canonical variables, shows that the quantities appearing in the left hand side of eqs.(40) and (42) can be identified provided $c_{\mathbf{k}r}$ and $c^*_{\mathbf{k}r}$ satisfy

$$c_{\mathbf{k}r} = \frac{1}{2\omega_k}(\omega_k x_{\mathbf{k}r} + i p_{\mathbf{k}r}) \quad , \quad c^*_{\mathbf{k}r} = \frac{1}{2\omega_k}(\omega_k x_{\mathbf{k}r} - i p_{\mathbf{k}r}) , \tag{43}$$

implying in turn the inverse relations

$$x_{\mathbf{k}r} = c_{\mathbf{k}r} + c^*_{\mathbf{k}r} \quad , \quad p_{\mathbf{k}r} = -i\omega_k (c_{\mathbf{k}r} - c^*_{\mathbf{k}r}) . \tag{44}$$

4. Quantization of the harmonic oscillator

Having pointed out the formal analogy between the energy of the classical radiation field and the energy of a system of classical harmonic oscillators, we now want to generalize to the quantum mechanical case.

Let us consider, for simplicity, a single harmonic oscillator of unit mass and angular frequency ω. Its quantum mechanical hamiltonian reads

$$h = \frac{1}{2}(p^2 + \omega^2 x^2) , \tag{45}$$

where x and p are now the position and momentum operators in Hilbert's space, satisfying the commutation rule (we use units such that $\hbar = 1$)

$$[x, p] = i \tag{46}$$

The hamiltonian of eq.(45) can be rewritten in terms of the operators a and $a^\dagger$, defined as

$$a = \frac{1}{\sqrt{2\omega}}(\omega x + ip) \quad , \quad a^\dagger = \frac{1}{\sqrt{2\omega}}(\omega x - ip) , \tag{47}$$

and satisfying the commutation rule that follows immediately from eq.(46)):

$$[a, a^\dagger] = 1 . \tag{48}$$

Using eqs.(45) and (48) we find

$$h = \frac{\omega}{2}(a^\dagger a + a a^\dagger) = \frac{\omega}{2}(2a^\dagger a + 1) = \omega \left(a^\dagger a + \frac{1}{2} \right) , \tag{49}$$

showing that the hamiltonian can be rewritten in terms of the hermitean operator $N = a^\dagger a$. The diagonal matrix elements of N satisfy the inequality

$$\langle \alpha | N | \alpha \rangle = \langle \alpha | a^\dagger a | \alpha \rangle = \sum_\beta \langle \alpha | a^\dagger | \beta \rangle \langle \beta | a | \alpha \rangle = \sum_\beta |\langle \beta | a | \alpha \rangle|^2 \geq 0 , \tag{50}$$

implying in turn that the lowest eigenvalue of N, n_0, is non negative. From the eigenvalue equation

$$N|n\rangle = a^\dagger a |n\rangle = n|n\rangle \tag{51}$$

and the commutation rule (48) it follows that

$$N a|n\rangle = (a^\dagger a) a|n\rangle = (a a^\dagger - 1) a|n\rangle = (n-1) a|n\rangle . \tag{52}$$

Hence, $a|n\rangle$ is an eigenstate of N belonging to the eigenvalue $(n-1)$. In the same fashion it is easy to show that

$$Na^\dagger|n\rangle = (n+1)a^\dagger|n\rangle \ , \tag{53}$$

i.e. that $a^\dagger|n\rangle$ is an eigenstate of N belonging to the eigenvalue $(n+1)$.

As n_0 is the minimum eigenvalue of N, the corresponding eigenstate must satisfy $a|n_0\rangle = 0$. As a consequence, the eigenvalue equation

$$N|n_0\rangle = a^\dagger a|n_0\rangle = n_0|n_0\rangle \tag{54}$$

necessarily implies $n_0 = 0$. Finally, it can be shown* that n can only be an integer.

The fact that the spectrum of N is composed of non-negative integers implies that the energy of the quantum mechanical harmonic oscillator is quantized. The eigenvalues of N give the number of oscillator quanta, while the normalized eigenstate belonging to any eigenvalue n can be obtained applying n times the operator $a^\dagger$ to the *vacuum state* $|0\rangle$. In fact, from

$$a|n\rangle = \sqrt{n}|n-1\rangle \qquad , \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \tag{55}$$

it follows that

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle \ . \tag{56}$$

In conclusion, the eigenvalue equation for the hamiltonian operator can be written

$$h|n\rangle = \omega\left(N + \frac{1}{2}\right)|n\rangle = \omega\left(n + \frac{1}{2}\right)|n\rangle \ . \tag{57}$$

In the Heisemberg picture, a generic operator O evolves in time according to

$$A(t) = \mathrm{e}^{iHt} \ O \ \mathrm{e}^{-iHt} \ , \tag{58}$$

H being the hamiltonian operator. The above definition leads immediately to Heisenberg's equation of motion

$$i\dot{O}(t) = [O(t), H] \ . \tag{59}$$

In the case of the harmonic oscillator, using the hamiltonian (49), we find that the time evolution of the operator $a(t)$ is dictated by the equation

$$i\dot{a}(t) = [a(t), h] = \omega[a(t), a^\dagger a] \ , \tag{60}$$

implying

$$\dot{a}(t) = -i\omega a(t) \ , \tag{61}$$

and

$$a(t) = a \ \mathrm{e}^{-i\omega t} \ . \tag{62}$$

5. Quantization of the electromagnetic field

Using the results of the previous section, we can write the hamiltonian of a collection of quantum mechanical harmonic oscillators as

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*the intersted reader can find a proof in: C. Cohen-Tannoudji, B. Diu and F. Laloë, *Quantum mechanics* (Wiley-Interscience, New York, 1977), p. 492.

$$H = \sum_{\mathbf{k}r} \frac{\omega_k}{2} \left(a^\dagger_{\mathbf{k}r} a_{\mathbf{k}r} + a_{\mathbf{k}r} a^\dagger_{\mathbf{k}r} \right) \ . \tag{63}$$

Comparison with eq.(40) immediately shows that the energy of the electromagnetic field can be identified with the above hamiltonian, provided the coefficients $c_{\mathbf{k}r}$ and $c^*_{\mathbf{k}r}$ appearing in the Fourier expansion of the vector potential $\mathbf{A}$ are identified with quantum mechanical operators, related to $a_{\mathbf{k}r}$ and $a^\dagger_{\mathbf{k}r}$ through

$$c_{\mathbf{k}r} = \frac{1}{\sqrt{2\omega_k}} \ a_{\mathbf{k}r} \quad , \quad c^*_{\mathbf{k}r} = \frac{1}{\sqrt{2\omega_k}} \ a^\dagger_{\mathbf{k}r} \ . \tag{64}$$

Within this picture, the vector potential has to be regarded as a quantum mechanical operator, defined in terms of the $a_{\mathbf{k}r}$ and $a^\dagger_{\mathbf{k}r}$ according to

$$\begin{aligned} \mathbf{A}(\mathbf{x},t) &= \sum_{\mathbf{k}r} \frac{1}{\sqrt{2V\omega_k}} \boldsymbol{\epsilon}_r \left[a_{\mathbf{k}r}(t) \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x})} + a^\dagger_{\mathbf{k}r}(t) \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x})} \right] \\ &= \sum_{\mathbf{k}r} \frac{1}{\sqrt{2V\omega_k}} \boldsymbol{\epsilon}_r \left[a_{\mathbf{k}r} \mathrm{e}^{-i(\omega_k t - \mathbf{k}\cdot\mathbf{x})} + a^\dagger_{\mathbf{k}r} \mathrm{e}^{i(\omega_k t - \mathbf{k}\cdot\mathbf{x})} \right] \qquad (65) \\ &= \mathbf{A}^+(\mathbf{x},t) + \mathbf{A}^-(\mathbf{x},t) \ . \qquad (66) \end{aligned}$$

The coefficients in the above expansion (and therefore the field $\mathbf{A}(\mathbf{x},t)$) are quantum mechanical operators acting in the Hilbert space whose state vectors are given by

$$|n_{\mathbf{k}_1 r_1}, n_{\mathbf{k}_2 r_2}, \dots n_{\mathbf{k}_n r_n} \dots \rangle \ , \tag{67}$$

where the integer $n_{\mathbf{k}r}$ denotes the number of quanta, called photons, oscillating in the mode specified by the wave vector $\mathbf{k}$ and the polarization vector $\boldsymbol{\epsilon}_{\mathbf{k}r}$. From the results of the previous section it follows that the generic state (67) can be obtained from the vacuum state, in which no pothons are present, through

$$|n_{\mathbf{k}_1 r_1}, n_{\mathbf{k}_2 r_2}, \dots \rangle = \prod_{\mathbf{k}_i r_i} \frac{(a^\dagger_{\mathbf{k}_i r_i})^{n_{\mathbf{k}_i r_i}}}{\sqrt{n_{\mathbf{k}_i r_i}!}} |0\rangle \ . \tag{68}$$

For example, the two photon state reads

$$|n_{\mathbf{k}_1 r_1}, n_{\mathbf{k}_2 r_2} \rangle = a^\dagger_{\mathbf{k}_1 r_1} a^\dagger_{\mathbf{k}_2 r_2} |0\rangle \ . \tag{69}$$

Note that we have split the series of eq.(66) in such a way that $\mathbf{A}^+(\mathbf{x},t)$ and $\mathbf{A}^-(\mathbf{x},t)$ contain only photon annihilation ($a_{\mathbf{k}r}$) or creation ($a^\dagger_{\mathbf{k}r}$) operators, respectivey, implying $\mathbf{A}^+(\mathbf{x},t)|0\rangle \ = \ 0$.

The electromagnetic hamiltonian

$$H = \sum_{\mathbf{k}r} \omega_k \left(N_{\mathbf{k}r} + \frac{1}{2} \right) \ , \tag{70}$$

with $N_{\mathbf{k}r} = a^\dagger_{\mathbf{k}r} a_{\mathbf{k}r}$, can be rewritten in a slightly different form exploiting the arbitrariness inherent in the choice of the energy scale. Choosing a scale in which the vacuum state has zero energy allows one to replace eq.(70) with

$$H = \sum_{\mathbf{k}r} = \omega_k \ N_{\mathbf{k}r} \ , \tag{71}$$

leading to the eigenvalue equation

$$H|n_{\mathbf{k}_1 r_1}, n_{\mathbf{k}_2 r_2}, \dots \rangle = \sum_{\mathbf{k}_i r_i} n_{\mathbf{k}_i r_i} \omega_{k_i} \ |n_{\mathbf{k}_1 r_1}, n_{\mathbf{k}_2 r_2}, \dots \rangle \ , \tag{72}$$

and

$$H|0\rangle = \sum_{\mathbf{k}_i r_i} \omega_{k_i} N_{\mathbf{k}_i r_i} \ |0\rangle = 0 \ . \tag{73}$$

As a final remark, note that, as $\mathbf{A}(\mathbf{x},t)$ is linear in the $a_{\mathbf{k}r}$ and $a^\dagger_{\mathbf{k}r}$, from

$$[a_{\mathbf{k}r}, N_{\mathbf{k}'r'}] = \left[a_{\mathbf{k}r}, a^\dagger_{\mathbf{k}'r'} a_{\mathbf{k}r}\right] = \delta_{rr'}\delta_{\mathbf{k}\mathbf{k}'} a_{\mathbf{k}r} \tag{74}$$

and

$$\left[a^\dagger_{\mathbf{k}r}, N_{\mathbf{k}'r'}\right] = \left[a^\dagger_{\mathbf{k}r}, a^\dagger_{\mathbf{k}'r'} a_{\mathbf{k}r}\right] = \delta_{rr'}\delta_{\mathbf{k}\mathbf{k}'} a^\dagger_{\mathbf{k}r} \ , \tag{75}$$

it follows that the operator $N_{\mathbf{k}r}$, whose eigenvalue yields the number of photons in the mode $\mathbf{k}r$, does not commute with the vector potential $\mathbf{A}(\mathbf{x},t)$. As a consequence, it does not commute with either the electric field $\mathbf{E}$ or the magnetic field $\mathbf{B}$. This result implies that the number of photons and the strengths of the physical fields cannot be simultaneously determined to arbitrary accuracy. Moreover, since the potential is linear in the photon creation and annihilation operators, the expectation values $\langle \mathbf{E} \rangle$ and $\langle \mathbf{B} \rangle$ in the state defined by eq.(67), containing a definite number of photons, vanish.

6. Spin of the photon

We have seen that the quanta of the electromagnetic field carry momentum $\mathbf{k}$ and energy $\omega_k = |\mathbf{k}|$, implying that their rest mass $k^2 = \omega_k^2 - \mathbf{k}^2$ vanishes. Let us now briefly discuss the photon spin $\mathbf{J}$.

The projection of the photon intrinsic angular momentum along the quantization axis is related to photon's polarization. It can be shown[†] that the operator associated with the projection of $\mathbf{J}$ along the direction of propagation satisfies the commutation rule

$$\left[J_3, a^\dagger_{\mathbf{k}r}\right] = i\left(\epsilon^1_{\mathbf{k}r} a^\dagger_{\mathbf{k}2} - \epsilon^2_{\mathbf{k}r} a^\dagger_{\mathbf{k}1}\right) \ , \tag{76}$$

where we have taken the z-axis in the direction of $\mathbf{k}$. Let us now define the new operators

$$a^\dagger_{\mathbf{k}R} = \frac{1}{\sqrt{2}}\left(a^\dagger_{\mathbf{k}1} + i a^\dagger_{\mathbf{k}2}\right) \quad , \quad a^\dagger_{\mathbf{k}L} = \frac{1}{\sqrt{2}}\left(a^\dagger_{\mathbf{k}1} - i a^\dagger_{\mathbf{k}2}\right) \ , \tag{77}$$

which create *circularly polarized* photons, i.e. photons whose polarization is described by the vectors

$$\epsilon_{\mathbf{k}R} = \frac{1}{\sqrt{2}}\left(\epsilon_{\mathbf{k}1} + i\epsilon_{\mathbf{k}2}\right) \quad , \quad \epsilon_{\mathbf{k}L} = \frac{1}{\sqrt{2}}\left(\epsilon_{\mathbf{k}1} - i\epsilon_{\mathbf{k}2}\right) \ . \tag{78}$$

Choosing $\epsilon_{\mathbf{k}1} \equiv (1,0,0)$ and $\epsilon_{\mathbf{k}2} \equiv (0,1,0)$ and rewriting eq.(76) in terms of the new operators we obtain

$$\left[J_3, a^\dagger_{\mathbf{k}R}\right] = a^\dagger_{\mathbf{k}R} \quad , \quad \left[J_3, a^\dagger_{\mathbf{k}L}\right] = -a^\dagger_{\mathbf{k}L} \ , \tag{79}$$

implying in turn

$$J_3 a^\dagger_{\mathbf{k}R}|0\rangle = \left[J_3, a^\dagger_{\mathbf{k}R}\right]|0\rangle = a^\dagger_{\mathbf{k}R}|0\rangle \ , \tag{80}$$

$$J_3 a^\dagger_{\mathbf{k}L}|0\rangle = \left[J_3, a^\dagger_{\mathbf{k}L}\right]|0\rangle = -a^\dagger_{\mathbf{k}L}|0\rangle \ . \tag{81}$$

The above equations show that the photon has spin $|\mathbf{J}| = 1$ and the two spin projections $J_3 = \pm 1$ correspond to circularly polarized states. While the value of $|\mathbf{J}|$ is dictated by the vector nature of the electromagnetic field, the absence of the $J_3 = 0$ state is a consequence of its property of being transverse, which is in turn a consequence of the fact that photons are massless.

[†] a detailed derivation can be found in: J.D. Bjorken and S.D. Drell, *Relativistic quantum fields* (McGraw-Hill, New York, 1965), chapt. 14.