Real-world options: smile and residual risk

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Abstract

We present a theory of option pricing and hedging, designed to address non-perfect arbitrage, market friction and the presence of 'fat' tails. An implied volatility 'smile' is predicted. We give precise estimates of the residual risk associated with optimal (but imperfect) hedging.

1 Introduction

In 1973, Black and Scholes [1] developed the mathematical theory of perfectly *hedged* options in an *arbitrage-free* lognormal random walk asset price model. Fundamental for the mathematical formulation and deeply rooted in the mind of both theoreticians and practitioners, evolved the notion that a general strategy of pricing and coverage of derivatives involves the *hedging away of all risks*. However, we do not live in a Black-Scholes world :

- asset prices do not follow a continuous time lognormal processes, but may exhibit large jumps and be distributed with fat tails, in addition to present subtle correlations between increments at different times (the markets are 'incomplete'),

- the replicating portfolio cannot be implemented exactly, since it involves continuous rebalancing: several 'imperfections' such as transaction costs, delays and lack of liquidity necessarily introduce a finite time scale for transactions.

Confronted with these practical problems, many authors have attempted to generalize the Black and Scholes strategy. Here, we present an intuitively appealing formalism which is flexible enough to address these problems efficiently and show how it can be implemented practically. Our method is based on a global (integral) representation of wealth balance, in contrast to Black-Scholes local (differential) approach. The mathematical tool-box is that of functional integration and derivation, in contrast to the standard Itô calculus and partial differential equation formalism. We propose two main new concepts :

- The minimization of the *non zero residual risk* as a criterion to fix an optimal strategy [2]. One should note in this respect that somewhat related ideas have previously been proposed in the mathematical literature in a rather formal way [3]. Furthermore, our hypothesis are more general and the resulting formulae ar e easily implemented numerically.

- For strongly fluctuating assets, option pricing and its hedging is obtained by a 'tail chiseling' of wealth distributions as a treatment of large risks [4].

One of our main thrust is to demonstrate how these ideas can be put in practical terms, using relatively simple mathematics.

Let us discuss it for the basic European call option pricing problem – generalisation to more complicated products does not involve any conceptual difficulties. Suppose that an operator wants to buy a given share, a certain time $t = T$ from now ($t = 0$ at which the current value is $x(t = 0) \equiv x_0$), at a fixed 'striking' price x_c. If the share value at $t = T$, $x(T)$, exceeds x_c, the operator 'exercises' his option, with an immediate profit difference $x(T) - x_c$. On the contrary, if $x(T) < x_c$ the operator may not buy the share. What is the price $\mathcal{C}(x_0, x_c, T)$ of this possibility given to the operator by -say- the "bank", and what trading strategy should be followed by the bank between now and T, depending on what the share value $x(t)$ actually does between $t = 0$ and $t = T$? These are the two questions answered by Black and Scholes in the context of an ideal complete market, and which we wish to consider from a somewhat different point of view adapted to real-world markets.

In common with Black and Scholes, the starting point is to write a wealth balance for the bank. However, instead of computing the *instantaneous* variation of the value of its portfolio (which assumes continuous time), we write a global balance at time T, and assume that time is discrete, with a certain 'microscopic' time scale τ below which trading is impossible (or, in the presence of transaction costs, very unfavorable – see below).

We shall assume for simplicity that the risk-free interest rate r is constant; more involved developments (including a random interest rate) are reported elsewhere. The *global* balance ΔW for the bank between $t = 0$ and $t = T$ is thus given by:

$$\Delta W = \mathcal{C}(x_0, x_c, T)\exp(rT) - \max(x(T) - x_c, 0) +$$

$$+ \sum_{t=0}^{T} \phi(x,t)\exp(r(T-t))[\Delta_\tau x - r x(t)] \qquad (1),$$

with x_c the striking price, x_0 the initial ($t = 0$) share price, and $\phi(x,t)$ is the number of shares owned by the bank at time t, knowing that the share price is x. $\Delta_\tau x$ is the difference of share values between times $t + \tau$ and t. $\mathcal{C}(x_0, x_c, T)$ is the looked for option price. Expression (1), which in the continuous case is in fact the time-integrated version of Black and Scholes's differential equation, has a very intuitive meaning :

- The first term is the gain from pocketing from the buyer the option price at $t = 0$, discounted at time T.
- The second term gives the potential loss equal to $-(x(T)-x_c)$ if $x(T) > x_c$ (i.e. if the option is exercized) and zero otherwise.
- The third term quantifies the effect of the trading between $t = 0$ and $t = T$: the true variation of wealth W between t and $t+\tau$ is only due to the fluctuations of the share price, i.e. $\phi(x,t)\Delta_\tau x$, corrected by the fact that $x\phi(x,t)$ has not benefited from the risk-free interest rate. (Note that the term $x\Delta_\tau \phi$ describes conversion of shares into other assets or the reverse, but not a real change of wealth).

The bank wealth variation ΔW depends a priori on the specific realization $\{x(t)\}|_{t=0-T}$ of the asset price. For a continuous log-normal process, the result of Black and Scholes is that ΔW is strictly vanishing for $\tau \to 0$ for a particular choice of $\phi(x,t)$ and $\mathcal{C}(x_0, x_c, T)$. Equivalently, both equalities $\langle \Delta W \rangle = 0$ and $\langle (\Delta W)^2 \rangle = 0$ hold, where $\langle ... \rangle$ denotes the average over all possible different realisations of the history $\{x(t)\}$. In other words, the averages $\langle ... \rangle$ are taken over the initial historical measure in which the drift is not in general equal to the risk-free interest rate. The first condition $\langle \Delta W \rangle = 0$ is the 'no free lunch' condition. The second expression simply writes that the square of the bank wealth volatility $\langle \Delta W^2 \rangle$ is zero which ensures the hedging away of all risk. This is just another view point to retrieve Black and Scholes results, holding true for any quasi-Gaussian, time-continuous processes [2].

2 Risk-corrected option prices

In the real world where ΔW cannot be made to vanish exactly, we propose to apply the spirit of Markowitz portfolio approach to the total bank wealth balance, viewed as an effective portfolio estimated at the time T of the death of the option. In this goal, one has to calculate the average return $\langle \Delta W \rangle$ and the second moment (related to the square of the bank wealth volatility) $\langle \Delta W^2 \rangle$ when they exist (the case of very strongly fluctuating 'Lévy processes' with ill-defined theoretical volatility will be addressed below). For the time being, we assume that both $\langle \Delta W \rangle$ and $\langle \Delta W^2 \rangle$ exist. The optimal strategy $\phi^*(x,t)$ that the bank must follow should certainly be such that the uncertainty on the outcome is minimum. The reason for this is that the bank will adjust the option price according to its risk aversion, for instance with the criterion

$$\langle \Delta W \rangle|_{\phi=\phi^*} = \lambda \sqrt{\mathcal{R}^*}, \tag{2}$$

where $\mathcal{R}^* \equiv \langle \Delta W^2 \rangle - \langle \Delta W \rangle^2|_{\phi=\phi^*}$ is the square of the volatility of the bank wealth. Eq. (2) simply means that the bank wishes to keep the probability of global loss incurred by delivering the option at a certain level – say 10%. The minimisation of $\mathcal{R}^*$ is thus necessary to keep the option price itself as low as possible. Alternatively, the knowledge of $\langle \Delta W \rangle|_{\phi=\phi^*}$ and $\sqrt{\mathcal{R}^*}$ allows one to extract from the market price of a given option the value of its 'λ' (a yet unused Greek letter). λ can be thought of as an objective (dimensionless) measure of the real option price, in units of the real risk associated to the underlying asset – high λ's corresponding to expensive options. Note that the term in the right hand side of eq.(2) allows the bank to introduce a bid-ask offer range around a central 'fair' price which itself is independent of λ. This bid-ask offer range is dependent upon the bank portfolio and its risk-aversion.

2.1 Independent increments of the share value

Let us suppose that the local slopes $\Delta_\tau x$ are statistically independent (but not necessarily Gaussian) for different times. In this case, both $\langle \Delta W \rangle$ and the 'risk' measured as $\mathcal{R}[\phi(x,t)] = \langle (\Delta W)^2 \rangle - \langle \Delta W \rangle^2$ can be explicitely calculated. We can then determine the optimal strategy $\phi^*(x,t)$ through a 'functional minimisation':

$$\frac{\partial \mathcal{R}[\phi(x,t)]}{\partial \phi(x,t)}|_{\phi=\phi^*} = 0. \tag{3}$$

from which a general expression of both $\phi^*(x,t)$ and of the residual risk $\mathcal{R}^* = \mathcal{R}[\phi^*(x,t)]$ can be derived [2] – see the 'Technical sheet' below). These results are valid for an *arbitrary* stochastic process with uncorrelated increments, including 'jump', or discrete-time, processes. The process can furthermore be explicitely time-dependent, with a variance which is a function of time, as for the much studied 'ARCH' processes, which are Gaussian processes with a t-dependent variance $\mathcal{D}(t)$.

The obtained formulae can be simplified in several cases [2] and allows us to retrieve Black and Scholes' result when $\Delta_\tau x$ are Gaussian variables and τ tends to zero (continuous process). In particular, one finds indeed that $\mathcal{R}^*$ vanishes exactly in the continuous time limit. One may also check that in that case, the results on the option price and the optimal strategy are indeed *independent* of the average return of the share.

The condition $\mathcal{R}^* = 0$ is very specific to the Gaussian can and does *not hold* for a more general stochastic process. This is the main difference between the present approach and that of Black and Scholes and subsequent workers (see however [3]):

1) we find that a vanishing residual risk cannot be achieved in the general case and are able to quantify it precisely (see Fig 2 below);

2) however, this does not imply that an optimal strategy does not exist. We have indeed found an optimal $\phi^*(x,t)$ which minimize the risk and which is a natural generalization of Black and Scholes result.

Note finally that other definitions of the risk are possible, for example through higher moments of the distribution of ΔW. The functional minimisation technique presented here can easily be adapted to these cases – although the calculations are more cumbersome.

2.2 Illustration: options on the MATIF

These findings are relevant to various concrete situations. In particular, strong deviations from a Gaussian behaviour ("leptokurtosis") are often observed in many situations. In this case, the above method allows one to estimate quantitatively the residual risk and correct the trading strategy and the option price accordingly. We present an illustration by comparing the option price obtained from the proposed functional quadratic minimization procedure and from Black and Scholes formula applied on MATIF options, with a 30 days maturity ($T = \frac{30}{365}$ days). In this goal, we have determined the historical conditional distribution $P(x',t'|x,t)$ from the MATIF daily quotation variations in the period 1990-1992. We have tested a certain degree of stationarity of the data by constructing the distribution of price increments over various time intervals. We then determined $P(x',t'|x,t)$ by scanning the data set and using the assumption of stationarity implying that

$P(x',t'|x,t) = P(x'-x,t'-t)$. In order to apply the Black and Scholes results, we have fitted $P(x'-x,t'-t=1)$ by a log-normal distribution, finding a volatility $\sigma^2 \simeq 10^{-5}$ per day. $P(x'-x,t'-t=1)$ and its log-normal fit is shown in Fig.1. We can also apply directly our formulae using the empirically determined distribution and obtain the option price $\mathcal{C}(x_0,x_c,T)$ as a function of the striking price x_c (Fig 2) in a risk insensitive world ($\lambda = 0$), and compare it to the Black-Scholes price with the historical volatility (dotted curve). As is well known, this procedure underestimates the 'true' price: the presence of 'tails' in the distribution induces a larger effective volatility. One can in fact invert the Black-Scholes formula and determine an implied volatility from our determination of the price. The result is given in the inset of Fig 2 and reproduces the well known 'smile': the effective volatility is stronger for 'out of the money' options. More importantly, the residual risk $\mathcal{R}^*$ is *non-zero*, it furthermore depends on the trading 'frequency', i.e. the number of 'inactive' days τ without rehedging. As expected, the larger this number of days, the larger the risk : we have plotted in Fig 3 the quantity $\sqrt{\mathcal{R}^*}$ as a function of τ, together with the total transaction costs $\mathcal{K}$ associated with these rehedging, assuming that each trading has a cost equal to 0.05% of the share's value. The full curve represents the sum of these two sources of extra-cost: $\overline{\mathcal{K}} = \sqrt{\mathcal{R}^*} + \mathcal{K}$ is the risk-aversion cost ($\lambda = 1$) plus the transaction costs. Interestingly, $\overline{\mathcal{K}}$ has a *minimum*, which is the optimal trading rate ($\tau \simeq 10$ days for this particular choice of parameters). Due to the impact of transaction costs, it appears that a daily rehedging strategy is not reasonable – unless λ is very large (strong risk aversion). Note that $\overline{\mathcal{K}}$ represents an appreciable fraction of the 'fair' price.

Finally, we have compared in Fig 3 (Inset) the residual risk for our optimal strategy with the residual risk obtained following the Black-Scholes strategy. As expected, the latter is larger: the risk can be *substantially reduced* using our 'optimal' strategy.

2.3 Rare events and fat tails

Our method can be adapted to many situations: more exotic options, or more complicated stochastic processes, such as correlated Brownian motions [2] or Lévy processes, which have been argued by many authors [5] to be adequate models for short enough time lags, when the kurtosis is large. The latter case is interesting since the notion of variance becomes ill-defined due to the presence of extremely large fluctuation (crashes). The above criterion fixing the optimal strategy $\phi(x,t)$, based on a minimization of the variance is thus meaningless. Intuitively, this comes from the fact that the variance is dramatically sensitive to large price variations. This indicates that moments are not sufficient anymore to capture the informations contained in the price and wealth distribution : one must study directly the distributions themselves, and more precisely the tails of the distribution $P(\Delta W)$ of the wealth variation ΔW. By the laws of composition of Lévy laws, one can show that $P(\Delta W)$ decays as $\frac{W_0^\mu[\phi(x,t)]}{|\Delta W|^{1+\mu}}$ for large losses ($\Delta W \longrightarrow -\infty$), with W_0 depending on $\phi(x,t)$. We then propose to determine the optimal strategy $\phi^*(x,t)$ by the condition $\frac{\delta W_0}{\delta\phi(x,t)} = 0$ - corresponding to a minimization of the 'catastrophic' risks, since W_0 controls the *scale* of the distribution of losses, i.e. their order of magnitude.

The full derivation of the solution of this minimisation problem will be presented elsewhere; here we restrict to the simple 'variational' ansatz where $\phi(x,t)$ is taken to be a constant ϕ, independent of x and t, to be optimized. Using Eq.(1), we need to calculate the distribution $P(\Delta W)$ of large losses. ΔW can take large negative values when

- $x(t)$ drops dramatically : the option is not realized ($\max(x(T)-x_c, 0) = 0$) but the bank looses $(x(T)-x_0)\phi$ due to its hold position. Taken alone, this favors $\phi \to 0$.

- $x(t)$ increases much above x_c : the bank has to produce the share and thus loses $-(x(T)-x_c)$ but partially compensates this loss by its holding of $(x(T)-x_0)\phi$, thus resulting in a net loss of $-(1-\phi)(x(T)-x_0)$. Taken alone, this situation favors $\phi \to 1$.

There is thus a trade-off between the two possibilities, leading to a non-trivial optimal ϕ.

In order to obtain it, we write the distribution $P(\Delta W)$ of large losses as the sum of these two independent processes and get

$$W_0^\mu = (1-\mathcal{P})C_-\phi^\mu + \mathcal{P}C_+(1-\phi)^\mu. \tag{4}$$

where $C_\pm$ describe the 'tails' of the historical distribution:

$$P(x,T|x_0,0) \simeq_{x-x_0\to\pm\infty} \frac{C_\pm}{|x-x_0|^{1+\mu}}.$$

$\mathcal{P}$ is the probability at $t = 0$ that the option will be exercised. Minimisation of W_0 with respect to ϕ yields

$$\phi^* = \frac{[C_+\mathcal{P}]^\zeta}{[C_+\mathcal{P}]^\zeta + [C_-(1-\mathcal{P})]^\zeta} \qquad \zeta \equiv \frac{1}{\mu - 1} \tag{5}$$

A more complete treatment with an arbitrary $\phi(x,t)$, in fact leads to the same result given by eq.(5), but with a *time dependent* $\mathcal{P} = \int_{x_c}^{\infty} dy' P_0(y', T|y, t)$, which is the probability at time t at which the stock price is y that the option will be exercized (at T).

If $(1-\mathcal{P})C_- \ll \mathcal{P}C_+$ (resp. $(1-\mathcal{P})C_- \gg \mathcal{P}C_+$), $\phi \to 1$ (resp. 0), as is natural since then one of the two histories dominate. If both have the same weight, the intuitive result $\phi^* = \frac{1}{2}$ is recovered. The resulting optimal 'scale' $W_0(\phi^*)$ is now the correct measure of the risk which must be added to the risk neutral price with a coefficient depending on risk aversion.

The case where $\mu < 1$, for which the mean becomes itself ill-defined, has been discussed in [2]. Let us simply mention that the optimal strategy in that case is either $\phi^* = 0$ or 1. Finally, in the border case $\mu = 2$ (asymptotically attracted to the 'Gaussian' stable law), with symmetric tails $C_+ = C_-$, we recover $\phi^*(x,t) = \mathcal{P}(x,t)$, which is precisely Black and Scholes's result.

3 Conclusion

We have thus proposed a pragmatic and flexible theory of option pricing, which naturally generalizes the Black-Scholes results to intrinsically risky markets. Our method allows one to decompose in readable way the option price into an irreducible part corresponding to a 'fair game' condition *plus an extra-cost* $\overline{\mathcal{K}}$ which includes the residual risk associated with trading and transaction costs. The risk part is minimized in a variational way with respect to the strategy. A well defined trading frequency appears, as a trade-off between risk and transaction costs. Our precedure is furthermore relatively easy to implement numerically, and we hope that it will prove useful to professionals.

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Figure Captions.

Fig. 1. : Historical distribution of the daily relative variation $\frac{\Delta x}{x}$ for the MATIF daily quotation variations in the period 1990-1992 and comparison with the best fit to a normal distribution (dotted curve). As is by now familiar, the empirical distribution has clearly visible 'fat tails' and a sharper maximum. For example, the kurtosis of the distribution is equal to 28.1 i nstead of 3 for a normal distribution.

Fig.2 : a) Comparison between the option price obtained from Eq.(8) with $\lambda = 0$ and assuming independent increments (full line) and from the Black and Scholes formula with the historical volatility (dotted line), applied on MATIF options with 30 days maturity (with striking price higher than the daily quotation $x_0 = 100$) and put (with striking price lower then $x_0 = 100$) options. As well known, Black and Scholes formula underestimates option prices when using the historical volatility. . The difference between the two curves is significantly larger than the error bars from our numerical implementation of the analytical formula based on historical data.

b) Inset: 'Implied volatility' obtained by inverting the Black-Scholes formula. The well known 'volatility smile' is reproduced.

Fig 3. : a) Risk aversion cost $\sqrt{\mathcal{R}^*}$ (dashed line) and total transaction cost $\mathcal{K}$ (dotted line) associated with rehedging as a function of trading time τ, for a 30 days option with $x_c = x_0$. We assume that trading costs 0.05% of the 'share' value. The sum of the two curves (full line) represents the total extra cost $\overline{\mathcal{K}}$ that should be considered to adjust the option price when the risk aversion coefficient $\lambda = 1$. Notice that $\overline{\mathcal{K}}$ has a minimun for $\tau \simeq 10$ days (indicated by the cross) which is, for this choice of parameters, the optimal trading time. Total extra cost at $\tau = 10$ is .77 which represents an appreciable fraction of the 'risk-neutral' price.

b) Inset: Comparison between the residual risk corresponding to our optimal strategy (full line) and the Black and Scholes strategy (dotted line). As expected the latter is larger.

Inset: Technical sheet.

In order not to overload the main text with mathematical formulae, we give in this separate inset the most important final equations used for our numerical implementation, where we have neglected interest rate effects (i.e. $r \equiv 0$). The optimal strategy $\phi^*(x,t)$ is given by:

$$\phi^*(x,t) = \int_{x_c}^{\infty} dx' \langle \frac{dx}{dt} \rangle_{(x,t)\longrightarrow(x',T)} \frac{(x'-x_c)}{\mathcal{D}(x)} P(x',T|x,t) \qquad (I.1)$$

where $P(x',t'|x,t)$ is the probability that the value of x' occurs (within dx') at t', knowing that it was x at $t \le t'$, $\langle \frac{dx}{dt} \rangle_{(x,t)\longrightarrow(x',T)}$ is the mean instantaneous increment conditioned to the initial condition (x,t) and a final condition (x',T), and $D(x)$ is the local volatility. The residual risk $\mathcal{R}^*$ then reads:

$$\mathcal{R}^* = \mathcal{R}_c - \sum_{t=0}^{T} \int_{-\infty}^{+\infty} dx \mathcal{D}(x) P(x,t|x_0,0) \phi^{*2}(x,t), \qquad (I.2)$$

where $\mathcal{R}_c$ is the "bare" risk which would prevail in the absence of trading ($\phi(x,t) \equiv 0$):

$$\mathcal{R}_c = \left[\int_{x_c}^{\infty} dx (x-x_c)^2 P(x,T|x_0,0)\right] - \left[\int_{x_c}^{\infty} dx (x-x_c) P(x,T|x_0,0)\right]^2 \quad (I.3)$$

. Finally, the option price is determined using Eq. (2) as:

$$\mathcal{C}(x_0,x_c,T) = \left[\int_{x_c}^{+\infty} dx' P(x',T|x_0,0)(x'-x_c) + \lambda\sqrt{\mathcal{R}^*}\right] \qquad (I.4)$$

which allows one to obtain numerically the option price, once $P(x',t'|x,t)$ is reconstructed, which we do using the histogram of daily variations plus the assumption that these daily increments are uncorrelated.

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