Growth of non-infinitesimal perturbations in turbulence

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We discuss the effects of finite perturbations in fully developed turbulence by introducing a measure of the chaoticity degree associated to a given scale of the velocity field. This allows one to determine the predictability time for non-infinitesimal perturbations, generalizing the usual concept of maximum Lyapunov exponent. We also determine the scaling law for our indicator in the framework of the multifractal approach. We find that the scaling exponent is not sensitive to intermittency corrections, but is an invariant of the multifractal models. A numerical test of the results is performed in the shell model for the turbulent energy cascade.

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The standard characterization of the chaotic behavior of a dynamical system is given by the maximum Lyapunov exponent $\lambda_{\max}$, which measures the typical exponential rate of growth of an infinitesimal disturbance [1]. It is thus expected that the predictability time is proportional to $\lambda_{\max}^{-1}$. The underlying point is that the growth of a perturbation is well described by the linear equations for the tangent vector even if this cannot be literally true for non-infinitesimal perturbations. There exist indeed many situations where the Lyapunov analysis has no relevance for the predictability problem and it is necessary to introduce indicators which are able to capture the essential features of a chaotic system. For instance, when two or more characteristic time scales are present a direct identification of the Lyapunov and predictability times leads to paradoxes as recently pointed out in Ref. [2].

In this letter, we introduce a measure of the chaoticity degree related to the average doubling time that extends the concept of Lyapunov exponent in the case of non-infinitesimal perturbations. Our indicator is a scale-dependent Lyapunov exponent which becomes particularly useful when there exists a hierarchy of characteristic times such as the eddy turn-over times in three dimensional fully developed turbulence [3]. Our work may also be viewed as an extension of the work of Lorenz, Leith, Kraichnan on predictability, motivated by atmospheric forecast [4,5] – see also [6] – with proper account for the multifractal character of fully developed three-dimensional turbulence.

In turbulent flows it is natural to argue that the maximum Lyapunov exponent is roughly proportional the turnover time τ of eddies of the size of the Kolmogorov length η (the viscous cut-off) that is the shortest characteristic time [7]. Denoting by V, L and $\tau_o = V/L$ the typical velocity, size and time of the energy containing eddies, respectively, the turnover time of an eddy of size ℓ is, by dimensional analysis, $\tau(\ell) \sim \tau_o(\ell/L)^{1-h}$, where h is the scaling exponent of the velocity difference in the eddy

$$v_\ell \equiv |\boldsymbol{v}(\boldsymbol{x}') - \boldsymbol{v}(\boldsymbol{x})| \sim V(\ell/L)^h, \qquad \ell = |\boldsymbol{x}' - \boldsymbol{x}|. \quad (1)$$

The viscous cut-off vanishes as a power of the Reynolds number Re, i.e. $\eta \sim L\mathrm{Re}^{-1/(1+h)}$. These relations imply that the maximum Lyapunov exponent should scale as

$$\lambda_{\max} \sim 1 \,/\, \tau(\eta) \sim \tau_o^{-1}\mathrm{Re}^{\alpha} \quad \text{with } \alpha = \frac{1-h}{1+h}. \quad (2)$$

In the Kolmogorov K41 theory [8], $h = 1/3$ for all space points $\boldsymbol{x}$ so that $\alpha = 1/2$, as first pointed out by Ruelle [7]. However, the intermittency of energy dissipation leads to the existence of a spectrum of possible scaling exponents h affecting the value of α. In the multifractal approach [9], the probability that the velocity difference on scale ℓ scales as $v_\ell \sim V(\ell/L)^h$ is assumed to be $P_\ell(h) \sim (\ell/L)^{3-D(h)}$. This *ansatz* can be tested by measuring the scaling of the structure functions

$$\langle v_\ell^p \rangle \sim V^p (\ell/L)^{\zeta_p}. \tag{3}$$

In the K41 theory [8,10] $\zeta_p = p/3$ while in the multifractal scenario [9] ζ_p is a non-linear function of p given by the Legendre transform of the function $D(h)$, $\zeta_p = \min_h [\, h\,p - D(h) + 3\,]$. Moreover, as a consequence of multifractality there is a spectrum of viscous cut-offs [11], since each h selects a different damping scale $\eta(h) \sim L Re^{-1/(1+h)}$, and hence a spectrum of turnover times $\tau(\,\eta(h)\,)$. To find the Lyapunov exponent, we have to integrate over the h-distribution [12]

$$\begin{aligned}\lambda_{\max} &\sim \int \tau^{-1}(\,\eta(h)\,)\, P_\eta(h)\, dh \\ &\sim \tau_o^{-1} \int (\eta/L)^{h-D(h)+2}\, dh \sim \tau_o^{-1} \mathrm{Re}^{\alpha}.\end{aligned} \tag{4}$$

In the limit $\mathrm{Re} \to \infty$, the integral can be estimated by the saddle point and gives $\alpha = \max_h \, [D(h) - 2 - h]/(1+h)$. By using the function $D(h)$ obtained with the random beta model fit [9,13] one has $\alpha = 0.459...$

In the predictability problem, we are interested in defining the growth of an error on the velocity field. As usual we consider the Euclidean norm in a box of volume $\mathcal{V}$

$$\delta v(t) = \left(\mathcal{V}^{-1} \int d^3x\, |\boldsymbol{v}'(\boldsymbol{x},t) - \boldsymbol{v}(\boldsymbol{x},t)|^2 \right)^{1/2}. \tag{5}$$

to introduce the notion of distance between two realizations of the velocity field, $\boldsymbol{v}$ and $\boldsymbol{v}'$.

Then, the predictability time T_{p} is the time necessary for an initial error $\delta v(0) \equiv \delta_0$ to become larger than a given but arbitrary threshold value Δ:

$$T_{\mathrm{p}} = \max\,[t \mid \delta v(t') \le \Delta \text{ for } t' < t]. \tag{6}$$

In a first approximation, neglecting the non-linear terms of the evolution equation for the error growth and assuming that both δ_0 and Δ are infinitesimal, one obtains

$$T_{\mathrm{p}} \sim \lambda_{\max}^{-1} \ln(\Delta/\delta_0) \approx \lambda_{\max}^{-1}. \tag{7}$$

In turbulence, such a relation would imply that $T_P \sim \tau_0 \mathrm{Re}^{-\alpha}$. This is contradictory with the quite intuitive expectation that the predictability time should be roughly proportional to the turn-over time of the energy containing eddies on the large scales, and so practically independent of the Reynolds number [4,5].

The paradox stems from assuming the validity of the Lyapunov analysis for perturbations δv that are much larger than the typical velocity difference $v_\eta \sim \eta/\tau(\eta)$ on the dissipative length scale η. In this case, the error growth is non-exponential as it can be understood by simple heuristic arguments [4,5]. The problem can be faced by generalizing the concept of maximum Lyapunov exponent to the case of non-infinitesimal perturbations. The generalization is particularly useful in systems with many characteristic time-scales.

For this purpose, it is convenient to consider the time $T_r(\delta v)$ necessary for a perturbation to grow from δv to $r\,\delta v$, for a generic $r > 1$. For $r = 2$ this is the doubling time of a perturbation. After performing an average over different realizations of the flow or, equivalently, a time average along a trajectory $\boldsymbol{v}(t)$ in the phase space, we introduce the scale-dependent Lyapunov exponent

$$\lambda(\delta v) = \left\langle \frac{1}{T_r(\delta v)} \right\rangle \ln r. \tag{8}$$

Such a definition is consistent with the request of recovering the maximum Lyapunov exponent in the limit of infinitesimal error, since

$$\lim_{\delta v \to 0} \lambda(\delta v) = \lambda_{\max}. \tag{9}$$

It is easy to estimate the scaling of $\lambda(\delta v)$ when the perturbation is in the inertial range $v_\eta \le \delta v \le V$. In this case, following the phenomenological ideas of Lorenz, the doubling time of an error of magnitude δv can be identified with the turn-over time $\tau(\ell)$ of an eddy with typical velocity difference $v_\ell \sim \delta v$. Since $\tau(\ell) \sim \tau_o(\ell/L)^{h-1} \sim \tau_o(v_\ell/V)^{1-1/h}$, one has

$$\lambda(\delta v) \sim \tau_o^{-1}(\delta v/V)^{-\beta}, \qquad \beta = 1/h - 1. \tag{10}$$

Neglecting intermittency, i.e. using the Kolmogorov value $h = 1/3$, gives $\beta = 2$. In the dissipative range $\delta v < v_\eta$, the error can be considered infinitesimal, implying $\lambda(\delta v) = \lambda_{\max}$.

The intermittency of energy dissipation reflects the dynamical intermittency of the chaoticity degree, so that our arguments based on dimensional analysis cannot be fully correct. In the framework of the multifractal approach, our indicator scales as

$$\lambda(\delta v) \sim \tau_o^{-1} \int dh\, (\delta v/V)^{[3-D(h)]/h}\, (\delta v/V)^{1-1/h} \tag{11}$$

where we have used arguments similar to those leading to (4) and the scaling factor $\ell \sim L(\delta v/V)^{1/h}$. From the inequality $D(h) \le 3h + 2$, which is the analogous for turbulence of the standard inequality $f(\alpha) \le \alpha$ in multifractals, we have

$$\frac{2 + h - D(h)}{h} \ge -2 \quad \text{for all } h. \tag{12}$$

Equality holds for $h = h_3$, the exponent that realizes the minimum in the Legendre transform for the exponent of the third-order structure function $\zeta_3 = \min_h [3h + 3 - D(h)] = 1$. Therefore a saddle point estimation of (11) gives

$$-\beta = \min_h \left[\frac{2 + h - D(h)}{h} \right] = -2. \tag{13}$$

An important consequence of multifractality follows from the existence of a spectrum of dissipative cut-offs $\eta(h)$ which reduces the effective inertial range where the scaling of $\lambda(\delta v)$ holds [11,14]. To be more specific, the multifractal approach leads to $\eta(h) \sim L\mathrm{Re}^{-1/(1+h)}$, and the integral (11) has to be performed for $\widetilde{h}(\delta v) \leq h \leq h_{\max}$, where $\widetilde{h}(\delta v)$ is given by

$$\delta v \sim V\mathrm{Re}^{-\tilde{h}(\delta v)/(1+\tilde{h}(\delta v))}. \tag{14}$$

As a consequence, the scaling $\lambda(\delta v) \sim \tau_o^{-1}(\delta v/V)^{-\beta}$ holds only for $\delta v > V\mathrm{Re}^{-h_3/(1+h_3)}$, i.e., the inertial range is reduced by intermittency. In the range $V\mathrm{Re}^{-1/4} < \delta v < V\mathrm{Re}^{-h_3/(1+h_3)}$ we expect a non-trivial shape of $\lambda(\delta v)$ depending on $D(h)$ [14].

In order to test our results we have numerically studied the GOY shell model [15,16] for the energy cascade in fully developed turbulence. This model mimics the Navier-Stokes dynamics in the Fourier space. It is obtained by dividing the Fourier space into shells of wavenumbers $k_n < |\boldsymbol{k}| < k_{n+1}$. A complex scalar u_n is associated with the n^{th} shell individuated by $k_n = k_0\, 2^n$. It represents the velocity difference over a length scale $\ell \sim k_n^{-1}$. Only the interactions of a shell with its nearest and next-nearest neighbors are taken into account. The GOY model is described by the set of N ordinary differential equations:

$$\begin{aligned}\frac{d}{dt} u_n = ik_n \ \big(& a_n u_{n+1}^* u_{n+2}^* + b_n u_{n-1}^* u_{n+1}^* \\ & + c_n u_{n-2}^* u_{n-1}^* \big) - \nu k_n^2 u_n + f\delta_{n,4}.\end{aligned} \tag{15}$$

The coefficients are $a_n = 1$, $b_n = -1/4$, $c_n = -1/8$ and $b_1 = b_N = c_1 = c_2 = a_{N-1} = a_N = 0$ on the boundary shells. That model exhibits non-linear exponents ζ_p for the structure functions [16], close to what found in experimental data [13]. We have determined the scale-dependent Lyapunov exponent by a numerical integration using the following procedure. First, we generate two sets of initial conditions $\{u_n\}$ and $\{u_n'\}$ which are close in the Euclidean distance. In practice, u_n' differs from u_n by a small fraction of $\langle |u_n|^2 \rangle^{1/2}$. We have also checked that the results do not change if we consider two fields that are different only on small scales, i.e. if $|u_n - u_n'|$ takes a small non-zero value only on the last shells. Then, we follow the evolution of $\{u_n(t)\}$ and $\{u_n'(t)\}$ until the Euclidean distance $\delta u(t) = (\sum_n |u_n - u_n'|^2)^{1/2}$, has reached a threshold δ_o small compared to the velocity on the dissipative scale, $v_\eta \sim V\mathrm{Re}^{-1/4}$. Further, we consider a sequence of thresholds $\delta_j = r^j \delta_o$ $(j = 1, 2, \ldots)$, and measure the time $T_r(\delta_j)$ needed to increase the distance δu from δ_j to δ_{j+1}. The procedure is repeated for different realizations of the trajectory $u_n(t)$, and average quantities are computed.

Figure 1 shows the scaling of $\langle 1/T_r(\delta v) \rangle$ (with $r = \sqrt{2}$) as a function of δv in the GOY model. For comparison we also plot the eddy turn-over times

$$\tau_n^{-1} = k_n \langle |u_n|^2 \rangle^{1/2}. \tag{16}$$

One sees that there is a large range of small scales where $\lambda(\delta u) = \lambda_{\max}$ while $\tau_n \sim \tau_o(u_n/V)^{-\beta}$. That is a consequence of the reduction of the inertial range for the scale dependent Lyapunov exponent. Note that in the GOY model $\lambda_{\max} \approx 10^{-2}\, \tau(\eta)^{-1}$, although the dependence of the two quantities on Re is the same.

To take into account the multifractal corrections we rescale the data at different $\mathrm{Re} = \nu^{-1}$ using the multiscaling procedure [14]. The results are shown in Fig. 2, where $\ln\langle 1/T_r(\delta u) \rangle / \ln(\mathrm{Re}/R_o)$ is reported as a function of $\ln(\delta u/V_o)/\ln(\mathrm{Re}/R_o)$, R_o and V_o being two fitting parameters. The data collapse is quite good. We stress that the scaling law $\lambda(\delta u) \sim \tau_o^{-1}(\delta u/V)^{-2}$ holds only in a small inertial range. In the intermediate dissipative range the behavior is non-trivial and depends on the shape of $D(h)$ and on finite Re corrections [14].

We conclude by noting that our scale-dependent Lyapunov exponent $\lambda(\delta v)$ has some similarity with the concept of the ϵ-entropy recently discussed by Gaspard and Wang [17–19] for the treatment of experimental data. However, since the maximum Lyapunov exponent is more easily computable than the Kolmogorov-Sinai entropy, we expect that also $\lambda(\delta v)$ is much more accessible quantity than the ϵ-entropy. Moreover, when one knows the evolution law and has not to analyze experimental data, there are no particular limitations for estimating $\lambda(\delta v)$, such as the number of degrees of freedom involved

In conclusion, when the perturbations are non-infinitesimal it is necessary to extend the definition of Lyapunov exponent to make it physically consistent. The generalization proposed in this letter is particular useful when many characteristic time scales are present. Our result allows one to get a quantitative control of the growth of perturbations which are non-infinitesimal, looking at the average of the inverse doubling time. By this definition one has the two advantages of maintaining the link with the forecast limitation of a system and of recovering the maximum Lyapunov exponent in the limit of infinitesimal perturbations. The scale-dependent Lyapunov exponent thus is an important tool of investigation of highly dimensional dynamical systems and, far from being limited to the predictability problem of turbulent flows in geophysics [4,5,20], it can assume a great relevance in the characterization of very different chaotic phenomena.

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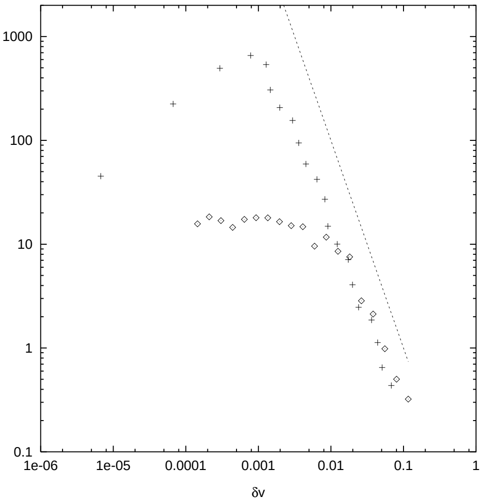

FIG. 1. $\langle 1/T_r(\delta u)\rangle$ (diamond) as a function of δv for the GOY model with $N = 27$, $k_0 = 0.05$, $f = (1+i) \times 0.005$ and $\nu = 10^{-9}$. The crosses are the inverse of the eddy turn-over times $\tau^{-1}(\delta v) = k_n \langle |u_n|^2\rangle^{1/2}$ versus $\delta v = \langle |u_n|^2\rangle^{1/2}$. The straight line has slope -2.

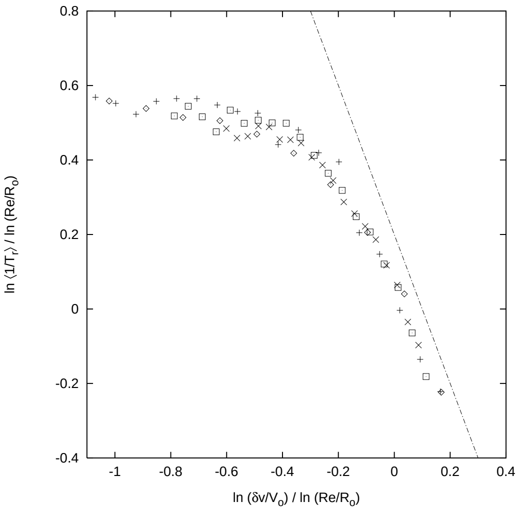

FIG. 2. Multiscaling data collapse (see text). The results are obtained in the GOY model for $k_0 = 0.05$, $f = (1+i) \times 0.005$ and: (diamond) $N = 24$ and $\nu = 10^{-8}$; (plus) $N = 27$ and $\nu = 10^{-9}$; (square) $N = 32$ and $\nu = 10^{-10}$; (cross) $N = 35$ and $\nu = 10^{-11}$. The straight line has slope -2. The fitting parameters are $R_o = 6 \times 10^6$, $V_o = 5 \times 10^{-2}$, and $\mathrm{Re} = \nu^{-1}$.