Crossover behaviour of a one dimensional Random Energy Model.

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In this note we formulate a finite dimensional generalization of the Random Energy Model (REM) where we introduce a geometry and spatial correlations between energies. We study the model in dimension one by transfer matrix techniques and we look at the crossover from one dimensional to mean-field behaviour. In a first version of the model the mean field limit reproduces the behaviour of the original REM, while a second version of the model exhibits a first order phase transition with a finite latent heat.

The Random Energy Model (REM) [1,2] is a very simple mean field spin glass model, that can be exactly solved. The model, in spite of its extreme simplicity, captures many features of other, more complicated, spin glass models. The REM belongs to the class of spin glasses which present a phase transition basically due to an entropy collapse phenomenon. In other words, at low enough temperature, the system finds no states with energies below a given value and remains stuck in the lowest state. Below a critical value of the temperature the model therefore freezes into a state of minimal energy and zero entropy. This transition is rather peculiar: even though it involves no latent heat, it does not show any precursor effect (no divergent susceptibility), sharing therefore some features typical of first order transitions and others typical of second order ones. This behaviour is qualitatively common to a whole class of mean field models of spin glasses (e.g. the $p>2$-spin model [3,4], the Potts model) [5,6] which show a discontinuous one-step replica symmetry breaking transition (1RSB) [7]. An interesting observation is that models of this class seem to be a good paradigm to describe structural glasses where no disorder is explicitly present in the Hamiltonian but there is an effective, self-induced, randomness [8–10]. The aim of this work is to provide some insight on what happens to these models when considered in finite dimension. Some work on this line of research has been contemporarily done on a short ranged p-spin glass model above the lower critical dimension [11–13]. In this note we shall introduce a finite dimensional generalization of the REM which includes a spatial dependence of the variables. We will study in detail the properties of the model in one dimension where no phase transition occurs. Nevertheless the model is defined in such a way that, in the limit of a parameter M to infinity, the mean field solution is recovered. So we will observe the crossover towards mean field behaviour as M is increased.

The model will be formulated in two versions, which have an interestingly different behaviour.

In the standard REM one considers a system of 2^N levels with energies which are random independent variables extracted from a Gaussian distribution

$$P(E) \sim \exp(-E^2/N). \qquad (1)$$

The energy levels of the system can be thought as correspondent to the configurations of N Ising spins.

Note that we have not specified any microscopic variable for this model. This is the natural consequence of the hypothesis that in this model the energies are totally uncorrelated from the microscopical configurations which are now only labels of the energy level and are here indicated by the index i.

Usually the limit $N \to \infty$ is considered, where the system freezes into a state of zero entropy. This is easily seen by the following argument. The average number of configurations with total energy between E and $(E+\delta E)$ is

$$\overline{n(E)} = 2^N e^{-\frac{E^2}{N}}, \qquad (2)$$

where the bar indicates the average over $P(E)$. In the large-N limit, for $|E| > E_0 = N\sqrt{\ln 2}$, the entropy of the system is

$$\ln\left(\overline{n(E)}\right).$$

For energies such that $|E| > E_0$ the exponent becomes negative and for large N there are no energy levels. In this case the system is frozen into its ground state with zero entropy. Introducing the temperature by

$$T \equiv \left(\frac{\partial S(E)}{\partial E}\right)^{-1}$$

and inverting this relation one obtains the free-energy

$$F = \begin{cases} -N\left(T\ln 2 + \frac{1}{4T}\right) & \text{for } T > T_c = \frac{1}{2\sqrt{\ln 2}} \\ -N\sqrt{\ln 2} & \text{for } T < T_c \end{cases} \qquad (3)$$

At the critical temperature the saddle point solution changes discontinuously and one would say that the transition is first order. Furthermore, there are no physical quantities that diverge at T_c. Nevertheless the free energy is differentiable at the critical temperature and no latent heat is involved in the transition.

In the following we shall try to learn if the features of this peculiar transition are a property of the adimensional case and how this transition appear when the model is generalized to finite dimension.

The REM can be generalized by introducing a geometry and a spatial correlation between the energy levels.

The Dimensional Random Energy Model (DREM) can be formulated in general dimension and the mean field solution can be found using analytical arguments. For finite M we analyze the model in $d=1$ by transfer matrix techniques and study the crossover from one dimensional to mean-field behaviour.

The DREM will be formulated in two versions, which have quite different mean field limit. An interesting question is whether a growing correlation length develops or not for increasing M.

A first version of the model shows, in the large-M limit, a transition with no latent heat similar to the case of the REM. A second version of the model exhibits a first order phase transition into a crystalline state with a discontinuity in the specific heat.

The model is defined in the following way: we consider a d-dimensional square lattice of side L with M spins on each site, in the limit of $L \to \infty$. So $V = L^d$ is the number of sites and MV is the total number of spins of the model.

Let us consider the link $(i \to i+\hat{\mu})$ between site i and a nearest-neighbour site $i+\hat{\mu}$ where $\hat{\mu}$ is a positive unit lattice vector.

To each of the 2^{2M} possible configurations of the spins at the edges of the link $i \to i+\hat{\mu}$ we associate a random energy extracted from the probability distribution

$$P(E(\sigma,\tau)) = \frac{1}{\sqrt{M\pi}} \exp\left[\frac{-E^2(\sigma,\tau)}{M}\right]. \tag{4}$$

The possible energy levels of a link are therefore 2^{2M} independent numbers extracted from a Gaussian distribution. The partition function of the model is

$$Z = \sum_{\{s\}} \exp\left(-\beta \sum_{i}^{V} \sum_{\hat{\mu}} E_{i,i+\hat{\mu}}(s_i, s_{i+\hat{\mu}})\right), \tag{5}$$

$E_{i,i+\hat{\mu}}(s_i, s_{i+\hat{\mu}})$ being the energy of link $i \to i+\hat{\mu}$. Note that, to avoid double counting, we for each site i we consider only the d nearest neighbours taken along the positive versus of each direction. A possible version of the model consists in taking the energies of different links as independent variables, in which case we have a non translational (NTI) invariant spin glass. This is very similar to the short range p-spin model introduced in [11–13] in the large-p limit, where the energies are uncorrelated. The only difference, which should not be very important, is that this model does not account for the interactions between spins which are on the same site. A second possible version is a translational invariant (TI) model *, in which the correspondence between the spin configurations and the possible energy levels is space independent. This means that for each sample one assigns a law $(\sigma,\tau) \to E(\sigma,\tau)$ extracting the values of the energy from (4).

Both versions of the model can be formulated in a symmetric and non-symmetric way *i.e.* one can impose or not impose the following symmetry condition which reduces by half the number of independent energy levels

$$E_{i,i+\hat{\mu}}(\sigma,\tau) = E_{i,i+\hat{\mu}}(\tau,\sigma). \tag{6}$$

In the following we will study in detail the NTI and the symmetric translationally invariant model (STI). While the first model is rather natural in a spin glass context, the second is more attractive in connection to the modeling of structural glasses. In this latter model, in fact, if the lattice is chessboard decomposable, the system has a crystalline ground state. Nevertheless, frustration due to the presence of the disorder, makes the minimization of the (free) energy a hard task, and the system may eventually fall in a glassy state.

Both versions of the model can be easily solved in the large-M limit. This is quite trivial in the case of the non-translationally invariant (NTI) DREM since one basically recovers the REM. This is clear because for large M one can consider the energies of the links essentially as uncorrelated so the average number of configurations with total energy between E and $(E+\delta E)$ is

$$\overline{n(E)} = 2^{MV} e^{-\frac{E^2}{MV}}. \tag{7}$$

Applying Derrida's standard microcanonical argument on the total energy of the system one obtains a critical temperature of $T_c = 1/(2\sqrt{\ln 2})$ and a ground state total energy $E_0 = -MV\sqrt{\ln 2}$. This is evidently the mean field solution of the model since, for large M, each spin interacts with a large number of nearest-neighbours. The high temperature free energy density of the model is therefore the same as that of the REM and at T_c the model freezes into its ground state.

The translational invariant model behaves quite differently even at mean field level if formulated with the further condition of symmetry (6).

*If $d>1$ the model would have to be also rotationally invariant for our following consideration to be true. With the "TI" we will therefore mean also rotational invariant if $d>1$.

The STI-DREM has, in fact, a state of lower energy than $-ML\sqrt{\ln 2}$. This can be understood by considering the possible energy levels of one single link. The set of the energies of each single link is a REM with 2^{2M} energy levels and has therefore a ground state energy $E_{i,i+\hat{\mu}}(\sigma_i^0, \tau_{i+\hat{\mu}}^0) = -M\sqrt{2\ln 2}$. This is true also for the NTI model or for the TI model without the symmetry condition, but the choice of the configuration $\tau_{i+\hat{\mu}}^0$ on site $i+\hat{\mu}$ to minimize $E_{i,i+\hat{\mu}}$, in general does not allow $E_{i+\hat{\mu},i+\hat{\mu}+\hat{\nu}}$ to be minimized. Here $i+\hat{\mu}+\hat{\nu}$ is a general nearest neighbour site of $i+\hat{\mu}$. The reason for this is that if the disorder is different from link to link and there is no condition assuring that the ground state of $E_{i+\hat{\mu},i+\hat{\nu}}$ will correspond to the configuration $\tau_{i+\hat{\mu}}^0$ on site $i+\hat{\mu}$. In fact the TI without the symmetry condition can crystallize on a periodic state only in those samples having $\tilde{E}_0 = E(\sigma_i^0, \sigma_{i+\hat{\mu}}^0)$ so the energy of each link may again be in the ground state. For the STI-DREM the picture is quite different: if the lattice is chessboard decomposable, the spins can always arrange themselves in a structure that alternates in space the minimizing configurations $\sigma_i^0, \tau_{i+\hat{\mu}}^0$. In this way, contrarily to the NTI model, every link is in its true ground state. This yields a total energy $\tilde{E}_0 = -MV\sqrt{2\ln 2}$ which is lower than E_0. The freezing into this true ground state will happen at a temperature $\tilde{T}_c$ at which the high temperature free energy reaches the value $\tilde{E}_0$. One has

$$\tilde{T}_c = \frac{1+\sqrt{2}}{2\sqrt{\ln 2}}. \tag{8}$$

This transition is first order and the latent heat is

$$C_{lat} = \sqrt{\ln 2}. \tag{9}$$

Below the lower critical dimension the phase transition disappears when M is finite. We do not know at present what the critical dimension is, but we know that it has to be larger than one. However, even in dimension one, it is interesting to study the crossover from smooth to sharp behaviour when M is increased.

For finite values of M we analyzed the model in one dimension by transfer matrix. For each link i of the model we have a $2^M \times 2^M$ transfer matrix $\hat{T}_i$. For the translationally invariant model one has

$$\hat{T}_i \equiv \hat{T},$$

and it is easy to show, by standard transfer matrix arguments, that, in the limit of an infinite chain ($L \to \infty$), one can calculate the free energy density and the correlation length by the following identities

$$-\beta F = \lim_{L\to\infty} \frac{1}{L} \ln(t_1) \equiv \lambda_1,$$
$$\xi = \left(\ln\left|\frac{t_1}{t_2}\right|\right)^{(-1)}, \tag{10}$$

where t_1 and t_2 are respectively the first and second largest (in modulus) eigenvalues of $\hat{T}$ and λ_1 is called *maximum Lyapunov exponent* .

In the case of the NTI model one has to consider the product of the sequence of L transfer matrices $\mathbf{P_L} = \prod_{i=1}^{L} \hat{T}_i$ and define the correspondent hermitian matrix

$$\mathbf{V_L} \doteq \left(\mathbf{P_L}^{+}\mathbf{P_L}\right).$$

For ($L \to \infty$) the free energy density of the model can be calculated by making use of some central limit theorems for products of random matrices.

More precisely one can say (Fustemberg theorem) that the following limit

$$-\beta F = \tilde{\lambda}_1 = \lim_{L\to\infty} \frac{1}{L} \overline{\ln||\mathbf{P_L}||} \tag{11}$$

exists with probability one.

$\tilde{\lambda}_1$ is called *maximum Lyapunov characteristic exponent* and is a positive non-random quantity [14]. One can define a whole set of characteristic Lyapunov exponents

$$\tilde{\lambda}_i \equiv \lim_{L\to\infty} \frac{1}{2L} \ln(\tilde{t}_i), \tag{12}$$

where the $\tilde{t}_i$ are the eigenvalues of $\mathbf{V_L}$. Similarly to the TI case a correlation length can then be defined

$$\xi = \left(\tilde{\lambda}_1 - \tilde{\lambda}_2\right)^{(-1)}. \tag{13}$$

We computed the first and second Lyapunov characteristic exponent by means of the method developed by Benettin et al. [14].

The results obtained by transfer matrix for finite M are summarized in figure (1-5). The values of the free energy per link $F(T)$ are slightly different from the $M \to$ infinity values above but one can verify that they are consistent with them. For finite M a given sample of the TI model is more likely to freeze at higher temperature than the mean field value, and the free energy is always well above the mean field curve.

In figure (1) and (2) we plot the free energy density per link of the STI-DREM and of the NTI-DREM in function of the temperatures and for different values of M.

In the first figure we plot the sample-averaged free energy, while the second needs no average in virtue of the Fustemberg theorem. In figure (1) the errors are of the same order of magnitude of the pointsize. In figure (2) there is a small numerical imprecision which, we reckon, is responsible for the slight wiggling of all the curves for the NTI case.

We note that the non-translational invariant models as well as the non symmetric translational invariant follow quite well the mean field theoretical prediction already for quite small M. In figure (1) and (2) we also potted the lowest energy state for different values of M. The

TI model succeeds in freezing right in the lowest energy by arranging itself on the configuration of period 2. The NTI model does not reach its lowest energy state and freezes only at lower temperatures with a higher value of the energy.

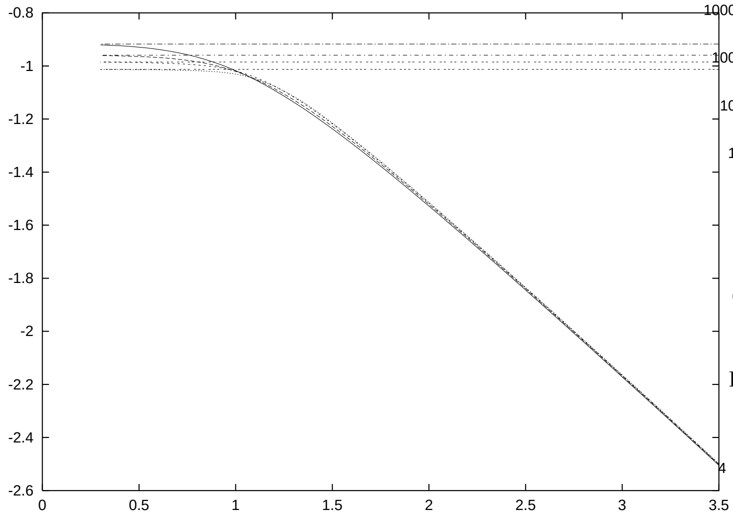

FIG. 1. Free energy of the TI $d = 1$ REM vs temperature for $M = 4, 5, 6, 7$. The horizontal lines are the ground state energies

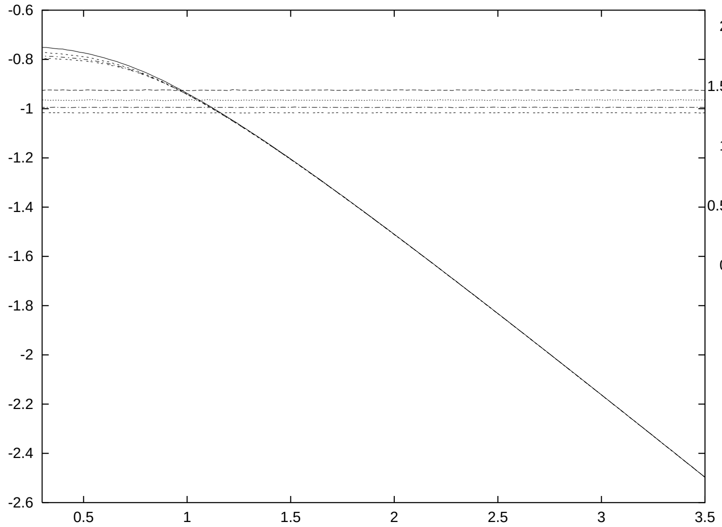

FIG. 2. Free energy of the NTI $d = 1$ REM vs temperature for $M = 4, 5, 6, 7$. The horizontal lines are the ground state energies.

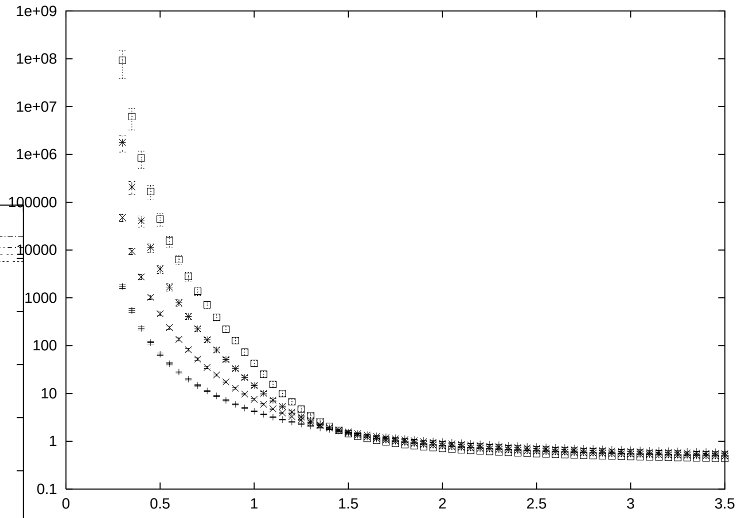

FIG. 3. Correlation length of the TI $d = 1$ REM vs temperature for $M = 4, 5, 6, 7$.

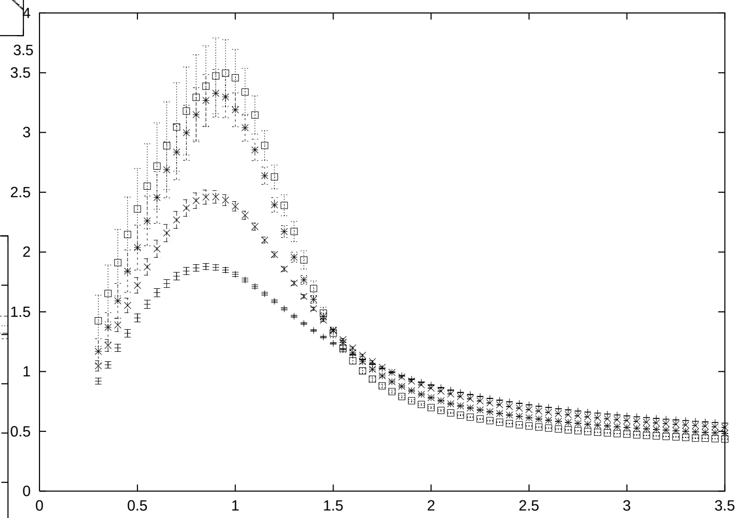

FIG. 4. Ferromagnetic correlation length of the of the TI $d = 1$ REM vs temperature for $M = 4, 5, 6, 7$.

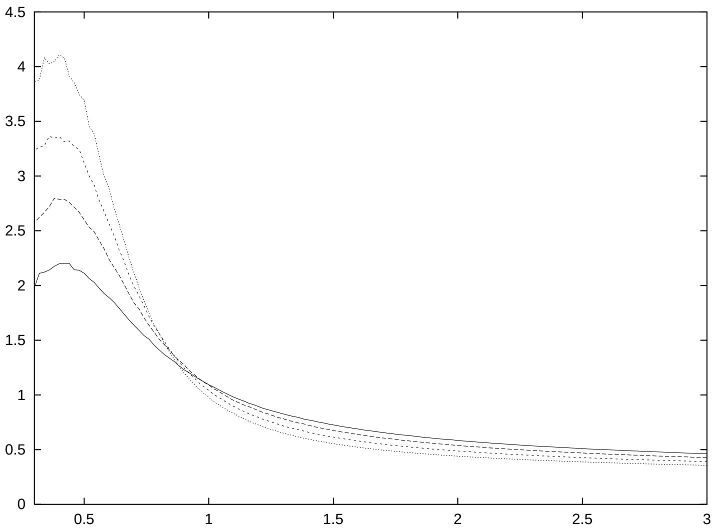

FIG. 5. Correlation length of the NTI $d = 1$ REM vs temperature for $M = 4, 5, 6, 7$.

In figure (3) and (5) we plot the correlation lengths of the models.

For the TI model we averaged the logarithm of the correlation length for different samples. One notices that at $\tilde{T}_c$ the curves for various values of M separate consistently and are steeper and steeper the larger M is. Here one sees the effect of the crossover with the mean field limit since one could imagine a discontinuity at $\tilde{T}_c$ for infinite M. For the TI model we also defined a 'ferromagnetic' correlation length by computing

$$\xi_f = \left(\ln\left|\frac{t_1}{t_f}\right|\right)^{(-1)}, \tag{14}$$

where t_f is the second maximum *positive* eigenvalue. At low temperatures ξ_f does not coincide with the real ξ because there is usually a negative eigenvalue than is larger in modulus than t_f.

It can be seen that $\xi_f \equiv \xi$ only in those samples (that occur with frequency $O(1/M)$) in which the ground states happens to be on the diagonal of the matrix $E(\sigma,\tau)$. The second eigenvalue for the most of the samples is negative because it detects an anti-ferromagnetic ordering. The reason of this is that $\tilde{E}_0 = E(k_0, k_0)$ implies that two nearby sites tend to be have the same configuration of spins in the lowest energy state. The opposite happens in the anti-ferromagnetic case. As it can be seen from figure (4), the ferromagnetic correlation length does not diverge at zero temperature but shows a peak, whose height grows with M, in correspondence of the transition temperature $\tilde{T}_c$. In figure (5) we plot the correlation length of the non-translationally invariant model. Consistently with our mean field predictions, the peak, which shows the crossover with the mean field behaviour, seems to predict the MF critical temperature T_c and not $\tilde{T}_c$. So this one-dimensional REM can give us a slight idea on what happens when the models that in MF present a discontinuous 1RSB transition are generalized to finite dimension.

The speculation that one could make from the results obtained in this work is the following: if the model is provided with an underlying crystalline ground state, the transition in finite dimension becomes a real first order transition with a finite latent heat; if there is no underlying crystalline state, the transition does not seem to show any discontinuity on the first derivative of the free energy as a second order transition (for the possibility of the arousal of divergent correlations see [11–13]).

The numerical exact solution in dimension $d = 1$ represents a complementary approach to what was done in [11] where one started from the MF solution. There is still much work that has to be done on the subject.

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