On the out of equilibrium order parameter in long-range spin-glasses.

A. Baldassarri, L. F. Cugliandolo *, J. Kurchan †, G. Parisi

Dipartimento di Fisica, Università di Roma I, La Sapienza, Roma, Italy,

INFN - Sezione di Roma I

Abstract

We show that the dynamical order parameters can be reexpressed in terms of the distribution of the staggered auto-correlation and response functions. We calculate these distributions for the out of equilibrium dynamics of the Sherrington-Kirpatrick model at long times. The results suggest that the landscape this model visits at different long times in an out of equilibrium relaxation process is, in a sense, self-similar. Furthermore, there is a similarity between the landscape seen out of equilibrium at long times and the equilibrium landscape.

The calculation is greatly simplified by making use of the superspace notation in the dynamical approach. This notation also highlights the rather mysterious formal connection between the dynamical and replica approaches.

We also perform numerical simulations which show good agreement with the analytical results for the out of equilibrium dynamics.

---

*Address after October 1st: Service de Physique de l'Etat Condensé, CEA, Saclay, France.

†Address after October 1st: LPTHE, Ecole Normale Supérieure, Paris, France.

I. INTRODUCTION

The partition function of mean-field spin-glasses is dominated by many states. The geometrical organization of these states, their relative weights in the Gibbs-Boltzmann measure, and the distribution of their mutual distances have been known for some time [1,2]. Of particular importance is the functional order parameter $P(q)$ giving the probability distribution of states with mutual overlap q.

The Gibbs-Boltzmann measure can be studied analytically using a dynamical approach [3]. For instance, the Langevin Dynamics

$$\Gamma_0^{-1}\,\partial_t \sigma_i(t) = -\beta \frac{\delta H}{\delta \sigma_i(t)} + \xi_i(t) \tag{I.1}$$

(Γ_0 determines the time scale and $\xi_i(t)$ is a Gaussian white noise with zero mean and variance 2), with the following order of large times and large N limits

$$\lim_{N\to\infty} \; \lim_{t\to\infty} \,, \tag{I.2}$$

guarantees ergodicity and leads the system to equilibrium. The equilibrium thermodynamical values of any operator O are then obtained as averages over the noise $\langle O \rangle_{eq} = \lim_{N\to\infty} \lim_{t\to\infty} \langle O(t) \rangle$.

A different situation, closer to the experimental settings, is to consider the relaxation of an infinite system at long but finite times. The time is measured from the initial time - the quenching time in experiments - which we take as zero. The order of limits is then

$$\lim_{t\to\infty} \; \lim_{N\to\infty} \;. \tag{I.3}$$

An analytical solution for mean-field spin-glasses in this regime has been recently developed [4–7]. It was there argued that in regime (I.3) mean-field spin-glass systems *below* the critical temperature never achieve equilibrium, not even within a restricted sector of phase-space. This is in agreement with experimental spin-glasses, for which the estimate is that aging effects take a few years to die away [8].

A relevant order parameter for the long-time asymptotics of the relaxation is the dynamical $P_d(q)$ defined as follows [4]: We add time-independent source terms $h_{i_1 \dots i_r}$ to the energy

$$H_h = H + \frac{1}{N^{r-1}} \sum_{i_1,\dots,i_r}^{N} h_{i_1 \dots i_r} \sigma_{i_1} \dots \sigma_{i_r} \ , \tag{I.4}$$

and then consider the generating functions of the generalized susceptibilities

$$\begin{aligned} Lim \quad & \left[1 - \frac{r}{N^r} \sum_{i_1 < \dots < i_r} \left. \frac{\partial \overline{< s_{i_1}(t) \dots s_{i_r}(t) >}}{\partial h_{i_1 \dots i_r}} \right|_{h=0} \right] \\ & = \int_0^1 dq' \, \frac{dX_{()}(q')}{dq'} \, q'^r = \int_0^1 dq' \, P_{()}(q') \, q'^r \ . \end{aligned} \tag{I.5}$$

If the sysmbol Lim stands for (I.2), it defines the usual Parisi order parameters $x(q)$ and $P(q)$ [1]. If it stands for (I.3) then Eq. (I.5) defines the dynamical parameters $X_d(q)$ and $P_d(q)$ [4].

The dynamical order parameters $X_d(q)$ and $P_d(q)$ - unlike their static counterparts - have not as yet been given a probabilistic interpretation. The main purpose of this paper is to show that $P_d(q)$ can be recast into a form that:

i. Makes its physical meaning more explicit.

ii. Shows for the Sherrington-Kirkpatrick (SK) model that their is a self-similarity in the landscape. Although at all long but finite times (limit (I.3)) the system is exploring regions of phase-space which it will eventually leave, never to return, some geometrical properties of these regions coincide with those of the equilibrium states.

iii. Is amenable to numerical simulations.

The Hamiltonian of the Sherrington-Kirpatrick model is given by

$$H(\sigma) = \frac{1}{N} \sum_{ij} J_{ij} \sigma_i \sigma_j + a \sum_i (\sigma_i^2 - 1)^2 \ . \tag{I.6}$$

The interactions J_{ij} are quenched random variables Gaussianly distributed with zero mean and variance $1/\sqrt{N}$. The last is a spin weight term and the hard-spin limit (± 1) is recovered taking $a \to \infty$.

The out of equilibrium dynamics of the SK model has been studied in Ref. [6]. A surprising outcome, obtained under a set of hypothesis described there in detail, is that the dynamical order parameters $P_d(q)$ and $X_d(q)$ coincide with the static order parameters $P(q)$ and $X(q)$ - even if the physical situations they describe are very different. The coincidence of dynamical and static order parameters does not hold for every model, for instance the p-spin spherical model behaves in a different way [4].

We shall use in this paper the results of Ref. [6] to compute analytically the staggered auto-correlation and response functions [9,10] in the limit of any two large times (limit (I.3)) for the SK model. All the information about the asymptotic large times solution is encoded in the order parameter $X_d(q)$ and in the 'triangle relation' f relating the correlation functions at any three long times (see Ref. [6]). We shall then compare analytical and numerical results for the staggered distributions. Their good agreement gives numerical support for the predicted equality of dynamical and static order parameters in this case.

In order to obtain the staggered distributions from $X_d(q)$ (or $P_d(q)$) and f we shall heavily use the *formal* relation between the static replica approach and the dynamical approach, which becomes transparent when the latter is formulated in terms of superspace variables [11,12]. (Although the underlying supersymmetry in this dynamics is partially broken by the 'boundary' - initial - conditions, it still has useful consequences.)

This proceeds in two steps: we firstly identify the dynamical - superspace - counterparts of the static - replica space - variables. We obtain, roughly speaking, the same formulæ with superspace integrals - including a time-integral - substituting sums over replicas.

Secondly, we look for the solution for the dynamical order parameter. Again, this solution has many points in common with the solution for the statics although they are not equivalent for every model and describe entirely different physical situations.

The paper is organised as follows. In Section II we introduce the staggered distributions.

Using the SUSY formalism, we find that they are related to the powers of the dynamical order parameter $\boldsymbol{Q}(1,2)$. In Section III we compute, in general, these powers of $\boldsymbol{Q}(1,2)$ and then specialise to the long-time asymptotics of the SK model using the results of Ref. [6]. In Section IV we obtain the staggered auto-correlation function for large times. In Section V we describe the numerical simulations and compare them to the analytical results. Finally, we discuss the physical picture in Section VI.

II. STAGGERED DISTRIBUTION FUNCTIONS

The staggered auto-correlation function is defined as

$$g(\lambda;t_1,t_2) \equiv \langle \sigma_\lambda(t_1)\ \sigma_\lambda(t_2)\rangle = \sum_{ij} \langle\lambda|i\rangle\ \langle\lambda|j\rangle\ \langle\sigma_i(t_1)\ \sigma_j(t_2)\rangle\ . \tag{II.1}$$

λ denotes the eigenvalues of the $N \times N$ random matrix J_{ij} associated with the eigenvectors $|\lambda\rangle$. $|\sigma(t)\rangle$ is the time-dependent N-dimensional vector of spins, $\sigma_i(t) \equiv \langle i|\sigma(t)\rangle$, and $\sigma_\lambda(t) \equiv \langle\lambda|\sigma(t)\rangle$ are the staggered spin states.

The staggered response function is

$$\hat{g}(\lambda;t_1,t_2) \equiv \left\langle \frac{\delta\sigma_\lambda(t_1)}{\delta h_\lambda(t_2)} \right\rangle = \sum_{ij} \langle\lambda|i\rangle\ \langle\lambda|j\rangle\ \left\langle \frac{\delta\sigma_i(t_1)}{\delta h_j(t_2)} \right\rangle\ . \tag{II.2}$$

The functions g and $\hat{g}$ are in turn related to the set of time-dependent two-point functions

$$E^{(k)}(t_1,t_2) \equiv \frac{1}{N}\sum_{ij} \overline{\left\langle \sigma_i(t_1)\ (J^k)_{ij}\ \sigma_j(t_2) \right\rangle}\ , \tag{II.3}$$

$$\hat{E}^{(k)}(t_1,t_2) \equiv \frac{1}{N}\sum_{ij} \overline{\left\langle (J^k)_{ij}\ \frac{\delta\sigma_i(t_1)}{\delta h_j(t_2)} \right\rangle}\ , \tag{II.4}$$

where $\overline{\cdot}$ represents the mean over the quenched disorder. In terms of g and $\hat{g}$ they read

$$E^{(k)}(t_1,t_2) \equiv \int d\lambda\, \rho(\lambda)\, \lambda^k\, g(\lambda;t_1,t_2)\ , \tag{II.5}$$

$$\hat{E}^{(k)}(t_1,t_2) \equiv \int d\lambda\, \rho(\lambda)\, \lambda^k\, \hat{g}(\lambda;t_1,t_2)\ . \tag{II.6}$$

$\rho(\lambda)$ is the eigenvalue distribution that, in the limit of large N, corresponds to the semicircle law $\rho(\lambda) = 1/(2\pi)\sqrt{4-\lambda^2}$ if the variance of the J_{ij} is finite [13]. In particular, if $t_1 = t_2$ and $k=1$ Eq. (II.5) gives the time-dependent energy density.

A. Supersymmetric Formalism

Following Ref. [11] we introduce the supersymmetric 'field' $\boldsymbol{\phi}_i(1)$, $i = 1, \ldots, N$,

$$\boldsymbol{\phi}_i(1) \equiv \sigma_i(t_1) + \eta_i(t_1)\,\overline{\theta}_1 + \theta_1\,\overline{\eta}_i(t_1) + \hat{\sigma}_i(t_1)\,\overline{\theta}_1\theta_1 \tag{II.7}$$

with $1 \equiv (t_1, \theta_1, \overline{\theta}_1)$.

The dynamical expectation value of a quantity O can then be written as

$$\begin{aligned}
\langle O(t_1)\rangle &= \int \Pi_i D[\boldsymbol{\phi}_i]\; O(t_1)\exp[-S_{KIN} - S_{POT}]\;, \\
S_{KIN} &= \Gamma_0^{-1}\int d\theta d\bar{\theta} dt\; \sum_i \frac{\partial \boldsymbol{\phi}_i}{\partial \theta}\left(\frac{\partial \boldsymbol{\phi}_i}{\partial \bar{\theta}} - \theta\frac{\partial \boldsymbol{\phi}_i}{\partial t}\right)\;, \\
S_{POT} &= \beta\int d\theta d\bar{\theta} dt\; H(\boldsymbol{\phi})\;.
\end{aligned} \tag{II.8}$$

As in the static replica approach, once the mean is taken over the couplings one ends up with a functional of the order parameters that can be calculated by saddle point. The dynamical order parameter is the 'supercorrelation' function defined as

$$\boldsymbol{Q}(1,2) \equiv \frac{1}{N}\sum_i \langle \boldsymbol{\phi}_i(1)\boldsymbol{\phi}_i(2)\rangle \tag{II.9}$$

that plays the same role as Q_{ab} in the statics. For the mean-field case that satisfies causality the saddle-point value of $\boldsymbol{Q}(1,2)$ can be written as

$$\boldsymbol{Q}(1,2) = C(t_1,t_2) + (\overline{\theta}_2 - \overline{\theta}_1)\;[\theta_2\;G(t_1,t_2) + \theta_1\;G(t_2,t_1)] \tag{II.10}$$

and it encodes the two-time functions C and G that are the stantard auto-correlation and response functions

$$C(t_1,t_2) \equiv \frac{1}{N}\sum_i \langle \sigma_i(t_1)\, \sigma_i(t_2)\rangle \ , \tag{II.11}$$

$$G(t_1,t_2) \equiv \frac{1}{N}\sum_i \left\langle \frac{\delta\sigma_i(t_1)}{\delta h_i(t_2)}\right\rangle \ , \tag{II.12}$$

respectively. Because of causality, $G(t_1,t_2)=0$ if $t_2>t_1$. In these formulæ we have ommited the mean over the disorder since C and G are self-averaging in the limit $N\to\infty$ *for finite times* as can be easily proven by considering the evolution of two independent copies of the system with the same couplings J_{ij}.

We shall need the definition of the operator powers of $\boldsymbol{Q}$

$$\boldsymbol{Q}^k(1,3) \equiv \int d2\ \boldsymbol{Q}^{k-1}(1,2)\, \boldsymbol{Q}(2,3)\ . \tag{II.13}$$

It is easy to see that $\boldsymbol{Q}^k$ conserves the form (II.10) with $C^{(k)}$ and $G^{(k)}$ given inductively by

$$C^{(k)}(t_1,t_3) = \int dt_2\ \left[C^{(k-1)}(t_1,t_2)\, G(t_3,t_2) + G^{(k-1)}(t_1,t_2)\, C(t_2,t_3)\right]\ , \tag{II.14}$$

$$G^{(k)}(t_1,t_3) = \int dt_2\ G^{(k-1)}(t_1,t_2)\, G(t_2,t_3)\ , \tag{II.15}$$

$t_1>t_3$. From now on supra-indices within parenthesis denote entries in the function $\boldsymbol{Q}^k$ while supra-indices without parenthesis denote ordinary powers.

B. Staggered auto-correlation and response functions

We now start the computation of the staggered auto-correlations and responses. With the superspace notation most of the manipulations of Ref. [10] carry through without change, just substituting replica indices by superspace variables. Defining

$$\boldsymbol{E}^{(k)}(1,2) = E^{(k)}(t_1,t_2) + (\overline{\theta}_2 - \overline{\theta}_1)\ \left[\theta_2\, \hat{E}^{(k)}(t_1,t_2) + \theta_1\, \hat{E}^{(k)}(t_2,t_1)\right] \tag{II.16}$$

we have that

$$\boldsymbol{E}^{(k)}(1,2) = \frac{1}{N}\sum_{ij} \overline{\left\langle \phi_i(1)\,(J^k)_{ij}\,\phi_j(2)\right\rangle}\ , \tag{II.17}$$

here '$\langle\,\cdot\,\rangle$' denotes mean with the measure (II.8). Correspondingly, the staggered distributions can be encoded as

$$\boldsymbol{g}(\lambda;1,2) \equiv g(\lambda;t_1,t_2) + (\overline{\theta}_2 - \overline{\theta}_1)\ \left[\theta_2\,\hat{g}(\lambda;t_1,t_2) + \theta_1\,\hat{g}(\lambda;t_2,t_1)\right]\ . \tag{II.18}$$

We shall use a related set of order parameters [1]

$$\boldsymbol{\mathcal{X}}^{(k)}(1,2) \equiv \sum_{r=0}^{k} S_{k,r}\ \boldsymbol{E}^{(r)}(1,2)\ , \tag{II.19}$$

where $S_k(z) = \sum_{r=0}^{k} S_{k,r}\ z^r$ are the Chebyshev polynomials of the second kind, generated by

$$\sum_{k=0}^{\infty} S_k(z)\ y^k = \frac{1}{(1 - yz + y^2)}\ . \tag{II.20}$$

In components, $\boldsymbol{\mathcal{X}}^{(k)}(1,2)$ reads

$$\boldsymbol{\mathcal{X}}^{(k)}(1,2) \equiv \mathcal{X}^{(k)}(t_1,t_2) + (\overline{\theta}_2 - \overline{\theta}_1)\ \left[\theta_2\,\hat{\mathcal{X}}^{(k)}(t_1,t_2) + \theta_1\,\hat{\mathcal{X}}^{(k)}(t_2,t_1)\right]\ . \tag{II.21}$$

Following exactly the same steps as in Ref. [10], one gets

$$\boldsymbol{\mathcal{X}}^{(k)}(1,2) = \frac{1}{\pi}\int_{-2}^{2} d\lambda\ \sqrt{1-\frac{\lambda^2}{4}}\ S_k(\lambda)\ \boldsymbol{g}(\lambda;1,2)\ , \tag{II.22}$$

i.e. each component $\mathcal{X}^{(k)}$ and $\hat{\mathcal{X}}^{(k)}$ is the coefficient of the expansion of g and $\hat{g}$ in the polynomials S_k.

One can now show [10] that the $\boldsymbol{\mathcal{X}}^{(k)}$ are obtained from

$$\boldsymbol{\mathcal{X}}^{(k)}(1,2) = \beta^k\ \boldsymbol{Q}^{k+1}(1,2)\ , \tag{II.23}$$

or disentagling the superspace notation

$$\mathcal{X}^{(k)}(t_1,t_2) = \frac{1}{\pi}\int_{-2}^{2} d\lambda\ \sqrt{1-\frac{\lambda^2}{4}}\ S_k(\lambda)\ g(\lambda;t_1,t_2) = \beta^k\, C^{(k+1)}(t_1,t_2) \tag{II.24}$$

$$\hat{\mathcal{X}}^{(k)}(t_1,t_2) = \frac{1}{\pi}\int_{-2}^{2} d\lambda\ \sqrt{1-\frac{\lambda^2}{4}}\ S_k(\lambda)\ \hat{g}(\lambda;t_1,t_2) = \beta^k\, G^{(k+1)}(t_1,t_2)\ . \tag{II.25}$$

---

[1]The functions $\boldsymbol{\mathcal{X}}^{(k)}(1,2)$ are the dynamical analogs of the functions X_k of Ref. [10].

We are left with the task of calculating the powers of the superorder parameter $\boldsymbol{Q}$. Before dealing with this, let us give a compact form for $\boldsymbol{g}$; using Eq. (II.20) and the orthogonality properties of the Chebyshev polynomials we obtain

$$\boldsymbol{g}(\lambda;1,2) = \left[\boldsymbol{Q}\left(\boldsymbol{\delta} - \beta\lambda\boldsymbol{Q} + \beta^2\boldsymbol{Q}^2\right)^{-1}\right](1,2)\,. \tag{II.26}$$

Products and inverses are as in Eq. (II.13) and the identity is defined as $\boldsymbol{\delta}(1-2) \equiv (\theta_2-\theta_1)(\overline{\theta}_2-\overline{\theta}_1)\delta(t_2-t_1)$. The relation (II.26) is valid for all times. It is purely a consequence of the mean-field limit and the (super)symmetries of the problem, we have not yet used at all the dynamical solution.

In the following sections we shall concentrate on the long-times regime (I.3). We shall express the results rather than in terms of the times, in terms of the value the auto-correlation function takes at those times.

III. POWERS OF Q

We now calculate the powers $\boldsymbol{Q}^k$ for large times. Until explicitly noted our calculation is not particular to the SK model but only relies on the assumptions made in Ref. [6] for the long times dynamics of mean-field spin-glasses.

For any three large times the auto-correlations satisfy 'triangle relations':

$$C(t_{max},t_{min}) = f\left(C(t_{max},t_{int}),C(t_{int},t_{min})\right)\,. \tag{III.1}$$

The function f is an associative composition law.

We also have that

$$G(t_1,t_2) = \frac{\partial F[C(t_1,t_2)]}{\partial t_2} = X_d[C(t_1,t_2)]\,\frac{\partial C(t_1,t_2)}{\partial t_2}\,. \tag{III.2}$$

Eq. (III.2) define $X_d[C]$ and $F[C]$ (the latter up to a constant). It says that the violation of the FDT theorem for the non-equilibrium dynamics of spin-glasses is determined by a function $X_d[C]$ that depends on the times exclusively through $C(t_1, t_2)$.

This scenario has been proposed to analyse the large-times dynamics of the mean-field spin-glass models. The solution of the dynamical problem for a particular model gives explicit expressions for X_d, F and f [4,6].

In the Appendices we shall show that the structure (III.1)-(III.2) carries through to $\boldsymbol{Q}^k$. The reasoning is general and does not depend on the model. The main steps are the following: We first show that $C^{(k)}$ depends on the times only through $C(t_1, t_2)$:

$$C^{(k)}(t_1, t_2) = C^{(k)}[C(t_1, t_2)] \tag{III.3}$$

The triangle relation for $C^{(k)}$ can be read from

$$C(C^{(k)}(t_{max}, t_{min})) = f\left(C(C^{(k)}(t_{max}, t_{int})), C(C^{(k)}(t_{int}, t_{min}))\right) \ , \tag{III.4}$$

i.e. the new triangle relation is isomorphic to the old one.

Relation (III.2) then maps into

$$G^{(k)}(t_1, t_2) = \frac{\partial F^{(k)}[C^{(k)}(t_1, t_2)]}{\partial t_2} = X_d^{(k)}[C^{(k)}(t_1, t_2)] \ \frac{\partial C^{(k)}(t_1, t_2)}{\partial t_2} \tag{III.5}$$

In the Appendices we also show for the SK model that $X_d^{(k)}$ is obtained through

$$X_d^{(k)}(C^{(k)}[C]) = X_d[C] \ . \tag{III.6}$$

Of particular importance are the values of the correlations $C = a_i^*$ that are 'fixed points' of f

$$f(a_i^*, a_i^*) = a_i^* \ . \tag{III.7}$$

Equation (III.4) implies that fixed points corresponding to C are mapped into fixed points corresponding to $C^{(k)}$.

The fixed points separate the range of auto-correlations in 'discrete scales' [6]. Under very general (model independent) assumptions, the relation f between two fixed points is ultrametrical

$$f(a_i^*, a_j^*) = \min(a_i^*, a_j^*) \tag{III.8}$$

but not so the relation between values of the auto-correlation that are not fixed points and belong to the same discrete scale.

In general, it turns out that the function $C^{(k)}(C)$, when evaluated in the fixed points a_i^* is related to the ultrametric ansatz in replica space as follows: Let Q_{ab} be an ultrametric matrix with elements q_r associated with blocks of sizes X_r. We compute the matrix power $[Q^k]_{ab}$, and consider its elements (say, $q_r^{(k)}$) associated with blocks of size X_r. Then, the functional $q^{(k)}[q]$ coincides with the dynamical functional $C^{(k)}[C]$.

This relationship (at this point purely kinematical) between powers of static and dynamical order parameters holds only for 'fixed point' values of C. The values of q that are not contained as entries of the ultrametric matrix correspond to values of C intermediate between fixed points, *i.e.* within discrete scales [2]; for these auto-correlation values there is no replica counterpart within the ultrametric ansatz.

---

[2]Let us note, in passing, that the correspondence we have just described is an example of a more general connection between static replica and dynamic SUSY treatments. Indeed this connection holds not only for powers of the order parameters, but for a wide class of functionals $H[\boldsymbol{Q}]$ [12].

A. SK Model

For the SK problem in zero magnetic field [3], the solution of the mean-field dynamical equations yields a dense set of fixed points of $f(C,C)$ in the interval $[0,q_{EA}]$, plus an isolated fixed point $C(t,t)=1$. The value q_{EA} is the Edwards-Anderson parameter, and the interval $(q_{EA},1]$ corresponds to the 'FDT' (discrete) scale. For times associated with autocorrelations in this interval $X_d(C)=1$ and FDT holds. Instead, for large times associated to C in $[0,q_{EA}]$, FDT is modified as in Eq. (III.2) by a non-trivial factor $X_d[C]$. The function $X_d[C]$ is part of the solution to the mean-field equations of motion.

To obtain the explicit form of the powers $\boldsymbol{Q}^k$ it is useful to separate the FDT discrete scale writing $\boldsymbol{Q}(1,2)$ as

$$\boldsymbol{Q}(1,2) = \boldsymbol{Q}_{FDT}(1,2) + \mathcal{Q}(1,2)\ . \tag{III.9}$$

The FDT term $\boldsymbol{Q}_{FDT}(1,2)$ has entries that satisfy

$$C_{FDT}(t_1,t_2) = C_{FDT}(t_1-t_2)\ , \tag{III.10}$$

$$G_{FDT}(t_1,t_2) = \frac{\partial C_{FDT}(t_1-t_2)}{\partial t_2}\ . \tag{III.11}$$

The function $C_{FDT}(\tau)$, $\tau \equiv t_1-t_2$, is a rapidly (with respect to the variation of $\mathcal{Q}$) decreasing function; $C_{FDT}(0) = 1-q_{EA}$ and $C_{FDT}(\infty)=0$. It is the output of the Sompolinsky-Zippelius dynamics 'within a valley' [3]. Operator powers of $\boldsymbol{Q}_{FDT}$ have entries that verify Eqs. (III.10) and (III.11) and are relevant in the same time-region.

The $\mathcal{Q}$ function varies slowly; $\mathcal{C}(t_1,t_1)=q_{EA}$ and $\mathcal{C}(t_1,t_f)=0$ if $t_1 >> t_f$.

---

[3]This solution has been obtained for T slightly below the critical temperature T_c. We expect it to hold for all temperatures below T_c.

In the operator product $\boldsymbol{Q}_{FDT}\ \boldsymbol{\mathcal{Q}}$, the operator $\boldsymbol{Q}_{FDT}$ acts as the identity $\boldsymbol{\delta}(1-2)$ times $1-q_{EA}$ [11].

The separation (III.9) is the dynamic counterpart of the separation of the - static - replica matrix Q_{ab},

$$Q_{ab} = (1-q_{EA})\delta_{ab} + \mathcal{Q}_{ab} \tag{III.12}$$

where $\mathcal{Q}_{ab}$ has q_{EA} in the diagonal.

In order to compute $\boldsymbol{\mathcal{X}}^{(k)}$ we use that, for long times,

$$\begin{aligned}
\boldsymbol{Q}^k(1,3) &= \sum_{l=0}^{k} \begin{pmatrix} k \\ l \end{pmatrix} \int d2\ \boldsymbol{Q}_{FDT}^{k-l}(1,2)\ \boldsymbol{\mathcal{Q}}^l(2,3) \\
&= \boldsymbol{Q}_{FDT}^k(1,3) + \left((1-q_{EA})\boldsymbol{\delta} + \boldsymbol{\mathcal{Q}}\right)^k(1,3) - ((1-q_{EA})\boldsymbol{\delta})^k(1,3)\ .
\end{aligned} \tag{III.13}$$

This relation allows to write $\left\{C^{(k)}, G^{(k)}\right\}$, the entries of $\boldsymbol{Q}^k$, in terms of $\left\{C_{FDT}^{(l)}, G_{FDT}^{(l)}\right\}$ and $\left\{\mathcal{C}^{(l)}, \mathcal{G}^{(l)}\right\}$, the entries of $\boldsymbol{Q}_{FDT}^l$ and $\boldsymbol{\mathcal{Q}}^l$, respectively.

In Appendix A we give expressions for the entries $C_{FDT}^{(k)}$ and $G_{FDT}^{(k)}$ of $\boldsymbol{Q}_{FDT}^k$. The explicit form of $C_{FDT}^{(k)}(t_1,t_2)$ has no analog in the ultrametric ansatz for the replica approach, except for the values at equal times and at times such that $C = q_{EA}$ (*i.e.* at the limits of the FDT 'discrete' scale). $C_{FDT}^{(k)}$ is also a rapidly decreasing function which falls from $(1-q_{EA})^k$ at equal times to zero at widely separated times.

In Appendix B we calculate the entries $\mathcal{C}^{(k)}$ and $\mathcal{G}^{(k)}$ of $\boldsymbol{\mathcal{Q}}^k$ for large times.

Using these results we are in a position to express $C^{(k)}$ and $G^{(k)}$ for all ranges of times:

- For large and widely separated times t_1, t_2 such that $C(t_1,t_2) < q_{EA}$, we compute the sum in Eq. (III.13) to get, in terms of $C(t_1,t_2)$,

$$\begin{aligned}
C^{(k)}[C] &= \frac{(1-q_{EA}-F[0])^k - (1-q_{EA}-F[C])^k}{X[\mathcal{C}]} \\
&\quad - \int_{X[0]}^{X[C]} \frac{dx'}{x'^2}\ \left[(1-q_{EA}-F[C(x')])^k - (1-q_{EA}-F[0])^k\right]
\end{aligned} \tag{III.14}$$

and

$$G^{(k)}(t_1,t_2) = X_d[C]\,\frac{\partial C^{(k)}(C)}{\partial t_2}\,. \tag{III.15}$$

- For large and close times such that $C > q_{EA}$

$$C^{(k)}(t_1,t_2) = C^{(k)}_{FDT}(t_1-t_2) + C^{(k)}(q_{EA}^-)\,, \tag{III.16}$$

$$G^{(k)}(t_1,t_2) = \frac{\partial C^{(k)}(t_1,t_2)}{\partial t_2}\,, \tag{III.17}$$

with $C^{(k)}(q_{EA}^-)$ from Eq. (III.14).

In particular we shall need the result for equal times

$$C^{(k)}(t_1,t_1) = (1-q_{EA})^k + C^{(k)}(q_{EA}^-)\,. \tag{III.18}$$

IV. EXPRESSIONS FOR THE STAGGERED AUTO-CORRELATION

Expressions (III.17) and (III.18), together with (III.14) and (III.15) are all that is needed to calculate the staggered auto-correlation and response functions at long times. In order to make contact with the results of Ref. [10] we make a change of variables:

$$\Delta[X_d] \equiv F[0] - F[C(X_d)]\;\theta(X_M - X_d) \tag{IV.1}$$

$$I \equiv F[0] + q_{EA} \tag{IV.2}$$

where $X_M = X_d[q_{EA}^-]$.

Inverting equation (II.22), using (II.20) and the low-temperature phase result $\beta(1-I) = 1$, after some algebra we obtain the staggered auto-correlation at long equal times

$$\begin{aligned} g(\lambda) &\equiv \lim_{t\to\infty}\lim_{N\to\infty} g(\lambda,t,t) \\ &= \frac{1}{\beta(2-\lambda)}\left[1 + \int_0^1 \frac{dX_d}{X_d^2}\left(\frac{(\beta\Delta(X_d))^2}{1-\lambda(1+\beta\Delta(X_d)) + (1+\beta\Delta(X_d))^2}\right)\right]\,. \end{aligned} \tag{IV.3}$$

The staggered auto-correlation $g(\lambda, C)$, between two large and widely separated times t_1, t_2 chosen such that $C(t_1,t_2) = \mathcal{C} < q_{EA}$ is given by

$$
\begin{aligned}
g(\lambda,\mathcal{C}) &\equiv \lim_{t_1\to\infty, C(t_1,t_2)=\mathcal{C}} \; \lim_{N\to\infty} g(\lambda,t_1,t_2) \\
&= g(\lambda) - \frac{1}{\beta(2-\lambda)} \left[1 + \frac{(1-\beta(1-q_{EA}))^2}{1-\lambda\beta(1-q_{EA})+\beta^2(1-q_{EA})^2} \right. \\
&\quad \left. + \int_{X_d(\mathcal{C})}^{X_d(q_{EA}^-)} \frac{dX'_d}{X'^2_d} \left(\frac{(\beta\Delta(X'_d))^2}{1-\lambda(1+\beta\Delta(X'_d))+(1+\beta\Delta(X'_d))^2} \right) \right] .
\end{aligned}
\qquad \text{(IV.4)}
$$

Both these last expressions are valid for the low-temperature phase.

We now note that *for the SK model* the functions $X_d(C)$ for the dynamics and the usual function $X(q)$ of the replica treatment coincide at all temperatures. Furthermore, the diagonal values Q^k_{aa} and $C^{(k)}(t_1,t_1)$ also coincide. This also implies the equality of the functions Δ and I.

Hence, we have just proven that for the long and equal times the dynamic staggered spin auto-correlation (IV.3) coincides with the static one obtained in Ref. [10].

Furthermore, the staggered auto-correlation $g(\lambda,\mathcal{C})$ coincides with the static one [4] computed with configurations belonging to two equilibrium states with mutual overlap $\mathcal{C}$.

Finally, let us show that, both statically and dynamically, $g(\lambda)$ contains all the information needed to reconstruct $P_{()}(q)$. To this end we define

$$
t(X) \equiv \frac{1+(1+\beta\Delta(X))^2}{1+\beta\Delta(X)} \qquad \text{(IV.5)}
$$

and

$$
h(\lambda) \equiv \beta\,(2-\lambda)\,g(\lambda) - 1 \; . \qquad \text{(IV.6)}
$$

---

[4]One can extend the equilibrium calculation of Ref. [10] to this case by considering the entry of the replica matrices ${(1-\beta\lambda Q+\beta^2Q^2)^{-1}}_{ab}$ corresponding to a pair of replicas having mutual overlap $Q_{ab} = \mathcal{C}$.

The functions $t(X)$ and $q(X)$ both have a plateau for the same values of $X \in (X_M, 1)$.

Equation (IV.3) becomes

$$h(\lambda) = \int_0^{X_M} \frac{dX}{X^2} \frac{t(X)-2}{t(X)-\lambda} + \left(\frac{1}{X_M} - 1\right) \frac{t(X_M)-2}{t(X_M)-\lambda} \,. \tag{IV.7}$$

Having excluded the plateau in $t(X)$, we can change variables in the integral to obtain

$$h(\lambda) = \int_2^{t_M} \frac{\mu(t')\,dt'}{t'-\lambda} + \left(\frac{1}{X_M} - 1\right) \frac{t_M - 2}{t_M - \lambda} \tag{IV.8}$$

where $t_M \equiv t(X_M)$ and $\mu(t) = (t-2)X^{-2}(t)\, dX/dt$.

This is an electrostatic problem with positive charges. The determination of $\mu(t), X_M, t_M$ can in principle be done in a unique way: the analytical continuation of $h(\lambda)$ from the interval $-2 < \lambda < 2$ yields the 'charge density' $\mu(t)$, the magnitude of the 'discrete charge' and its position t_M.

The knowledge of $\mu(t), X_M, t_M$ then allows to calculate $X(t)$ as

$$\frac{1}{X(t)} = \int_t^{t_M} \frac{\mu(t')}{t'-2} dt' + \frac{1}{X_M(t_M)} \,. \tag{IV.9}$$

Hence, we have shown that $g(\lambda)$ contains all the information needed to obtain $X(q)$.

V. NUMERICAL SIMULATIONS

We have performed Montecarlo simulation of the dynamics of a system with $N = 996$ spins at temperature $T = 0.3$.

We have calculated the distribution of overlaps for two copies of the system relaxing from different initial configurations, at times $t = 600, 2000, 10000$ Montecarlo sweeps. At these times the system is well out of equilibrium as shown by the form of the overlap distribution. This eventually takes the form of the static $P(q)$ (except for finite-size corrections) at equilibrium, but is only bell-shaped at the times considered (see Fig. 1).

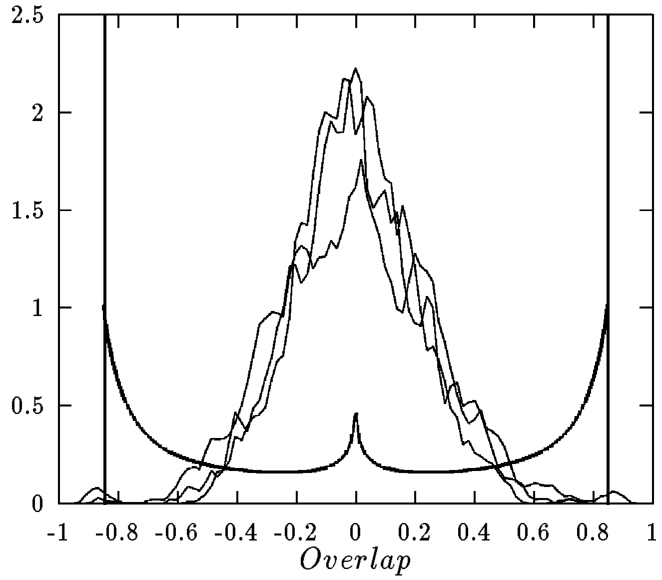

Figure 1. Overlap distributions for $t = 600, 2000, 10000$ Montecarlo sweeps. The dashed line shows the analytical equilibrium $P(q)$.

Figure 2 shows the equal-times staggered auto-correlation times the density of eigenvalues, $\rho(\lambda)g(\lambda, t, t)$, at times $t = 600, 2000, 10000$, together with the analytical result for the equilibrium $\rho(\lambda)g(\lambda)$. We notice that the convergence to a curve that coincides with the equilibrium one is very fast, even in a situation manifestly out of equilibrium (*cfr.* Fig.1).

In particular, the time dependent energy density is given by

$$e(t) = \int d\lambda\, \lambda\, \rho(\lambda)\, g(\lambda; t, t)\ . \tag{V.1}$$

Hence, the equivalence $\lim_{t\to\infty} g(\lambda; t, t) = g(\lambda)$ ensures the equivalence of the asymptotic energy and the equilibrium one, a result that we have also checked numerically.

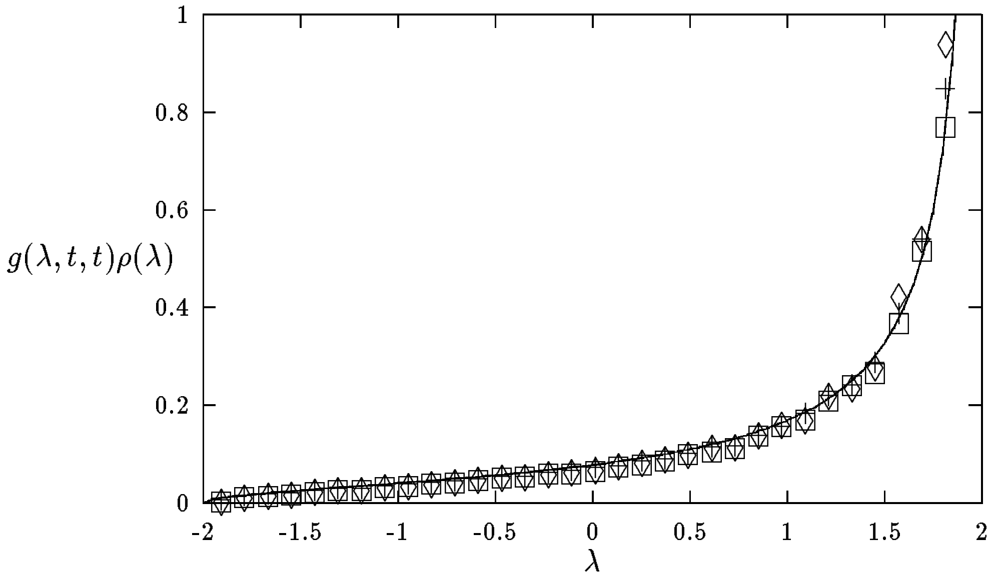

Figure 2. The equal-times staggered auto-correlation distribution $g(\lambda; t, t)$ times the density of eigenvalues $\rho(\lambda)$, for $t = 600$ (diamonds), $t = 2000$ (crosses) and $t = 10000$ (squares) Montecarlo sweeps. The continuous line shows the analytical result for the statics and for long but finite times.

VI. DISCUSSION

The partition function of the SK model is dominated by the low-lying states. The out of equilibrium dynamics never reaches any of these states: there is never a situation of 'effective' dynamical equilibrium in which the system is trapped forever in one of these states ignoring the rest of the phase-space and satisfying FDT and time-translational invariance.

Indeed, as time passes, the top evolution of the system slows down more and more but it is never completely trapped. In Ref. [6] it was pointed out that the equality $P_d(q) = P(q)$ implied that an infinite SK system has an energy density which goes asymptotically to the equilibrium energy density. Furthermore, the 'width' of the region in which the system has a fast relaxation at long times coincides with the 'size' q_{EA} of the equilibrium states.

This already points to a similarity between the long-time landscape and the (different) region that dominates the partition function. The results in this paper suggest that this similarity is much deeper: Consider the relaxation at two large times (t_1, t_2). Because of weak ergodicity breaking [14], given t_1 we can always choose $t_2 > t_1$ such that the auto-correlation $C(t_2, t_1)$ between the configurations $\sigma_i(t_1)$ and $\sigma_i(t_2)$ at those times is any given value $\mathcal{C}$. If we now compute the staggered auto-correlation distribution $g(\lambda, t_1, t_2)$ for those configurations we obtain the same distribution we would have obtained with configurations chosen from two equilibrium states at mutual distance $\mathcal{C}$.

This result is quite surprising, since we know that the system *is not in any equilibrium state* at times t_1 or t_2, however long; it will eventually leave the neighbourhood of $\sigma_i(t_1)$ and $\sigma_i(t_2)$ never to return.

If we now keep the configuration at times t_2 and let the system evolve up to a time t_3 such that again $C(t_2, t_3) = \mathcal{C}$ we obtain the same form for the staggered autocorrelation $g(\lambda, t_2, t_3)$. Note however, that because the system slows down, $t_2 - t_1 < t_3 - t_2$ if $\mathcal{C} < q_{EA}$.

The picture that this seems to suggest is that the geometry of phase space seen at different long-times is similar in every respect, except that the relevant barriers found at larger times are higher, thus slowing down the system.

We expect that this similarity between the equilibrium and the long-time out of equilibrium regions of phase-space will hold for models that do not have a 'threshold' level below which the system cannot go. More precisely, we expect this similarity to hold for models with a continuous set of correlation scales; *e.g.* the SK model and the model studied in Refs. [5,7], but we do not expect it hold in the p-spin spherical model of Ref. [4].

Finally, let us remark that the good agreement between the numerical calculation of $g(\lambda, t, t)$ for large t and the static $g(\lambda)$ constitues a rather detailed test of the solution of the out of equilibrium dynamics for this model [15].

Appendix A

In this appendix we give an expression for $\boldsymbol{Q}^k_{FDT}$. Let us first note that the power of a FDT - supersymmetric - operator is itself FDT [11]. From Eq. (II.14) we then have

$$\begin{aligned} C^{(k)}_{FDT}(t_1-t_3) &= \left[C^{(k-1)}_{FDT}(t_1-t_2)\, C_{FDT}(t_3-t_2)\right]_0^{t_3} \\ &\quad + \int_{t_3}^{t_1} dt_2\; C_{FDT}(t_2-t_3)\, \frac{\partial C^{(k-1)}_{FDT}(t_1-t_2)}{\partial t_2}\,. \end{aligned} \tag{A.1}$$

Beacause the time-difference t_1-t_3 is in a one to one relation with $C_{FDT}(t_1-t_3)$, Eq. (A.1) proves Eq. (III.3) for the FDT regime.

The value of $G^{(k)}_{FDT}(t_1-t_3)$ is obtained as

$$G^{(k)}_{FDT}(t_1-t_3) = \frac{\partial C^{(k)}_{FDT}(t_1-t_3)}{\partial t_3}\,. \tag{A.2}$$

This says that $X_d^{k)}=1$ for C in the FDT regime, and hence is of the form (III.5).

In this paper we only need the value of $C^{(k)}_{FDT}(t_1,t_1)=C^{(k)}_{FDT}(0)$. This is easily obtained by putting $t_3=t_1$ in Eq. (A.1). In this way we obtain

$$C^{(k)}_{FDT}(t_1,t_1) = \left[C_{FDT}(t_1,t_1)\right]^k = (1-q_{EA})^k\,. \tag{A.3}$$

From (A.1) it is easy to see that $\boldsymbol{Q}^{(k)}_{FDT}$ is also small for very different times.

Appendix B

In this Appendix we analyse the properties of $\boldsymbol{Q}^k$ for long and widely separated times, for which $\boldsymbol{Q}=\boldsymbol{\mathcal{Q}}$ though $\boldsymbol{Q}^k \neq \boldsymbol{\mathcal{Q}}^k$. We first study $\boldsymbol{\mathcal{Q}}^k$ and then the properties for $\boldsymbol{Q}^k$ will follow from linearity (see Eq. (III.13)).

We analyse $\mathcal{C}^{(k)}$, $X_d^{(k)}$ and $F^{(k)}$. We first demonstrate by induction that $\mathcal{C}^{(k+1)}$ depends exclusively on $\mathcal{C}$

$$\mathcal{C}^{(k+1)}(t_1,t_2) = \mathcal{C}^{(k+1)}(\mathcal{C}(t_1,t_2)) \ . \tag{B.1}$$

Then we show also by induction that the relation between $\mathcal{G}^{(k+1)}$ and $\mathcal{C}^{(k+1)}$ maintains the form (III.5); there exist $F^{(k+1)}$ and $X_d^{(k+1)}$ that verify

$$\mathcal{G}^{(k+1)}(t_1,t_2) = \frac{\partial F^{(k+1)}[\mathcal{C}(t_1,t_2)]}{\partial t_2} = X_d^{(k+1)}[\mathcal{C}(t_1,t_2)] \ \frac{\partial \mathcal{C}^{(k+1)}(t_1,t_2)}{\partial t_2} \ . \tag{B.2}$$

Finally we explicitly compute $\mathcal{C}^{(k)}$, $X_d^{(k)}$ and $F^{(k)}$ in terms of $\mathcal{C}$ for the SK model. The $F^{(k)}$ are defined up to a constant, we fix it by imposing $F^{(k)}(\mathcal{C}(t_1,t_1)) = 0$ (*e.g.* for $k=1$, $F(q_{EA})=0$).

Let us define the (rather badly behaved) 'inverse' of f

$$\mathcal{C}(t_{max},t_{min}) = f\left(\mathcal{C}(t_{max},t_{int}),\mathcal{C}(t_{int},t_{min})\right) \Longrightarrow \mathcal{C}(t_{int},t_{min}) = \overline{f}\left(\mathcal{C}(t_{max},t_{int}),\mathcal{C}(t_{max},t_{min})\right) \tag{B.3}$$

The function $\overline{f}$ is written in such a way that its second argument is always smaller than the first one.

We start by assuming

$$\mathcal{C}^{(k)}(t_1,t_2) \ , \ = \mathcal{C}^{(k)}(\mathcal{C}(t_1,t_2)) \tag{B.4}$$

$$\mathcal{G}^{(k)}(t_1,t_2) = \frac{\partial F^{(k)}[\mathcal{C}^{(k)}(t_1,t_2)]}{\partial t_2} = X_d^{(k)}[\mathcal{C}^{(k)}(t_1,t_2)] \ \frac{\partial \mathcal{C}^{(k)}(t_1,t_2)}{\partial t_2} \ , \tag{B.5}$$

with $F^{(k)}[\mathcal{C}^{(k)}(t_1,t_1)] = 0$. Eq. (II.14) then reads

$$\begin{aligned}\mathcal{C}^{(k+1)}(t_1,t_3) = &-\mathcal{C}^{(k)}\left[\mathcal{C}(t_1,0)\right] \ F\left[\mathcal{C}(t_3,0)\right] - \int_{\mathcal{C}(t_1,0)}^{\mathcal{C}} dy \ \frac{\partial \mathcal{C}^{(k)}(y)}{\partial y} \ F\left[\overline{f}(\mathcal{C},y)\right] \\ &+ \int_{\mathcal{C}(t_1,0)}^{\mathcal{C}} dy \ X_d^{(k)}\left[\mathcal{C}^{(k)}(y)\right] \ \frac{\partial \mathcal{C}^{(k)}(y)}{\partial y} \ \overline{f}(\mathcal{C},y) \\ &+ \int_{\mathcal{C}}^{q_{EA}} dy \ X_d^{(k)}\left[\mathcal{C}^{(k)}(y)\right] \ \frac{\partial \mathcal{C}^{(k)}(y)}{\partial y} \ \overline{f}(y,\mathcal{C})\end{aligned} \tag{B.6}$$

where $\mathcal{C} \equiv \mathcal{C}(t_1,t_3)$.

It is easy to see that, for all $\overline{f}$ such that $\overline{f}(x,y) \propto y$, $C(t_1,0)=0 \Rightarrow C^{(k)}(t_1,0)=0$, $\forall k$. In these cases $\mathcal{C}^{(k+1)}$ reads

$$\mathcal{C}^{(k+1)}(\mathcal{C}) = -\int_0^{\mathcal{C}} dy\ \frac{\partial \mathcal{C}^{(k)}(y)}{\partial y}\ F\left[\overline{f}(\mathcal{C},y)\right] + \int_0^{\mathcal{C}} dy\ X_d^{(k)}\left[\mathcal{C}^{(k)}(y)\right]\ \frac{\partial \mathcal{C}^{(k)}(y)}{\partial y}\ \overline{f}(\mathcal{C},y)$$
$$+ \int_{\mathcal{C}}^{q_{EA}} dy\ X_d^{(k)}\left[\mathcal{C}^{(k)}(y)\right]\ \frac{\partial \mathcal{C}^{(k)}(y)}{\partial y}\ \overline{f}(y,\mathcal{C})\ . \tag{B.7}$$

This is a time-reparametrization invariant equality and $\mathcal{C}^{(k)}$ depends on t_1 and t_3 only through $\mathcal{C}$.

We now demonstrate that Eq. (B.2) holds $\forall k$. Eq. (II.15) implies

$$G^{(k+1)}(t_1,t_3) = \frac{\partial}{\partial t_3}\int_{t_3}^{t_1} dt_2\ \frac{\partial F^{(k)}\left[\mathcal{C}^{(k)}(t_1,t_2)\right]}{\partial t_2}\ F\left[\overline{f}\left(\mathcal{C}(t_1,t_2),\mathcal{C}(t_1,t_3)\right)\right] \tag{B.8}$$

Then, we can identify

$$F^{(k+1)}(t_1,t_3) = \int_{t_3}^{t_1} dt_2\ \frac{\partial F^{(k)}\left[\mathcal{C}^{(k)}(t_1,t_2)\right]}{\partial t_2}\ F\left[\overline{f}\left(\mathcal{C}(t_1,t_2),\mathcal{C}(t_1,t_3)\right)\right] \tag{B.9}$$

choosing the integration constant to be zero. Now, using $\mathcal{C}^{(k)}(t_1,t_2) = \mathcal{C}^{(k)}(\mathcal{C}(t_1,t_2))$

$$F^{(k+1)}(t_1,t_3) = \int_{\mathcal{C}}^{q_{EA}} dy\ \frac{\partial F^{(k)}\left[\mathcal{C}^{(k)}\right]}{\partial \mathcal{C}^{(k)}}\ \frac{\partial \mathcal{C}^{(k)}(y)}{\partial y}\ F\left[\overline{f}\left(y,\mathcal{C}\right)\right]\ . \tag{B.10}$$

The rhs only depends on $\mathcal{C}$ and then $F^{(k+1)}(t_1,t_3) = F^{(k+1)}[\mathcal{C}^{(k+1)}]$ and $X_d^{(k)}(t_1,t_3) = X_d^{(k+1)}[\mathcal{C}^{(k+1)}]$.

The derivation up to this point is general, it only depends on the assumptions of Ref. [6]. For the SK model $\mathcal{C}(t_1,0)=0$ for long enough time t_1 and the ultrametric dynamical relation between auto-correlation functions

$$\overline{f}(x,y) = \min(x,y) = y \tag{B.11}$$

holds [6]. Then $C^{(k)}(t_1,0)=0$, $\forall k$ and

$$\mathcal{C}^{(k+1)}(\mathcal{C}) = -\int_0^{\mathcal{C}} dy\ \left(\frac{\partial \mathcal{C}^{(k)}(y)}{\partial y}\ F\left[y\right] + F^{(k)}\left[\mathcal{C}^{(k)}(y)\right]\right)\ . \tag{B.12}$$

We can also solve Eq. (B.10) using the ultrametric relation (B.11). We obtain

$$F^{(k)}\left[\mathcal{C}^{(k)}(\mathcal{C})\right] = -\left(-F\left[\mathcal{C}\right]\right)^{k} \tag{B.13}$$

We now solve Eq. (B.12) for $\mathcal{C}^{(k+1)}$. Its derivative w.r.t. $\mathcal{C}$ is

$$\frac{\partial\mathcal{C}^{(k+1)}(\mathcal{C})}{\partial\mathcal{C}} = -\frac{\partial\mathcal{C}^{(k)}(\mathcal{C})}{\partial\mathcal{C}}\, F[\mathcal{C}] - F^{(k-1)}[\mathcal{C}^{(k-1)}]\;. \tag{B.14}$$

Using Eq.(B.13) we get the recursive equation

$$w^{(k+1)} = w^{(k)} + 1 \tag{B.15}$$

with $w^{(k)} \equiv (-F[\mathcal{C}])^{k-1}\, \partial\mathcal{C}^{(k)}(\mathcal{C})/\partial\mathcal{C}$. The solution is

$$\frac{\partial\mathcal{C}^{(k)}(\mathcal{C})}{\partial\mathcal{C}} = k\,(-F[\mathcal{C}])^{k-1} \tag{B.16}$$

$$\mathcal{C}^{(k)}(\mathcal{C}) = -\int_0^{\mathcal{C}} dy\; \frac{\partial(-F[y])^k}{\partial y}\frac{1}{X_d[y]} \tag{B.17}$$

We now obtain $X_d^{(k)}[\mathcal{C}^{(k)}(\mathcal{C})]$ in terms of $X_d[\mathcal{C}]$. Derivating Eq. (B.13) w.r.t. $\mathcal{C}$

$$X_d^{(k)}[\mathcal{C}^{(k)}(\mathcal{C})]\;\frac{\partial\mathcal{C}^{(k)}(\mathcal{C})}{\partial\mathcal{C}} = k\;\left(-F[\mathcal{C}]\right)^{k-1}\; X_d[\mathcal{C}] \tag{B.18}$$

and inserting the result in Eq. (B.16) we obtain

$$X_d^{(k)}[\mathcal{C}^{(k)}(\mathcal{C})] = X_d[\mathcal{C}]\;. \tag{B.19}$$

We have obtained these results $\mathcal{C}^{(k)} = \mathcal{C}^{(k)}(\mathcal{C})$ and Eq. (B.19) for the powers $\mathcal{Q}^k$. Since these hold for every power k and $C^{(k)}$ is a linear combination of these powers (see Eq. (III.13)) we have proven Eqs. (III.3) and (III.6).

REFERENCES

[1] G. Parisi; J. Phys. **A13**, L115 (1980); *ibid.* 1101 (1980); *ibid.* 1887 (1980).

[2] M. Mézard, G. Parisi and M.A. Virasoro; *Spin Glass Theory and Beyond* (World Scientific, Singapore, 1987).

[3] H. Sompolinsky; Phys. Rev. Lett. **47**, 935 (1981).
H. Sompolinsky and A. Zippelius; Phys. Rev. Lett. **45**, 359 (1981); Phys. Rev. **B25** 274 (1982).

[4] L.F. Cugliandolo and J. Kurchan; Phys. Rev. Lett. **71**, 173 (1993); *Weak ergodicity breaking in mean-field spin-glass models*, cond-mat **9403040**, Phil. Mag. (to be published).

[5] S. Franz and M. Mézard; Europhys. Lett. **26**, 209 (1994)

[6] L.F. Cugliandolo and J. Kurchan; J. Phys. **A27**, 5749 (1994).

[7] S. Franz and M. Mézard; Physica **A209**, 1 (1994).

[8] J-P Bouchaud, E. Vincent and J. Hammann; J. Physique France **4**, 139 (1994).

[9] C. Dasgupta and H. Sompolinsky, Phys. Rev. **B27**, 4511 (1983).

[10] M. Mézard and G. Parisi; J. Physique Lett. **45**, L-707 (1984).

[11] J. Kurchan, J. Phys. France **2**, 1333 (1992).

[12] L. F. Cugliandolo, S. Franz, J. Kurchan and M. Mézard, in preparation.

[13] M. L. Mehta; *Random matrices* (Academic Press, 1967).

[14] J-P Bouchaud; J. Physique **2**, 1705 (1992).
J-P Bouchaud and D. S. Dean; *'Aging on Parisi's tree'*, Saclay preprint, sept. 1994.

[15] An independent numerical check of the out of equilibrium dynamics of the SK model using the Eissfeller-Opper dynamics [16] is under study. G. Ferraro; work in progress.

[16] H. Eissfeller and M. Opper; Phys. Rev. Lett. **68**, 2094 (1992).